Sets of uniqueness for the Gevrey classes

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0. Introduction

Let $G_\alpha = G_\alpha(\mathbf{C}_+)$ $(\alpha > 0)$ be the set of all functions f, bounded and infinitely differentiable in the closed upper halfplane $\overline{\mathbf{C}}_+$ $(\mathbf{C}_+ \stackrel{\text{def}}{=} \{t \in \mathbf{C} : \operatorname{Im} t > 0\})$, analytic in $\mathbf{C}_+$, and satisfying the inequalities

$$|f^{(n)}(z)| \leqq C_f \cdot Q_f^n \cdot n! \cdot n^{\frac{n}{\alpha}} \tag{0.1}$$

for all z, $z \in \overline{\mathbf{C}}_+$, and $n = 0, 1, \ldots$. We call the set G_α the Gevrey class of order α.

Definition. A compact subset $\mathbf{E}$ of the real line $\mathbf{R}$ is said to be a set of uniqueness for G_α if there is no nonzero function $f, f \in G_\alpha$, that vanishes on $\mathbf{E}$ together with all its derivatives. The system of all sets of uniqueness for G_α will be denoted by $\mathcal{E}_\alpha$.

The main purpose of this work is a complete description of $\mathcal{E}_\alpha$ for all positive α. The results are stated in the next paragraph. Here we give a brief survery of earlier results concerning uniqueness problems for G_α. All results of this article can be reformulated (via conformal mapping) for classes $G_\alpha(\mathbf{D})$ of functions analytic in the unit disc $\mathbf{D}$ and analogous to $G_\alpha(\mathbf{C}_+)$. We prefer the half-plane to the disc because the formulas are simpler in the case of $\mathbf{C}_+$.

The results of this article were announced in the note [1].

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0.1. It is well known that when $\alpha \geqq 1$ every nonvoid subset of the real line is a set of uniqueness for G_α. In other words the class G_α is quasianalytic if $\alpha \geqq 1$. So if $0 < \alpha < 1$ the class G_α is not far from being quasianalytic. This is the main obstacle in the problem of describing $\mathcal{E}_\alpha$ for $0 < \alpha < 1$ and influences the final result in an essential way.

On the other hand the complete description of the sets of uniqueness for the class C_A^∞ of all $C^\infty(\overline{\mathbf{C}}_+)$-functions analytic and bounded in $\mathbf{C}_+$ has been known for a long time ([2], [3], [4], [5]). $\mathbf{E}(\mathbf{E}\subset\mathbf{R})$ is a set of uniqueness for C_A^∞ if and only if

$$\int_{\mathbf{R}}\frac{\log\varrho(x,\mathbf{E})}{1+x^2}\,dx=-\infty, \tag{0.2}$$

$\varrho(x,E)$ being the distance from x to $\mathbf{E}$.

Moreover B. A. Taylor and D. L. Williams [6] have described the sets of uniqueness for a class of functions analytic in $\mathbf{D}$ somewhat larger then $G_\alpha(\mathbf{D})$. This class arises when $n^{\frac{n}{\alpha}}$ in (0.1) is replaced by $\exp(n^p)$, $p>1$. The subset $\mathbf{E}$ of the unit circle $\mathbf{T}$ is a set of uniqueness for this class exactly when

$$\int_{-\pi}^{\pi}\log^q\frac{2}{\varrho(e^{i\theta},\mathbf{E})}\,d\theta=+\infty, \tag{0.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

As to the classes G_α $(0<\alpha<1)$ things are more complicated here. Simple necessary conditions for a set $\mathbf{E}$ not to belong to $\mathcal{E}_\alpha$ were pointed out by Carleson in the classical work [2]:

$$\mathbf{E}\subset\mathbf{R},\mathbf{E}\notin\mathcal{E}_\alpha\Rightarrow|\mathbf{E}|=0,\quad \sum_\nu l_\nu^{1-\alpha}<+\infty,$$

$|\mathbf{E}|$ being the Lebesgue measure of $\mathbf{E}$ and (l_ν) the sequence of lengths of all finite complementary intervals of $\mathbf{E}$.

These two conditions can be rewritten as follows

$$\int_{\mathbf{R}}\frac{1}{\varrho(x,E)^\alpha}\cdot\frac{dx}{1+x^2}<+\infty. \tag{0.4}$$

The necessity of (0.4) for $\mathbf{E}\notin\mathcal{E}_\alpha$ can be proved in the following way. If $f\in G_\alpha$, $f\not\equiv 0$, then $\log|f|\in L^1\left(\frac{dx}{1+x^2}\right)$ $\left(\text{this is to say } \int_{\mathbf{R}}\frac{|\log|f(x)||}{1+x^2}\,dx<+\infty\right)$. For $x\in\mathbf{R}$ the Taylor formula implies

$$|f(x)|\leqq\sup_{t\in\mathbf{R}}|f^{(n)}(t)|\cdot\frac{|x-c|^n}{n!}$$

c being the element of $\mathbf{E}$ nearest to x. Minimizing the right hand side with respect to n we conclude from the definition of the class G_α that

$$|f(x)|\leqq\exp\left\{-\frac{\text{const.}}{\varrho(x,\mathbf{E})^\alpha}\right\} \tag{0.5}$$

and so the proof is finished.

The same method is applicable to obtain necessary conditions (for non-uniqueness) for other spaces of analytic functions. The proof of the sufficiency of such a necessary condition usually requires the construction of an outer function whose modulus (on $\mathbf{R}$) behaves like the quantity corresponding to the right hand side of (0.5). That is why the efforts of all succeeding authors were concentrated on the search for sufficient conditions approaching (0.4). The proofs followed the scheme sketched above. At first A. Chollet [7] showed that

$$\mathbf{E} \subset \mathbf{R},\ |\mathbf{E}| = 0, \quad \sum_\nu l_\nu^{1-\alpha} \cdot \left(\frac{1}{l_\nu}\right)^{\alpha \cdot \frac{1+\alpha}{1-\alpha}} < +\infty \Rightarrow \mathbf{E} \notin \mathcal{E}_\alpha.$$

Then Chollet [8] and Pavlov and Suturin [9], [10], obtained a weaker sufficient condition

$$\sum_\nu l_\nu^{1-\alpha} \cdot \left(\frac{1}{l_\nu}\right)^{\alpha \cdot \frac{1-\alpha}{1+\alpha}} < +\infty.$$

In these articles [9], [10], also a totally different sufficient condition was given, namely $\mathbf{E} \notin \mathcal{E}_\alpha$, if

$$\int_{Cl_x} \frac{1}{\varrho(t, \mathbf{E})^\alpha} \cdot \frac{dt}{(x-t)^2} \leqq \frac{\text{const.}}{\varrho(x, E)^{1+\alpha}} \tag{0.6}$$

for every x, $x \in \mathbf{R} \setminus \mathbf{E}$, Cl_x being the complement of the complementary interval l_x of $\mathbf{E}$ containing x.

Simultaneously Korolevič and Pogorelyĭ [11] showed that $\mathbf{E} \notin \mathcal{E}_\alpha$ if $|\mathbf{E}| = 0$ and

$$\sum_\nu l_\nu^{1-\alpha} \cdot l_\nu^{-\varepsilon} < +\infty \tag{0.7}$$

for a positive number ε. The article [11] differs from the preceding ones as to the construction of the corresponding outer function. Its modulus on the interval l_ν is here no longer $\exp\left[-\frac{1}{\varrho(x, E)^\alpha}\right]$; the more complementary intervals of E are situated near l_ν the smaller this modulus is.

The best (but unpublished) result in this direction is due to S. A. Vinogradov. Regularizing the choice of the outer function by means of a conformal mapping of $\mathbf{D}$ onto a suitable domain S. A. Vinogradov proved that $\mathbf{E} \notin \mathcal{E}_\alpha$ if $|\mathbf{E}| = 0$ and

$$\sum_\nu l_\nu^{1-\alpha} \left(\log^+ \frac{1}{l_\nu}\right)^{\alpha+\varepsilon} < +\infty. \tag{0.8}$$

for a positive ε.

In spite of this progress the question of whether the Carleson condition (0.4) is sufficient or not remained open.

In this work this question is answered in the negative. We obtain a necessary and sufficient condition which is close to (0.6). Although more cumbersome than (0.4) our criterion is suitable to work with. In particular we are able to deduce from it that Vinogradov's result is almost best possible and to give another proof of this result. Our criterion enables us to construct two closed subsets of $\mathbf{R}$ with the same sequence of lengths of complementary intervals one of which belongs to $\mathcal{E}_\alpha$ and the other does not.

And to complete our survey we will mention interesting applications of the unicity theorems to the investigation of spectral properties of nonselfadjoint Schrödinger operators with a decreasing complex potential. This approach was developed by B. S. Pavlov in [12], (see a brief exposition in [13], [14]).

0.2. Now we list the principal symbols to be used in this article.

$\mathbf{R}$ — the set of all real numbers viewed as a subset of the complex plane $\mathbf{C}$: $\mathbf{R}=\{\xi\in\mathbf{C}\colon \operatorname{Im}\xi=0\}$;

$\mathbf{Z}$ — the set of all integers;

$\mathbf{N}$ — the set of all positive integers: $\mathbf{N}=\mathbf{Z}\cap(0,+\infty)$;

$\mathbf{R}_+\overset{\text{def}}{=}[0,+\infty)$; $\mathbf{Z}_+\overset{\text{def}}{=}\mathbf{Z}\cap\mathbf{R}_+$.

The letter $\mathbf{E}$ will be always used as a notation for a compact subset of the line $\mathbf{R}$, (l_ν) will denote the set of all bounded complementary intervals of $\mathbf{E}$. The length of the interval l will be denoted by the same letter l. Let $x\in\mathbf{R}\setminus\mathbf{E}$. Then l_x will mean the complementary interval of $\mathbf{E}$ containing x. The symbol $L^1\left(\frac{dx}{1+x^2}\right)$ will denote the space of all complex measurable functions f on $\mathbf{R}$ with

$$\int_{\mathbf{R}}\frac{|f(x)|}{1+x^2}\,dx<+\infty.$$

If a set $\mathbf{E}$ satisfies a condition (γ) (or has a property (γ)) we will write $\mathbf{E}\in(\gamma)$. The symbol Const. will denote a constant depending only on parameters which remain unchanged in the problem under consideration. The condition (0.4) (see above) complemented by the condition $|\mathbf{E}|=0$ will be noted by (C_α) and will be called the Carleson condition. The condition (0.6) will be denoted by (PS_α) using the first letters of its authors names. At last the condition (0.8) will be denoted by V_α.

1. Statements of results. Discussions

Theorems stated in this paragraph are numbered by means of single figures. Their proof requires (as a rule) a whole paragraph (for each of them).

Theorem 1. Let $\alpha\in(0,1)$. The set E is not a set of uniqueness: $E\notin\mathcal{E}_\alpha$ if and only if there exists a function f_E, $f_E\in L^1\left(\frac{dx}{1+x^2}\right)$ such that

$$(a)\quad \frac{1}{\varrho(x,E)^\alpha} \leqq f_E(x), \quad x\in\mathbf{R};$$
$$(b)\quad \int_{Cl_x} \frac{f_E(t)}{(t-x)^2}\,dt \leqq \text{const.}\, f_E(x)^{1+\frac{1}{\alpha}}, \quad x\in\mathbf{R}. \qquad (\alpha)$$

The condition (*b*) with f_E replaced by $(\varrho(x,E))^{-\alpha}$ becomes (0.6).

This theorem is the main result of the article. Its proof consists of two parts and occupies §§ 2—4. In § 2 it is proved that

$$E\notin\mathscr{E}_\alpha \Rightarrow E\in(\alpha).$$

This implication is proved with the essential use of a property of functions of the class C_A^n (n times continuously differentiable in the closed unit disc and analytic in its interior) discovered by **B.** Korenblum in his well known work [15] (Lemma 4.3)*) Roughly speaking this property means that the rates of vanishing of $|f(e^{i\theta})|$ when $\theta\to d-0$ and $\theta\to\alpha+0$ are interdependent if $f(e^{i\alpha})=\ldots=f^{(n)}(e^{i\alpha})=0$. The influence of this phenomenon on functions of the Gevrey class G_α is much more significant and is the cause of the non-sufficiency of the Carleson condition for G_α. The proof of this part of Theorem 1 is a modification of Korenblum's lemma and uses the same ideas.

In § 3 a more constructive reformulation of (α) is given. To every set E, $E\in(\alpha)$ corresponds a solution U_E of the non-linear equation

$$U_E(x)^{1+\frac{1}{\alpha}} = \int_{Cl_x} \frac{U_E(t)}{(t-x)^2}\,dt, \quad x\in\mathbf{R}\setminus E,$$

satisfying the inequality $U_E(x)\geqq\dfrac{1}{\varrho(x,E)^\alpha}$ $(x\in\mathbf{R}\setminus E)$. The reasoning of § 3 shows that in order to obtain U_E we have to increase the function $x\to(\varrho(x,E))^{-\alpha}|l_\nu$, the rate of increase depending on the amount of complementary intervals of E neighbouring l_ν. Thus we must construct an extremal function U_E which will be called the equilibrium function (by analogy with potential theory). The paragraph is concluded by the investigation of the "regularity" properties of U_E needed in the sequel.

In § 4 we show that the outer (with respect to the upper halfplane) function

$$f(z) = \exp\left\{\frac{i}{\pi}\int_{\mathbf{R}} \frac{U_E(t)}{(t-z)}\,dt\right\}, \quad z\in\mathbf{C}_+,$$

belongs to the Gevrey class G_α and that its boundary values on E vanish together with all its derivatives. Clearly $f\not\equiv 0$ $(f^{-1}(0)\cap\mathbf{R}=E)$ and so $E\notin\mathscr{E}_\alpha$. The extremal

*) In [1] this lemma is erroreously called "Lemma 4.1".

property of U_E is easily reformulated in terms of f. If $g\in G_\alpha$, $g^{(n)}|E\equiv 0$, $n\in\mathbf{Z}_+$, then there is a constant $\gamma>1$ such that

$$|g(x)|^\gamma \leqq |f(x)|, \quad x\in\mathbf{R}.$$

Roughly speaking the outer function f corresponding to U_E majorizes (on $\mathbf{R}$) the moduli of all functions belonging to G_α and vanishing on E with all their derivatives.

Unfortunately it is not easy to apply the condition (α) to decide whether a given set E satisfies it. That is why most of the remaining part of the article is devoted to the study of this condition.

The following result is obtained in § 5.

Theorem 2. (S. A. Vinogradov). If $|E|=0$ and for some ε, $\varepsilon>0$,

$$\sum_\nu l_\nu^{1-\alpha}\left(\log^+\frac{1}{l_\nu}\right)^{\alpha+\varepsilon} <+\infty,$$

then $E\notin\mathscr{E}_\alpha$.

For the sake of simplicity we gave here a slightly weaker assertion then the theorem proved in § 5. The original proof of this theorem (reproduced here by the permission of its author) displays a very interesting connection with the distortion estimates of conformal maps.

In § 6 it is shown that Theorem 2 is almost sharp.

Theorem 3. There exists a set E, $|E|=0$, $E\in\mathscr{E}_\alpha$, such that

$$\sum_\nu l_\nu^{1-\alpha}\left(\log^+\frac{1}{l_\nu}\right)^{\alpha-\varepsilon} <+\infty$$

for arbitrary ε, $\varepsilon>0$.

The set E can be chosen to be a Cantor type set with the non-constant ratio of dissection.

Theorem 3 raises the question of whether the Carleson condition (C_α) is sharp. In § 7 we give a positive answer to this question.

Theorem 4. There exists a set E, $E\notin\mathscr{E}_\alpha$, such that for every ε, $\varepsilon>0$,

$$\sum_\nu l_\nu^{1-\alpha}\left(\log^+\frac{1}{l_\nu}\right)^{\varepsilon} =+\infty.$$

The needed example is constructed as a countable set with single limit point.

The last two theorems can be intuitively interpreted as follows. If the set E is perfect (i.e. has no isolated points) then the condition of the non-uniqueness ($E\notin\mathscr{E}_\alpha$) is so to say attracted by the Vinogradov condition (V_α). If the set E is countable then the gravitation center is the Carleson condition (C_α). This interpretation is of course very vague and its more precise expression constitutes an interesting problem.

At the end of § 7 the following theorem is proved.

Theorem 5. There exist two compact subsets of $\mathbf{R}$ whose Lebesgue measure equals zero with the same family of lengths of complementary intervals and one of which belongs to $\mathcal{E}_\alpha$ and the other does not.

This theorem exhibits a very interesting property of the non-uniqueness sets for G_α. If the lengths of complementary intervals decrease rapidly enough (as for example in the condition (V_α)) their mutual situation plays no role and $E \notin \mathcal{E}_\alpha$. But if

$$\sum_\nu l_\nu^{1-\alpha}\left(\log\frac{1}{l_\nu}\right)^\alpha = +\infty,$$

then the mutual situation of the complementary intervals becomes important.

§ 8 contains various conditions implying (α) and the analysis of their interconnections. These conditions involve the Hardy—Littlewood maximal function. The aim of the investigation of § 8 is to estimate "the amount" of non-uniqueness sets not satisfying (V_α). Here we give a new interpretation of the condition (PS_α) (Theorem 8.2).

In § 9 we discuss the relation of the interpolation sets for G_α to the non-uniqueness sets and the connection between the condition (α) and the well-known condition of Muckenhoupt [16]. We show that an interpolation set E for G_α is an interpolation set for $G_{\alpha+\varepsilon}$, ε being positive and sufficiently small (depending on E). This implies that the interpolation sets for G_α form a relatively small part of the class of all non-uniqueness sets for G_α. For other details concerning the Muckenhoupt condition see in § 9.

