Multiparameter spectral theory

A. Källström and B. D. Sleeman

0. Introduction

Let $H_1, \ldots, H_n$ be separable Hilbert spaces and let $H=\otimes_{i=1}^n H_i$ be their tensor product. In each space H_i we assume we have operators A_i, $S_{ij}, j=1, \ldots, n$ enjoying the property,

(i) A_i, S_{ij}: $H_i \to H_i$, $i,j=1, \ldots, n$ are Hermitian and continuous.

In addition we shall require a certain "definiteness" condition which may be described as follows: Let $f=f_1\otimes\ldots\otimes f_n$ be a decomposed element of H with $f_i \in H_i$, $i=1, \ldots, n$ and let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be a given set of real numbers not all zero. Then the operators Δ_i: $H\to H$, $i=1, \ldots, n$, may be defined by the equation

$$Af = \sum_{i=0}^n \alpha_i \Delta_i f = \det \begin{vmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_n \\ -A_1 f_1 & S_{11} f_1 & \ldots & S_{1n} f_1 \\ \vdots & \vdots & & \vdots \\ -A_n f_n & S_{n1} f_n & \ldots & S_{nn} f_n \end{vmatrix}, \tag{0.1}$$

where the determinant is to be expanded formally using the tensor product. This defines $\Delta_i f$ for decomposable $f\in H$ and we can extend the definition to arbitrary $f \in H$ by linearity and continuity. The definiteness condition referred to above can now be stated as

(ii) A: $H\to H$ is positive definite, that is

$$(Af, f) \geqq C\|f\|^2 \tag{0.2}$$

for some constant $C>0$ and all $f\in H$. Here $(\cdot, \cdot)$ denotes the inner product in H and $\|\cdot\|$ the corresponding norm. Note that for a decomposable element $f=f_1\otimes\ldots$ $\ldots\otimes f_n$ in H we have

$$(Af, f) = \det \begin{vmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_n \\ (-A_1 f_1, f_1)_1 & (S_{11} f_1, f_1)_1 & \ldots & (S_{1n} f_1, f_1)_1 \\ \vdots & & & \\ (-A_n f_n, f_n)_n & (S_{n1} f_n, f_n)_n & \ldots & (S_{nn} f_n, f_n)_n \end{vmatrix} \geqq C\|f_1\|_1^2 \ldots \|f_n\|_n^2$$

where $(\cdot, \cdot)_i (\|\cdot\|_i)$ denotes the inner product (norm) in H_i, $i=1, \ldots, n$.

The system of operators $\{A_i, S_{ij}\}$, $i,j=1, \dots, n$ having the properties (i) (ii) above have formed the basis for multi-parameter spectral theory, firstly by Atkinson [1] and Browne [2] when property (ii) is specialized to the case $\alpha_i=0$, $i=1, \dots, n$, and secondly by the authors in [3, 4] when $\alpha_0=0$ and the operators A_i are assumed to be positive on H_i and at least one is positive definite. In fact the theories in [3, 4] allow the operators A_i to be self adjoint and not necessarily Hermitian, but in addition they must satisfy a certain "compactness" criterion. In this paper we dispense with any compactness requirements. Indeed a fundamental purpose of this paper is to show that each of the above special cases may be subsumed into a unified theory.

Each of the operators A_i, S_{ij}: $H_i \to H_i$, $i=1, \dots, n$ induces corresponding operators in H. The induced operators will be denoted by A_i^+, S_{ij}^+. For example, given any decomposed element $f=f_1 \otimes \dots \otimes f_n \in H$, $S_{ij}^+ f$ is defined by

$$S_{ij}^+ f = f_1 \otimes \dots \otimes f_{i-1} \otimes S_{ij} f_i \otimes f_{i+1} \otimes \dots \otimes f_n. \tag{0.3}$$

S_{ij}^+ is then extended to the whole of H by linearity and continuity.

