Mean oscillation and commutators of singular integral operators

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0. Introduction

Let T be a Calderón—Zygmund transform

$$Tg(x) = \text{P.V.} \int_{R^d} K(x-y)\, g(y)\, dy$$

where the kernel K is homogeneous of degree $-d$, i.e. $K(x)=|x|^{-d}K(x/|x|)$, $\int_{S^{d-1}} K=0$ and K satisfies some smoothness condition. $K\in C^\infty(S^{d-1})$ will always be sufficient. For the theory of these transforms, see e.g. Stein [7]. We need the result that T is bounded on L^p, $1<p<\infty$. K and T will be fixed throughout the paper and not identically zero.

Let f be a function on R^d, and let it also denote the operation of pointwise multiplication with f. We will study the commutator $[f, T]$ denoted by C_f.

Formally

$$\begin{aligned} C_f g(x) &= fTg(x) - Tfg(x) \\ &= f(x)\int K(x-y)\, g(y)\, dy - \int K(x-y) f(y)\, g(y)\, dy \\ &= \int (f(x)-f(y))K(x-y)\, g(y)\, dy. \end{aligned}$$

For these formulas to make sense, f has to be locally integrable. $C_f g$ is then defined a.e. as a principal value for g bounded and with compact support. C_f may be extended to all of L^p when we have proved it to be continuous. $C_f g$ is clearly bilinear.

Let Q be any cube in R^d. We define f_Q, the mean value of f on Q, as

$$|Q|^{-1}\int_Q f(x)\, dx$$

and $\Omega(f, Q)$, the mean oscillation of f on Q, as

$$|Q|^{-1}\int_Q |f-f_Q|\, dx.$$

$|Q|$ is the Lebesgue measure. BMO is the space of all functions of bounded mean oscillation, i.e. $f \in BMO$ if and only if $\Omega(f, Q) \leqq C$ for every Q ([4]). More generally, let φ be a non-decreasing positive function and define BMO_φ as the space of all functions f, with $\Omega(f, Q) \leqq C\varphi(r)$ whenever Q is a cube with edge-length r ([6], [3]). The norms are defined as the least possible constants C in the inequalities and the spaces are Banach spaces.

Coifman, Rochberg and Weiss [1] have proved that if $f \in BMO$, C_f is a bounded operator from L^p to itself, $1<p<\infty$. They also proved a partial converse, viz. if $[f, R_j]$ is bounded on L^p for every Riesz transform R_j, then f belongs to BMO. The purpose of this paper is to show that it suffices to assume the boundedness of one of these commutators, or of any commutator C_f. More generally $f \in BMO_\varphi$ if and only if C_f is a bounded operator from L^p to a suitable Orlicz space.

1. Notation and basic lemmas

C denotes different positive constants. $Q(x_0, r)$ denotes the cube with center x_0 and edge-length r. nQ denotes the cube with the same center as Q, but enlarged n times, i.e. $nQ(x_0, r)=Q(x_0, nr)$.

We state some lemmas without proofs. Cf. [3], [4], [6].

Lemma 1. $\Omega(f, Q) \leqq 2|Q|^{-1} \int_Q |f(x)-a|\,dx$ *for every* a.

Lemma 2. *If* $f \in BMO$, *then* $|f_Q - f_{nQ}| \leqq C\|f\|_{BMO} \log n$.

Lemma 3. *If* $f \in BMO$ *and* $p<\infty$, *then* $|Q|^{-1} \int_Q |f(x)-f_Q|^p\,dx \leqq C\|f\|^p_{BMO}$.

Let Λ_α, $0<\alpha\leqq 1$, be the space of Lipschitz continuous functions, possibly unbounded, $\Lambda_\alpha = \{f;\ |f(x)-f(y)| \leqq C|x-y|^\alpha\}$.

Lemma 4. $BMO_{t^\alpha} = \Lambda_\alpha$.

Let η be an infinitely differentiable function with compact support such that $\int \eta = 1$. Define $f_r(x)$ as $\int f(x-ry)\eta(y)\,dy$.