Finally in § 10 the non-uniqueness sets for the general Carleman classes are discussed. The proofs here are given in concise form. They are analogous to the corresponding proofs for the Gevrey classes. The mai nresult generalizes Theorem 1 to Carleman classes satisfying usual regularity requirements.

§ 2. Theorem 1. Proof of the necessity

Suppose that $E \notin \mathcal{E}_\alpha$. Then there is a function $f, f \in G_\alpha$, such that

$$f^{(n)}|E \equiv 0, \quad n \in \mathbf{Z}_+.$$

Without loss of generality we may assume the constant C_f to be arbitrarily small and Q_f to be arbitrarily great (see (0.1)). We are going to show that the function

$$f_E(x) = -L \cdot \log|f(x)|, \quad x \in \mathbf{R},$$

satisfies the conditions (a), (b) of Theorem 1 if L is large enough. The Jensen inequality implies

$$f_E \in L^1\left(\frac{dx}{1+x^2}\right).$$

Let us verify (a).

Lemma 2.1. For every pair (α, y) of positive numbers there exists a number $n, n\in\mathbf{Z}_+$, satisfying the inequalities

$$\exp\left\{-\frac{1}{\alpha e y^\alpha}\right\} \leqq y^n\cdot n^{\frac{n}{\alpha}} \leqq e^{\frac{1}{2\alpha}}\exp\left\{-\frac{1}{\alpha e y^\alpha}\right\}$$

and

$$n \leqq \frac{1}{ey^\alpha}+\frac{1}{2}.$$

Proof. Straightforward computations show that the minimum of the function $t\to y^t\cdot t^{t/\alpha}$ on the half-axis $(0, +\infty)$ equals $\exp\left\{-\frac{1}{\alpha e y^\alpha}\right\}$ and is attained at the point $t=\frac{1}{ey^\alpha}$. It is easy to see that the integer m, $m\in\left[\frac{1}{ey^\alpha}-\frac{1}{2}, \frac{1}{ey^\alpha}+\frac{1}{2}\right]$ has the necessary properties.

Let $x\in\mathbf{R}\setminus E$. Considering the Taylor series of f (as we have done in § 0 (see [2]) and using the Lemma 2.1. we obtain

$$|f(x)| \leqq C_f\cdot e^{\frac{1}{2\alpha}}\cdot\exp\left\{-\frac{1}{\alpha e Q_f^\alpha\cdot \varrho(x, E)^\alpha}\right\} \tag{2.1}$$

Without loss of generality we can assume $c_f\cdot e^{\frac{1}{2\alpha}}<1$. The condition (a) is obviously satisfied if $L=\alpha e Q_f^\alpha$.

To verify (b) apply an analogue of the already mentioned Lemma 4.3 by Korenblum (see [15]). Let

$$a_f(x)\overset{\text{def}}{=}\int_{Cl_x}\frac{-\log|f(t)|}{(t-x)^2}\,dt, \quad x\in\mathbf{R}\setminus E$$

The inequality $|f|<1$ implies $a_f>0$.

Lemma 2.2. Let $f\in G_\alpha$ and $c_f\leqq 1$ (see (0.1)). Then there is a constant $c_1>0$ such that

$$|f(x)| \leqq e^{1+\frac{1}{2\alpha}}\cdot\exp\left\{-c_1\cdot a_f(x)^{\frac{\alpha}{1+\alpha}}\right\}. \tag{2.2}$$

Proof. Let $x\in\mathbf{R}\setminus E$ and fix the point $z=x+iy$, $y>0$. We will proceed as follows: using the partial sums (with variable number n of terms) of the Taylor series of f with center at z we obtain an estimate of $|f(x)|$. Then employing rough estimates of the Taylor coefficients $\frac{f^{(k)}(z)}{k!}$ and minimizing with respect to n and y we come to the assertion of the lemma. Thus we begin with the inequality

$$|f(x)| \leqq \sum_{k=0}^{n-1}\frac{|f^{(k)}(z)|}{k!}\,y^k+Q_f^n\cdot y^n\cdot n^{\frac{n}{\alpha}} \quad (n\in\mathbf{N}). \tag{2.3}$$

Let $\mathbf{T}_z$ be the circle centered at z with radius $y/2$. The Cauchy formula implies

$$|f^{(k)}(z)| \leqq \frac{2^k}{y^k}\cdot k!\cdot \max_{\zeta\in \mathbf{T}_z} |f(\zeta)|. \tag{2.4}$$

Now we are going to estimate the maximum in the right side of (2.4) Using the Jensen inequality for the upper half-plane we obtain

$$\log|f(\xi)| \leqq -\frac{1}{\pi}\int_{\mathbf{R}} \frac{\operatorname{Im}\xi}{(\operatorname{Re}\xi-t)^2+(\operatorname{Im}\xi)^2}\left(-\log|f(t)|\right)dt.$$

We can replace here $\int_{\mathbf{R}}$ by $\int_{Cl_x}$ because $|f|\leqq 1$. Taking this into account and letting y satisfy

$$0<y<\varrho(x,E) \tag{2.5}$$

elementary estimates of the Poisson kernel give

$$\log|f(\zeta)| \leqq -\frac{y}{9\pi}a_f(x) \quad (\zeta\in\mathbf{T}_z).$$

This inequality and (2.4) lead to an estimate of the Taylor coefficients in (2.3) which in turn gives

$$|f(x)| \leqq 2^n \exp\left\{-\frac{y}{9\pi}a_f(x)\right\}+Q_f^n\cdot y^n\cdot n^{\frac{n}{\alpha}}. \tag{2.6}$$

Here $n\in\mathbf{Z}_+$, and y satisfies (2.5). It is clear that (2.6) is correct when $n=0$ (for $|f(x)|<1$).

Taking infimum in the second term of (2.6) (with respect to n) and using Lemma 2.1 we obtain

$$|f(x)| \leqq 2^{\frac{1}{2}}\exp\left\{-\frac{y}{9\pi}a_f(x)+\frac{\log 2}{ey^\alpha Q_f^\alpha}\right\}+e^{\frac{1}{2\alpha}}\exp\left\{-\frac{1}{\alpha e y^\alpha Q_f^\alpha}\right\} \tag{2.7}$$

Now it is natural to choose y as a root of the equation

$$ya_f(x) = y^{-\alpha}.$$

If $0<a_f^{-\frac{1}{1+\alpha}}<\varrho$ we can set $y=[a_f(x)]^{-\frac{1}{1+\alpha}}$ in (2.7). Without loss of generality we may suppose Q_f to be sufficiently large so that say $-\frac{1}{9\pi}+\frac{\log 2}{eQ_f^\alpha}<-\frac{1}{18\pi}$. Then we have only to put $c_1=\min\left(\frac{1}{9\pi}-\frac{\log 2}{eQ_f^\alpha},\ \frac{1}{\alpha e Q_f^\alpha}\right)>0$.

If $\varrho\leqq[a_f(x)]^{-\frac{1}{1+\alpha}}$ then we employ the inequality (2.1) replacing $\varrho^{-\alpha}$ by $[a_f(x)]^{\frac{\alpha}{1+\alpha}}$. This yields the required inequality.

And now we can deduce (b). Taking logarithms in (2.2) we obtain

$$c_1 a_f(x)^{\frac{\alpha}{1+\alpha}} \leqq -\log|f(x)|+\left(1+\frac{1}{2\alpha}\right).$$

Without loss of generality suppose that $c_f \cdot e^{1+\frac{1}{2\alpha}} < 1$ so that

$$a_f(x)^{\frac{\alpha}{1+\alpha}} \leqq -\frac{2}{c_1}\log|f(x)|$$

and

$$\left(\int_{Cl_x}\frac{f_E(t)}{(t-x)^2}\,dt\right)^{\frac{\alpha}{1+\alpha}} = L^{\frac{\alpha}{1+\alpha}}\, a_f^{\frac{\alpha}{1+\alpha}}(x) \leqq \frac{2}{c_1 L^{\frac{1}{1+\alpha}}} f_E(x).$$

§ 3. The equilibrium functions U_E

In this paragraph we give an equivalent reformulation of the condition (α) and show that to every set E satisfying (α) corresponds a function U_E defined on $\mathbf{R}$ and behaving regularly enough. In § 4 we will prove that the outer function with the modulus $\exp(-U_E)$ (on $\mathbf{R}$) belongs to G_α and vanishes on E with all its derivatives.

Let E be a compact subset of the line $\mathbf{R}$, $|E|=0$. Define the linear operator T_E mapping the space $L^1\left(\frac{dx}{1+x^2}\right)$ into the space of all functions (Lebesgue) measurable on $\mathbf{R}$:

$$T_E f(x) = \int_{Cl_x}\frac{f(t)}{(t-x)^2}\,dt, \quad x\in\mathbf{R}\setminus E.$$

Lemma 3.1. Let $U_0(x)=[\varrho(x,E)]^{-\alpha}$ $(x\in\mathbf{R})$ and suppose $U_0\in L^1\left(\frac{dx}{1+x^2}\right)$. Then for every x, $x\in CE=\mathbf{R}\setminus E$,

$$[T_E U_0(x)]^{\frac{\alpha}{1+\alpha}} > U_0(x). \tag{3.1}$$

Proof. Let $x\in l_x=(a,b)$. We suppose that the interval (a,b) is bounded, the case of $b=+\infty$ or $a=-\infty$ being analogous. For every $t\notin l_x$ we have $\varrho(t,l_x)\geqq \varrho(t,E)$. Therefore

$$T_E U_0(x) \geqq \int_{-\infty}^{a}\frac{1}{(t-x)^2}\cdot\frac{dt}{(a-t)^\alpha}+\int_b^{+\infty}\frac{1}{(t-x)^2}\cdot\frac{dt}{(t-b)^\alpha}$$

$$=\left\{\frac{1}{(x-a)^{1+\alpha}}+\frac{1}{(b-x)^{1+\alpha}}\right\}\cdot\int_0^\infty\frac{dt}{(1+t)^2t^\alpha}.$$

It is easy to see that the function $\alpha\to\int_0^\infty \frac{dt}{(1+t)^2t^\alpha}$ is increasing on [0, 1) so that for every $\alpha\in(0,1)$

$$\int_0^\infty \frac{dt}{(1+t)^2t^\alpha} > \int_0^\infty \frac{dt}{(1+t)^2} = 1,$$

and

$$[T_E U_0(x)]^{\frac{\alpha}{1+\alpha}} > \max\left\{\frac{1}{(x-a)^{1+\alpha}}, \frac{1}{(b-x)^{1+\alpha}}\right\}^{\frac{\alpha}{1+\alpha}} = U_0(x).$$

Now we want to give a criterion enabling us to verify the non-constructive condition (α). We begin with the following remark. With no loss of generality we may assume the constant C in (α) is equal to one. Let indeed $p=Q\cdot f_E$, where the choice of the number Q will be made a little later (f_E being a function satisfying (a) and (b) of (α) with Const$=C>1$). Clearly

$$\frac{1}{\varrho(x,E)^\alpha} < p(x), \quad x\in\mathbf{R}\setminus E,$$

and

$$[T_E p]^{\frac{\alpha}{1+\alpha}} \leqq C\cdot Q^{-\frac{1}{1+\alpha}}\cdot p.$$

Now we only have to put $C\cdot Q^{-\frac{1}{1+\alpha}}=1$.

Let (as above) $U_0(x)=\varrho(x,E)^{-\alpha}$ and

$$U_{n+1}=[T_E U_n]^{\frac{\alpha}{1+\alpha}} \tag{3.2}$$

for $n\in\mathbf{Z}_+$, $|E|=0$.

Lemma 3.2. I. The set E satisfies (α) if and only if

$$\int_{\mathbf{R}} \frac{U_n(x)}{1+x^2}\,dx = O(1), \quad n\to+\infty. \tag{3.3}$$

II. If $E\in(\alpha)$ there exists a function U_E such that $U_E\geqq U_0$, $U_E\in L^1\left(\frac{dx}{1+x^2}\right)$ and

$$U_E=[T_E U_E]^{\frac{\alpha}{1+\alpha}}, \tag{3.4}$$

Proof. The assertion II will follow automatically from the proof of I. Suppose that $E\in(\alpha)$. According to the remark preceding the statement of the lemma we may assume that the constant in (α) is equal to one. If $U_n<f_E$ the T_E-transform of this inequality gives

$$U_{n+1}=[T_E U_n]^{\frac{\alpha}{1+\alpha}} < [T_E f_E]^{\frac{\alpha}{1+\alpha}} < f_E.$$

Because $U_0 < f_E$ (see (α)) induction shows that

$$U_n < f_E, \quad n \in \mathbf{Z}_+,$$

and (3.3) is true because $f_E \in L^1\left(\dfrac{dx}{1+x^2}\right)$.

If (3.3) is fulfilled we will verify (α) constructing the function U_E whose existence is asserted in II. Note that the sequence $(U_n)_{n\geqq 0}$ is increasing, i.e.

$$U_n \leqq U_{n+1}, \quad n \in \mathbf{Z}_+.$$

This was already proved for $n=0$ in Lemma 3.1. Applying T_E to the n-th inequality and taking its $\dfrac{\alpha}{1+\alpha}$-th power we pass from n to $n+1$. Using the B. Levi theorem we conclude that the function $U_E \overset{\text{def}}{=} \lim_{n\to\infty} U_n$ belongs to $L^1\left(\dfrac{dx}{1+x^2}\right)$, Taking the limit in (3.2) gives (3.4).

The equality (3.4) shows the analyticity of U_E on every complementary interval of E. We will prove below that many "regularity" properties of U_0 are shared by U_E.

Lemma 3.3. The function U_E is convex (i.e. $U_E''>0$) on every complementary interval of E and

$$\lim U_E(x) = +\infty$$

as $\varrho(x, E)\to 0$.

Proof. The convexity of U_E is verified by explicit computation of the second derivative:

$$U_E''(x) = -\frac{4\alpha}{(1+\alpha)^2}(T_E U_E)^{-\frac{1}{1+\alpha}-1}\left\{\int_{Cl_x}\frac{U_E(t)}{(t-x)^3}\,dt\right\}^2 +$$

$$+\frac{6\alpha}{1+\alpha}\cdot(T_E U_E)^{-\frac{1}{1+\alpha}}\cdot\int_{Cl_x}\frac{U_E(t)}{(t-x)^4}\,dt$$

and the condition $U_E''(x)>0$ is equivalent to the inequality

$$\frac{2}{3(1+\alpha)}\left(\int_{Cl_x}\frac{U_E(t)}{(t-x)^3}\,dt\right)^2 < \int_{Cl_x}\frac{U_E(t)}{(t-x)^2}\,dt\cdot\int_{Cl_x}\frac{U_E(t)}{(t-x)^4}\,dt$$

implied by the Cauchy—Bunyakovski inequality (I am grateful to E. M. Dynkin for this remark) applied to the product of functions

$$t \to \sqrt{U_E(t)}\cdot(t-x)^{-1}, \quad t \to \sqrt{U_E(t)}\cdot(t-x)^{-2}.$$

The last assertion of the lemma follows from the inequality $U_E > U_0$.

Lemma 3.4. Let $x \in l_x = (a, b)$, l_x being a bounded complementary interval of E and denote by x^* the middle of the interval (a, x) or (x, b). Then

$$U_E(x^*) < 4^{\frac{\alpha}{1+\alpha}} \cdot U_E(x). \tag{3.5}$$

Proof. Suppose that $x^* = \dfrac{a+x}{2}$. The equality (3.4) becomes.

$$U_E(x^*) = \left\{\int_{-\infty}^{a} \frac{U_E(t)}{(x^*-t)^2}\,dt + \int_{b}^{+\infty} \frac{U_E(t)}{(t-x^*)^2}\,dt\right\}^{\frac{\alpha}{1+\alpha}}.$$

If $t > b$ then $t - x^* > t - x$. If $t < a$ then $x^* - t > \dfrac{1}{2}\,(x-t)$. Using these inequalities the last formula gives (3.5).

