On exceptional sets at the boundary for subharmonic functions

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1. Introduction

In this paper we shall discuss the following problem. Suppose u is subharmonic in a domain $D\subset R^n$, $n\geqq 3$. Let $E\subset\partial D$ be a closed set and suppose that $\limsup_{P\to Q} u(P)\leqq 0$ for all $Q\in\partial D-E$. In what way must the growth of u near ∂D be related to the size of E in order that it should follow that $u\leqq 0$? In the case when E consists of a single point this is answered by the Phragmén—Lindelöf theorems (for a treatment of these, see [6]). In the case when u is bounded from above it follows from [4] that if D is a Lipschitz domain and E is of vanishing $(n-1)$-dimensional Hausdorff measure then $u\leqq 0$. The case when $n=2$ can by the conformal mapping technique be reduced to a study of the situation in the unit disc, for which more can be said, see [5]. Therefore we assume from now on that $n\geqq 3$.

We recall that a bounded domain $D\subset R^n$ is called a Lipschitz domain if to each point $Q\in\partial D$ there is a coordinate system (ξ,η), $\xi\in R^{n-1}$, $\eta\in R$, a Lipschitz function φ in R^{n-1} (i.e. $\sup_{x\neq y}|x-y|^{-1}|\varphi(x)-\varphi(y)|<\infty$) and a neighbourhood V of Q such that $D\cap V=\{(\xi,\eta):\varphi(\xi)<\eta\}\cap V$. If $E\subset R^n$ we denote by $\omega(\cdot, E)$ the harmonic measure of the set $E\cap\partial D$ with respect to D. For the properties of ω see [8, Chapter 8]. If $Q\in D$ we put

$$\Lambda(\varrho)=\sup\left\{\omega(Q, B(P,\varrho))\colon\ P\in R^n\right\}.$$

(Sometimes we will write $\Lambda(\varrho, Q, D)$). Notice that if $K\subset D$ is a compact set then it follows from Harnack's inequality that there is a number $C_K<\infty$ such that $\sup\{\Lambda(\varrho,Q_1)/\Lambda(\varrho,Q_2)\colon Q_1,Q_2\in K\}\leqq C_K$ for all $\varrho>0$. In § 4 we give estimates of Λ. Let $d(P)$ denote the distance from P to ∂D. If u is a function in D we define

$$M(\varrho)=\sup\,\{u^+(P)\colon\ d(P)>\varrho\},$$

where $u^+=\max(u,0)$.

Theorem. *Let D be a Lipschitz domain in R^n, $n \geqq 3$, and let $F \subset \partial D$ be a closed set of vanishing α-dimensional Haudorff measure, where $0 < \alpha < n-1$. Let u be subharmonic in D and suppose* $\limsup_{P \to Q} u(P) \leqq 0$ *for all* $Q \in \partial D - F$. *If*

(1.1) $$\Lambda(\varrho) M(\varrho) = O(\varrho^\alpha) \quad \text{as} \quad \varrho \to 0$$

then $u \leqq 0$.

We remark that for sufficiently regular domains (see § 4) we have the estimate $c_1 \varrho^{n-1} \leqq \Lambda(\varrho) \leqq c_2 \varrho^{n-1}$ where $c_1 > 0$. Hence in this case condition (1.1) equivalent to the condition $M(\varrho) = O(\varrho^{\alpha+1-n})$ as $\varrho \to 0$.

In this case the theorem is sharp as the following proposition shows.

Proposition. *Let B be the unit ball in $R^n, n \geqq 2$. If $0 < \alpha < n-1$ and $E \subset \partial B$ is a closed set of positive α-dimensional Hausdorff measure then there is a harmonic function u in B such that $u(0) = 1$,* $\lim_{P \to Q} u(P) = 0$ *for all* $Q \in \partial B - E$ *and*

$$M(\varrho) = O(\varrho^{\alpha+1-n}).$$

2. Technical preliminaries

We start with the following observation. There is a number $C = C(n)$ such that each ball in R^n of radius 2ϱ can be covered by $C(n)$ balls of radius ϱ. From the definition of Λ it follows that

(2.1) $$\Lambda(2\varrho, Q) \leqq C\Lambda(\varrho, Q) \quad \text{for all} \quad Q \in D.$$

We will need the following elementary estimate for harmonic measure.