Lemma 5. *If* $\|f\|_{BMO_\varphi} \leqq 1$, *then* $\|f-f_r\|_{BMO} \leqq C\varphi(r)$.

Lemma 6. *If* $\|f\|_{BMO_\varphi} \leqq 1$, *then* $|f_r(x)-f_r(y)| \leqq C\dfrac{\varphi(r)}{r}|x-y|$ *and*

$$|f_r(x)-f_r(y)| \leqq C\int_r^{r+|x-y|} \frac{\varphi(t)}{t}\,dt.$$

This gives the following estimate of the Lipschitz norm.

Lemma 7. *If* $0<\alpha<1$ *and* $t^{-\alpha}\varphi(t)$ *is decreasing, or if* $\alpha=1$, *then* $\|f_r\|_{A_\alpha}\leqq Cr^{-\alpha}\varphi(r)\|f\|_{BMO_\varphi}$.

Let ψ be a non-decreasing convex function on R^+ with $\psi(0)=0$. ψ^{-1} denotes the inverse function. The Orlicz space L_ψ is defined as the set of functions f such that $\int\psi(\lambda|f|)<\infty$ for some $\lambda>0$. ([5], [8]). The norm is given by $\|f\|_{L_\psi}=\inf\frac{1}{\lambda}\left(1+\int\psi(\lambda|f|)\right)$.

Lemma 8. *If* $f\in L_\psi$ *and* E *is a set of finite measure, then* $\left|\int_E f(x)\,dx\right|\leqq\|f\|_{L_\psi}|E|\psi^{-1}(|E|^{-1})$.

We also need a result for maximal functions.

For $q\geqq 1$ define

$$M_q g(x)=\sup_{x\in Q}\left(|Q|^{-1}\int_Q |g|^q\,dx\right)^{1/q}.$$

$M_q g\geqq M_r g$ if $q\geqq r$. M_1 is bounded on L^p, $1<p<\infty$, see Stein [7]. Since $M_q=(M_1|g|^q)^{1/q}$, this gives

Lemma 9. *M_q is bounded on L^p, $q<p<\infty$.*

m_f denotes the distribution function. $m_f(t)=|\{x;\,|f(x)|>t\}|$.

We have the following Marcinkiewicz-type interpolation theorem.

Lemma 10. *Suppose* $1\leqq p_2<p<p_1<\infty$, ϱ *is a non-increasing function,* A *is a linear operator such that* $m_{Ag}\left(t^{1/p_1}\cdot\varrho(t)\right)\leqq\frac{C}{t}$, *if* $\|g\|_{p_1}\leqq 1$, *and* $m_{Ag}\left(t^{1/p_2}\cdot\varrho(t)\right)\leqq\frac{C}{t}$, *if* $\|g\|_{p_2}\leqq 1$. *Then* $\int_0^\infty m_{Ag}\left(2t^{1/p}\varrho(t)\right)\leqq C$, *if* $\|g\|_p\leqq(p/p_1)^{1/p}$.

Proof. Fix t for the moment. Set $u=t^{1/p}$. Set $g_1(x)=\min\left(|g(x)|,u\right)\cdot\operatorname{sgn} g(x)$ and $g_2=g-g_1$. Let $m(s)$ denote $m_g(s)$. Then

$$m_{g_1}(s)=\begin{cases} m(s), & s<u\\ 0, & s\geqq u\end{cases}\quad\text{and}\quad m_{g_2}(s)=m(s+u).$$

Thus

$$\|g_1\|_{p_1}^{p_1}=p_1\int_0^u s^{p_1-1}m(s)\,ds$$

and

$$\|g_2\|_{p_2}^{p_2}=p_2\int_0^\infty s^{p_2-1}m(s+u)\,ds\leqq p_2\int_u^\infty s^{p_2-1}m(s)\,ds.$$

