Fredholm representations of uniform subgroups

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Introduction

In [8] Mishchenko defined the notion of a Fredholm representation of a discrete group Γ and a map

$$\alpha\colon R(\Gamma) \to K(B\Gamma)$$

from the set of Fredholm representations of Γ to the K-theory of the classifying space $B\Gamma$. For the special case when $B\Gamma$ is homotopy equivalent to a compact manifold of negative curvature, it was proved that the image of α generates $K(B\Gamma)\otimes\mathbf{Q}$. This led to a proof of a conjecture of Novikov concerning rational homotopy invariants.

We give an extension of this result for the case when Γ is a torsionless, uniform subgroup of a non-compact, semisimple Lie group. By enlarging the class $R(\Gamma)$ to include representations which become unitary after projecting to the Calkin algebra, it can be proved that the map is surjective.

1. Fredholm representations

Let H, H_1, H_2 be Hilbert spaces. The symbol $B(H_1, H_2)$ will be used to denote the space of bounded linear operators from H_1 to H_2, and $A(H)$, the Calkin C^*-algebra of H. A representation ϱ of a group Γ on H will be said to become unitary in the Calkin algebra if

$$\varrho(\gamma)^* - \varrho(\gamma^{-1})$$

is compact for all $\gamma\in\Gamma$.

A Fredholm representation of a discrete group Γ is a triple

$$((H_1, \varrho_1), (H_2, \varrho_2), F)$$

where (H_1, ϱ_1) and (H_2, ϱ_2) are representations of Γ on H_1 and H_2, resp., that become unitary in their respective Calkin algebras and $F\colon H_1\to H_2$ is a Fredholm operator such that

$$F\varrho_1(\gamma)-\varrho_2(\gamma)F$$

is compact for all $\gamma\in\Gamma$.

Two Fredholm representations $((H_1, \varrho_1), (H_2, \varrho_2), F)$ and $((H_1', \varrho_1'), (H_2', \varrho_2'), F')$ will be said to be equivalent if there exist invertible $A_1\in B(H_1, H_2')$ and $A_2\in B(H_2, H_2')$ such that

(i) $A_1\varrho_1(\gamma)=\varrho_1'(\gamma)A_1, \quad A_2\varrho_2(\gamma)=\varrho_2'(\gamma)A_2$ for all $\gamma\in\Gamma$,
(ii) $F'A_1-A_2F$ is compact.

Then $R(\Gamma)$ will denote the set of equivalence classes of Fredholm representations of Γ.

If Γ has the property that $B\Gamma$ has a triangulation, then Mishchenko (see [8]) has shown how to define a map

$$\alpha\colon R(\Gamma)\to K(B\Gamma).$$

A uniformly continuous family $\mathfrak{J}\colon \widetilde{B\Gamma}\to B(H_1, H_2)$ parametrized by the universal covering space of $B\Gamma$ is said to be associated to a Fredholm representation $((H_1, \beta_1), (H_2, \beta_2), F)$ of Γ if

(i) $F-\mathfrak{J}(x)$ is compact for all $x\in\widetilde{B\Gamma}$,
(ii) $\mathfrak{J}(\gamma x)=\varrho_2(\gamma)\mathfrak{J}(x)\varrho_1(\gamma^{-1})$ for all $\gamma\in\Gamma$, $x\in\widetilde{B\Gamma}$

where Γ acts on $\widetilde{B\Gamma}$ by deck transformation. For each Fredholm representation, an associated family may be constructed by selecting a Γ-invariant triangulation of $\widetilde{B\Gamma}$, constructing an associated family on the zero skeleton and extending it to $\widetilde{B\Gamma}$ by linear interpolation on the higher dimensional simplices. Dividing out by the action of Γ gives a family of Fredholm operators on $B\Gamma$, which in turn determines an element of $K(B\Gamma)$ (see [1]).

2. *The basic operator*

In this section we examine the properties of an operator constructed by Hörmander (see [4]). It was suggested by Professor M. F. Atiyah that this operator might be used to generate Fredholm representations.

Let G be a non-compact semisimple Lie group with Lie algebra $\mathbf{g}_0$. Let $\mathbf{g}_0=\mathbf{k}_0+\mathbf{p}_0$ be a Cartan decomposition, θ the associated involution, K the corresponding maximal compact subgroup and G/K the non-compact symmetric space of maximal compact subgroups. Let B denote the Killing form on $\mathbf{g}_0$ and A the positive definite form given by

$$A(X, Y)=-B(X, \theta Y).$$

The left invariant metric on G induced by A is invariant under the action of right translation by K.

The Killing form induces a G-invariant metric on G/K. Let $r(x, y)$ denote the geodesic distance between x and y in G/K and $r(x)$ the distance from x to the coset eK. Let d denote exterior differentation on forms and d^* its metric adjoint. Let e and i denote the exterior and interior products of forms.