Lemma 3.5. Let $x \in \mathbf{R} \setminus E$ and let c denote the point of E nearest to x. Then

$$\left|\int_c^x U_E(t)\,dt\right| \leqq \text{Const.}\ \varrho(x, E) \cdot U_E(x). \tag{3.6}$$

Proof. Let $l_x = (a, b)$, $c = a$. Consider the sequence $(x_n)_{n \geqq 0}$ of points of the interval (a, x) defined by

$$x_0 = x, \quad x_{n+1} = \frac{a+x_n}{2} \quad (n \geqq 0).$$

By the Lemma 3.4 we have

$$U_E(x_n) \leqq 4^{\frac{\alpha}{1+\alpha}}\, U_E(x_{n-1}) \leqq \ldots \leqq 4^{\frac{n\alpha}{1+\alpha}}\, U_E(x).$$

The convexity of U_E on (a, b) implies

$$U_E(t) \leqq 4^{\frac{(n+1)\alpha}{1+\alpha}} \cdot U_E(x), \quad t \in [x_{n+1}, x_n].$$

Therefore

$$\int_a^x U_E(t)\,dt = \sum_{n=0}^{\infty} \int_{x_{n+1}}^{x_n} U_E(t)\,dt \leqq U_E(x) \sum_{n=0}^{\infty} 4^{\frac{(n+1)\alpha}{1+\alpha}}\,(x_n - x_{n+1}).$$

On account of the equalities $x_n - x_{n+1} = \dfrac{x-a}{2^{n+1}} = \dfrac{\varrho(x, E)}{2^{n+1}}$ we obtain

$$\int_a^x U_E(t)\,dt \leqq \varrho(x, E)\, U_E(x) \sum_{n=1}^{\infty} 2^{-n\frac{1-\alpha}{1+\alpha}} = \text{Const.}\ \varrho(x, E)\, U_E(x).$$

Lemma 3.6. Let $x \in \mathbf{R} \setminus E$, $\omega_x \overset{\text{def}}{=} (x-\delta, x+\delta)$, where $\delta = \dfrac{1}{5}\,\varrho(x, E)$. Then

$$\int_{l_x \setminus w_x} \frac{U_E(t)}{(t-x)^2}\,dt \leqq \text{Const.}\ U_E(x)^{1+\frac{1}{\alpha}}. \tag{3.7}$$

Proof. Suppose first that $l_x = (a, b)$ is bounded. Assume for the sake of simplicity that $x - a = \varrho(x, E)$. Clearly $l_x \setminus \omega_x = (a, x-\delta] \cup [x+\delta, b)$ Estimate now the contribution of the interval $(a, x-\delta)$ in the integral (3.7):

$$\int_a^{x-\delta} \frac{U_E(t)}{(t-x)^2}\,dt \leqq \frac{25}{\varrho(x,E)^2}\int_a^x U_E(t)\,dt \leqq \text{Const.}\,\frac{U_E(x)}{\varrho(x,E)}$$
$$\leqq \text{Const.}\,U_E(x)^{1+\frac{1}{\alpha}}.$$

(here we have used Lemma 3.5 and the inequality $U_E > U_0$).

Turn now to the contribution of $(x+\delta, b)$. Let y denote the middle of $(x+\delta, b)$. It is clear that $b-y<y-a$, $t-x>b-y$ if $t\in[y, b]$. Therefore

$$\int_y^b \frac{U_E(t)}{(t-x)^2}\,dt \leqq \frac{1}{\varrho(y,E)^2}\int_y^b U_E(t)\,dt \leqq \text{Const.}\ U_E(y)^{1+\frac{1}{\alpha}}$$

(see Lemma 3.5). Consider now the mid-point y_1 of (x, b), and the mid-point y_2 of (y_1, b). Obviously $y\in(x, y_2)$. It follows from Lemma 3.4 that

$$U_E(t) \leqq 16^{\frac{\alpha}{1+\alpha}}\, U_E(x)$$

if $t\in(x, y_2)$. Thus

$$\int_{x+\delta}^{y} \frac{U_E(t)}{(t-x)^2}\,dt \leqq 16^{\frac{\alpha}{1+\alpha}}\cdot U_E(x)\int_{x+\delta}^{y}\frac{dt}{(t-x)^2} \leqq 5\cdot 16^{\frac{\alpha}{1+\alpha}}\cdot U_E(x)^{1+\frac{1}{\alpha}}.$$

Putting all these estimates together we obtain the inequality (3.7)

Finally consider the case of the unbounded interval l_x. Suppose for instance that $l_x=(\alpha, +\infty)$. The function U_E decreasing on l_x we have

$$\int_{x+\delta}^{+\infty} \frac{U_E(t)}{(t-x)^2}\,dt \leqq U_E(x)\int_{x+\delta}^{+\infty}\frac{dt}{(t-x)^2} = \frac{5}{\varrho(x,E)}\cdot U_E(x).$$

The integral over $(x-\delta, x)$ is estimated in the same way as above. The lemma is proved.

Lemma 3.7. For every $x, x\in\mathbf{R}\setminus E$, the function U_E admits a complex-analytic continuation to the open disc $\mathbf{D}_x$ centered at x with radius $\delta_x=\delta=\frac{1}{5}\,\varrho(x, E)$, and $(z\in\mathbf{D}_x)$

$$U_E(z) = \left(\int_{Cl_x} \frac{U_E(t)}{(t-z)^2}\,dt\right)^{\frac{\alpha}{1+\alpha}}, \tag{3.8}$$

$$(1/3)^{\frac{\alpha}{1+\alpha}}\cdot U_E(x) \leqq |U_E(z)| \leqq (5/3)^{\frac{\alpha}{1+\alpha}} U_E(x), \tag{3.9}$$

$$\operatorname{Re} U_E(z) > (2/3)\cdot(1/3)^{\frac{\alpha}{1+\alpha}}\, U_E(x). \tag{3.10}$$

Proof. Elementary computations show that for $z\in\mathbf{D}_x$, $t\in Cl_x$

$$\frac{1}{(t-z)^2}=\frac{1}{(t-x)^2}\cdot\zeta_t,$$

where $|\zeta_t-1|<\frac{2}{3}$. Thus

$$\mathrm{Re}\int_{Cl_x}\frac{U_E(t)}{(t-z)^2}\,dt=\int_{Cl_x}\frac{U_E(t)}{(t-x)^2}\,\mathrm{Re}\zeta_t dt\geqq\frac{1}{3}\cdot\int_{Cl_x}\frac{U_E(t)}{(t-x)^2}\,dt.$$

This shows in particular that the function

$$U(z)=\int_{Cl_x}\frac{U_E(t)}{(t-z)^2}\,dt$$

has no zeros in $\mathbf{D}_x$, and so we can continue U_E analytically using (3.8). The same inequality implies the left inequality (3.9). The right inequality (3.9) is deduced by carrying the modulus under the integral sign and using the estimate $|\zeta_t|<5/3$. Observe now (turning to the proof of (3.10)) that the function $U|\mathbf{D}_x$ takes its values in the angle $|\arg z|<\theta_0=\arc\sin\frac{2}{3}$ (because of the inequality $|\arg\zeta_t|<\theta_0$). Thus

$$\mathrm{Re}U_E(z)\geqq|U_E(z)|\cos\left(\theta_0\frac{\alpha}{1+\alpha}\right)>|U_E(z)|\cdot\frac{\sqrt{5}}{3}>|U_E(z)|\frac{2}{3}$$

$$\geqq\frac{2}{3}\cdot\left(\frac{1}{3}\right)^{\frac{\alpha}{1+\alpha}}U_E(x).$$

§ 4. Theorem 1. Proof of sufficiency

Let the set E satisfy (α). Consider the auxillary function U_E (defined on $\mathbf{R}$) constructed with the aid of Lemma 3.2. Set

$$f(z)=\exp\left\{\frac{i}{\pi}\int_{\mathbf{R}}\frac{U_E(t)}{t-z}\,dt\right\}\quad(z\in\mathbf{C}_+)\tag{4.1}$$

and define $f(x)\overset{\mathrm{def}}{=}\lim_{y\to 0+}f(x+iy)$, $x\in\mathbf{R}$, (this limit exists almost everywhere on $\mathbf{R}$ because $|f|<1$ in the upper half-plane).

We begin by showing that f is analytically extendable across every complementary interval of E. Consider the contour Γ consisting of two rays of the real axis and of the lower half of the circle centered at $x(x\in\mathbf{R}\setminus E)$ with the radius $\delta_x=\frac{1}{5}\varrho(x,E)$ (fig. 1).

The function U_E being analytically extendable into $\mathbf{D}_x$ (see Lemma 3.7) we have

$$f(z)=\exp\left\{\frac{i}{\pi}\int_\Gamma \frac{U_E(\zeta)}{\zeta-z}\,d\zeta\right\}\quad (z\in \mathbf{D}_x) \tag{4.2}$$

This implies in particular that the function $x\to f(x)$ belongs to $C^\infty(\mathbf{R}\setminus E)$.

$(a, b) = l_x$

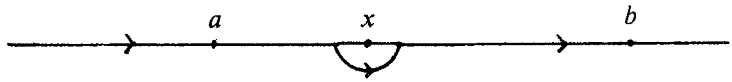

Fig. 1

Lemma 4.1. There is a positive constant Q such that for every $x, x\in \mathbf{R}\setminus E$, and $n\in \mathbf{Z}_+$

$$|f^{(n)}(z)|\leqq Q^n\cdot n!\cdot U_E(x)^{\frac{n}{\alpha}}\exp\left(-\frac{1}{2}U_E(x)\right). \tag{4.3}$$

Now we will finish The proof of Theorem 1 admitting this lemma to be true. Consider the function F defined on $\mathbf{R}$ by the following equalities: $F|E\equiv 0$, $F|(\mathbf{R}\setminus E)=f|(\mathbf{R}\setminus E)$. The estimate (4.3) and the equality $\lim_{\varrho(x,E)\to 0} U_E(x)=+\infty$ (Lemma 3.3) imply $F\in C^\infty(\mathbf{R})$, $F^{(n)}|E\equiv 0$, $n\in \mathbf{Z}_+$. But $F=f$ a.e. (recall that $|E|=0$, see (α)) and f is bounded in the upper half-plane and therefore coincides there with its Poisson integral:

$$f(z)=\frac{1}{\pi}\int_{\mathbf{R}}\frac{y}{(x-t)^2+y^2}F(t)\,dt,\quad z=x+iy,\ y>0. \tag{4.4}$$

This equality shows that $f=F$ everywhere on $\mathbf{R}$ and the continuous extension of f from the open upper half-plane onto its closure belongs to $C^\infty(\overline{\mathbf{C}}_+)$ and is analytic in the half-plane $\mathbf{C}_+$.

Taking the supremum in (4.3) with respect to x we obtain

$$|f^{(n)}(x)|\leqq Q^n\cdot n!\sup_{y>0} y^{\frac{n}{\alpha}}e^{-\frac{y}{2}}=\left\{Q\cdot\left(\frac{2}{e\alpha}\right)^{\frac{1}{\alpha}}\right\}^n\cdot n!\cdot n^{\frac{n}{\alpha}}.$$

Set $Q_f=Q\cdot\left(\frac{2}{e\alpha}\right)^{1/\alpha}$. Then by the maximum principle (see (4.4))

$$|f^{(n)}(z)|\leqq Q_f^n\cdot n!\cdot n^{\frac{n}{\alpha}}$$

for every z, $\operatorname{Im} z\geqq 0$ and $n\in \mathbf{Z}_+$. It is clear that $f\not\equiv 0$, $f^{(n)}|E\equiv 0$, $n\in\mathbf{Z}_+$ Hence $E\notin \mathscr{E}_\alpha$.

Proof of Lemma 4.1. Fix x, $x\in\mathbf{R}\setminus E$. We have already remarked that f is analytic in $\mathbf{D}_x$. Let $r>0$ and denote by C_r the circle centered at x with the radius r. If $r<\delta_x=\frac{1}{5}\,\varrho(x,E)$ then $C_r\subset\mathbf{D}_x$ and by the Cauchy formula

$$f^{(n)}(x)=\frac{n!}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta-x)^{n+1}}\,d\zeta$$

whence

$$|f^{(n)}(x)|\leqq\frac{n!}{r^n}\cdot\max_{\zeta\in C_r}|f(\zeta)|.$$

Consequently the required estimate will be proved if we show that

$$\max_{\zeta\in C_r}\log|f(\zeta)|\leqq-\frac{1}{2}U_E(x) \tag{4.5}$$

for every $r^{-1/\alpha}$, $0<r<\varepsilon\big(U_E(x)\big)^{-1/\alpha}$, where ε is a small positive number (we will choose it later). Let us remark now that inclusion $C_r\subset\mathbf{D}_x$ is ensured by the requirement $\varepsilon<1/5$ (remember that $U_E(x)>\big(\varrho(x,E)\big)^{-\alpha}$, see Lemma 3.2).

We begin the proof of (4.5) by remarking that the function $\log f$ is analytic in $\mathbf{D}_x$ and $\log|f|=\mathrm{Re}\log f$. This and the well known formula

$$\log f(x)=-U_E(x)+\frac{i}{\pi}\,(\text{p.v.})\int_{\mathbf{R}}\frac{U_E(t)}{t-x}\,dt$$

shows that $\log|f(x)|=-U_E(x)$.

For $z\in\mathbf{D}_x$ the Taylor formula implies

$$\log f(z)=\log f(x)+\textstyle\sum_{n\geqq1}a_n(z-x)^n$$

and

$$\log|f(z)|\leqq-U_E(x)+\textstyle\sum_{n\geqq1}|a_n|\cdot r^n$$

if $|z-x|<r$. Here $(a_n)_{n\geqq1}$ is the sequence of the Taylor coefficients of $\log f$ at x. Now if there exists a number ε, $0<\varepsilon<1/5$, such that

$$|a_n|\cdot\frac{\varepsilon^n}{U_E(x)^{\frac{n}{\alpha}}}\leqq2^{-n-1}\cdot U_E(x)\quad(n\in\mathbf{N}) \tag{4.6}$$

then (4.5) is proved. But we are going to prove a stronger inequality

$$|a_n|\leqq\text{const.}\left(\frac{5}{\varrho(x,E)}\right)^{n-1}\cdot U_E(x)^{1+\frac{1}{\alpha}}\quad(n\in\mathbf{N}) \tag{4.7}$$

which implies (4.6) for all sufficiently small positiv ε (it is useful to remark once more that $U_E(x)>\big(\varrho(x,E)\big)^{-\alpha}$). Clearly

$$a_n=\frac{(\log f)^{(n)}(x)}{n!}=\frac{i}{\pi}\int_\Gamma\frac{U_E(\zeta)}{(\zeta-x)^{n+1}}\,d\zeta$$

(in this connection see (4.2)). Therefore

$$|a_n| \leqq \frac{1}{\pi}\int_{|t-x|\geqq\delta_x}\frac{U_E(t)}{|t-x|^{n+1}}\,dt+\frac{1}{\delta_x^n}\cdot\left(\frac{5}{3}\right)^{\frac{\alpha}{1+\alpha}}\cdot U_E(x)$$

(see (3.9)). Further, using the important equality (3.4) and Lemma 3.6 we obtain

$$\int_{|t-x|\geqq\delta_x}\frac{U_E(t)}{|t-x|^{n+1}}\,dt \leqq \frac{1}{\delta_x^{n-1}}\left\{\int_{Cl_x}\frac{U_E(t)}{(t-x)^2}\,dt+\int_{l_x\setminus\omega_x}\frac{U_E(t)}{(t-x)^2}\,dt\right\}$$
$$\leqq \frac{1}{\delta_x^{n-1}}\left\{U_E(x)^{1+\frac{1}{\alpha}}+\text{Const.}\,U_E(x)^{1+\frac{1}{\alpha}}\right\}.$$

Thus,

$$|a_n| \leqq \frac{\text{Const.}}{\delta_x^{n-1}}\cdot U_E(x)^{1+\frac{1}{\alpha}}+\frac{1}{\delta_x^n}\cdot\left(\frac{5}{3}\right)^{\frac{\alpha}{1+\alpha}}\cdot U_E(x).$$

The inequality (4.7) now follows by means of elementary transformations (we have to take in to account $5\,\delta_x=\varrho(x,E)$ and $(\varrho(x,E))^{-\alpha}<U_E(x)$).

§ 5. The restriction imposed on the complementary intervals. Proof of Theorem 2

Consider an increasing unbounded function ω defined on the half-axis $[0,+\infty)$ and such that $\omega(0)>0$ and

$$\int_1^{+\infty}\frac{dx}{x\omega(x)}<+\infty \tag{5.1}$$

Theorem 2'. Let E be a compact subset of $\mathbf{R}$, such that $|E|=0$ and

$$\int_{\mathbf{R}}\left[\frac{1}{\varrho(x,E)}\cdot\omega\left(\frac{1}{\varrho(x,E)}\right)\right]^{\alpha}\frac{dx}{1+x^2}<+\infty. \tag{5.2}$$

Then $E\notin\mathscr{E}_\alpha$.

Remark. It is easy to see that if $\omega(x)=(\max(1,\log x))^{1+\varepsilon}$ $(x\geqq 0)$ then (5.2) is equivalent to (V_α) with a corresponding ε, $\varepsilon>0$. The method of S. A. Vinogradov can be used to prove this theorem (stronger than Theorem 2).

Lemma 5.1. Let E, be a compact subset of $\mathbf{R}$, $E\in(C_\alpha)$, and T_E the operator corresponding to it (see § 3). Suppose there exists a constant C, $C>0$, such that

$$\left(\int_{\mathbf{R}}\frac{|T_E f(x)|^{\frac{\alpha}{1+\alpha}}}{1+x^2}\,dx\right)^{\frac{1+\alpha}{\alpha}} \leqq C\int_{\mathbf{R}}\frac{|f(x)|}{1+x^2}\,dx \tag{5.3}$$

for every f, $f\in L^1\left(\frac{dx}{1+x^2}\right)$, $\left(\text{i.e. } T_E \text{ is of strong type } \left(1,\frac{\alpha}{1+\alpha}\right)\right)$. Then $E\notin\mathscr{E}_\alpha$.