We have

$$p_1\int_0^u u^{p-p_1}s^{p_1-1}m(s)\,ds\leqq p_1\int_0^u s^{p-1}m(s)\,ds\leqq\frac{p_1}{p}\|g\|_p^p\leqq 1.$$

Thus

$$u^p\leqq u^{p_1}\|g_1\|_p^{-p_1}\quad\text{and}\quad\varrho(u^p)\geqq\varrho(u^{p_1}\|g_1\|_p^{-p_1}).$$

We apply the assumptions to $\frac{g_1}{\|g_1\|_{p_1}}$ and obtain

$$m_{Ag_1}(u\varrho(u^p)) \leqq m_{Ag_1}(u\varrho(u^{p_1}\|g_1\|_p^{-p_1}))$$

$$= m_{A\frac{g_1}{\|g_1\|}}(u\|g_1\|_{p_1}^{-1}\varrho(u^{p_1}\|g_1\|_{p_1}^{-p_1})) \leqq Cu^{-p_1}\|g_1\|_{p_1}^{p_1} = Cu^{-p_1}\int_0^u s^{p_1-1}m(s)\,ds.$$

Similarly

$$m_{Ag_2}(u\varrho(u^p)) \leqq Cu^{-p_2}\int_u^\infty s^{p_2-1}m(s)\,ds.$$

Thus we have

$$\int_0^\infty m_{Ag}(2t^{1/p}\varrho(t))\,dt = p\int_0^\infty u^{p-1}m_{Ag}(2u\varrho(u^p))\,du$$

$$\leqq C\int_0^\infty\int_0^u u^{p-1-p_1}s^{p_1-1}m(s)\,ds\,du + C\int_0^\infty\int_u^\infty u^{p-1-p_2}s^{p_2-1}m(s)\,ds\,du$$

$$= C\int_0^\infty\int_s^\infty u^{p-1-p_1}\,du\,s^{p_1-1}m(s)\,ds + C\int_0^\infty\int_0^s u^{p-1-p_2}\,du\,s^{p_2-1}m(s)\,ds$$

$$= C\int_0^\infty s^{p-1}m(s)\,ds \leqq C.$$

2. The main result

Theorem. *Let* $1<p<\infty$, *and let* φ *and* ψ *be two non-decreasing positive functions on* R^+ *connected by the relation* $\varphi(r)=r^{d/q}\psi^{-1}(r^{-d})$, *or equivalently* $\psi^{-1}(t)=t^{1/p}\varphi(t^{-1/d})$. *We assume that* ψ *is convex,* $\psi(0)=0$ *and* $\psi(2t)\leqq C\psi(t)$. *Then* f *belongs to* BMO_φ *if and only if* C_f *maps* L^p *boundedly into* L_ψ.

Remark. By duality, f belongs to BMO_φ if and only if C_f maps L_{ψ^*} into $L^{p'}$. Also, the proof may be generalized to show that f belongs to BMO_φ if and only if C_f maps L_{ψ_1} into L_{ψ_2} with

$$\varphi(r) = \frac{\psi_2^{-1}(r^{-d})}{\psi_1^{-1}(r^{-d})},$$

under suitable conditions on ψ_1 and ψ_2.

Proof. We first prove that the condition is sufficient. Assume that C_f maps L^p into L_ψ.

$\frac{1}{K(z)}$ is many times infinitely differentiable in an open set. Consequently, we may choose $z_0\neq 0$ and $\delta>0$ such that $\frac{1}{K(z)}$ can be expressed in the neighborhood $|z-z_0|<\sqrt{d}\delta$ as an absolutely convergent Fourier series, $\frac{1}{K(z)} = \Sigma\, a_n e^{iv_n\cdot z}$. (The exact form of the vectors v_n is irrelevant.)