We consider the (unbounded) operator

$$D\colon L^2(\Omega^*(G/K)) \to L^2(\Omega^*(G/K))$$

on square-integrable differential forms defined by the formula

$$D = d + d^* + e(d(r^2/2)) + i(d(r^2/2))$$

and taking the closure of the operator on smooth forms of compact support.

Lemma 2.1. *The operator D is self-adjoint.*

The proof is standard (see [4]).

Let $\tilde{H}^1(\Omega^*(G/K))$ denote the completion of the space of smooth forms of compact support with respect to the graph norm associated to D.

Theorem 2.2. *The restricted operator*

$$D\colon \tilde{H}^1(\Omega^{ev}(G/K)) \to L^2(\Omega^{odd}(G/K))$$

is surjective with a one dimensional kernel generated by the function $e^{-r^2/2}$.

It is easy to check that the kernel of the restriction of D to 0-forms, i.e., functions, is generated by $e^{-r^2/2}$. The proof is then completed by proving the existence of positive constants C_q, $q = 1, 2, \ldots, \dim G/K$, such that

$$\|Df\|_{L^2} \geqq C_q \|f\|_{L^2} \quad \text{for all} \quad f \in \tilde{H}^1(\Omega^q(G/K)).$$

Our method is a generalization of that given by Hörmander in [4]. If $\pi\colon G \to G/K$ denotes the canonical projection, clearly, it suffices to find positive C_q such that

$$\|\pi^* Df\| \geqq C_q \|\pi^* f\|, \quad q > 0.$$

Let $X_1, \ldots, X_r$ be an orthonormal basis for $\mathbf{p}_0$ and $X_{r+1}, \ldots, X_n$ an orthonormal basis for $\mathbf{k}_0$. We will use the same notation for the corresponding left invariant vector fields on G and $w_1, \ldots, w_n$ for the dual Maurer—Cartan forms. If f is a k-form on G/K then $\pi^* f = \sum_I f_I w^I$ where $I = \{i_1 \leqq i_2 \leqq \ldots \leqq i_k\}$ is a subset of the indices $1, \ldots, r$ and $w^I = w^{i_1} \wedge \ldots \wedge w^{i_k}$. Further

$$\pi^*(df) = \sum_I \sum_{j=1}^r X^j(f^I) w^j \wedge w^I,$$

$$\pi^*(d^* f) = -\sum_{I,j} X^j(f_I) i(w^j)(w^I).$$

Let $\pi^*(d(r^2/2))=\sum x_i w^i$. Then

$$\|\pi^* Df\|^2 = \sum_I \sum_{j=1}^r \|(x_j+X_j)f_I\|^2 + \sum_{j,l}\sum_{|L|=q-1}([x_l+X_l,\, x_j-X_j]f_{jL}, f_{lL})$$

where $[x_l+X_l, x_j-X_j]=X_j(x_l)+X_l(x_j)+[X_j, X_l]$. The derivation property $d^2=0$ implies that $X_l(x_j)=X^j(x_l)$ and the theorem is proved by computing

$$\sum\{2(X_j(x_l)f_{jL}, f_{lL})+([X_l, X_j]f_{jL}, f_{lL})\}.$$

Notice that in the Euclidean case, $G=R^n$, $[X_l, X_j]=0$ and $X_j(x_l)=\delta_{jl}$ see ([4]).

Proposition 2.3. *The Lie Bracket term satisfies*

$$\|\pi^*(d+d^*)f\|^2-\sum_{j,\,|I|=q}\|X_j(f_I)\|^2+\sum_{|L|=q-1}\sum_{j,l}([X_j, X_l]f_{jL}, f_{lL}) =$$
$$= -(q/2)\|f\|^2+2(q-1)^{-1}\sum_{|I|=q-2,\,\alpha=r+1}^{n}\left\|\sum_{m,\,j=1}^{r} C^\alpha_{mj} f_{jmJ}\right\|^2$$

where the C^α_{mj} *are the constants of structure given by*

$$[X_m, X_j] = \sum_{\alpha=r+1}^{n} C^\alpha_{mj} X_\alpha.$$

Proof. This may be computed directly but is better understood by considering the Weitzenböck formula for the Laplacian on forms, namely, if ∇ denotes the connection associated to the metric, then

$$\Delta = -\mathrm{tr}\,\nabla\circ\nabla + D^q R$$

where $R\in\Gamma(T\otimes T^*\otimes T\otimes T^*)$ denotes the curvature and $D^q R$, the linear operator induced on q-forms, the derivation extension of R (see [10]).

The operator $\sum_{j,l}([X_j, X_l]f_{jL}, f_{lL})$ is precisely $(D^q R(f), f)$.