Proof. Use Lemma 3.2 (assertion I) and Theorem 1. Let $(U_n)_{n\geqq 0}$ be the sequence from § 3 and set for the sake of brevity

$$\|f\| = \int_{\mathbf{R}} \frac{|f(x)|}{1+x^2}\,dx.$$

Then

$$\|U_{n+1}\| = \|(T_E U_n)^{\frac{\alpha}{1+\alpha}}\| \leqq C^{\frac{\alpha}{1+\alpha}} \|U_n\|^{\frac{\alpha}{1+\alpha}} \leqq C^{\frac{\alpha}{1+\alpha}+\left(\frac{\alpha}{1+\alpha}\right)^2} \|U_{n-1}\|^{\left(\frac{\alpha}{1+\alpha}\right)^2}$$

$$\leqq C^{\frac{\alpha}{1+\alpha}+\left(\frac{\alpha}{1+\alpha}\right)^2+\dots+\left(\frac{\alpha}{1+\alpha}\right)^{n+1}} \cdot \|U_0\|^{\left(\frac{\alpha}{1+\alpha}\right)^{n+1}}.$$

But

$$\frac{\alpha}{1+\alpha}+\left(\frac{\alpha}{1+\alpha}\right)^2+\dots+\left(\frac{\alpha}{1+\alpha}\right)^n < \alpha \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{\alpha}{1+\alpha}\right)^n = 0.$$

Hence

$$\|U_n\| = O(1), \quad n \to +\infty,$$

and $E \notin \mathscr{E}_\alpha$ (by Lemma 3.2, see (3.3)).

Define now the auxiliary operator S_E:

$$S_E f(x) \overset{\text{def}}{=} \frac{\varrho(x, E)}{\omega\left(\dfrac{1}{\varrho(x, E)}\right)} \cdot T_E f(x),$$

where $f \in L^1\left(\dfrac{dx}{1+x^2}\right)$ and $x \in \mathbf{R} \setminus E$.

Lemma 5.2. If S_E is continuous in the space $L^1\left(\dfrac{dx}{1+x^2}\right)$ and $E \in$ (5.2), then $E \notin \mathscr{E}_\alpha$.

Proof. We will prove that condition (5.3) of the preceding lemma is satisfied. Apply the Hölder inequality with the exponents $p = \dfrac{1+\alpha}{\alpha}$, $q = 1+\alpha$. We have

$$\int_{\mathbf{R}} |T_E f(x)|^{\frac{\alpha}{1+\alpha}} \frac{dx}{1+x^2} = \int_{\mathbf{R}} |S_E f(x)|^{\frac{\alpha}{1+\alpha}} \cdot \left[\frac{1}{\varrho}\,\omega\left(\frac{1}{\varrho}\right)\right]^{\frac{\alpha}{1+\alpha}} \cdot \frac{dx}{1+x^2}$$

$$\leqq \left\{\int_{\mathbf{R}} |S_E f(x)| \frac{dx}{1+x^2}\right\}^{\frac{\alpha}{1+\alpha}} \cdot \left\{\int_{\mathbf{R}} \left[\frac{1}{\varrho}\,\omega\left(\frac{1}{\varrho}\right)\right]^{\alpha} \frac{dx}{1+x^2}\right\}^{\frac{1}{1+\alpha}}$$

The integral in the second bracket is finite because of (5.2). The lemma is proved.

Proof of Theorem 2. It is sufficient to prove that $S_E f \in L^1\left(\dfrac{dx}{1+x^2}\right)$ for every non negative function $f, f \in L^1\left(\frac{dx}{1+x^2}\right)$ (see Lemma 5.2). We have the following obvious identity:

$$\int_{\mathbf{R}} S_E f(x) \frac{dx}{1+x^2} = \int_{\mathbf{R}} f(t)\,dt \int_{Cl_t} \frac{1}{(t-x)^2} \cdot \frac{\varrho(x,E)}{\omega\left(\frac{1}{\varrho(x,E)}\right)} \cdot \frac{dx}{1+x^2} \tag{5.4}$$

For the sake of brevity we denote by $g(t)$ the interior integral in the right side of (5.4). The set E being compact it easy to see that

$$g(t) = O\left(\frac{1}{t^2}\right), \quad |t| \to +\infty. \tag{5.5}$$

Let $t \in l_t = (a, b)$. Then obviously

$$g(t) \leqq \int_b^{+\infty} \frac{1}{(x-b)^2} \cdot \frac{\varrho(x,E)}{\omega\left(\frac{1}{\varrho(x,E)}\right)} \cdot \frac{dx}{1+x^2} + \int_{-\infty}^{\alpha} \frac{1}{(a-x)^2} \cdot \frac{\varrho(x,E)}{\omega\left(\frac{1}{\varrho(x,E)}\right)} \cdot \frac{dx}{1+x^2}. \tag{5.6}$$

Estimate now the first integral in (5.6) (the second is estimated analogously). Note that $\varrho(x,E) < x-b\,(x>b)$ and so $\omega^{-1}(\varrho^{-1}(x,E)) < \omega^{-1}((x-b)^{-1})$. Thus this integral does not exceed

$$\int_0^{+\infty} \frac{dx}{x\omega\left(\frac{1}{x}\right)(1+(x+b)^2)} = I(b) < +\infty.$$

It is clear that the function $b \to I(b)$ is continuous. The compactness of E implies $\sup_{t \in E} I(t) < +\infty$ and $g \in L^\infty(\mathbf{R})$. This, (5.5) and (5.4) prove that $S_E f \in L^1\left(\frac{dx}{1+x^2}\right)$.

We finally give S. A. Vinogradov's proof of Theorem 2. This proof is interesting in itself since it exhibits the connection of the problems under consideration with distortion problems for conformal mappings. The proof is published with the permission of its author.

Let δ be an increasing bounded function defined on $\mathbf{R}_+$ such that

$$\delta(0) = 0;$$

$$\int_0^\infty \frac{\delta(t)}{t}\,dt < +\infty, \tag{5.7}$$

$$0 \leqq \delta(y) - \delta(x) \leqq \delta(y-x) \quad \text{if} \quad 0 < x \leqq y, \tag{5.8}$$

$$\lim_{t \to 0+} \frac{t\delta'(t)}{\delta(t)} = 0. \tag{5.9}$$

Remark. According to previous notations $\delta(x)=\left[\omega\left(\frac{1}{x}\right)\right]^{-1}$. If we only want to prove Theorem 2 then we may choose δ so that $\delta(x)=\left(\log\frac{1}{x}\right)^{-1-\varepsilon}$, $\varepsilon>0$, for all small values of x.

Now we will construct an auxiliary domain G containing the lower half-plane $\operatorname{Im} z>0$. We define G as the subgraph of a non negative function Δ_E vanishing exactly on E. We will seek Δ_E in the form

$$\Delta_E(x) = \int_{-\infty}^{x} \psi(t)\,dt.$$

Set $\psi|E\equiv 0$. Let l be a bounded complementary interval of E and let F the three-point set consisting of the end-points and the mid-point of l. Set $\psi(x)=\delta(\varrho(x, F))$ if x belongs to the left half of l, $\psi(x)=-\delta(\varrho(x, F))$ if x belongs to the right half of l. It is clear that $\psi|l$ is continuous, that its integral taken over l is equal to zero and that the continuity modulus of $\psi|l$ does not exceed the function δ (verifying the last assertion one must use (5.8) and the symmetry of $\psi|l$). On both the half-bounded complementary intervals we define ψ so as to preserve the mentioned properties. The graph of Δ_E is represented on the figure 2 below (in the case when E consists of four points).

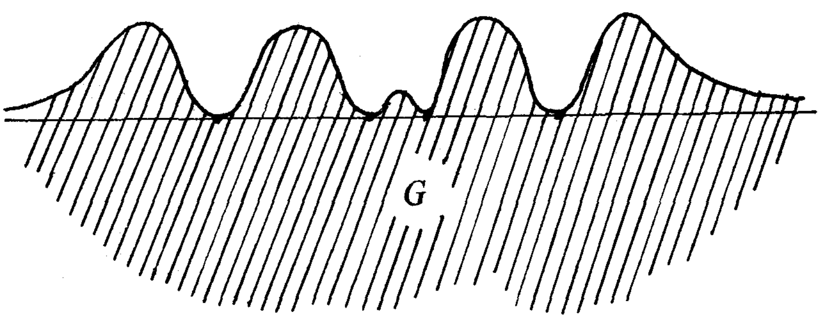

Fig. 2

Lemma 5.3. Let w be the conformal homeomorphism of the upper half plane onto G. Then w is smooth up to the boundary and is distortion-free, i.e.

$$c|w(J)| \leqq |I| \leqq C|w(J)|, \quad c, C > 0,$$

for every interval J, $J\subset\mathbf{R}$.

Proof. The continuity modulus of Δ_E' does not exceed the function δ. It is easy to see that the angle formed by the tangent to the boundary ∂G of G and the line $\mathbf{R}$ considered as a function of the arc-length parameter on ∂G has a continuity modulus with the same estimate. The lemma now follows from the Kellog's theorem [17],

p. 411 (remember that δ satisfies (5.7)). Strictly speaking this theorem is proved in [17] only for $\delta(t)=t^\alpha$, $0<\alpha<1$, but the proof is easily generalized to the general case (see [18]).

Lemma 5.4. Let $\alpha\in(0,1)$ and $\Delta(x)\overset{\text{def}}{=}\int_0^x\delta(t)\,dt$, $x\geqq 0$. Then

$$\int_0^x \frac{dt}{\Delta^\alpha(t)} \sim (1-\alpha)^{-1}\frac{x}{\Delta^\alpha(x)}, \quad x\to 0+.$$

Proof. Use de l'Hospital's rule twice and apply (5.9).

The proof of Theorem 2. Let W be the inverse of the mapping w (Lemma 5.3). Set $E^*=W(E)$ and construct the function Δ_{E^*}. Consider the outer function f_0 in the upper half-plane which is defined by

$$\log|f_0(t)| = -\frac{1}{(\Delta_{E^*}(t))^\alpha}, \quad t\in\mathbf{R}, \quad \alpha\in(0,1).$$

Then $f=f_0\circ W\in G_\alpha$ (in the lower half-plane) and vanishes with all its derivatives on E. Indeed if $\zeta\in\partial G$ and ζ^* is the point of E nearest to ζ (with respect to ∂G) then

$$|f(\zeta)| = |f_0\circ W(\zeta)| = \exp\left[-\frac{1}{\Delta_{E^*}(W(\zeta))^\alpha}\right]$$

$$= \exp\left[-\frac{1}{\Delta(|W(\zeta)-W(\zeta^*)|)^\alpha}\right] \leqq \exp\left[-\frac{1}{\Delta(|\zeta-\zeta^*|\cdot\text{Const.})^\alpha}\right],$$

where $\Delta(x)=\int_0^x\delta(t)\,dt$ (see Lemma 5.4). If now $x\in\mathbf{R}\setminus E$, then by the Cauchy formula $(n\geqq 1)$:

$$|f^{(n)}(x)| = \frac{n!}{2\pi}\left|\int_{\partial G}\frac{f(\zeta)}{(\zeta-x)^{n+1}}\,d\zeta\right| \leqq \frac{n!}{2\pi}\cdot\text{Const.}\cdot\sup_{\zeta\in\partial G}\left|\frac{f(\zeta)}{(\zeta-x)^{n-1}}\right|$$

$$\leqq \frac{n!}{2\pi}\cdot\text{Const.}\cdot\sup_{\zeta\in\partial G}|\zeta-x|^{-(n-1)}\cdot\exp\left[-(\Delta(|\zeta-\zeta^*|\cdot\text{Const.}))^{-\alpha}\right]$$

$$\leqq \frac{n!}{2\pi}\cdot(\text{Const.})^n\cdot\sup_{\zeta\in\partial G}\frac{\exp\left[-(\Delta(|\zeta-\zeta^*|)\cdot\text{Const.})^{-\alpha}\right]}{(\text{Const.})^{n-1}(\Delta(|\zeta-\zeta^*|))^{n-1}}$$

$$\leqq (\text{Const.})^n\cdot n!\sup_{t>0} t^{-n}\cdot\exp(-t^{-\alpha}) \leqq (\text{Const.})^n\cdot n!\cdot n^{\frac{n}{\alpha}}.$$

§ 6. Cantor sets. Theorem 3

Let $(l_n)_{n\geqq 0}$ be a decreasing sequence of positive numbers such that

$$\sum_{n=0}^{\infty} 2^n l_n = 1. \tag{6.1}$$

Remove the open interval of length l_0 centered at 1/2 from the segment [0, 1]. Denote by E_0 the union of two remaining segments. Remove from each of them the concentric interval of length l_1. The union of the four mutually disjoint remaining segments is denoted by E_1. Condition (6.1) makes it possible to continue this procedure indefinitely. The union of the 2^{n+1} closed mutually disjoint segments arising at the n-th step (we call them "the segments of the rank n") will be denoted by E_n. The open intervals of length l_n removed at the n-th step will be called "intervals of the rank n". The set

$$E = \bigcap_{n\geqq 0} E_n$$

is called the Cantor type set corresponding to the sequence $(l_n)_{n\geqq 0}$. It is obvious that $|E|=0$. In this paragraph we will prove

Theorem 3′. There is a sequence $(l_n)_{n\geqq 0}$ of positive numbers satisfying (6.1) and such that the Cantor type set E corresponding to it has the following properties:

a) $E\in\mathscr{E}_\alpha$;

b) $\sum_{n=0}^{+\infty} 2^n l_n^{1-\alpha}\left(\log\frac{1}{l_n}\right)^{\alpha-\varepsilon} < +\infty$

for an arbitrary $\varepsilon>0$.

The proof of the theorem will be preceded by two lemmas.

Lemma 6.1. There is a sequence $(l_n)_{n\geqq 0}$ of positive numbers such that:

(i) $l_n > l_{n+1}$, $n\in\mathbf{Z}_+$;

(ii) the identity (6.1) holds;

(iii) there is a constant $C>0$ for which

$$\sum_{k=n+1}^{\infty} 2^k l_k < C\cdot 2^n l_n, \quad n\in\mathbf{Z}_+;$$

(iv) there is a constant $a>0$ for which

$$\frac{1}{2^{n+1} l_n^{1-\alpha}} \sum_{k=n+1}^{\infty} 2^k l_k^{1-2} > a\log\frac{1}{l_n}.$$

Proof. Let $\mathcal{N}$ be a large positive integer. We define l_n by means of the equation

$$2^n l_n^{1-\alpha} = \frac{1}{n^{1+\alpha}} \tag{6.2}$$

if $n>\mathcal{N}$. It is possible to choose the values $l_0>l_1>\ldots>l_N$ satisfying (6.1) (because of $2^n l_n=o\left(\frac{1}{n}\right)$, $n\to+\infty$). Then conditions (i) and (ii) will be satisfied. For $n>\mathcal{N}$ we have

$$\frac{2^{n+1}l_{n+1}}{2^n l_n}=2\cdot\frac{l_{n+1}}{l_n}=2^{1-\frac{n+1}{1-\alpha}}\cdot(n+1)^{-\frac{1+\alpha}{1-\alpha}}\cdot 2^{\frac{n}{1-\alpha}}\cdot n^{\frac{1+\alpha}{1-\alpha}}<2\cdot 2^{-\frac{1}{1-\alpha}}<1.$$

Thus (iii) holds also. Let us verify (iv). Suppose $n>\mathcal{N}$. Then

$$\frac{1}{2^{n+1}l_n^{1-\alpha}}\sum_{k=n+1}^{\infty}2^k l_k^{1-\alpha}=\frac{n^{1+\alpha}}{2}\cdot\sum_{k=n+1}^{\infty}\frac{1}{k^{1+\alpha}}>\frac{n^{1+\alpha}}{2}\cdot\int_{n+1}^{\infty}\frac{dx}{x^{1+\alpha}}$$

$$=\frac{1}{2\alpha}\cdot\frac{n^{1+\alpha}}{(n+1)^{\alpha}}=\frac{n}{2\alpha}\cdot\left(\frac{n}{n+1}\right)^{\alpha}.$$

But (6.2) implies

$$\log\frac{1}{l_n}=\frac{\log 2}{1-\alpha}\cdot n+\frac{1+\alpha}{1-\alpha}\log n.$$

Therefore we can take $a=\dfrac{1-\alpha}{4\alpha\log 2}$ if $\mathcal{N}$ is sufficiently large. Such an a satisfies (iv) for all $n>\mathcal{N}$. Diminishing a if necessary we obtain (iv) for all n, $n\in\mathbf{Z}_+$.

Lemma 6.2. Let $(C_n)_{n\geqq 0}$ be an increasing sequence of positive numbers and suppose the function f satisfies

$$f(x)\geqq\frac{C_n}{l_n^{\alpha}}$$

for every x from every interval ω_n of rank n. Then

$$[T_E f(x)]^{\frac{\alpha}{1+\alpha}}>\left(\frac{a}{C^2}\right)^{\frac{\alpha}{1+\alpha}}\cdot C_n^{\frac{\alpha}{1+\alpha}}\cdot\left(\log\frac{1}{l_n}\right)^{\frac{\alpha}{1+\alpha}}\cdot\frac{1}{l_n^{\alpha}}$$

for all x, $x\in\omega_n$.

Proof. Let Δ_n be the segment of rank n neighbouring ω_n by the property (iii) we have

$$|\Delta_n|<2^{-n-1}\cdot C\cdot 2^n\cdot l_n=\frac{1}{2}C\cdot l_n$$

because $|\omega_n|=l_n$ and

$$|\Delta_n|=\frac{1}{2^{n+1}}\cdot\sum_{K=n+1}^{\infty}2^k l_k.$$

Thus for $x \in \omega_n$ and $t \in \Delta_n$

$$(t-x)^2 \leqq \left(1+\frac{c}{2}\right)^2 l_n^2 < C^2 l_n^2$$

because without loss of generality we may assume $C>2$. Consequently for all x, $x \in \omega_n$,

$$T_E f(x) \geqq \int_{\Delta_n} \frac{f(t)}{(t-x)^2}\, dt \geqq \frac{1}{C^2 l_n^2} \int_{\Delta_n} f(t)\, dt.$$

Estimate the last integral from below. It equals the sum of integrals taken on all intervals of rank $>n$ which are contained in Δ_n. Inequality (6.3) being true for all intervals of rank n we have

$$\int_{\Delta_n} f(t)\, dt \geqq \frac{1}{2^{n+1}} \sum_{k=n+1}^{\infty} 2^k l_k \cdot \frac{C_k}{l_k^\alpha}.$$

Using the monotonicity of the sequence $(C_n)_{n\geqq 0}$ and property (IV) we obtain the inequality

$$T_E f(x) > \frac{1}{C^2} \cdot \frac{1}{l_n^{1+\alpha}} \cdot C_n \cdot a \cdot \log \frac{1}{l_n}$$

for every $x \in \omega_n$. All that remains now is to take the $\dfrac{\alpha}{1+\alpha}$-th power in both parts of the last estimate.