Set $z_1=\delta^{-1}z_0$. If $|z-z_1|<\sqrt{d}$, we have the expansion

$$\frac{1}{K(z)}=\frac{\delta^{-d}}{K(\delta z)}=\delta^{-d}\sum a_n e^{iv_n\cdot\delta z}.$$

Choose now any cube $Q=Q(x_0, r)$. Set $y_0=x_0-rz_1$ and $Q'=Q(y_0, r)$ Thus, if $x\in Q$ and $y\in Q'$,

$$\left|\frac{x-y}{r}-z_1\right|\leqq\left|\frac{x-x_0}{r}\right|+\left|\frac{y-y_0}{r}\right|\leqq\sqrt{d}.$$

Denote $\operatorname{sgn}(f(x)-f_{Q'})$ by $s(x)$. This gives us

$$\int_Q |f(x)-f_{Q'}|\,dx=\int_Q (f(x)-f_{Q'})s(x)\,dx=|Q'|^{-1}\int_Q\int_{Q'}(f(x)-f(y))s(x)\,dy\,dx$$

$$=r^{-d}\int_{R^d}\int_{R^d}(f(x)-f(y))\frac{r^dK(x-y)}{K\left(\frac{x-y}{r}\right)}s(x)\chi_Q(x)\chi_{Q'}(y)\,dy\,dx$$

$$=C\iint (f(x)-f(y))K(x-y)\sum a_n e^{iv_n\cdot\delta\frac{x-y}{r}}s(x)\chi_Q(x)\chi_{Q'}(y)\,dy\,dx$$

$$=C\sum a_n\iint (f(x)-f(y))K(x-y)e^{i\frac{\delta}{r}v_n\cdot x}s(x)\chi_Q(x)e^{-i\frac{\delta}{r}v_n\cdot y}\chi_{Q'}(y)\,dy\,dx.$$

If we introduce

$$g_n(y)=e^{-i\frac{\delta}{r}v_n\cdot y}\chi_{Q'}(y)$$

and

$$h_n(x)=e^{i\frac{\delta}{r}v_n\cdot x}s(x)\chi_Q(x)$$

we have obtained

$$\int_Q |f(x)-f_{Q'}|\,dx=C\sum a_n\iint (f(x)-f(y))K(x-y)g_n(y)h_n(x)\,dy\,dx$$

$$=C\sum a_n\int C_f g_n(x)h_n(x)\,dx\leqq C\sum |a_n|\int |C_f g_n||h_n|\,dx$$

$$=C\sum |a_n|\int_Q |C_f g_n|\,dx.$$

However, g_n belongs to L^p, and its norm is $|Q|^{1/p}=r^{d/p}$. Consequently, $\|C_f g_n\|_{L^\psi}\leqq Cr^{d/p}$ and, by Lemma 8,

$$\int_Q |C_f g_n|\leqq Cr^{d/p}|Q|\psi^{-1}(|Q|^{-1}).$$

Thus we have obtained

$$\int_Q |f(x)-f_{Q'}|\,dx\leqq C\sum |a_n|r^{d/p}|Q|\psi^{-1}(|Q|^{-1})=C|Q|r^{d/p}\psi^{-1}(r^{-d})=C|Q|\varphi(r),$$

and $\Omega(f, Q)\leqq C\varphi(r)$ by Lemma 1.

We prove the converse in several steps and begin with two special cases.

Lemma 11. *If* $\|f\|_{BMO}\leqq 1$ *and* $\|g\|_p\leqq 1$, $1<p<\infty$, *then* $\|C_f g\|_p\leqq C$.

This is proved in [1]. The following simpler proof was suggested to the author by Jan-Olov Strömberg.