The Weitzenböck formula gives a decomposition of $D^q R(f)$ into two parts: the operator induced by the Ricci curvature and the difference. The Ricci curvature $R_{jk}=\sum_i R_{ijki}$ of a non-compact symmetric space is $1/2$ (Riemannian metric) (see [7]) and hence the first term $-(q/2)\|f\|^2$, the order q of the form enters as repetition coming from the number of ways of expressing f_I as f_{jL}, $|L|=q-1$.

We explain the remaining term more carefully.

Let $(R_2 f, f)$ denote the remaining term. Then

$$(R_2 f)_{i_1\dots i_q} = 2\sum_{\mu<\nu}(-1)^{\mu+\nu} R^{a\,b}_{i_\nu i_\mu} f_{abi_1\dots\hat{i}_\mu\dots\hat{i}_\nu\dots i_q}$$

where we are using the orthonormal basis $w_1, \dots, w_r$ for $\mathbf{p}_0^*$ and $R_{b\nu a\mu}$ denotes the curvature coefficient with respect to this basis. Then

$$(R_2 f, f) = -(q-1)^{-1}\sum_{|L|=p-2,\;a,b,\nu,\mu}(R_{b\nu a\mu} f_{abL}, f_{\nu\mu L}) =$$
$$= (q-1)^{-1}\sum(R_{ba\mu\nu} f_{abL}, f_{\mu\nu L}) + (q-1)^{-1}\sum(R_{b\mu\nu a} f_{abL}, f_{\mu\nu L}) =$$
$$= (q-1)^{-1}\sum(R_{ba\mu\nu} f_{abL}, f_{\mu\nu L}) - (q-1)^{-1}\sum(R_{b\mu a\nu} f_{abL}, f_{\nu\mu L}) =$$
$$= (q-1)^{-1}\sum(R_{ba\mu\nu} f_{abL}, f_{\mu\nu L}) - (R_2 f, f).$$

However,

$$R_{ba\mu\nu} = -B([[X_\mu, X_\nu], X_a], X_b) = -B([X_\mu, X_\nu], [X_a, X_b]) = \sum_\alpha C^\alpha_{\mu\nu} C^\alpha_{ab}.$$

Hence

$$(R_2 f, f) = 2(q-1)^{-1} \sum_{\alpha,\, a, b, \mu, \nu} C^\alpha_{\mu\nu} C^\alpha_{ab} (f_{abL}, f_{\mu\nu L}) =$$

$$= 2(q-1)^{-1} \sum_\alpha \left\| \sum_{a,b} C^\alpha_{ab} f_{abL} \right\|^2.$$

Proposition 2.4. *If f is a square integrable q-form on G/K, then*

$$\sum_{j,l} \sum_{|L|=q-1} (X_j(x_l) f_{jL}, f_{lL}) \geqq q \sum_{|I|=q} \|f_I\|^2.$$

Proof. We consider the bilinear form with coefficients $X_j(x_l)$. Note that

$$X_j(x_l) = B([X_j, \sum x_i X_i], X_l)$$

where $\sum x_i X_i$ is the vector field on G dual to the form

$$\sum x_i w^i = \pi^*(d(r^2/2)).$$

In terms of the Cartan decomposition $G = \exp \mathfrak{p}_0 \cdot K$, we define a vector valued map $\varphi\colon G \to \mathfrak{p}_0$ by

$$\varphi(g) = \operatorname{ad}(k^{-1})(P) \quad \text{where} \quad g = \exp P \cdot k = k \cdot \exp \varphi(g)$$

and then

$$X_j(x_l) = B(X_j(\varphi), X_l).$$

The function φ is left K-invariant, $\varphi(kg) = \varphi(g)$, so

$$B(X_j(\varphi), X_l)(kg) = B(X_j(\varphi), X_l)(g)$$

and hence $X_j(x_l)$ is left K-invariant. It now suffices to calculate the form on $\exp \mathfrak{p}_0$.

The map φ factors $\pi \cdot \tau\colon G \to K \times \mathfrak{p}_0 \to \mathfrak{p}_0$ where $\tau(k \exp P) = (k, P)$ is a diffeomorphism and π the projection onto the second factor in the product.

We compute

$$(L_{\exp P} Z)(\varphi) = Z(\varphi \cdot L_{\exp P}) \quad \text{where} \quad Z \in \mathfrak{p}_0.$$

Firstly

$$(L_{\exp P} Z)(\varphi) = d\varphi(L_{\exp P} Z) = d\pi \cdot d\tau(L_{\exp P} Z)$$

where $d\varphi \in \Gamma(T^*(G) \otimes p_0)$ is the vector valued exterior derivative.