The theory to be developed here is based, as are the theories of Atkinson [1] and Browne [2] on the solvability of certain systems of linear operator equations. Let $f \in H$ be given; we seek elements $f_i \in H$, $i=0, 1, \dots, n$, satisfying the system of equations

$$\sum_{i=0}^{n} \alpha_i f_i = f, \tag{0.4}$$

$$-A_i^+ f_0 + \sum_{j=1}^{n} S_{ij}^+ f_j = 0, \quad i = 1, \dots, n.$$

It has been established by Källström and Sleeman [5] that the system (0.4) subject to the condition (ii) is uniquely solvable for any $f \in H$ and the solution is given by Cramer's rule, that is

$$f_i = (\Delta^+)^{-1} \Delta_i^+ f, \quad i = 0, 1, \dots, n, \tag{0.5}$$

where the operators Δ^+, Δ_i^+: $H \to H$, $i=0, 1, \dots, n$ are the operators induced by Δ, Δ_i as defined in (0.1). Note: because of condition (ii) $(\Delta^+)^{-1}$ exists as a bounded operator.

The operators Γ_i: $H \to H$, $i=0, 1, \dots, n$ defined by

$$\Gamma_i = (\Delta^+)^{-1} \Delta_i, \quad i = 0, 1, \dots, n \tag{0.6}$$

are basic for the theory to be developed.

The plan of this paper is as follows. In Section 1 we reconsider the solvability of the system (0.4) and establish some commutativity properties enjoyed by the operators A_i, S_{ij}. Section 2 develops the spectral theory based on the operators Γ_i defined in (0.6) while Section 3 discusses the concepts of "homogeneous" and "inhomogeneous" eigenvalues.

1. Commutativity in operator equations

For convenience we write

$$\alpha_0 \equiv -A_0^+, \quad -A_i^+ \equiv S_{i0}^+, \quad \alpha_j \equiv S_{0j}^+, \quad j = 1, \dots, n \tag{1.1}$$

and consider the system

$$\sum_{j=0}^{n} S_{ij}^+ f_j = g_i, \quad i = 0, 1, \dots, n \tag{1.2}$$

where $g_0 = f$ and $g_i \in H$, $i = 1, \dots, n$ are arbitrary. Furthermore since A defined in (0.1) is positive definite there is no loss in generality in assuming it has at least one positive definite cofactor. This follows from [5, Lemma 1]. Thus, as in [5], the system (1.2) is uniquely solvable for $f_i \in H$, $i = 0, 1, \dots, n$ and the solution is given by Cramer's rule, i.e.

$$f_j = (A^+)^{-1} \sum_{i=0}^{n} \hat{S}_{ij}^+ g_i \quad j = 0, 1, \dots, n \tag{1.3}$$

where $\hat{S}_{ij}^+$ is the cofactor of S_{ij}^+ in the determinant A.

First we note that S_{ij}^+ commutes with $\hat{S}_{ik}^+$ for $j, k = 0, 1, \dots, n$. This follows because $\hat{S}_{ik}^+$ contains no elements from the i-th row. Secondly the f_j given by (1.3) must satisfy (1.2). Thus on substitution we find,

$$\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1} \sum_{k=0}^{n} \hat{S}_{kj}^+ g_k = g_i, \quad i = 0, 1, \dots, n,$$

i.e.

$$\sum_{j=0}^{n} \sum_{k=0}^{n} S_{ij}^+ (A^+)^{-1} \hat{S}_{kj}^+ g_k = g_i, \quad i = 0, 1, \dots, n. \tag{1.4}$$

However, this must be true for all $g_i \in H$, $i = 0, 1, \dots, n$, and so on equating coefficients of g_i in (1.4) we find

$$\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1} \hat{S}_{ij}^+ = I, \quad i = 0, 1, \dots, n \tag{1.5}$$

where I denotes the identity in H and

$$\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1} \hat{S}_{kj}^+ = 0, \quad k \neq i, \quad i, k = 0, 1, \dots, n. \tag{1.6}$$

In particular with $i=0$, in (1.5, 1.6) we have

$$\sum_{j=0}^{n} \alpha_j (A^+)^{-1} \hat{\alpha}_j = I,$$

$$\sum_{j=0}^{n} \alpha_j (A^+)^{-1} \hat{S}_{kj}^+ = 0, \quad k = 1, \dots, n, \qquad (1.7a, b, c)$$

and

$$\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1} \hat{\alpha}_j = 0, \quad i = 1, \dots, n.$$