Lemma 1. *Let D be a Lipschitz domain in R^n, $n \geqq 3$. Then there is a number $C = C(D) > 0$ such that if $P \in \partial D$, $\varrho > 0$ and $Q \in B(P, \varrho) \cap D$ we have $\omega(Q, B(P, 2\varrho)) \geqq C$.*

Proof. Since D is a Lipschitz domain there are numbers R and α, $R > 0$, $0 < \alpha < \pi/2$ such that to each point $P \in \partial D$ there exists a cone K_P with vertex at P, congruent to $K = \{x = (x_1, \dots, x_n) \in R^n : x_1 \geqq (\cos \alpha)\, |x|\}$ with the property that $K_P \cap \overline{B(P, R)} \subset R^n - D$. For $0 < \varrho < \frac{1}{2} R$, let $D(P, \varrho) = B(P, 2\varrho) - K_P$. If ω' denotes the harmonic measure of $\partial D(P, \varrho) \cap B(P, 2\varrho)$ with respect to $D(P, \varrho)$ then the maximum principle implies that $\omega'(Q) \leqq \omega(Q, B(P, 2\varrho))$ for all $Q \in B(P, \varrho) \cap D$. A change of scale shows that $\inf \{\omega'(Q) : Q \in B(P, \varrho) \cap D(P, \varrho)\}$ is independent of P and ϱ and hence the lemma follows.

We shall need an estimate for the Green function of D.

Lemma 2. *Let D be as in Lemma 1 and let G be the Green function of D. If $P'\in D$ there is a number $C=C(P', D)$ such that if $0<\varrho<\frac{1}{3}d(P')$ then*

$$\varrho^{n-2}\sup\{G(P, P'):\ d(P)\leqq\varrho\}\leqq C\Lambda(\varrho, P').$$

Proof. Put $B'(P)=B\left(P, \frac{1}{2}d(P)\right)$ for $P\in D$. Since $G(P, Q)\leqq|P-Q|^{2-n}$ it follows that $d(P)^{n-2}\sup\{G(P, Q): Q\in\partial B'(P)\}\leqq 2^{n-2}$. Pick a point $P^*\in\partial D$ such that $d(P)=|P-P^*|$. Since $B(P^*, 2d(P))\supset\overline{B'(P)}$, there is by Lemma 1 a number $C_1=C_1(D)>0$ such that $\omega(Q, B(P^*, 4d(P)))\geqq C_1$ for $Q\in\overline{B'(P)}$. The maximum principle now implies $C_1 d(P)^{n-2}G(P, Q)\leqq 2^{n-2}\omega(Q, B(P^*, 4d(P)))$ for all $Q\in D-\overline{B'(P)}$ and the lemma follows.

We will need estimates for the harmonic measure of certain sets, which we shall now describe. For $m>0$ let $L(m)$ be the set of all functions $\varphi: R^{n-1}\to R$ such that $\varphi(0)=0$ and $|\varphi(x)-\varphi(y)|\leqq m|x-y|$. For $a>0$, $r>0$ let $\Sigma=\Sigma(\varphi, r, a)=\{(x, y): \varphi(x)<y<\varphi(x)+a(|x|-r),\quad r<|x|<2r\}$. Let $\Gamma=\Gamma(\varphi, r, a)=\partial\Sigma\cap\cap\{(x, y): |x|=2r\}$.

Lemma 3. *If $m>0$ and $a>0$ are given, then there are numbers $C=C(a, m)$ and $\lambda=\lambda(a)$ with the following properties. If $\varphi\in L(m)$, $r>0$ and $r<\varrho<2r$ then*

$$\sup\{\omega(P):\ P=(x, y)\in\Sigma(\varphi, r, a)\ \text{and}\ |x|=\varrho\}\leqq C(\varrho r^{-1}-1)^{\lambda},$$

where ω is the harmonic measure of $\Gamma(\varphi, r, a)$ with respect to $\Sigma(\varphi, r, a)$. In addition $\lim_{a\to 0}\lambda(a)=\infty$.