A tangent vector at (k, P) has the form $(L_k T, Y)$ where $T \in \mathbf{k}_0$, $Y \in \mathbf{p}_0$ and

$$(d\tau^{-1})_{(k,P)}(L_k T, Y) = L_{k \exp P}\left\{ e^{-\operatorname{ad}(P)} T + \frac{1 - e^{-\operatorname{ad}(P)}}{\operatorname{ad}(P)}(Y) \right\}.$$

According to the Cartan decomposition $\mathbf{g}_0=\mathbf{k}_0\oplus\mathbf{p}_0$, the $\mathbf{k}_0$ component is given by

$$\cosh \operatorname{ad}(P)T+\frac{1-\cosh \operatorname{ad}(P)}{\operatorname{ad}(P)}Y \tag{1}$$

and the $\mathbf{p}_0$ component by

$$-\sinh \operatorname{ad}(P)T+\frac{\sinh \operatorname{ad}(P)}{\operatorname{ad}(P)}Y. \tag{2}$$

Equating compact and non-compact components in

$$\big((d\tau)_{\exp P}\big)^{-1}(L_{\exp P}Y) = (d\tau^{-1})_{(e,P)}(L_{\exp P}Y) = L_{\exp P}Z,$$

we have (1)=0 and (2)$=Z$. Hence

$$Z=-\sinh \operatorname{ad}(P)T+\frac{\sinh \operatorname{ad}(P)}{\operatorname{ad}(P)}Y = A(P)\big(Y-\operatorname{ad}(P)T\big)$$

where $A(P)=\dfrac{\sinh ad(P)}{ad(P)}$.

The map $A(P)$: $\mathbf{p}_0\to\mathbf{p}_0$ is well known to be invertible: if exp: $\mathbf{p}_0\to G/K$, then $(d\exp)_P=L_{\exp P}A(P)$. Hence $Y-\operatorname{ad}(P)T=A(P)^{-1}Z$. From (1) we get

$$T+\frac{\cosh \operatorname{ad}(P)-1}{\operatorname{ad}(P)}\big(\operatorname{ad}(P)T-Y\big)=0$$

and hence

$$T=\frac{\cosh \operatorname{ad}(P)-1}{\operatorname{ad}(P)}A(P)^{-1}Z.$$

Finally, $Y-\operatorname{ad}(P)T=A(P)^{-1}Z$, $Y=\cosh \operatorname{ad}(P)\big(A(P)^{-1}Z\big)$,

$$\text{i.e., } (d\varphi)(L_{\exp P}Z)=\cosh \operatorname{ad}(P)\big(A(P)^{-1}Z\big).$$

We now choose an orthonormal basis $X_1,\dots,X_r$ for $\mathbf{p}_0$ by taking $X_1=P/\|P\|$ and X_i, $2\leqq i\leqq r$ to be eigenvectors of the self-adjoint transformation $\operatorname{ad}(P)^2$: $\mathbf{p}_0\to\to\mathbf{p}_0$ with $\operatorname{ad}(P)^2X_i=\lambda_iX_i$. The symmetric space G/K has negative sectional curvature and it follows that $\lambda_i\geqq 0$, $1\leqq i\leqq r$. Finally,

$$X_j(x_l)=B\big(d\varphi(L_{\exp P}X_j), X_l\big)=B\big(\cosh \operatorname{ad}(P)(A(P)^{-1}X_j), X_l\big)=$$

$$=\begin{cases}\dfrac{\sqrt{\lambda_j}}{\tanh\sqrt{\lambda_j}}\,\delta_{jl} & \text{if } \lambda_j\neq 0\\ \delta_{jl} & \text{if } \lambda_j=0\end{cases}$$

and $\dfrac{\sqrt{\lambda_j}}{\tanh\sqrt{\lambda_j}}\geqq 1$.

Hence

$$\sum_{j,l,L}\big(X_j(x_l)f_{jL}, f_{lL}\big)\geqq q\sum_I\|f_I\|^2.$$

The two lemmas together imply that

$$\|\pi^* Df\|^2 \geqq (3/2)q\|\pi^* f\|^2.$$

Remark. If we had chosen the weight $e^{-\varepsilon r^2}$ instead of e^{-r^2} i.e., the form $d(\varepsilon r^2/2)$, then the same method would yield positive constants C_q whenever $\varepsilon > 1/4$.

3. *The space* $\tilde{H}^1(\Omega^*(G/K))$

The space $\tilde{H}^1(\Omega^*(G/K))$ is the completion of the smooth forms of compact support on G/K, with respect to the graph norm associated to D.

Lemma 3.1. *The space* $\tilde{H}^1(\Omega^*(G/K))$ *may be characterized by*

$$\tilde{H}^1(\Omega^*(G/K)) = \{f \in L^2(\Omega^*(G/K)) \mid df,\ d^*f,\ |r|f \in L^2(\Omega^*(G/K))\}.$$

Proof. $\|Df\|^2 = \|df\|^2 + \|d^*f\|^2 + \|e(rdr)f\|^2 + \|i(rdr)f\|^2 + (Rf, f) + (f, Rf)$ where R denotes the Lie derivative operation with respect to the radial vector field rd/dr.