Proof of Theorem 3'. Define by induction the family $i \to (C_{n,i})_{n\geqq 0}$, $i \in \mathbf{Z}_+$ of sequences $(C_{n,i})_{n\geqq 0}$. Let

$$C_{n,0} = 1, \quad n \in \mathbf{Z}_+.$$

Note that for $x \in \omega_n$

$$U_0(x) = \frac{1}{\varrho(x, E)^\alpha} > \frac{1}{l_n^\alpha}. \tag{6.4}$$

Set for $i>0$ inductively

$$C_{n,i} = \left(\frac{a}{C^2}\right)^{\frac{\alpha}{1+\alpha}} \cdot \left(\log \frac{1}{l_n}\right)^{\frac{\alpha}{1+\alpha}} \cdot C_{n,i-1}^{\frac{\alpha}{1+\alpha}} \tag{6.5}$$

It is obvious (see Lemmas 6.2 and 6.4) that

$$U_i(x) > \frac{C_{n,i}}{l_n^\alpha}$$

for every $x \in \omega_n$. The equality (6.5) implies

$$C_{n,i} = \left[\frac{a}{C^2} \cdot \log \frac{1}{l_n}\right]^{\frac{\alpha}{1+\alpha} + \dots + \left(\frac{\alpha}{1+\alpha}\right)^i}$$

Since the function U majorizes every U_i we have

$$U(x) \geqq \frac{1}{l_n^\alpha}\left[\frac{a}{C^2}\log\frac{1}{l_n}\right]^\alpha \quad (\in C\omega_n).$$

Consequently

$$\int_0^1 U(x)\,dx = \left(\frac{a}{C^2}\right)^\alpha \cdot \sum_{n=0}^\infty 2^n l_n^{1-\alpha}\left(\log\frac{1}{l_n}\right)^\alpha.$$

But the series to the left diverges because

$$2^n l_n^{1-\alpha}\left(\log\frac{1}{l_n}\right)^\alpha \sim \frac{\text{Const.}}{n}, \quad n \to +\infty.$$

Therefore $E\in\mathscr{E}_\alpha$ by Lemma 3.2. On the other hand it is easy to see that condition b) of Theorem 3 holds.

Unfortunately we do not know whether $E\notin\mathscr{E}_\alpha$ or not, if

$$\sum l_\nu^{1-\alpha}\left(\log\frac{1}{l_\nu}\right)^\alpha < +\infty.$$

§ 7. Countable sets with a single limit point

Let the decreasing sequence $(x_n)_{n\geqq 0}$ of positive numbers tend to zero and consider the compact set $E=\{0, x_0, x_1, \dots\}$. We number its complementary intervals as follows:

$$\Delta_{-\infty} \overset{\text{def}}{=} (-\infty, 0), \quad \Delta_{+\infty} \overset{\text{def}}{=} (x_0, +\infty),$$

$$\Delta_n \overset{\text{def}}{=} (x_n, x_{n-1}), \quad n\in\mathbf{N}.$$

The length $l_n=l_n(E)$ of the interval Δ_n $(n\in\mathbf{N})$ equals $x_{n-1}-x_n$. Suppose that the sequence (l_n) satisfies the following conditions

$$0 < l_{n+1} \leqq l_n \quad (n\in\mathbf{N}), \tag{7.1}$$

$$\varliminf_{n\to+\infty} n^2 l_n^{1-\alpha} > 0, \tag{7.2}$$

$$\varliminf_{n\to+\infty} \frac{l_{2n}}{l_n} > 0, \tag{7.3}$$

$$\sum_{n=1}^\infty l_n^{1-\alpha} < +\infty. \tag{7.4}$$

Theorem 4′. $E\notin\mathscr{E}_\alpha$.

Remarks 1. Conditions (7.1) and (7.3) require that the sequence (l_n) tends to zero with a certain regularity while the condition (7.2) restricts the rapidity of convergence (from above). If $l_n^{1-\alpha}=0(n^{-2})$, then $E\notin\mathscr{E}_\alpha$ by Theorem 2. However the method of proof of Theorem 4 does not allow to drop condition (7.2).

2. Since conditions (7.2) and (7.3) being purely technical the following assertion seems to be true.

Conjecture. A countable set E with the single limit point does not belong to $\mathscr{E}_\alpha$ if and only if $E\in(C_\alpha)$.

Now we are going to deduce Theorems 4 and 5 from Theorem 4′.

Proof of Theorem 4. Construct the set E so that for every n, $n\in\mathbf{N}$,

$$l_n^{1-\alpha}=\frac{1}{n\log(1+n)(\log\log(2+n))^2}.$$

It is easy to see that (7.1)—(7.4) hold and thus $E\notin\mathscr{E}_\alpha$. Moreover

$$\log\frac{1}{l_n}\sim\frac{1}{1-\alpha}\log n,\quad n\to+\infty.$$

Therefore for every ε, $\varepsilon>0$, we have

$$\sum_{n=1}^{\infty}l_n^{1-\alpha}\left(\log\frac{1}{l_n}\right)^{\varepsilon}=+\infty.$$

Remark. Let $p\in\mathbf{N}$ and ${}_p\log x\overset{\text{def}}{=}\underbrace{\log\ldots\log}_{p\text{ times}}x$. Then for an arbitrary p there exists a set E, $E\notin\mathscr{E}_\alpha$, such that

$$\sum_{n=1}^{\infty}l_n^{1-\alpha}\left({}_p\log\frac{1}{l_n}\right)^{\varepsilon}=+\infty$$

for every ε, $\varepsilon>0$. The proof requires only obvious modifications of the formula defining the sequence $(l_n^{1-\alpha})_{n\geq1}$.

Proof of Theorem 5. Let E_1 be the Cantor type set constructed in § 6 and denote by m_n the length of interval of rank n. If n is large enough then

$$2^n\cdot m_n^{1-\alpha}=n^{-(1+\alpha)}\tag{7.5}$$

(see (6.2)). Set $l_n=m_k$ for $2^k\leq n<2^{k+1}$ $(k\in\mathbf{Z}_+)$ and construct the set E_2 with a single limit point such that $l_n(E_2)=l_n$, $n\in\mathbf{N}$. We have $E_1\in\mathscr{E}_\alpha$ by Theorem 3′. It is clear that E_1 and E_2 have the same family of lengths of complementary intervals (counting multiplicities). We have only to prove that $E_2\notin\mathscr{E}_\alpha$. To do this we check conditions (7.1)—(7.4) with respect to $(l_n)_{n\geq1}$. The condition (7.1) obviously holds and (7.4) follows from (7.5). If now $2^k\leq n<2^{k+1}$ then

$$n^2l_n^{1-\alpha}\geq2^{2k}m_k^{1-\alpha}=2^k\cdot k^{-(1+\alpha)}$$

and

$$\frac{l_{2n}}{l_n} = \frac{m_{k+1}}{m_k} = 2^{-\frac{1}{1-\alpha}} \cdot \left(\frac{k}{k+1}\right)^{\frac{1+\alpha}{1-\alpha}}$$

which implies (7.2) and (7.3).

The remaining part of the paragraph is devoted to the proof of Theorem 4′ which consists of verifying the condition (α). Define the auxiliary function

$$f_E(x) = \begin{cases} \max\left(1, \varrho(x, E)^{-\frac{2\alpha}{1+\alpha}}\right) & \text{if} \quad x < 0 \\ \varrho(x, E)^{-\alpha} & \text{if} \quad 0 < x < x_0 \\ \max\left(1, \varrho(x, E)^{-\alpha}\right) & \text{if} \quad x_0 < x \end{cases}$$

and show that for every x, $x \in \mathbf{R} \setminus E$,

$$T_E f_E(x) \leqq \text{Const.}\, f_E(x)^{1+\frac{1}{\alpha}}. \tag{7.6}$$

Then $E \notin \mathscr{E}_\alpha$ will follow from theorem 1 because $f_E(x) \geqq \varrho(x, E)^{-\alpha}$ and $f_E \in L^1\left(\frac{dx}{1+x^2}\right)$.

Lemma 7.1. The inequality (7.6) holds if $x < 0$.

Proof. It is sufficient to prove (7.6) near the origin and near infinity because $(-\infty, 0) \cap E = \emptyset$. If $x \to -\infty$ then

$$\lim T_E f_E = 0, \quad \lim f_E(x)^{1+\frac{1}{\alpha}} = 1.$$

Let now $x \to 0-$. Then

$$T_E f_E(x) = \int_0^{+\infty} \frac{f_E(t)}{(t-x)^2}\, dt \leqq \frac{1}{\varrho(x, E)^2} \int_0^1 f_E(t)\, dt + \int_1^{+\infty} \frac{f_E(t)}{t^2}\, dt$$

$$= O\left(\frac{1}{\varrho(x, E)^2}\right).$$

It remains only to notice that $f_E(x)^{1+\frac{1}{\alpha}} = \varrho(x, E)^{-2}$ if $-1 < x < 0$.

Consider now $x > 0$. Estimate first the contribution of the interval $\Delta_{-\infty}$ to the integral $T_E f_E$.

Lemma 7.2. If $x > 0$ then

$$\int_{-\infty}^0 \frac{f_E(t)}{(t-x)^2}\, dt \leqq \text{Const.}\, f_E(x)^{1+\frac{1}{\alpha}}.$$

Proof. Elementary computations show that

$$\int_{-\infty}^{0} \frac{f_E(t)}{(t-x)^2}\,dt = \int_0^1 \frac{dt}{(t+x)^2 t^{\frac{2\alpha}{1+\alpha}}} + \int_0^1 \frac{dt}{(t+x)^2} \leqq \frac{\text{Const.}}{x^{1+\frac{2\alpha}{1+\alpha}}}$$

if $0<x<x_0+1$. If $x \geqq x_0+1$ then the required inequality becomes obvious, because in this case $f_E(x)^{1+\frac{1}{\alpha}} \equiv 1$. It is therefore sufficient to show that

$$x^{-1-\frac{2\alpha}{1+\alpha}} \leqq \text{Const.}\, \varrho(x, E)^{-1-\alpha} \quad (0 < x < x_0+1). \tag{7.7}$$

If $x_0 \leqq x$ we can obtain (7.7) by choosing Const. sufficiently large. Suppose now $x \in \Delta_n$ $(n \in \mathbf{N})$ It is then obvious that

$$\sum_{k>n} l_k < x \quad \text{and} \quad l_n^{-1-\alpha} < \varrho(x, E)^{-1-\alpha}.$$

Consequently (7.7) is implied by the inequality

$$l_n^{1+\alpha} \leqq \text{Const.}\, \Big(\sum_{k>n} l_k\Big)^{1+\frac{2\alpha}{1+\alpha}} \quad (n \in \mathbf{N}). \tag{7.8}$$

Condition (7.3) implies

$$\sum_{k>n} l_k > \sum_{k=n+1}^{2n} l_k \geqq \text{Const.}\, n l_n$$

and (7.8) follows from the estimate

$$l_n^{1+\alpha} \leqq \text{Const.}\, (n l_n)^{1+\frac{2\alpha}{1+\alpha}}$$

which holds because of (7.2).

So we only have to prove that for every x, $x>0$,

$$\int_{\mathbf{R}_+ \setminus I_x} \frac{f_E(t)}{(t-x)^2}\,dt \leqq \text{Const.}\, f_E(x)^{1+\frac{1}{\alpha}}. \tag{7.9}$$

The proof of (7.9) is an easy matter if $x \in \Delta_\infty$. Therefore we will assume that $0<x<$ $<x_0$. We recall that in this case

$$f_E(x) = \varrho(x, E)^{-\alpha}.$$

The next lemma allows us to estimate the above integral over a bounded complementary interval of E.

Lemma 7.3. Let $\Delta = (a, b)$ be a complementary interval of E, $l = b-a$, $x \notin \Delta$. Denote by $d = d(x)$ the distance from the point x to the interval Δ. Then two positive numbers C_1, C_2 exist, depending only on α, and such that

$$\frac{c_1}{d^{1+\alpha}} \min\left\{1, \left(\frac{l}{d}\right)^{1-\alpha}\right\} \leqq \int_\Delta \frac{\varrho(t, E)^{-\alpha}}{(t-x)^2}\,dt \leqq \frac{c_2}{d^{1+\alpha}} \min\left\{1, \left(\frac{l}{d}\right)^{1-\alpha}\right\}.$$

Proof. Without loss of generality we may assume $\Delta=(0, l)$, $x<0$. Denoting the integral to be estimated by $I(x)$ we have

$$I(x) \geqq \int_0^{l/2} \frac{dt}{t^\alpha (t+d)^2} = \frac{1}{d^{1+\alpha}} \int_0^{l/2d} \frac{dt}{t^\alpha (1+t^2)}.$$

It is also obvious that $I(x)$ does not exceed the integral in the right side of the preceding inequality multiplied by two. Now the completion of the proof is an elementary task.

Let $x \in \Delta_n$ $(n \in \mathbf{N})$. Lemma 7.3 implies that

$$\int_{\Delta_{n\pm 1}} \frac{f_E(t)}{(t-x)^2} dt \leqq \frac{\text{Const.}}{\varrho(x, E)^{1+\alpha}}$$

(we assume here $\Delta_0 \overset{\text{def}}{=} \Delta_\infty$). First we discuss a finite family of intervals Δ_k $(k=n-2, \ldots, 0)$ and estimate their contribution to the integral $T_E f_E$. By Lemma 7.3 we have

$$\sum_{k=0}^{n-2} \int_{\Delta_k} \frac{f_E(t)}{(t-x)^2} dt \leqq \text{Const.} \sum_{k=0}^{n-2} \frac{1}{(l_{n-1}+\ldots+l_{k+1})^{1+\alpha}} =$$

$$= \text{Const.} \sum_{k=0}^{n-2} \frac{1}{l_n^{1+\alpha}} \cdot \frac{1}{\left(\frac{l_{n-1}}{l_n}+\ldots+\frac{l_{k+1}}{l_n}\right)^{1+\alpha}} \leqq \frac{\text{Const.}}{l_n^{1+\alpha}} \cdot \sum_{k=0}^{n-2} \frac{1}{(n+k-1)^{1+\alpha}}$$

(see (7.1)). Since the inequality $\varrho(x, E) \leqq \frac{l_n}{2}$ holds on Δ_n this contribution does not exceed

$$\text{Const.}\, \varrho(x, E)^{-1-\alpha}.$$

Finally we estimate the sum

$$S_n(x) \overset{\text{def}}{=} \sum_{k>n+1} \int_{\Delta_k} \frac{f_E(t)}{(t-x)^2} dt.$$

It follows from lemma 7.3 that

$$S_n(x) \leqq \text{Const.} \sum_{k>n+1} \frac{l_k^{1-\alpha}}{(l_{k-1}+\ldots+l_{n+1})^2},$$

because the length of Δ_k is less than the distance from x to Δ_k if $k>n+1$. Divide the last sum into two parts and estimate each of them separately. Using (7.3) we have

$$\sum_{k=n+2}^{2n} \frac{l_k^{1-\alpha}}{(l_{K-1}+\ldots+l_{n+1})^2} \leqq \sum_{k=n+2}^{2n} \frac{l_n^{1-\alpha}}{l_n^2 \left(\frac{l_{k-1}}{l_n}+\ldots+\frac{l_{n+1}}{l_n}\right)^2}$$

$$< \frac{\text{Const.}}{l_n^{1+\alpha}} \cdot \sum_{k=1}^{n} \frac{1}{k^2} \leqq \frac{\text{Const.}}{l_n^{1+\alpha}}.$$

And

$$\sum_{k>2n} \frac{l_k^{1-\alpha}}{(l_{k-1}+\ldots+l_{n+1})^2} = \sum_{k>2n} \frac{l_k^{1-\alpha}}{l_n^2\left(\frac{l_{k-1}}{l_n}+\ldots+\frac{l_{n+1}}{l_n}\right)^2}$$

$$\leqq \frac{\text{Const.}}{n^2 l_n^2} \cdot \sum_{k>2n} l_k^{1-\alpha} \leqq \frac{\text{Const.}}{l_n^{1+\alpha}} \cdot \frac{1}{n^2 l_n^{1-\alpha}} \leqq \frac{\text{Const.}}{l_n^{1+\alpha}}.$$

(the first inequality depends on (7.3) and the last depends on (7.2)). The proof of the theorem is now finished.