Proof. We will estimate $(C_f g)^{\#}(x)=\sup_{x\in Q}\Omega(C_f g, Q)$. Choose q and r greater than 1 such that $p>qr$. Let x and $Q=Q(x_0, s)$ be fixed with $x\in Q$. Set $g_1=g\cdot\chi_{2Q}$ and $g_2=g-g_1$. This gives

$$C_f g = C_{f-f_Q}g = (f-f_Q)Tg-T(f-f_Q)g_1-T(f-f_Q)g_2.$$

We estimate the mean oscillation on Q of each of these functions separately. Hölder's inequality and Lemma 3 give

$$|Q|^{-1}\int_Q|f-f_Q||Tg| \leqq \left(|Q|^{-1}\int_Q|f-f_Q|^{q'}\right)^{1/q'}\left(|Q|^{-1}\int_Q|Tg|^q\right)^{1/q}\leqq CM_qTg(x).$$

We also have

$$|Q|^{-1}\int_{R^d}|f-f_Q|^r|g_1|^r = |Q|^{-1}\int_{2Q}|f-f_Q|^r|g|^r$$

$$\leqq \left(|Q|^{-1}\int_{2Q}|f-f_Q|^{rq'}\right)^{1/q'}\left(|Q|^{-1}\int_{2Q}|g|^{rq}\right)^{1/q}\leqq C(M_{rq}g(x))^r.$$

Thus

$$\|(f-f_Q)g_1\|_r\leqq C|Q|^{1/r}M_{rq}(g)(x)$$

and consequently

$$|Q|^{-1}\int_Q|T(f-f_Q)g_1|\leqq|Q|^{-1/r}\|T(f-f_Q)g_1\|_r\leqq C|Q|^{-1/r}\|(f-f_Q)g_1\|_r\leqq CM_{rq}g(x).$$

For the last term we have for any $y\in Q$

$$|T(f-f_Q)g_2(y)-T(f-f_Q)g_2(x_0)| = \left|\int(K(y-z)-K(x_0-z))(f(z)-f_Q)g_2(z)\,dz\right|$$

$$\leqq\int_{\complement 2Q}|K(y-z)-K(x_0-z)||f(z)-f_Q||g(z)|\,dz$$

$$\leqq C\int_{\complement 2Q}\frac{|y-x_0|}{|x_0-z|^{d+1}}|f(z)-f_Q||g(z)|\,dz$$

$$\leqq C\sum_{n=2}^{\infty}\int_{2^nQ\setminus 2^{n-1}Q}2^{-n}|2^nQ|^{-1}(|f(z)-f_{2^nQ}|+|f_{2^nQ}-f_Q|)|g(z)|\,dz$$

$$\leqq C\sum 2^{-n}|2^nQ|^{-1}\int_{2^nQ}|f(z)-f_{2^nQ}||g(z)|\,dz+C\sum 2^{-n}n|2^nQ|^{-1}\int_{2^nQ}|g(z)|\,dz$$

$$\leqq C\sum 2^{-n}\left(|2^nQ|^{-1}\int_{2^nQ}|f(z)-f_{2^nQ}|^{q'}dz\right)^{1/q'}\left(|2^nQ|^{-1}\int_{2^nQ}|g(z)|^q dz\right)^{1/q}+CMg(x)$$

$$\leqq CM_q g(x)+CMg(x).$$

These estimates give

$$\Omega(C_f g, Q) \leqq 2|Q|^{-1}\int_Q |C_f g(z) - T(f-f_Q)g_2(x_0)|\,dz$$
$$\leqq CM_q Tg(x) + CM_{rq}g(x) + CM_q g(x) + CM_1 g(x) \leqq C(M_q Tg(x) + M_{rq}g(x)).$$

This holds for every Q containing x, and thus

$$(C_f g)^{\#} \leqq C(M_q Tg + M_{rq}g) \in L^p.$$

This, however, implies

$$\|C_f g\|_p \leqq C\|(C_f g)^{\#}\|_p \leqq C\|M_q Tg\|_p + C\|M_{rq}g\|_p \leqq C\|g\|_p,$$

see [2].

Lemma 12. *If $f \in \Lambda_\alpha$ and $g \in L^p$, $1 < p < \frac{d}{\alpha}$, then $\|C_f g\|_q \leqq C\|f\|_{\Lambda_\alpha}\|g\|_p$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$.*

Proof.

$$|C_f g(x)| \leqq \int |f(x)-f(y)|\,|K(x-y)|\,|g(y)|\,dy$$
$$\leqq C\|f\|_{\Lambda_\alpha}\int |x-y|^\alpha |x-y|^{-n}|g(y)|\,dy = C\|f\|_{\Lambda_\alpha} I_\alpha(|g|)(x).$$

The theorem of fractional integration [7, p. 119] shows that this Riesz potentiat exists a.e. and belongs to L^q with the right norm.