A routine calculation using spherical coordinates shows that

$$|(Rf, f) + (f, Rf)| \leqq C(\|f\|^2 + \|\sqrt{r}f\|^2), \quad C > 0$$

(see Appendix).

Theorem 3.2. *The inclusion* $\tilde{H}^1(\Omega^*(G/K)) \hookrightarrow L^2(\Omega^*(G/K))$ *is a compact operator.*

Proof. Let $C \subseteqq G/K$ be compact and let

$$\tilde{H}^1(C) = \{f \in \tilde{H}^1(\Omega^*(G/K)) \mid \operatorname{supp} f \subseteqq C\}.$$

Then the inclusion $\tilde{H}^1(C) \subseteqq L^2(\Omega^*(G/K))$ is well known to be compact (see [3]).

Let $\varepsilon > 0$ be small and $\chi\colon G/K \to [0, 1]$ be such that

$$\begin{aligned}\chi(x) &= 1 \quad \text{if} \quad B(x, x) \leqq \varepsilon^{-2}\\ &= 0 \quad \text{if} \quad B(x, x) \geqq (\varepsilon/2)^{-2}\end{aligned}$$

and $\|d\chi\| \leqq 2$. This can be done by defining χ as a function of r.

Let

$$B = \{f \mid \|f\|,\ \|df\|,\ \|d^*f\|,\ \|rf\| \leqq 1\}$$

and $f \in B$. Then

$$\|f - \chi \cdot f\|_{L^2} \leqq \int_{\|P\| \geqq \varepsilon^{-1}} |f(P)|^2 |\det A(P)|\, dP.$$

But $\int |r \cdot f|^2 \leqq 1$ and hence $\int_{\|P\| > \varepsilon^{-1}} |f|^2 \leqq \varepsilon$.

Hence $\|f - \chi \cdot f\| \leqq \varepsilon$.

The composition

$$\tilde{H}^1(\Omega^*(G/K)) \to \tilde{H}^1(C) \hookrightarrow L^2(\Omega^*(\Omega/K))$$

given by $f \mapsto \chi \cdot f$ is compact. Hence the inclusion $\tilde{H}^1(\Omega^*(G/K)) \hookrightarrow L^2(\Omega^*(G/K))$ can be uniformly approximated by compact operators and is hence compact.

Next we examine the natural action of G on $\tilde{H}^1(\Omega^*(G/K))$. It is clear that this action is isometric on $L^2(\Omega^*(G/K))$ and preserves the operators d and d^*. To check that it preserves $\tilde{H}^1(\Omega^*(G/K))$, we need only observe that $r - g \cdot r$ is bounded by the geodesic distance from gK to eK, i.e., $r(g \cdot 0, 0)$.

A proof that bypasses the characterization of $\tilde{H}^1(\Omega^*(G/K))$ in Lemma 3.1 involves proving the following result.

Lemma 3.3. *The form*

$$w(x) = d\big(r(0, x)^2 - r(g \cdot 0, x)^2\big)$$

is bounded with respect to the metric on $T^*(G/K)$.

Proof. Decomposing the form as a difference of squares, we get

$$w(x) = d\big(r(0, x) + r(g \cdot 0, x)\big) \cdot \big(r(0, x) - r(g \cdot 0, x)\big) +$$
$$+ \big(r(0, x) + r(g \cdot 0, x)\big) \cdot d\big(r(0, x) - r(g \cdot 0, x)\big).$$

The form $d(r(0, x))$ has norm 1 everywhere except at 0. The homogeneity of G/K gives the same result for $d(r(g \cdot 0, x))$. Hence the first term is bounded.

Put $r(0, x) = R$ and $r(g \cdot 0, x) = \varepsilon$. In terms of the metric induced on $\mathbf{p}_0$ by $\exp\colon \mathbf{p}_0 \to G/K$, the sine of the angle between the geodesics from x to 0 and x to $g \cdot 0$ is less than or equal to ε/R. The cotangent vectors $d(r(0, x))$ and $d(r(g \cdot 0, x))$ are dual to the unit tangent vectors along these geodesics and hence

$$|d(r(0, x)) - d(r(g \cdot 0, x))| \leqq \sin^{-1}(\varepsilon/R) = 0(1/R).$$

Hence the second term is bounded.

This lemma implies that $[D, g]$, the commutator of the differential operator D and an isometry $g \in G$, preserves $\tilde{H}^1(\Omega^*(G/K))$.

Lemma 3.4. *Let* $g \in G$, *then* $[D, g]\colon \tilde{H}^1(\Omega^*(G/K)) \to L^2(\Omega^*(G/K))$ *is a compact operator.*

Notice that if $g \in K$, then $[D, g] = 0$ and hence K acts on $\tilde{H}^1(\Omega^*(G/K))$ isometrically. Further, the action of G becomes unitary in the Calkin algebra of $\tilde{H}^1(\Omega^*(G/K))$. This follows from the following general result.