§ 8. Sufficient conditions

We have already seen that the testing of the condition (α) is not an easy matter. It is natural therefore to sacrifice the generality and seek simpler sufficient conditions. This is partly done in § 0 and § 5. Here we give some sufficient conditions expressed in terms of the maximal function of Hardy and Littlewood. On the other side this will enable us to estimate the gap between conditions (C_α) of Carleson and (V_α) of Vinogradov. Moreover we will give a new interpretation of the condition (PS_2) of Pavlov and Suturin (see § 0). For technical reasons it will be convenient here to replace the half-plane by the disc. Now G_α will stand for the Gevrey class in the open unit disc $\mathbf{D}$, $\mathbf{T}$ for the unit circle, m for the normalized Lebesgue measure on $\mathbf{T}$. We will identify $\mathbf{T}$ with the group $\mathbf{R}/2\pi\mathbf{Z}$ and use symbol x to denote the identity map of $\mathbf{R}/2\pi\mathbf{Z}$. Let $K(t)=t^{-2}$ ($|t|\leqq\pi$) and continue K 2π-periodically onto the whole axis $\mathbf{R}$. For $f\in L^1(\mathbf{T})$, $f\geqq 0$, we consider

$$\mathcal{M}f(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} f(t)\,dt$$

the Hardy—Littlewood maximal function. The symbol f^* will be used to denote the non-increasing rearrangement of a non-negative function f. Recall that f^* is defined on the segment [0, 1] and is decreasing there and that for every λ, $\lambda\geqq 0$,

$$m\{f \geqq \lambda\} = \sup\{t : f^*(t) \geqq \lambda\}.$$

In the case of the circle the condition (α) gets the following form:

There is a function f_E, $f_E\in L^1(\mathbf{T})$, such that

(i) $\varrho(x, E)^{-\alpha} \leqq f_E(x)$;

(ii) $\int_{Cl_x} K(x-t) f_E(t)\,dm(t) \leqq \text{Const.}\, f_E(x)^{1+\frac{1}{\alpha}}$. $\qquad(\alpha)$

A well known estimate of the Poisson integral (see [19], p. 77—80) shows that

$$\varrho(x, E)\cdot\int_{Cl_x} K(x-t) f_E(t)\, dm(t) \leqq \text{Const.}\, Mf_E(x).$$

Using this inequality we can weaken (α) to (α^*):

There is a function f_E, $f_E \in L^1(\mathbf{T})$ and $f_E > 0$ such that

$$\mathcal{M} f_E(x) \leqq \text{Const.}\, \varrho(x, E)\cdot f_E(x)^{1+\frac{1}{\alpha}}. \tag{α^*}$$

Notice that the inequality $f_E(x) \geqq \text{Const.}\, \varrho(x, E)^{-\alpha}$ follows from (α^*) and from the trivial estimate $f_E \leqq \mathcal{M} f_E$, and so $(\alpha^*) \Rightarrow (\alpha)$.

Replacing $\varrho(x, E)\cdot f_E(x)^{1/\alpha}$ by one in the right side of (α^*) we obtain the condition (α^{**}) which obviously implies (α^*):

There is a function f_E, $f_E \in L^1(\mathbf{T})$, such that

$$\left.\begin{aligned} &\text{(i)}\ \ \varrho(x, E)^{-\alpha} \leqq f_E(x);\\ &\text{(ii)}\ \ \mathcal{M} f_E(x) \leqq \text{Const.}\, f_E(x). \end{aligned}\right. \tag{α^{**}}$$

Our first aim is to show that $(V_\alpha) \Rightarrow (\alpha^*)$. Let F_0 be the decreasing rearrangement of the function $x \to \varrho(x, E)^{-1}$, $E_\delta = \{t \in \mathbf{T} : \varrho(t, E) \leqq \delta\}$ and ω the function from § 5 satisfying the condition (5.1).

Theorem 8.1. Each of the following conditions:

$$\int_{\mathbf{T}} \left[\frac{1}{\varrho(x, E)}\cdot \omega\left(\frac{1}{\varrho(x, E)}\right)\right]^{\alpha} dm < +\infty \tag{8.1}$$

$$\int_0^1 \left(\frac{F_0(t)}{t}\right)^{\frac{\alpha}{1+\alpha}} dt < +\infty \tag{8.2}$$

$$\int_0^1 \left(\frac{mE_\delta}{\delta^\alpha}\right)^{\frac{1}{1+\alpha}} \frac{d\delta}{\delta} < +\infty \tag{8.3}$$

implies $E \in (\alpha^*)$ (and consequently $E \notin \mathcal{E}_\alpha$).

Remark. It is easy to see that (8.2) and (8.3) are equivalent. Moreover $(8.1) \Rightarrow (8.2)$ (see the proof of Lemma (5.2)). It is therefore sufficient to prove that $(8.2) \Rightarrow E \in (\alpha^*)$.

It will be convenient to divide the proof of the theorem into two lemmas.

Lemma 8.1. The inclusion $E \in (\alpha^*)$ holds if it is possible to find a decreasing function F defined and summable on $(0, 1]$ and such that

$$\left[F_0(t)\cdot\frac{1}{t}\int_0^t F(S)\, dS\right]^{\frac{\alpha}{1+\alpha}} = F(t) \quad (t \in (0, 1]). \tag{8.4}$$

Proof. It is easy to see (compare with Lemma 3.2) that $E \in (\alpha^*)$ if and only if the sequence $(f_n)_{n \geqq 0}$:

$$f_0(x) \stackrel{\text{def}}{=} \varrho(x, E)^{-\alpha}, \quad f_{n+1} \stackrel{\text{def}}{=} [f_0^{1/\alpha} \cdot \mathcal{M} f_n]^{\frac{\alpha}{1+\alpha}} \quad (n \in \mathbf{Z}_+) \tag{8.5}$$

is bounded in $L^1(\mathbf{T})$. It is well known (see [20]) that

$$(f \cdot g)^* \leqq f^* \cdot g^*$$

and

$$\varphi(f)^* = \varphi(f^*)$$

for an arbitrary increasing function φ. By the theorem of Hardy and Littlewood (see [20])

$$(\mathcal{M} f)^*(t) \leqq \frac{2}{t} \int_0^t f^*(S)\, dS.$$

Using the definition of the sequence $(f_n)_{n \geqq 0}$ and the facts mentioned above we conclude that

$$f_0^* = F_0^\alpha, \quad f_{n+1}^*(t) \leqq \left[F_0 \cdot \frac{2}{t} \int_0^t f_n^*(S)\, dS\right]^{\frac{\alpha}{1+\alpha}}. \tag{8.6}$$

The function F decreases and so

$$F(t) \leqq \frac{1}{t} \int_0^t F(S)\, dS.$$

Thus it follows from (8.4) that $F_0^\alpha \leqq F$. From this and (8.6) we deduce inductively that

$$f_n^* \leqq 2^\alpha F \quad (n \in \mathbf{Z}_+).$$

Hence

$$\int_{\mathbf{T}} f_n\, dm = \int_0^1 f_n^*\, dt \leqq 2^\alpha \int_0^1 F(t)\, dt.$$

Lemma 8.2. The equation (8.4) admits a decreasing solution F, $F \in L^1(0, 1)$, if and only if

$$\int_0^1 F_0^{\frac{\alpha}{1+\alpha}}(t) \cdot t^{-\frac{\alpha}{1+\alpha}}\, dt < +\infty. \tag{8.7}$$

Proof. It follows from (8.4) that

$$\left[\frac{F_0(t)}{t}\right]^{\frac{\alpha}{1+\alpha}} = \frac{F(t)}{\left[\int_0^t F(S)\, dS\right]^{\frac{\alpha}{1+\alpha}}}.$$

Integrating this equality from t_0 to t $(0<t_0<1)$ we obtain

$$\int_{t_0}^{1}\left[\frac{F_0(t)}{t}\right]^{\frac{\alpha}{1+\alpha}}dt=\int_{t_0}^{1}\frac{d\left(\int_0^t F(S)\,dS\right)}{\left[\int_0^t F(S)\,dS\right]^{\frac{\alpha}{1+\alpha}}}$$

$$=(1+\alpha)\left(\int_0^1 F(S)\,dS\right)^{\frac{1}{1+\alpha}}-(1+\alpha)\left(\int_0^{t_0} F(S)\,dS\right)^{\frac{1}{1+\alpha}}.$$

Therefore

$$\int_0^t F(S)\,dS=\left\{\left(\int_0^1 F(S)\,dS\right)^{\frac{1}{1+\alpha}}-\frac{1}{1+\alpha}\int_t^1\left[\frac{F_0(S)}{S}\right]^{\frac{\alpha}{1+\alpha}}dS\right\}^{1+\alpha}. \tag{8.8}$$

Hence the summability of F implies (8.7). Suppose now that (8.7) holds. Differentiating formally the equality (8.8) we define F by means of the formula

$$F(t)=\left\{\frac{1}{1+\alpha}\int_0^t\left[\frac{F_0(S)}{S}\right]^{\frac{\alpha}{1+\alpha}}dS\right\}^{\alpha}\cdot\left\{\frac{F_0(t)}{t}\right\}^{\frac{\alpha}{1+\alpha}}. \tag{8.9}$$

The function F is obviously summable and satisfies (8.4). To prove that F is decreasing notice first of all that F is continuous on $(0, 1]$. This follows from (8.9) and from the continuity of F_0. The function $t\to\left[\frac{F_0(t)}{t}\right]^{\frac{\alpha}{1+\alpha}}$ decreases on $(0, 1]$. Thus (8.9) implies the inequality $F\geqq\left(\frac{1}{1+\alpha}\right)^{\alpha}F_0^{\alpha}$, and consequently

$$\lim_{t\to 0+}F(t)=+\infty.$$

Suppose F is not decreasing. Then two values $t_1, t_2, 0<t_1<t_2$ exist such that

$$F(t_1)<F(t_2).$$

Without loss of generality we may assume that

$$F(t_1)=\inf_{0<t<t_2}F(t).$$

Since F is continuous it is possible to find for every y, $F(t_1)<y<F(t_2)$, numbers $a_y, a_y<t_1$, and b_y, $b_y,>t_1$, such that

$$F(a_y)=F(b_y)\quad\text{and}\quad F(t)<y$$

for every t, $t\in(a_y, b_y)$. But F_0 decreases and F satisfies (8.4). Hence

$$\frac{1}{a_y}\int_y^{a_y}F(S)\,dS\leqq\frac{1}{b_y}\int_0^{b_y}F(S)\,dS$$

and so

$$\frac{1}{a_y}\int_0^{a_y} F(S)\,dS \leqq \frac{1}{b_y-a_y}\int_{a_y}^{b_y} F(S)\,dS < y.$$

If now $y \downarrow F(t_1)$ then $a_y \uparrow a$, $a \leqq t_1$. Therefore

$$\frac{1}{a}\int_0^a F(S)\,dS \leqq F(t_1).$$

But this is impossible because $F(S) \geqq F(t_1)$ if $S \in (0, a)$, and $F(S) > F(t_1)$, if S is small enough. This contradiction completes the proof of the lemma.

Corollary. $(V_\alpha) \Rightarrow (\alpha^*)$.

Proof. Set $\omega(x) = (\log x)^{1+\varepsilon}$ $(\varepsilon > 0)$ in (8.1).

The following example shows that (α^*) does not follow from (α).

Example. Let E be the set with a single limit point constructed in § 7 and let

$$l_n^{1-\alpha} = \frac{C_1}{n\big(\log(1+n)\big)^{1+\alpha}}, \quad n \in \mathbf{N},$$

where the constant C_1, $C_1 > 0$, is defined by the equality

$$\sum_{n \geqq 1} l_n = \pi.$$

Denote the image of the set E under the standard map $\mathbf{R} \to \mathbf{R}/2\pi\mathbf{Z}$ by the same letter. It is not hard to see that

$$\frac{C_2}{x\left(\log \dfrac{2\pi}{x}\right)^{1+\alpha}} \leqq \frac{1}{\varrho(x, E)^\alpha} \tag{8.10}$$

$(C_2 > 0)$ if $0 < x < \pi \pmod{2\pi}$. It follows from Theorem 4′ that $E \notin \mathscr{E}_\alpha$. We have to prove now that $E \notin (\alpha^*)$. This will be shown if we prove that

$$\lim_{n \to +\infty} \int_{\mathbf{T}} f_n\,dm = +\infty$$

(the definition of f_n is given in (8.5)). Using (8.10) and the definition of f_1 it is easy to verify that

$$f_1(x) \geqq \frac{C_2}{2x\left(\log \dfrac{2\pi}{x}\right)^{1+\alpha\cdot\frac{\alpha}{1+\alpha}}}.$$

From this we deduce inductively that

$$f_n(x) \geqq \frac{C_2}{2x\left(\log \dfrac{2\pi}{x}\right)^{1+\alpha\cdot\left(\frac{\alpha}{1+\alpha}\right)^n}}, \quad n \in \mathbf{N}.$$

It remains only to notice that

$$\int_0^{\pi} \frac{dx}{x \log \frac{2\pi}{x}} = +\infty.$$

In conclusion we will discuss the condition (PS_α). Remember that $E \in (PS_\alpha)$ (see § 0) if for every x, $x \in \mathbf{R} \setminus E$, we have

$$\int_{Cl_x} \frac{1}{(t-x)^2} \cdot \frac{dt}{\varrho(xt, E)^\alpha} \leqq \frac{\text{Const.}}{\varrho(x, E)^{1+\alpha}} \qquad (PS_\alpha).$$

Here it will be convenient to return again to the case of the half-plane.

Theorem 8.2. Let E be a compact subset of the real line $\mathbf{R}$. A function f, $f \in G_\alpha$, with the property

$$-\log |f(x)| \asymp \frac{1}{\varrho(x, E)^\alpha} \;{}^{*)} \qquad (8.11)$$

exists if and only if $E \in (PS_\alpha)$.

Proof. If $E \in (PS_\alpha)$ then the existence of an outer function $f \in G_\alpha$, satisfying (8.11) is proved in [10] (or can be deduced from Theorem 1). The inverse follows from Theorem 1.

§ 9. Condition (α^{**}), the Muckenhoupt condition and interpolation sets for Gevrey classes

9.1. The *Muckenhoupt condition.* Let W, $0 < W \leqq +\infty$, be a measurable function on the circumference $\mathbf{T}$ and let $L^p(W, \mathbf{T})$ be the space of all functions f whose p-th power is summable with the weight W:

$$\int_{\mathbf{T}} |f|^p \cdot W dm < +\infty.$$

Consider a linear (or even sublinear) operator T in the space $L^p(W, \mathbf{T})$ defined on a dense set. Condition (A_p) we are dealing with in this section turns out to be necessary and sufficient for T to be of strong type (p, p) provided T belongs to a suitable class of operators (for instance T is the Hilbert transform or the Hardy—Littlewood maximal transform).

This fact was first discovered by Muckenhoupt in [16] for the Hardy—Littlewood maximal transform. Following Muckenhoupt we say that the weight W

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*) The symbol $a \asymp b$ means that there are numbers $C_1, C_2 > 0$ such that $C_1 \cdot b \leqq a \leqq C_2 \cdot b$.

satisfies condition (A_p) $(1<p<+\infty)$ if there exists a constant C_p, $C_p>0$, such that for every interval I, $I\subset \mathbf{T}$, the following inequality holds

$$\left(\frac{1}{|I|}\int_I W dm\right)\cdot\left(\frac{1}{|I|}\int_I W^{-\frac{1}{p-1}} dm\right)^{p-1} \leqq C_p. \qquad (A_p)$$

In the case $p=1$ this condition becomes

$$\frac{1}{|I|}\int_I W dm \leqq C_1 \cdot \operatorname{ess\,inf}_{x\in I} W(x). \qquad (A_1)$$

It is easy to see that (A_1) is equivalent with

$$\mathscr{M}W(x) \leqq C_1 \cdot W(x)$$

for almost all $x, x\in \mathbf{T}$ (may be with another constant C_1, $C_1>0$). We will use (A_1) in this last form.

By the Minkowski inequality $(A_p)\Rightarrow(A_r)$ if $r>p$. Muckenhoupt found the highly nontrivial and deep result that $(A_p)\Rightarrow(A_{p-\varepsilon})$ for a positive ε, $\varepsilon>0$, if $p>1$ (more precisely: for every $W\in(A_p)$ there is a positive number ε such that $W\in(A_{p-\varepsilon})$). We need two main lemmas of the article [16] (see Lemma 3 and 4 respectively).

Lemma A. Let $1\leqq p<+\infty$, $W\geqq 0$, I be a fixed interval, $I\subset\mathbf{T}$, such that for every interval J, $J\subset I$,

$$\left(\frac{1}{|J|}\int_J W dm\right)\left(\frac{1}{|J|}\int_J W^{-\frac{1}{p-1}} dm\right)^{p-1} \leqq K.$$

Then for every S, $0<S<\frac{|I|}{20}$ we have

$$\int_0^S W^*(t)\,dt \leqq 20\, K\, 3^{p-1} S W^*(S)$$

(recall that W^* denotes the decreasing rearrangement of the function $W\,|I$).

Lemma B. Let h, $h\geqq 0$, be a decreasing function defined on $(0, l)$ and suppose there is a constant D, $D>1$, such that

$$\int_0^S h(t)\,dt \leqq DSh(S)$$

for every S, $0<S<\frac{l}{20}$. Then

$$\left(\frac{1}{l}\int_0^l h^r(t)\,dt\right)^{1/r} \leqq \frac{20}{\left(1-r\,\frac{D-1}{D}\right)^{1/r}}\cdot\frac{1}{l}\int_0^l h(t)\,dt$$

if $1\leqq r<\frac{D}{D-1}$.

9.2. *Conditions* $(C_{\alpha+\varepsilon})$ *and* (α^{**}). Let $U, 0<U\leqq+\infty$ be a measurable function on the circumference $\mathbf{T}$.