These results may be conveniently summarized in

Lemma 1. *The operators appearing in the system* (1.2) *enjoy the following commutativity properties.*

$$\sum_{j=0}^{n} \alpha_j (A^+)^{-1} \hat{\alpha}_j = I,$$

$$\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1} \hat{S}_{ij} = I, \quad i = 1, \dots, n,$$

$$\sum_{j=0}^{n} \alpha_j (A^+)^{-1} \hat{S}_{kj} = 0, \quad k = 1, \dots, n,$$

$$\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1} \hat{\alpha}_j = 0, \quad i = 1, \dots, n,$$

$$\sum_{j=0}^{n} S_{ij}^+ (A^+)^{-1} \hat{S}_{kj}^+ = 0, \quad k \neq i, \quad i, k = 0, 1, \dots, n.$$

We now establish a fundamental result.

Theorem 1. *The solution operators* Γ_i, $i=0, 1, \dots, n$, *defined by* (0.6) *or equivalently from* (1.3) *by* $\Gamma_i=(A^+)^{-1}\hat{S}_{0i}^+$, $i=0, 1, \dots, n$ *commute.*

Proof. In the same way as in [1, Theorem 6.7.2] we show that for any $f \in H$,

$$\Delta_i (A^+)^{-1} \Delta_j f = \Delta_j (A^+)^{-1} \Delta_i f, \quad i \neq j$$

and an application of $(A^+)^{-1}$ establishes the result.

2. Multiparameter spectral theory

Rather than use the inner product $(\cdot, \cdot)$ in H generated by the inner products $(\cdot, \cdot)_i$ in H_i, we use the inner product given by $(A^+ \cdot, \cdot)$ which will be denoted by $[\cdot, \cdot]$. The norms induced by these inner products are equivalent and so topological concepts such as continuity of operators and convergence of sequences may be discussed unambiguously without reference to a particular inner product. Algebraic concepts however may depend on the inner product. For L: $H \to H$ we denote by $L^{\#}$ the adjoint of L with respect to $[\cdot, \cdot]$, i.e. for all $f, g \in H$ we have

$$[Lf, g] = [f, L^{\#} g]. \tag{2.1}$$

For the operators $\Gamma_i : H \to H$, $i=0, 1, \ldots, n$ defined by (0.6) we have

Theorem 2.

$$\Gamma_i^{\#} = \Gamma_i, \qquad i = 0, 1, \ldots, n.$$

The proof of this is an immediate consequence of our definition of adjoint.

Working with the inner product $[\cdot, \cdot]$ in H the operators Γ_i, $i=0, 1, \ldots, n$ form a family of $(n+1)$ commuting Hermitian operators. Let $\sigma(\Gamma_i)$ denote the spectrum of Γ_i and $\sigma_0 = \times_{0 \leq i \leq n} \sigma(\Gamma_i)$ the Cartesian product of the $\sigma(\Gamma_i)$, $i=0, 1, \ldots, n$. Then since $\sigma(\Gamma_i)$ is a non-empty compact subset of $\mathbf{R}$ it follows that σ_0 is a non-empty compact subset of $\mathbf{R}^{n+1}$.

Let $E_i(\cdot)$ denote the resolution of the identity for the operator Γ_i and let $M_i \subset \mathbf{R}$ be a Borel set, $i=0, 1, \ldots, n$. We then define $E(M_0 \times M_1 \times \ldots \times M_n) = \prod_{i=0}^{n} E_i(M_i)$. Notice that the projections $E_i(\cdot)$ will commute since the operators Γ_i commute. Thus in this way we obtain a spectral measure $E(\cdot)$ on the Borel subsets of $\mathbf{R}^{n+1}$ which vanishes outside σ_0. Thus for each $f, g \in H$ $[E(\cdot)f, g]$ is a complex valued Borel measure vanishing outside σ_0. Measures of the form $[E(\cdot)f, f]$ will be non-negative finite Borel measures vanishing outside σ_0.