Proof. Since the assertion is invariant under changes of scale, it is sufficient to prove it for the case $r=1$. We extend ω to all of R^n by putting $\omega=0$ outside Σ. Let S be the unit sphere in R^{n-2}. We now define

$$m(s)=\int_{-\infty}^{\infty}\int_S\omega^2(s\theta, y)\,d\theta dy.$$

We claim there is a function $\lambda': (0, \infty)\to(0, \infty)$ such that

$$m(s)\leqq A(s-1)^{\lambda'(a)},\quad 1<s<2, \tag{2.2}$$

where A is the area of Γ. We will show (2.2) by using the Carleman method, see [6]. We first make the assumption that φ is C^∞ in $\{x: |x|<3\}$. From [1] follows that $\omega|\Sigma$ has a smooth extension across $\partial\Sigma-\bar\Gamma$. Hence we can differentiate m and we find by the Green formula:

$$m'(s)=2\int_{-\infty}^{\infty}\int_S[(\partial/\partial s)\omega(s\theta, y)]\omega(s\theta, y)\,d\theta dy=$$
$$=2\int_1^s\int_{-\infty}^{\infty}\int_S|\nabla\omega(t\theta, y)|^2\,d\theta dy\,dt.$$

Here $\nabla\omega$ denotes the gradient of ω. Therefore

$$m''(s) = 2\int_{-\infty}^{\infty}\int_S |\nabla\omega(s\theta, y)|^2 d\theta dy$$

$$\geqq 2\int_{-\infty}^{\infty}\int_S [(\partial/\partial y)\omega(s\theta, y)]^2 d\theta dy +$$

$$+2\int_{-\infty}^{\infty}\int_S [(\partial/\partial s)\omega(s\theta, y)]^2 d\theta dy = B_1(s)+B_2(s).$$

From Hölder's inequality we obtain $(m'(s))^2 \leqq 2m(s)B_2(s)$. Since the function $y\to\omega(s\theta, y)$, $1<s<2$, $\theta\in S$, equals zero outside an interval of length $a(s-1)$, it follows from Wirtinger's inequality [7, Chapter 7] that $B_1(s)\geqq 2\pi^2(s-1)^{-2}a^{-2}m(s)$. Using these estimates we find $2m''(s)/m(s)\geqq 4\pi^2a^{-2}(s-1)^{-2}+(m'(s)/m(s))^2$, which implies

$$m''(s) \geqq 2\pi a^{-1}(s-1)^{-1}m'(s), \quad 1 < s < 2.$$

We notice $\lim_{s\to 1} m(s)=0$ and $\lim_{s\to 2} m(s)=A$. Hence $m(s)\leqq A(s-1)^{\lambda'(a)}$, where $\lambda'(a)=1+2\pi a^{-1}$ and inequality (2.2) is proved for the case when φ is C^∞ in $\{|x|<3\}$.

If $\varphi\in L(m)$ and not assumed C^∞ we can pick functions $\varphi_i\in C^\infty(R^{n-1})$ such that $\sup\{|\nabla\varphi_i(x)|: x\in R^{n-1},\ i=1, 2, \dots\}<\infty$, $\varphi_i(0)=0$, and φ_i converges to φ uniformly on compact sets. If A_i is the area of $\Gamma(\varphi_i, 1, a)$ and ω_i denotes the harmonic measure of $\Gamma(\varphi_i, 1, a)$ with respect to $\Sigma(\varphi_i, 1, a)$ then $A_i\to A$ and $\omega_i(P)\to\omega(P)$ for each $P\in\Sigma(\varphi, 1, a)$. Hence (2.2) follows.

Let $M(s)=\max\{\omega(x, y): (x, y)\in\Sigma \text{ and } |x|=s\}$ where $1<s<2$. We notice that we find a number c', $0<c'<1/2$, only depending on m and such that if $\xi\in R^{n-1}$ and $1<|\xi|<3/2$ then $B'_\xi\subset\Sigma(\varphi, 1, a)$, where B'_ξ is the ball with center in $P_\xi= =(\xi, 1/2a(|\xi|-1))$ and radius $c'(|\xi|-1)$. We next choose a number c, $0<c<c'$ such that $D_\xi=\{(x, y): |x-\xi|<c(|\xi|-1), \varphi(x)<y<\varphi(x)+a(|x|-1)\}$ is star-shaped with respect to P_ξ. This number can be taken to depend only on a and m. Hence it follows from [9, Lemma 2] that there is a number C, only depending on a and m such that if u is a non-negative harmonic function in D_ξ, with vanishing boundary values on $\partial D_\xi\cap\{(x, y): |x-\xi|<c(|\xi|-1)\}$ then $\sup\{u(\xi, t): \varphi(\xi)<t<\varphi(\xi)+ +a(|\xi|-1)\}\leqq Cu(P_\xi)$.