Theorem 9.1. A majorant $W, W\geqq U$ satisfying (A_1) exists if and only if $U\in L^p(\mathbf{T})$ for a $p, p>1$.

Proof. Suppose such a majorant exists. Apply Lemma A to the function W and to the interval $I, I=\mathbf{T}$, and then Lemma B to the function $h, h=W^*$, and to the interval (0,1). Hence W (and U) belongs to $L^p(\mathbf{T})$ for a $p, p>1$.

Suppose now that $U\in L^p(\mathbf{T})$, $p>1$, and let

$$\|f\|_p\overset{\text{def}}{=}\left(\int_{\mathbf{T}}|f|^p dm\right)^{1/p}$$

be the standard norm in $L^p(\mathbf{T})$. By the theorem of Hardy—Littlewood on the maximal function (see [19], p. 15)

$$\|\mathcal{M}f\|_p\leqq C_p\|f\|_p$$

for every f, $f\in L^p(\mathbf{T})$. Define the sequence of elements of $L^p(\mathbf{T})$ inductively:

$$U_0\overset{\text{def}}{=}U,\quad U_{n+1}\overset{\text{def}}{=}MU_n,\quad n\in\mathbf{Z}_+.$$

Consider the function W,

$$W\overset{\text{def}}{=}\sum\nolimits_{n\geqq 0}\frac{1}{(2C_p)^n}U_n.$$

This function belongs to $L^p(\mathbf{T})$ (because of the inequality $\|U_n\|\leqq C_p^n\|U\|$) and moreover

$$\mathcal{M}W\leqq\sum\nolimits_{n\geqq 0}\frac{1}{(2C_p)^n}\cdot\mathcal{M}U_n=2C_p(W-U)$$

whence

$$U\leqq W\leqq\mathcal{M}W\leqq 2C_pW$$

i.e. W majorizes U and $W\in(A_1)$.

Corollary. The set E satisfies (α^{**}) if and only if $E\in(C_{\alpha+\varepsilon})$ for a positive ε.

Proof. Notice that $E\in(\alpha^{**})$ if and only if there exist W, $W\in(A_1)$, majorizing the function $x\to(\varrho(x,E))^{-\alpha}$ and we only have to apply the preceding theorem with this function playing the role of U.

9.3. *Interpolation sets for the class* G_α. Let $\mathcal{G}_\alpha$ be the set of all C^∞-functions on $\mathbf{T}$ satisfying the inequalities

$$|f^{(n)}(x)|\leqq C_f\cdot Q_f^n\cdot n!\cdot n^{\frac{n}{\alpha}},\quad n\in\mathbf{Z}_+.$$

Clearly $\mathcal{G}_\alpha\supset G_\alpha$.

Definition. A closed set $E, E\subset \mathbf{T}$, is said to be an interpolation set for G_α if

$$G_\alpha | E = \mathcal{G}_\alpha | E.$$

Interpolation sets for G_α and for other Carleman classes were described in the articles [21], [22].

Denote the class of all interpolation sets for G_α by $\mathcal{I}_\alpha$. It turns out that $E\in\mathcal{I}_\alpha$ if and only if for every interval $I, I\subset \mathbf{T}$, the following inequality is valid:

$$\frac{1}{|I|}\int_I \frac{dm}{\varrho(x,E)^\alpha} \leqq \text{Const.}\, |I|^{-\alpha}. \tag{9.1}$$

Thus the inclusion $E\in\mathcal{I}_\alpha$ implies $E\in(\alpha^{**})$ and so $E\notin\mathcal{E}_\alpha$ (the last being incidentally obvious).

The following theorem shows that the part of the class of all non-uniqueness sets for G_α occupied by $\mathcal{I}_\alpha$ is not too large.

Theorem 9.2. For every $E\in\mathcal{I}_\alpha$ there is a positive number ε such that $E\in\mathcal{I}_{\alpha+\varepsilon}$.

Remark. It is not hard to deduce from (9.1) that $\mathcal{I}_\alpha\supset\mathcal{I}_\beta$ if $\alpha<\beta$. Thus the theorem asserts that for every closed set $E, E\subset\mathbf{T}$, there is a number $\alpha_E, \alpha_E\geqq 0$ such that $\{\alpha\colon E\in\mathcal{I}_\alpha\}=(0,\alpha_E)$.

Proof of the theorem. Let $f_0\stackrel{\text{def}}{=}\varrho(x,E)^{-\alpha}$, $W=f_0|I$ and apply lemmas A and B. Notice that in the case under consideration the constant K occurring in Lemma A does not depend on I but is determined only by the constant from condition (9.1). Hence there is a number $r, r>1$, such that

$$\left(\frac{1}{|I|}\int_0^{|I|} W^{*r}\,dt\right)^{1/r} \leqq \text{Const.}\frac{1}{|I|}\int_0^{|I|} W^*\,dt,$$

or

$$\left(\frac{1}{|I|}\int_I f_0^r\,dm\right)^{1/r} \leqq \text{Const.}\frac{1}{|I|}\int_I f_0\,dm,$$

whence

$$\frac{1}{|I|}\int_I f_0^r\,dm \leqq \frac{\text{Const.}}{|I|^{r\alpha}}.$$

$$\varepsilon=(r-1)\alpha.$$

In conclusion we indicate a simple proposition characterizing the interrelation between conditions (A_p), (PS_α) and $E\in(I_\alpha)$.

Proposition 9.1.1. If $E\in\mathcal{I}_\alpha$, then $f_0=(\varrho(x,E))^{-\alpha}\in(A_1)$.
2. If $f_0\in(A_1)$ then $E\in(PS_\alpha)$.
The proof is not hard and we leave it to the reader.

§ 10. Carleman classes

In this paragraph we prove the analogue of Theorem 1 for the general Carleman classes $C_A\{M_n\}$, i.e. the set of all functions f infinitely differentiable in the closed upper half-plane $\operatorname{Im} z \geqq 0$, analytic in the open half-plane $\operatorname{Im} z > 0$, and such that

$$\sup_{\operatorname{Im} z \geqq 0} |f^{(n)}(z)| \leqq C_f \cdot Q_f^n \cdot n! \cdot M_n, \quad n \in \mathbf{Z}_+,$$

$(M_n)_{n\geqq 0}$ being a preassigned sequence of positive numbers. Suppose that $(M_n)_{n\geqq 0}$ satisfies the usual regularity conditions:

(1) the sequence $(M_n)_{n\geqq 0}$ is increasing;

(2) $\lim_{n\to+\infty} \dfrac{x^n}{M_n} = 0$ for every $x, x \geqq 1$;

(3) the sequence $(\log M_n)_{n\geqq 0}$ is convex;

(4) $M_0 \leqq e^{-1}$

(the last condition is introduced only for technical reasons).

The sequence $(M_n)_{n\geqq 0}$ generates a function φ_M (the so called characteristic of $(M_n)_{n\geqq 0}$) defined on $(0, +\infty)$, namely

$$\varphi_M(x) \stackrel{\text{def}}{=} \sup_{n\geqq 0} \left(n \log \frac{1}{x} - \log M_n \right).$$

The sequence $(M_n)_{n\geqq 0}$ grows faster than every geometric progression (see (2)) and so $\varphi_M(x) < +\infty$ for every $x, x > 0$, and the supremum occurring in the definition of $\varphi_M(x)$ is attained at a point $n = n(x) \in \mathbf{Z}_+$. Note that φ_M is non increasing, $\lim_{x\to 0+} \varphi_M(x) = +\infty$ and $\varphi_M \geqq 1$ (see (4)).

We have to impose two supplementary conditions on the sequence $(M_n)_{n\geqq 0}$:

(5) there exists a constant $C, C > 0$, with

$$\int_0^x \varphi_M(t)\,dt \leqq C \cdot x \cdot \varphi_M(x), \quad x > 0;$$

(6) the function $x \to x \cdot \varphi_M(x)$ is increasing in a neighbourhood of the origin, and

$$\lim_{x\to 0+} x \cdot \varphi_M(x) = 0.$$

The condition (6) is not restrictive, for if $\int_0^1 \varphi_M(t)\,dt = +\infty$ then the class $C_A\{M_n\}$ is quasianalytic. As to the condition (5), it restricts our consideration to the Carleman classes containing some Gevrey class G_α with $\alpha < 1$. Thus we don't succeed in describing the sets of uniqueness for the non-quasianalytic classes $C_A\{M_n\}$ satisfying

$$\bigcap_{0<\alpha<1} G_\alpha \supset C_A\{M_n\} \supset G_1.$$

Note that a necessary and sufficient condition for the quasianalyticity of the class $C_A\{M_n\}$ was obtained in [23].

It is possible to reformulate conditions (5) and (6) in terms involving the sequence $(M_n)_{n\geqq 0}$ only. We will use the following condition equivalent to (6):

(6a) All sufficiently large positive integers n satisfy $\frac{1}{n}+\frac{\log M_n}{n}<\frac{\log M_{n+1}}{n+1}$.

Remarks. 1. If $0<\alpha<1$ and $M_n=n^{n/\alpha}$, $n\geqq 1$, then $C_A\{M_n\}=G_\alpha$ and $\varphi_M(x)\asymp x^{-\alpha}$, $x\to 0+$. It is not hard to see that all conditions (1)—(6) hold.

2. Set $p>1$ and $M_n=e^{n^p}$, $n\geqq 1$. Then $\varphi_M(x)\asymp\left(\log\frac{1}{x}\right)^q$, $x\to 0+$, where $q=\frac{p}{p-1}$. Conditions (1)—(6) hold again.

Now we are going to construct an auxiliary function Φ_M that will occur in the statement of the theorem below. The function

$$x\to\frac{\varphi_M(x)}{x}$$

is strictly decreasing from infinity to zero when x is increasing from zero to infinity. This function being continuous has the inverse function, Ψ_M say. It is clear that

$$\Psi_M\left(\frac{\varphi_M(x)}{x}\right)=x,\quad x>0, \tag{10.1}$$

Let now $\Phi_M(y)\overset{\text{def}}{=}y\cdot\Psi_M(y)$, $y>0$.

The following elementary identity will be useful in the sequel

$$\Phi_M(y)=\varphi_M(\Psi_M(y)),\quad y>0. \tag{10.2}$$

This identity follows from (10.1) $\left(\text{we only have to put } y=\frac{\varphi_M(x)}{x}\right)$ and is equivalent to the equality

$$\Phi_M(y)=\varphi_M(x),\quad y=\frac{\varphi_M(x)}{x}.$$

Let us discuss the simplest properties of Φ_M. The function φ_M being constant in a neighbourhood of infinity, so Φ_M is constant in a neighbourhood of zero. Moreover Φ_M is not decreasing on the half-axis $(0,+\infty)$ and $\lim_{y\to+\infty}\Phi_M(y)=+\infty$. Further, if $Q>1$, then $\Phi_M(Qy)=Qy\Psi_M(Qy)<Qy\Psi_M(y)=Q\cdot\Phi_M(y)$ (we must take into account the monotonicity of Ψ_M) and so

$$\Phi_M(Qy)<Q\Phi_M(y),\quad y>0,\quad Q>1. \tag{10.3}$$

Definition. A compact subset E of the real line $\mathbf{R}$ is said to be a set of uniqueness for the class $C_A\{M_n\}$ if there exists no nonzero function $f, f\in C_A\{M_n\}$, satisfying $f^{(n)}|E\equiv 0$, $n=0,1,\ldots$. The collection of all sets of uniqueness for the class $C_A\{M_n\}$ will be denoted by $\mathscr{E}\{M_n\}$.

Theorem 6. A set E, $E\subset\mathbf{R}$ is not a set of uniqueness for $C_A\{M_n\}$ if and only if there exists a function f_E, $f_E\in L^1\left(\dfrac{dx}{1+x^2}\right)$, such that

(a) $\varphi_M(\varrho(x, E))\leqq f_E(x), \quad x\in\mathbf{R};$

(b) $\Phi_M\left(\int_{Cl_x}\dfrac{f_E(t)}{(t-x)^2}\,dt\right)\leqq \text{Const.}\, f_E(x), \quad x\in\mathbf{R}.$ $\qquad (M)$

Remark. Condition (a) implies that

$$\int_{\mathbf{R}}\frac{\varphi_M(\varrho(x, E))}{1+x^2}\,dx<+\infty. \tag{$\mathrm{C_M}$}$$

Let us now ask under what condition imposed on φ_M, ($\mathrm{C_M}$) is equivalent to $E\notin\mathscr{E}\{M\}$? It turns out that such a condition is

$$\varphi_M(x)=O(\varphi_M(\sqrt{x})), \quad x\to 0+. \tag{10.4}$$

We will prove this supposing that Theorem 1 has been proved. Let $f_E=\varphi_M(\varrho(x\ E))$. It follows from ($\mathrm{C_M}$) that $f_E\in L^1\left(\dfrac{dx}{1+x^2}\right)$. Clearly,

$$\int_{Cl_x}\frac{f_E(t)}{(t-x)^2}\,dt\leqq\frac{\text{Const.}}{\varrho(x, E)^2}$$

and so the inequality (M), (b) follows from the estimate

$$\Phi_M\left(\frac{1}{x^2}\right)=O(\varphi_M(x)), \quad x\to 0+.$$

Now change the variable: $\dfrac{1}{x^2}=\dfrac{\varphi_M(t)}{t}$. Then

$$\varphi_M(t)=O\left(\varphi_M\left(\sqrt{\frac{t}{\varphi_M(t)}}\right)\right).$$

But $\varphi_M\geqq 1$ and therefore $\sqrt{t}\geqq\sqrt{\dfrac{t}{\varphi_M(t)}}$. Now we have only to use the monotonicity of φ_M and to apply the estimate (10.4).

If, in particular, $M_n=e^{n^p}$, $p>1$, then φ_M obviously satisfies (10.4) and we obtain the result of Taylor and Williams mentioned in the introduction.

10.1 The proof of the necessity of (M) follows the scheme exposed in § 2. We retain here the notation from § 2.

Suppose $E\notin\mathscr{E}\{M\}$ and take a function $f, f\in C_A\{M_n\}$, $f^{(n)}|E\equiv 0$, $n=0, 1, 2, \ldots$ but $f\not\equiv 0$. For every $x, x\in\mathbf{R}$, we have

$$|f(x)| \leqq 2^n \exp\left\{-\frac{y}{9\pi}\cdot a_f(x)\right\}+(Q_f y)^n M_n \tag{10.5}$$

(see (2.6)).

The following lemma is analogous to Lemma 2.1.

Lemma 10.1. Let m be an integer satisfying

$$x^m M_m = \inf_n x^n M_n = \exp\left(-\varphi_M(x)\right).$$

Then $m\leqq\varphi_M(x)$ for all sufficiently small positive x.

Proof. We see, letting $y=\log\frac{1}{x}$, that the supremum

$$M(y) = \sup_{n\geqq 0}(ny-\log M_n)$$

is attained for $m=n$. The function $y\to M(y)$ is convex being the upper envelope of a family of linear functions. It is easy to see (using the convexity of the sequence $(\log M_n)_{n\geqq 0}$) that

$$M(y) = ny-\log M_n$$

if

$$\log M_n-\log M_{n-1} < y < \log M_{n+1}-\log M_n,$$

Thus m is the positive integer n satisfying the above inequality for $y=\frac{1}{x}$.

To finish the proof we only have to verify the inequality

$$m \leqq M(y).$$

But this inequality obviously holds if

$$m < \left(m(\log M_m-\log M_{m-1})-\log M_m\right),$$

or

$$\frac{1}{m-1}+\frac{\log M_{m-1}}{m-1} < \frac{\log M_m}{m}.$$

It remains to apply condition (6a).