The spectrum σ of the system $\{A_i, S_{ij}\}$ may be defined as the support of the operator valued measure $E(\cdot)$, i.e. σ is the smallest closed set outside of which $E(\cdot)$ vanishes or alternatively σ is the smallest closed set with the property $E(M) = E(M \cap \sigma)$ for all Borel sets $M \subset \mathbf{R}^{n+1}$. Thus σ is a compact subset of $\mathbf{R}^{n+1}$ and if $\lambda \in \sigma$, then for all non-degenerate closed rectangles M with $\lambda \in M$, $E(M) \neq 0$. Thus the measures $[E(M)f, g], f, g \in H$ actually vanish outside σ.

We are now in a position to state our main result namely the Parseval equality and eigenvector expansion

Theorem 3. *Let $f \in H$. Then*

(i) $(A^+ f, f) = \int_\sigma [E(d\lambda)f, f] = \int_\sigma (E(d\lambda)f, A^+ f)$.

(ii) $$f = \int_\sigma E(d\lambda) f,$$

where this integral converges in the norm of H.

This theorem is an easy consequence of the theory of functions of several commuting Hermitian operators. See for example Prugovečki [6, pp. 270—285].

3. Eigenvalues

In this section we discuss the eigenvalues of the system $\{A_i, S_{ij}\}$. A "homogeneous" eigenvalue is defined to be an $(n+1)$-tuple of complex numbers $\lambda=(\lambda_0, \lambda_1, \dots, \lambda_n)$ for which there exists a non-zero decomposable element $u=u_1\otimes\dots\otimes u_n\in H$ such that

$$\sum_{i=0}^{n} \alpha_i\lambda_i = 1$$

and (3.1)

$$-\lambda_0 A_i u_i+\sum_{j=1}^{n}\lambda_j S_{ij}u_i = 0, \quad i = 1, \dots, n.$$

If λ is an eigenvalue then because of (0.2) and the self adjointness of the A_i it is well known that each λ_i is real. It then follows

Theorem 4. [2] *If $\lambda\in\sigma$ is such that $E(\{\lambda\})\neq 0$, then λ is an eigenvalue. Conversely if λ is an eigenvalue then $\lambda\in\sigma$ and $E(\{\lambda\})\neq 0$.*

It is appropriate to note here that if $\alpha_0=1$ and $\alpha_i=0$, $i=1, \dots, n$ then $\lambda_0=1$ and the results of Theorem 3 and Theorem 4 reduce to those of Browne [2].

If, as is usual, we go over to the "inhomogeneous" concept of spectrum and eigenvalue, then necessarily we must have $\lambda_0\neq 0$. That is we require

$$0\notin\sigma(\Gamma_0) = \sigma(A^{-1}S)$$

where A is defined by (0.1) and $S=\det\{S_{ij}^{+}\}$ in (0.1). Now $0\in\sigma(A^{-1}S)$ if and only if $f\in H_A(\infty)$ where

$$H_A(\infty) = \{f\in H \mid Sf = 0\}. \tag{3.2}$$

Thus if we define

$$\sigma^* = \{\lambda\in\sigma \mid \lambda_0 = 0\}$$

then for the "inhomogeneous" concept of spectrum we have in analogy with Theorem 3

Theorem 5. *Let $f\in H\ominus H_A(\infty)$. Then*

(i) $(A^+f, f) = \int_{\sigma-\sigma^*}(E(d\lambda)f, A^+f)$

(ii) $f = \int_{\sigma-\sigma^*} E(d\lambda)f.$

Theorem 5 generalizes, for bounded operators, the Parseval equality and eigenvector expansion of [3, 4]. Again if $\alpha_i=0$, $i=1, \dots, n$ then (0.2) reduces to the condition S is positive definite. Consequently $\sigma^*=\emptyset$ and Theorem 5 coincides with that of Browne [2].

Acknowledgement. The research for this paper was supported by a grant from the Science Research Council.

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Received February 25, 1976

A. Källström
Institut Mittag—Leffler
Auravägen 17
S-18262 Djursholm
Sweden
and
B. D. Sleeman
Department of Mathematics
The University
Dundee
Scotland