Letting $1<s<3/2$, let us now choose $\xi\in R^{n-1}$, $|\xi|=s$, such that $M(s)= \omega(\xi, \eta)$ for some η, $\varphi(\xi)<\eta>\varphi(\xi)+a(s-1)$. From the reasoning above it follows that $m(s)\leqq C\omega(P_\xi)$, where C can be taken to depend only on a and m. Let B_ξ be the ball with center P_ξ and radius $c(|\xi|-1)$. Then $\bar{B}_\xi\subset D_\xi\subset\Sigma$.

Since ω^2 is subharmonic in Σ, it follows there is a constant $C=C(a, m)$ such that

$$M^2(s) \leqq C(s-1)^{-n} \int_{B_\xi} \omega^2(P)\,dP$$
$$\leqq C(s-1)^{-1} \int_{|t-s|\leqq c(s-1)} t^{n-2} m(t)\,dt \leqq C(s-1)^{-n+\lambda'(a)+1}$$

and the lemma is proved.

3. The main result

We can now prove our main result.

Proof of the theorem. Let u and D be as in the theorem. We start with the following observation. Since D is a Lipschitz domain we can find a finite number of open sets $V_1, \dots, V_N$ such that $\partial D \subset \bigcup V_i$ and to each i there is an coordinate system (ξ, η) $\xi \in R^{n-1}$, $\eta \in R$, a Lipschitz function φ_i in R^{n-1} such that $D \cap V_i' = \{(\xi, \eta)\colon \varphi_i(\xi) < \eta\} \cap V_i'$ where V_i' is an open set such that $V_i' \supset \bar{V}_i$. For $Q \in \partial D$ we let $I(Q)$ denote the largest index j for which $Q \in V_j$. If $I(Q) = i$ we define for $a > 0$, $r > 0$ the open set $N(Q, r, a)$ in the following way. Let $Q = (\xi_0, \varphi_i(\xi_0))$. We now put $M(Q, r, a) = \{(\xi, \eta)\colon |\xi - \xi_0| \leqq 2r,\ \ \varphi_i(\xi) + a(|\xi - \xi_0| - r)^+ < \eta < \varphi(\xi) + ar\}$. Under our assumptions there is a number r_0 such that $M(Q, r, a) \subset D$ for all r, $0 < r < r_0 = r_0(a, D)$. For $0 < r < r_0$ we define $N(Q, r, a) = D - \overline{M(Q, r, a)}$. For an integer $m \geqq 2$ we also define $E(m) = E(m, Q, r, a)$ as the set $\{(\xi, \eta)\colon 2^{-m-1} r \leqq |\xi - \xi_0| - r < 2^{-m} r,\ \varphi(\xi) + a(|\xi - \xi_0| - r)$. Finally, let ω_m denote the harmonic measure of $E(m)$ with respect to $N(Q, r, a)$. We claim that if $P_0 \in D$ then there are numbers $C = C(a, D, P_0)$, $r_1 = r_1(a, D, P_0)$ and a function $\sigma\colon R^+ \to R^+$ such that

$$\omega_m(P_0) \leqq C\, 2^{-m\sigma(a)} A(r, P_0) \quad \text{for} \quad 0 < r < r_1, \quad \text{and} \quad \lim_{a \to 0} \sigma(a) = \infty. \tag{3.1}$$

To prove (3.1) we note there is no loss of generality in assuming $\xi_0 = 0$ and $\varphi_i(0) = 0$. We put for $r < |\xi| < 2r$, $Q_\xi = (\varphi_i(\xi),\ a(|\xi| - r))$. An inspection now shows there are numbers $c = c(a, D)$ and $r_2 = r_2(a, D)$ such that if $0 < r < r_2$ and if $Q_\xi \in E(m, Q, r, a)$, $r \geqq 2$, then $B(Q_\xi, 2cr2^{-m}) \subset N(Q, r, 2a)$. Letting G' denote the Green function of $N(Q, r, 2a)$ we now see there is a number $C = C(a, D)$ such that if $|P - Q_\xi| \leqq cr2^{-m}$ then