Lemma 3.5. *Let* H *be a Hilbert space,* $D\colon H \to H$ *a densely defined operator with a closed graph,* $A\colon H \to H$ *a unitary operator such that* A (domain of D) = = domain of D *and such that the commutator* $[D, A]$ *is compact. Then the restriction of* A *to the domain of* D, *equipped with the graph norm, is unitary modulo a compact operator.*

Proof. This is proved by calculating the polar decomposition of $A|\text{domain of } D$. When the domain of D is identified with the graph of D, the action of A becomes

$$(h, Dh) \mapsto (Ah, DAh) = (Ah, ADh + [A, H]h).$$

In terms of the inclusion i: graph $D \hookrightarrow H \times H$, we have that $A|\text{domain of } D$ becomes

$$\begin{pmatrix} A & [A, D] \\ 0 & A \end{pmatrix} \Bigg|_{\text{graph}\, D}$$

Let $\mathbf{P}$ denote the orthogonal projection onto the graph of D, then,

$$A|\text{domain of } D = \mathbf{P} \begin{pmatrix} A & [A, D] \\ 0 & A \end{pmatrix} i$$

and hence

(1) $$(A|\text{domain of } D)^*(A|\text{domain of } D) =$$

$$= \mathbf{P} \begin{pmatrix} A^{-1} & 0 \\ [A, D]^* & A^{-1} \end{pmatrix} i\mathbf{P} \begin{pmatrix} A & [A, D] \\ 0 & A \end{pmatrix} i.$$

However,

$$\mathbf{P} \begin{pmatrix} A & [A, D] \\ 0 & A \end{pmatrix} i = \begin{pmatrix} A & A, D \\ 0 & A \end{pmatrix} i.$$

Hence

$$(1) = \mathbf{P} \begin{pmatrix} A^{-1} & 0 \\ [A, D]^* & A^{-1} \end{pmatrix} \begin{pmatrix} A & [A, D] \\ 0 & A \end{pmatrix} i = \mathbf{P} \begin{pmatrix} 1 & A^{-1}[A, D] \\ [A, D]^* A & 1 \end{pmatrix} i.$$

Hence $(A|\text{domain of } D)^*(A|\text{domain of } D)$ has the form $1 +$ self-adjoint compact operator. The spectral decomposition of this operator implies that the positive square root will have the same form.

4. Hilbert bundles

Let Γ be a discrete, torsionless subgroup of a non-compact, semi-simple Lie group G, ξ a smooth vector vector bundle on $\Gamma \backslash G/K$, and $\xi \leqq \mathbf{C}^N$ a fixed smooth inclusion in a trivial bundle with $\mathbf{P}$ the orthogonal projection of $\mathbf{C}^N$ onto ξ. We define the densely defined operator

$$D_\xi \colon L^2(\Omega^*(G/K) \otimes \xi) \to L^2(\Omega^*(G/K) \otimes \xi)$$

by the composition

$$L^2(\Omega^*(G/K) \otimes \xi) \hookrightarrow L^2(\Omega^*(G/K) \otimes \mathbf{C}^N) \xrightarrow{\oplus D} L^2(\Omega^*(G/K) \otimes \mathbf{C}^N) \xrightarrow{\mathbf{P}} L(\Omega^*(G/K) \otimes \xi)$$

where $L^2(\Omega^*(G/K) \otimes \xi)$ denotes the square-integrable forms taking values in the bundle ξ pulled back onto G/K. The space $\tilde{H}^1(\Omega^*(G/K)\otimes\xi)$ is defined to be the domain of the closure of the operator D_ξ equipped with the graph norm. The bundle ξ on G/K has a canonical Γ-action and hence induces an action on $L^2(\Omega^*(G/K)\otimes\xi)$ and $\tilde{H}^1(\Omega^*(G/K)\otimes\xi)$.

We will prove that when **P** is constant outside a compact subset of $\Gamma\backslash G/K$ that the triple $(\tilde{H}^1, L^2, D_\xi)$ is mapped by α to the class defined by $[\xi]$ in $K(\Gamma\backslash G/K)$.

Fix $y\in G/K$ and let

$$D_y = d+d^*+(e+i)(d(r(y,x)^2/2)).$$

Then D_y-D is compact and if $g\cdot 0=y$ $(0=e\cdot K)$ then $D_y=g\cdot D$. Let H and L denote the trivial Hilbert bundles over G/K with fibres $\tilde{H}^1(\Omega^*(G/K))$ and $L^2(\Omega^*(G/K))$, respectively.

Lemma 4.1. *The family*

$$\mathscr{D} = \{D_y\}_{y\in G/K}\colon G/K \to B(H, L)$$

is a G-invariant uniformly continuous family of bounded operators.

The proof of this lemma is similar to the proof of Lemma 3.3.