To complete the proof of the theorem, let us assume that $\|f\|_{BMO_\varphi} \leqq 1$. We note that there exists a $q < \infty$ such that $(2t)^{-q}\psi(2t) < t^{-q}\psi(t)$. Thus, replacing ψ by an equivalent Orlicz function if necessary, $t^{-q}\psi(t)$ is decreasing. Consequently $t^{-1/q}\psi^{-1}(t)$ is increasing and $r^{d(1/q-1/p)}\varphi(r)$ is decreasing.

Let α be the minimum of $d(\frac{1}{p} - \frac{1}{q})$ and 1. Assume that $1 < p_i < \frac{d}{\alpha}$, and thal $\|g\|_{p_i} \leqq 1$. Lemma 7 shows that $\|f_r\|_{\Lambda_\alpha} \leqq Cr^{-\alpha}\varphi(r)$, and Lemma 12 gives

$$\|C_{f_r} g\|_{q_i} \leqq Cr^{-\alpha}\varphi(r), \quad \text{where} \quad \frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{d}.$$

Lemmas 5 and 11 give

$$\|C_{f-f_r} g\|_{p_i} \leqq C\varphi(r).$$

We set in these formulas $r = t^{-1/d}$ and obtain a weak estimate.

$$m_{C_f g}(t^{1/p_i}\varphi(t^{-1/d})) \leqq \left(\frac{2C\varphi(r)}{t^{1/p_i}\varphi(r)}\right)^{p_i} + \left(\frac{2Cr^{-\alpha}\varphi(r)}{t^{1/p_i}\varphi(r)}\right)^{q_i} = \frac{C}{t} + \frac{C}{t^{(\frac{1}{p_i}-\frac{\alpha}{d})q_i}} = \frac{C}{t}.$$

Choose $1 < p_2 < p < p_1 < \frac{d}{\alpha}$. Let $\varrho(t)$ be $\varphi(t^{-1/d})$ and let A be C_f. We have just proved that the conditions in Lemma 10 are fulfilled. Thus, if $\|g\|_p \leqq (p/p_1)^{1/p}$,

$$\int \psi\left(\frac{1}{2}|C_f g|\right) = \int_0^\infty m_{C_f g}(2\psi^{-1}(t))\,dt \leqq C.$$

That is, $\|C_f g\|_{L_\psi} \leqq C$.

3. Examples

1. $\varphi\equiv 1$. We may take any $1<p<\infty$ and $\psi(t)=t^p$. Thus C_f maps L^p into L^p if and only if $f\in BMO$, as asserted in the introduction.

2. $\psi(t)=t^q$, $1<p<q<\infty$. $\varphi(r)=r^{d/p}r^{-d/q}$. Thus, by Lemma 4, C_f maps L^p into L^q if and only if $f\in\Lambda_{d(\frac{1}{p}-\frac{1}{q})}$. This holds even if $d(\frac{1}{p}-\frac{1}{q})>1$, then f has to be a constant.

3. $\psi(t)=t^p(1+\log^+ t)^a$, $1<p<\infty$, $a>0$. $\psi^{-1}(t)\sim t^{1/p}(1+\log^+ t)^{-a/p}$ i.e. $\varphi(r)\sim(1+\log^+\frac{1}{r})^{-a/p}$. Thus $f\in BMO_{(1+\log^+ 1/r)^{-a/p}}$ if and only if C_f maps L^p into "$L^p(1+\log^+ L)^a$".

Added im proof. There is an overlap between the results of this paper and those of A. UCHIYAMA, Compactness of operators of Hankel Type. *Tôhoku Math. J.* **30** (1978), 163—171.

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