$$Cr^{n-2} 2^{-m(n-2)} G'(P, Q_\xi) \geqq 1. \tag{3.2}$$

If h denotes the harmonic measure of $B(Q_\xi, c2^{-m} r) \cap \partial N(Q, r, a)$ with respect to $N(Q, r, a)$ it follows from (3.2) and the maximum principle

$$h(P_0) \leqq Cr^{n-2} 2^{-m(n-2)} G'(P_0, Q_\xi). \tag{3.3}$$

From Lemma 3 follows $G'(P_0, Q_\xi) \leqq C2^{-m\lambda(a)} m(r)$ where $\lambda(a) \to \infty$ as $a \to 0$ and $m(r) = \sup\,\{G'(P_0, Q)\colon Q \in \Gamma(\varphi_i, 2r, a)\}$. Since $G' \leqq G$, where G is the Green function of D, it follows from Lemma 2 that $r^{n-2}m(r) \leqq CA(r, P_0)$ for r sufficiently small. We note there is a constant C, such that to all $m \geqq 2$ we can find points ξ_i, $2^{-m}r \leqq |\xi_i| < 2^{1-m}r$, $1 \leqq i \leqq C2^{m(n-2)}$ such that $E(m, Q, r, a) \subset \cup B(Q_{\xi_i}, c2^{-m}r)$. This yields (3.1).

We can now complete the proof of the Theorem. Our assumptions mean that to all $\varepsilon > 0$ we can find points $Q_1, \ldots, Q_M$ in F and numbers ε_i, $0 < \varepsilon_i < \varepsilon$ such that $F \subset \bigcup_1^M B(Q_i, \varepsilon_i)$ and

$$\sum_{i=1}^{M} \varepsilon_i^\alpha \leqq \varepsilon. \tag{3.4}$$

We put $D' = \bigcap_1^M N(Q_i, \varepsilon_i, a)$ and let $P_0 \in D$, where we will choose a later. If ε is sufficiently small then $P_0 \in D'$. It is now convenient to split $\partial D'$ into different parts. Let for $m \geqq 2$, $1 \leqq i \leqq M$, $A_{m,i} = \partial D' \cap E(m, Q_i, \varepsilon_i, a)$, $A_{1,i} = \partial D' \cap \cap M(Q_i, \varepsilon_i, a) - (\bigcup_{m=2}^\infty A_{m,i})$. We put $\mu_{m,i} = \sup \{u^+(P) \colon P \in A_{m,i}\}$ and let $h_{m,i}$ denote the harmonic measure of $A_{m,i}$ with respect to $N(Q_i, \varepsilon_i, a)$. Since u is bounded from above in D' the maximum principle gives

$$u^+(P_0) \leqq \sum_{i=1}^M \sum_{m=1}^\infty \mu_{m,i} h_{m,i}(P_0). \tag{3.5}$$

It is easy to see there is a number $c = \beta = \beta(a, D)$ such that the distance between $A_{m,i}$ and ∂D is greater that $\beta 2^{-m}\varepsilon_i$. From Lemma 2 and (3.1) follows the existence of a constant $C = C(a, D, P_0)$ such that $h_{m,i}(P_0) \leqq C2^{-m\sigma(a)}\Lambda(\varepsilon_i, P_0)$, Using (2.1) we find $h_{m,i}(P_0) \leqq c^{m+1}2^{-m\sigma(a)}\Lambda(\beta 2^{-m}\varepsilon_i, P_0)$. From this and our assumption on u we obtain

$$u^+(P_0) \leqq \sum_{i=1}^M \sum_{m=1}^\infty c^{m+1} 2^{-m\sigma(a)} M(\beta 2^{-m}\varepsilon_i) \Lambda(\beta 2^{-m}\varepsilon_i, P_0)$$

$$\leqq C \sum_{i=1}^M \varepsilon_i^\alpha \sum_{m=1}^\infty c^m 2^{-m\sigma(a) - m\alpha}.$$

We now pick a so small that the last sum converges. With this choice of a it follows from (3.4) that $u^+(P_0) \leqq C\varepsilon$ for all $\varepsilon > 0$. Since P_0 was arbitrary in D it follows that $u \leqq 0$ and the theorem is proved.

We shall now prove Proposition 1.