Similarly, we define a Γ-invariant family

$$\mathscr{D}_\xi^1\colon G/K \to B(\tilde{H}^1(\Omega^{\mathrm{ev}}(G/K)(\otimes\xi), L^2(\Omega^{\mathrm{odd}}(G/K)\otimes\xi)$$

by using the inclusion and orthogonal projection associated to ξ.

Theorem 4.2. *Let Γ be a discrete, torsionless subgroup of a non-compact semi-simple Lie group G and ξ a smooth vector bundle on G/K which is trivial on the complement of a compact subset then ξ and the family $\mathscr{D}_\xi^1$ define the same element of $K(\Gamma\backslash G/K)$.*

Consider the family

$$\mathscr{D}_\xi^0\colon \tilde{H}^1(\Omega^{\mathrm{ev}}(G/K))\otimes\xi \to L^2(\Omega^{\mathrm{odd}}(G/K))\otimes\xi$$

given by $\mathscr{D}_\xi^0(y)=D_y\otimes 1$. At each point $y\in G/K$, the corresponding operator is surjective with kernel

$$\langle e^{-r(y,x)^2/2}\rangle\otimes\xi_y.$$

Hence $\mathscr{D}_\xi^0$ defines the same lement as ξ in $K(\Gamma\backslash G/K)$.

The proof of this theorem consists of giving an invariant homotopy between the families $\mathscr{D}_\xi^1$ and $\mathscr{D}_\xi^0$. Unfortunately, it is not uniformly continuous but Hörmander has developed a theory of strongly continuous homotopy which can be used here (see [4]).

We summarize the results needed from this paper of Hörmander.

Let I be a compact space, E and F Banach spaces and $P\colon I\to B(E, F)$.

Then P is called a closed family if the graph

$$\{(t, u, f)|t\in I,\ u\in E,\ f=P(t)u\} \leqq I\times E\times F \quad \text{is a closed set.}$$

If the map

$$(t, u)\mapsto P(t)u\colon\ I\times E\to F$$

is continuous, then P is said to be a strongly continuous family. Let B_E denote the unit ball in E. If the image of $I\times B_E$ is relatively compact in F, we say that P is a compact family. If $\{E_t\}_{t\in I}$ is a family of subspaces of E indexed by I, we say that it is locally compact if $\{(t, e)|e\in E_t\}$ is a locally compact subset of $I\times E$.

Lemma. *If $\{E_t\}_{t\in I}$ is a locally compact family of subspaces, then*

(i) $\dim E_t$ *is a finite, upper semi-continuous function,*
(ii) *if* $\dim E_t$ *is constant, then the spaces*

$\{E_t\}$ with the topology induced from $I\times E$ form a vector bundle over I.

Proposition. *Let $\{P_t\}_{t\in I}$ be a closed family of operators from E to F. Then if*

(i) *the family is almost left invertible in the sense that there exists a strongly continuous family $\{Q_t\}_{t\in I}$ from F to E and a compact family $\{K_t\}_{t\in I}$ from E to such that for every $t\in I$, $Q_tP_t=1_E+K_t$,*

(ii) $\dim \operatorname{Ker} P_t$ *is a finite upper semi-continuous function of t, the range of P_t is closed and index P_t is upper semi-continuous.*

The homotopy we use is as follows. Let $y\in G/K$, $t\in[0, 1]$. Then $\exp\colon T_y(G/K)\to\to G/K$ is a diffeomorphism. Let $C_{y,t}\colon G/K\to G/K$ be the map induced on G/K by multiplication by t in $T_y(G/K)$. The family $C_{y,t}$, $t\in[0, 1]$ consists of contractions along the geodesics radiating from y. Let $\xi_{y,t}$ be the pullback of ξ by $C_{y,t}$, i.e., $\xi_{y,t}(x)=\xi(C_{y,t}(x))$. Then $\xi_{y,1}=\xi$ and $\xi_{y,0}=\xi(y)$, the trivial bundle with fibre $\xi(y)$.

The inclusion $\xi\hookrightarrow \mathbf{C}^N$ induces inclusions $\xi_{y,t}\hookrightarrow\mathbf{C}^N$ and orthogonal projections $\mathbf{P}_{y,t}\colon \mathbf{C}^N\to\xi_{y,t}$. We have now two families of operators $\mathscr{D}_\xi$ and $\mathscr{D}_{\xi\perp}$ parametrized by $G/K\times[0, 1]$

$$\mathscr{D}_\xi(y, t)\colon \tilde{H}^1(\Omega^*(G/K)\otimes\xi_{y,t})\to L^2(\Omega^*(G/K)\otimes\xi_{y,t}),$$
$$\mathscr{D}_{\xi\perp}(y, t)\colon \tilde{H}^1(\Omega^*(G/K)\otimes\xi^\perp_{y,t})\to L^2(\Omega^*(G/K)\otimes\xi^\perp_{y,t})$$

where $\xi^\perp_{y,t}$ is the orthogonal complement to $\xi_{y,t}$ in $\mathbf{C}^N$. Altogether,

$$\mathscr{D}_\xi\oplus\mathscr{D}_{\xi\perp}\colon G/K\times[0, 1]\to B(\tilde{H}^1(\Omega^*(G/K))^N,\ L^2(\Omega^*(G/K))^N).$$

We show that the family $\mathscr{D}_\xi\oplus\mathscr{D}_{\xi\perp}$ satisfies the conditions of Hörmander's proposition locally.