Taking now the infimum (with respect to n) in (10.5) and using the lemma we have just proved we obtain

$$|f(x)| \leqq \exp\left\{-\frac{y}{9\pi}\cdot a_f(x)+(\log 2)\varphi_M(Q_f y)\right\}+\exp\{-\varphi_M(Q_f y)\}.$$

We recall that (10.5) is proved under the assumption $0<y\leqq\varrho(x, E)$. Let us forget this restriction for a moment and pick a number y to make both exponents in the

right hand part of the last inequality equal:

$$\frac{y}{9\pi} a_f(x) = (\log 2e) \cdot \varphi_M(Q_f y)$$

whence

$$Q_f y = \Psi_M\left(\frac{a_f}{9\pi(\log 2e)Q_f}\right).$$

If the inequality $y \leqq \varrho(x, E)$ turns out to be true such a choice of y will be admissible and we obtain

$$|f(x)| \leqq 2\exp\left\{-\Phi_M\left(\frac{a_f}{9\pi(\log 2e)Q_f}\right)\right\}. \tag{10.6}$$

If, on the other hand $y > \varrho(x, E)$ then we apply a simpler estimate

$$|f(x)| \leqq \exp\{-\varphi_M(Q_f \varrho(x, E))\}. \tag{10.7}$$

But $Q_f \cdot \varrho < Q_f y$ and so $-\varphi_M(Q_f \varrho) < -\varphi_M(Q_f y)$, whence

$$|f(x)| \leqq \exp\left\{-\Phi_M\left(\frac{a_f}{9\pi(\log 2e)Q_f}\right)\right\}.$$

Let

$$f_E = CQ_f \cdot (-\log|f|)$$

$\left(\text{here } C \text{ denotes a constant from (5) and } |f| < \frac{1}{2}\right)$. We have

$$-\log|f| \geqq \varphi_M(Q_f \varrho) \geqq \frac{1}{CQ_f} \varphi_M(\varrho)$$

(see (10.7) and (5)). Thus

$$\varphi_M(\varrho(x, E)) \leqq f_E(x), \quad x \in \mathbf{R},$$

i.e. f_E satisfies (M), (a). We may assume without loss of generality that

$$-\log\frac{1}{2}|f| < f_E.$$

This inequality and (10.6) imply

$$\Phi_M\left(\frac{1}{9\pi(\log 2e)CQ_f^2} \cdot C \cdot Q_f \cdot a_f\right) \leqq f_E$$

or

$$\Phi_M\left(\int_{Cl_x} \frac{f_E(t)}{(t-x)^2}\,dt\right) \leqq 9\pi(\log 2e) \cdot C \cdot Q_f^2 f_E.$$

10.2. To prove the sufficiency of (M) for $E \notin \mathcal{E}$ we will use the scheme of reasoning of §§ 3—4. However the consideration of general Carleman classes, unlike the Gevrey classes, is connected with some technical difficulties. In §§ 3—4 replaced the function Φ_M $(M_n = n^{n/\alpha})$ by the function $y \to y^{\frac{\alpha}{1+\alpha}}$ with the same order of growth at infinity. After that we used the analytic continuability of the latter function into the right half-plane and some properties of the continuation. Thus we begin the general sufficiency proof with the construction of a function $\mathbf{\Phi}$ analytic in the right half-plane and such that $\mathbf{\Phi}(y) \asymp \Phi_M(y)$, $y > 0$. Recall that Φ_M was defined by means of φ_M. It is natural therefore to find a suitable analytic substitute for φ_M. Let

$$\theta(z) \overset{\text{def}}{=} \int_0^{+\infty} \frac{z}{z^2+t^2} \varphi_M(t)\, dt, \quad \operatorname{Re} z > 0.$$

Lemma 10.2. The function θ is analytic in the half-plane $\operatorname{Re} z > 0$. The inequalities

$$\frac{\pi}{4} \varphi_M(x) \leqq \theta(x) \leqq 2C\varphi_M(x)$$

hold for every $x, x > 0$, (C is the constant occurring in (5)). If $|z-x| \leqq \frac{x}{2}$ then

$$|\theta(z)| \leqq 6\theta(x).$$

For every $x, x > 0$, and $n, n \in \mathbf{Z}_+$ we have

$$|\theta^{(n)}(x)| \leqq 6 \cdot 2^n \cdot n! \cdot \frac{\theta(x)}{x^n}.$$

The proof of the lemma is elementary and we leave it to the reader. See an analogous assertion in [22].

To build an analytic equivalent for Ψ_M we have to construct the inverse function of the function $z \to \frac{\theta(z)}{z} \overset{\text{def}}{=} \vartheta(z)$.

Lemma 10.3. The function $z \to \vartheta(z)$ is analytic and univalent in the angle $|\arg z| < \pi/8$.

Proof. The analyticity of ϑ in the right half-plane is obvious. If $|\arg z_K| < \frac{\pi}{8}$, $k = 1, 2$, $z_1 \neq z_2$, $\vartheta(z_1) = \vartheta(z_2)$ then

$$\int_0^\infty \frac{1}{(t^2+z_1^2)(t^2+z_2^2)} \varphi_M(t)\, dt = 0.$$

But

$$\operatorname{Re}(t^2+z_1^2)^{-1}(t^2+z_2^2)^{-1} = |t^2+z_1^2|^{-2}|t^2+z_2^2|^{-2} \cdot \operatorname{Re}(t^2+\bar{z}_1^{2})(t^2+\bar{z}_2^{2}) > 0$$

and the proof is finished.

Now we are able to define the function Φ on the ϑ-image of the angle $|\arg z|<\frac{\pi}{8}$:

$$\Phi(\vartheta(z)) = \theta(z).$$

Its analyticity follows from the identity $\Phi(w)=w\cdot\vartheta^{-1}(w)$. We have to prove that the set $\vartheta\left(\left\{z:|\arg z|<\frac{\pi}{8}\right\}\right)$ contains the angle $|\arg z|<\varepsilon_1$ for a sufficiently small positive ε_1. We are going to apply the distortion theorem of Koebe.

Koebe's one quarter theorem: Let f be a conformal and univalent mapping of the disc $\mathbf{D}(x, R)$. Then the f-image of this disc contains $\mathbf{D}\left(f(x), \frac{1}{4}R|f'(x)|\right)$.

See [24] p. 455 for the proof.

Put $f=\vartheta$, $R=c\cdot x$, where $C, C>0$, is chosen so that the disc $\mathbf{D}(x, cx)$ is contained in the angle $|\arg z|<\frac{\pi}{8}$. Then

$$|\vartheta'(x)| = 2\int_0^{+\infty}\frac{x}{(x^2+t^2)^2}\varphi_M(t)\,dt \overset{t=Sx}{=}$$
$$= \frac{2}{x^2}\int_0^{+\infty}\frac{1}{(1+S^2)^2}\cdot\varphi_M(Sx)\,dS \geqq \frac{2}{x^2}\varphi_M(x)\cdot\int_0^1\frac{dt}{(1+t^2)^2}.$$

By Koebes theorem the ϑ-image of $\mathbf{D}(x, cx)$ covers the disc centered at $\frac{\theta(x)}{x}$ with radius not smaller than

$$\frac{1}{4}cx\cdot\frac{2}{x^2}\varphi_M(x)\int_0^1\frac{dt}{1+t^2} \geqq \text{Const.}\frac{\theta(x)}{x} = c_1\frac{\theta(x)}{x}. \tag{10.8}$$

Since the function ϑ maps the real axis onto itself it is possible to find a positive ε_1 such that

$$\{w:|\arg w|<\varepsilon_1\}\subset\vartheta\left\{z:|\arg z|<\frac{\pi}{8}\right\}.$$

The function Φ is therefore analytic at least in the angle $|\arg w|<\varepsilon_1$.

Note that Φ is increasing on the half-axis $\mathbf{R}_+$, $\lim_{y\to\infty}\Phi(y)=+\infty$ and $\lim_{y\to 0}\Phi(y)=0$.

Lemma 10.4. Let $y>0$, let c_1 be the constant from (10.8) and let $w\in\mathbf{D}(y, c_1y)$. Then

$$\text{Const.}\,\Phi(y) \leqq \operatorname{Re}\Phi(w) \leqq |\Phi(w)| \leqq \text{Const.}\,\Phi(y). \tag{10.9}$$

Proof. Let $y=\vartheta(x)$. We have seen that $\vartheta(\mathbf{D}(x, cx))\supset\mathbf{D}(y, c_1y)$. The equality $\Phi(\vartheta(z))=\theta(z)$ shows that it is sufficient to prove the following chain of inequalities

$$\text{const.}\,\theta(x) \leqq \operatorname{Re}\theta(z) \leqq |\theta(z)| \leqq \text{const.}\,\theta(x)$$

in the disc $\mathbf{D}(x, cx)$. This is not hard to do using the integral representation of θ.

Remark. Lemma 10.4 is a substitute for Lemma 3.7.

We can now proceed to the construction of the equilibrium function U_E following the scheme of § 3. For the convenience of the reader recall that the roles of the functions $y \to y^{\frac{-\alpha}{1+\alpha}}$, $x \to x^{-\alpha}$ are now played by Φ and θ respectively. The function U_E was defined as the limit of the recurrent sequence $(U_n)_{n \geqq 0}$ whose monotonicity was the consequence of Lemma 3.1. Put

$$U_0(x) = \varepsilon\theta(\varrho(x, E)),$$

where ε, $\varepsilon > 0$, is chosen so that

$$\Phi(T_E U_0) > U_0.$$

Such a choice of ε is indeed possible. It is easy to see (compare with the proof of Lemma 3.1) that

$$T_E U_0(x) \geqq \varepsilon \int_0^\infty \frac{\theta(S)}{(\delta+S)^2}\, dS,$$

where $\delta = \varrho(x, E)$. The above integral is not smaller than $\frac{1}{2}\, \vartheta(\delta)$ (restrict the integration to $(0, \delta)$ and apply the monotonicity of θ). Thus

$$\Phi(T_E U_0) \geqq \Phi\left(\frac{\varepsilon}{2}\vartheta(\delta)\right) \tag{10.10}$$

(Φ is increasing). We will prove below that

$$\Phi(ax) < \sqrt{a}\,\Phi(x)$$

for every a, $a > 1$. Putting $x = \frac{\varepsilon\vartheta(\delta)}{2}$ and $a = \frac{2}{\varepsilon}$ $(a > 1$ if $\varepsilon < 2)$ we obtain

$$\theta(\delta) = \Phi(ax) < \sqrt{\frac{2}{\varepsilon}}\,\Phi\left(\frac{\varepsilon}{2}\vartheta(\delta)\right).$$

Hence

$$\Phi(T_E U_0) > \sqrt{\frac{\varepsilon}{2}}\,\theta(\delta) > U_0$$

if $\varepsilon < \frac{1}{2}$.

Now we put, of course,

$$U_{n+1} = \Phi(T_E U_n), \quad n = 0, 1, \ldots,$$

and the last inequality shows that U_n is increasing with n. Using (10.10) we verify (as in § 3) that without loss of generality we can take Const $= 1$ in (M). Thus there exists a function U_E, $U_E > U_0$ such that

$$\Phi(T_E U_E) = U_E,$$

this equality implies the analyticity of U_E on each complementary interval of E.

The convexity of U_E on these intervals was used in the earlier variant of the proof only to prove that U_E has exactly one minimum on every bounded complementary interval and is monotone on the unbounded ones. But now it is easy to deduce these properties directly by considering $T_E U_E$ and using the monotonicity of Φ.

The following proposition will be needed to reproduce our reasonings used in the proof of lemmas 3.4—3.6.

Proposition. There is a number δ, $\delta \in \left(0, \frac{1}{2}\right)$, such that

$$\Phi(ay) \leqq a^{\delta} \Phi(y), \quad y > 0,$$

for every $a, a > 1$.

Proof. The identity $\Phi(x) = x \cdot \vartheta^{-1}(x)$ shows that the inequality we want to prove is equivalent to the following estimate

$$a^{1-\delta} \vartheta^{-1}(ay) \leqq \vartheta^{-1}(y)$$

which is in its turn equivalent to

$$a\vartheta(t) \geqq \vartheta\left(\frac{t}{a^{1-\delta}}\right) \vartheta(bx)$$

(we set $y = \vartheta(t)$ and use the monotone decreasing of ϑ). At last putting $t = xb$, $b^{\frac{1}{1+\delta}} = a$ reduces our inequality to the inequality

$$\vartheta(x) \leqq b^{\frac{1}{1-\delta}} \vartheta(bx)$$

or

$$\theta(x) \leqq b^{\gamma} \theta(bx) \tag{10.11}$$

where $b > 1$, $\gamma = \frac{\delta}{1-\delta}$. The function $\delta \to \frac{\delta}{1-\delta}$ maps $\left(0, \frac{1}{2}\right)$ onto $(0, 1)$. Thus it is sufficient to prove the existence of a number γ, $\gamma \in (0, 1)$ such that (10.11) holds for every $b, b < 1$, and $x, x > 0$. We have (using the integral representation of θ and condition (5), see (10.0)).

$$\theta(x) = \int_0^{\infty} \frac{1}{1+t^2} \varphi(xt)\, dt = \int_0^{\infty} \left\{ \int_0^t \varphi(xS)\, dS \right\} \frac{2t}{1+t^2}\, dt \tag{10.12}$$

(here we omit the index M of φ for the sake of simplicity). On the other hand condition (5) implies that

$$\frac{1}{x} \leqq C \cdot \frac{\varphi(x)}{\int_0^x \varphi(t)\,dt}.$$

Integrating this inequality (we use here a reasoning of [16], see Lemma B) we obtain

$$\left(\frac{x_2}{x_1}\right)^{1/C} \leqq \frac{\int_0^{x_2} \varphi\,dt}{\int_0^{x_1} \varphi\,dt}, \quad 0 < x_1 < x_2.$$

Now put $x_1 = tx$, $x_2 = tbx$. Then

$$b^{1/C} \int_0^{tx} \varphi(S)\,dS \leqq \int_0^{tbx} \varphi(S)\,dS$$

or changing the variable,

$$b^{1/C} \int_0^t \varphi(xS)\,dS \leqq b \int_0^t \varphi(bxS)\,dS.$$

Substituting this inequality into the expression (10.12) for θ we obtain (10.11) with $\gamma = 1 - 1/C$ (note that the constant C in (5) is greater than one because of the monotonicity of φ).

The analogue of Lemma 3.6 in the present situation is

Lemma 10.5. Let $x \in CE$, and $\omega_x \overset{\text{def}}{=} (x-\delta, x+\delta)$, where $\delta = c_1 \varrho(x, E)$ $(0 < c_1 < 1$, c_1 the constant from the inequality (10.8)$)$. Then

$$\int_{l_x \setminus \omega_x} \frac{U_E(t)}{(t-x)^2}\,dt \leqq \text{Const.}\, \mathbf{\Phi}^{-1}(U_E(x)).$$

Proof. See § 3 and the preceding proposition. Define the sought function f by the formula (4.2) $(\text{now } \mathbf{D}_x = \mathbf{D}(x, c_1\varrho))$.

Lemma 10.6. There is a constant Q, $Q > 0$, such that for every x, $x \in \mathbf{R} \setminus E$, and $n = 0, 1, \ldots$

$$|f^{(n)}(x)| \leqq Q^n \cdot n! \left[\frac{1}{\theta^{-1}(U_E(x))}\right]^n \exp\left(-\frac{1}{2} U_E(x)\right).$$

Proof. See the proof of Lemma 4.1.

It remains only to deduce the inclusion $f \in C_A\{M_n\}$ from the preceding lemma. But

$$\sup_{x \in \mathbf{R}} |f^{(n)}(x)| \leqq Q^n \cdot n! \sup_{y > 0} \frac{1}{y^n} \exp\left(-\frac{1}{2}\theta(y)\right).$$

Note that $-\dfrac{1}{2}\theta \leqq -\dfrac{\pi}{8}\varphi_M \leqq -\dfrac{1}{3}\varphi_M$ and hence

$$\sup_{x \in \mathbf{R}} |f^{(n)}(x)| \leqq Q^n \cdot n! \sup_{y > 0} \frac{1}{y^n} \exp\left\{-\frac{1}{3}\left(3n \log\frac{1}{y} - \log M_{3n}\right)\right\} = Q^n \cdot n! \cdot M_{3n}^{1/3}.$$

Thus $f \in C_A\{M_{3n}^{1/3}\}$. The transformation $(M_n)_{n \geq 0} \to (M_{3n}^{1/3})$ does not change the corresponding Gevrey class but it is not true for general Carleman classes. Nevertheless it is not hard to finish the proof of Theorem 6. Let us state the above result as follows.

Proposition. $E \notin \mathscr{E}(M_{3n}^{1/3})$ if $E \in (M)$.

Thus it is sufficient to construct a sequence $N = (N_n)_{n \geq 0}$ such that $E \in (N)$ and $N_{3n}^{1/3} = M_n$ for $n = 0, 1, \dots$. Put $\log N_n = 3 \log M_K$ if $n = 3k$ and continue the function $n \to \log N_n$ onto $\mathbf{Z}_+$ linearly. For φ_N the supremum

$$\sup_{n \geq 0} \left(n \log \frac{1}{x} - \log N_n \right)$$

is attained at $n = 3k$ and is thus equal to

$$\sup_{K \geq 0} \left(3K \log \frac{1}{x} - \log N_{3K} \right) = 3 \sup_{K \geq 0} \left(K \log \frac{1}{x} - \log M_K \right) = 3 \cdot \varphi_M(x).$$

Hence $E \in (N)$, and Theorem 6 is proved (it is trivial to verify that the sequence $(N_n)_{n \geq 0}$ satisfies conditions (1)—(6)).

Let λ be a function increasing on $[1, +\infty)$, $\lambda(1) = 1$. Suppose that λ satisfies the following natural conditions concerning its regularity and growth:

(a) the function $y \to \log \lambda(e^y)$ is convex on $[0, +\infty)$;

(b) the function $x \to x \log \lambda\left(\frac{1}{x}\right)$ is increasing near the origin and

$$\lim_{x \to 0} x \log \lambda \left(\frac{1}{x} \right) = 0;$$

(c) $\int_0^\delta \log \lambda \left(\frac{1}{x} \right) dx \leq B\delta \log \lambda \left(\frac{1}{\delta} \right)$ for all sufficiently small positive δ.

(d) $\lim_{x \to +\infty} \dfrac{x^\alpha}{\lambda(x)} = 0$ for every α, $\alpha > 0$.

Consider the class $A(\lambda)$ of all functions analytic in $\mathbf{D}$ and satisfying the estimate

$$f(z) = o\left(\lambda \left(\frac{1}{1 - |z|} \right) \right), \quad |z| \to 1 - 0.$$

The norm $\|f\|_\lambda = \sup_{z \in \mathbf{D}} |f(z)| \cdot \lambda^{-1}((1 - |z|)^{-1})$ makes $A(\lambda)$ a separable Banach space. The well-known duality between uniqueness and approximation gives the following

Theorem 7. Let $E=\bar{E}\subset\mathbf{T}$. Then the set of all rational functions with poles on E is dense in $A(\lambda)$ if and only if $E\in\mathscr{E}(\Lambda)$ where

$$\Lambda=(\Lambda_n)_{n\geqq 0};\quad \Lambda_n=\sup_{x\geqq 1}\frac{x^n}{\lambda(x)},\quad n=0,1,\ldots.$$

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