Proof of Proposition 1. Let B be the unit ball of R^n and let $P(\cdot, y)$ be the Poisson kernel for B with pole at $y \in \partial B$. If $x \in B - \{0\}$ let $x^* = \frac{x}{|x|}$. From the explicit representation of P, see [8, Chapter 1] if follows that

$$P(x, y) \leqq Cd(x)/(|y - x^*| + d(x))^n.$$

If $E \subset \partial B$ is a closed set of positive α-dimensional Hausdorff measure, $0 < \alpha < n-1$, if follows from [3, p. 7] that there is a probability measure μ with support in E such that $\mu(B(x, r)) \leqq Cr^\alpha$ for all $x \in R^n$ and $r > 0$. Let $v(x) = \int P(x, y)\, d\mu(y)$. Then

v is non-negative and harmonic in B, and $\lim_{P\to Q} v(P)=0$ for all $Q\in\partial B-E$. If $x\neq 0$ then

$$u(x) \leqq Cd(x)\int (|y-x^*|+d(x))^{-n}\, d\mu(y).$$

Putting $g(t)=\mu(B(x^*, r))$, an integration by parts shows

$$u(x) = Cd(x)\int_0^\infty (t+d(x))^{-n-1} g(t)dt \leqq Cd(x)\int_0^\infty (t+d(x))^{-n-1} t^\alpha dt$$

$$= C(n,\alpha)d(x)^{\alpha-n+1}$$

and the proposition is proved.

4. Concluding remarks

In this section we shall discuss estimates of $\Lambda(\varrho)$. To begin with we notice that if D is a Lipschitz domain, then to each $P\in D$ there is a constant $c=c(D,P)>0$ such that

$$\Lambda(\varrho, P) \geqq c\varrho^{n-1}, \quad 0<\varrho<1. \tag{4.1}$$

For otherwise $\liminf_{\varrho\to 0} \varrho^{1-n}\Lambda(\varrho, P)=0$. Letting σ denote the surface measure of ∂D it is easity seen that there is a $c>0$ such that if $Q\in\partial D$ and $0<r<1$ then $\sigma(B(Q,r))\geqq cr^{n-1}$. Hence we would have $\liminf_{r\to 0}\dfrac{\omega(P,B(Q,r))}{\sigma(B(Q,r))}=0$ for all $Q\in\partial D$. Arguing as in [10, Theorem 14.5] this would mean $\omega=0$. This contradiction establishes (4.1).

Let $0<\theta<\pi/2$ and put $K_\theta=\{x=(x_1,\dots,x_n)\colon x_1\geqq |x|\cos\theta\}$. We say that a Lipschitz domain is θ-regular if for all points $Q\in\partial D$ there is a cone Γ_Q congruent to K_θ and with vertex at Q such that $\Gamma_Q\subset R^n-D$. Let $\lambda_\theta(r)=\omega(e, B(0,r), R^n-K_\theta)$, where $e=(-1,0,\dots,0)$. From Lemma 2 and the maximum principle it now follows that $\omega(P, B(Q,r), D)\leqq C\omega(P, B(Q,2r), R^n-K_\theta\}$ for all $Q\in\partial D$ and all $P\in D$. Harnack's inequality now shows that

$$\Lambda(\varrho, P, D) \leqq C\lambda_\theta(\varrho) \tag{4.2}$$

where C can be taken to depend only on P, D and θ. Estimates for λ_θ can be read off from the estimates for Green functions for cones in [2]. We omit the details but it follows there is to each θ, $0<\theta<\pi/2$ a number $h(\theta)<n-1$ such that

$$\lambda_\theta(\varrho) = O(\varrho^{h(\theta)}) \quad \text{as} \quad \varrho\to 0$$

and $h(\theta)\to n-1$ as $\theta\to\pi/2$.

If there is a number $R>0$ such that to each point $Q\in\partial D$ there is a closed ball B_Q with the property that $B_Q\subset R^n-D$ and $B_Q\cap\partial D\supset\{Q\}$ we find, using the arguments leading to (4.2)

$$\Lambda(\varrho, P, D) \leqq C\lambda(\varrho)$$

where $\lambda(\varrho)=\omega(e, B(0,r), B')$, $e=(-1,0,\dots,0)$, $B'=\{P:|P+e|<1\}$. Since $\lambda(\varrho)\leqq C\varrho^{n-1}$ it follows that

$$\Lambda(\varrho, P, D) \leqq C\varrho^{n-1}. \tag{4.3}$$

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