The first step is to produce a family of parametrices. The operator

$$D\colon \tilde{H}^1(\Omega^*(G/K))\to L^2(\Omega^*(G/K))$$

is Fredholm, the kernel and cokernel have dimension 1 and hence their exists a bounded operator P

$$P\colon L^2(\Omega^*(G/K)) \to \tilde{H}^1(\Omega^*(G/K))$$

(mapping forms even to odd and odd to even) such that PD and DP have the form $1+$compact operator. Further, $D-D_y$ is compact and hence P is also a parametrix for D_y. The parametrix P induces a parametrix

$$P\colon L^2(\Omega^*(G/K))^N \to \tilde{H}^1(\Omega^*(G/K))^N$$

by taking N copies. Let

$$P_{y,t}\colon L^2(\Omega^*(G/K)\otimes\xi_{y,t}) \to \tilde{H}^1(\Omega^*(G/K)\otimes\xi_{y,t})$$

be given by $\mathbf{P}_{y,t}P\,\mathbf{P}_{y,t}$ and $\mathscr{P}_\xi$ be the family parametrized by $G/K\times[0,1]$ where $\mathscr{P}_\xi(y,t)=P_{y,t}$ and $\mathscr{P}=\mathscr{P}_\xi\oplus\mathscr{P}_{\xi\perp}$. Then $\mathscr{P}$ is our candidate for the family of parametrices for $\mathscr{D}$.

Lemma 4.3. *The projections $\mathbf{P}_{y,t}$ induce a strongly continuous families of operators in both $\tilde{H}^1(\Omega^*(G/K))^N$ and $L^2(\Omega^*(G/K))^N$.*

Proof. We prove the $\tilde{H}^1$ case, the proof of the L^2 case is included.

Let $(x_n, t_n, f_n)\to(x, t, f)$ be a convergent sequence in $G/K\times[0,1]\times\tilde{H}^1(\Omega^*(G/K))^N$. We must show that $\mathbf{P}_{y_n,t_n}(f_n)\to\mathbf{P}_{y,t}(f)$ and $D(\mathbf{P}_{y_n,t_n}(f_n))\to D(\mathbf{P}_{y,t}(f))$ in $L^2(\Omega^*(G/K))^N$. The difference

$$\|\mathbf{P}_{y_n,t_n}(f_n)-\mathbf{P}_{y,t}(f)\| \leqq \|\mathbf{P}_{y_n,t_n}(f_n)-\mathbf{P}_{y_n,t_n}(f)\|+\|(\mathbf{P}_{y_n,t_n}-\mathbf{P}_{y,t})(f)\| \leqq$$
$$\leqq \|f_n-f\|+\|(\mathbf{P}_{y_n,t_n}-\mathbf{P}_{y,t})(f)\|.$$

Considering $\mathbf{P}_{y,t}$ as a matrix valued function on G/K, $\mathbf{P}_{y_n,t_n}\to\mathbf{P}_{y,t}$ uniformly on compacta. Choose R such that

$$\int_{|x|\geqq R}\|f\|^2<\varepsilon$$

and such that $\|\mathbf{P}_{y_n,t_n}(x)-\mathbf{P}_{y,t}(x)\|\leqq\varepsilon$ for all $x\in R$. Then

$$\|(\mathbf{P}_{y_n,t_n}-\mathbf{P}_{y,t})(f)\| \leqq \varepsilon\|f\|+2\varepsilon.$$

It follows that $\mathbf{P}_{y_n,t_n}(f_n)\to\mathbf{P}_{y,t}(f)$ in L^2.

The case involving D involves one further step,

$$\big\|D(\mathbf{P}_{y_n,t_n}(f_n)-\mathbf{P}_{y,t}(f))\big\| \leqq \big\|D(\mathbf{P}_{y_n,t_n}(f_n-f))\big\|+\big\|D((\mathbf{P}_{y_n,t_n}-\mathbf{P}_{y,t})(f))\big\| \leqq$$
$$\leqq \|[D,\mathbf{P}_{y_n,t_n}](f_n-f)\|+\|D(f_n-f)\|+\|[D,\mathbf{P}_{y_n,t_n}-\mathbf{P}_{y,t}](f)\|+\|(\mathbf{P}_{y_n,t_n}-\mathbf{P}_{y,t})(f)\|.$$

The last term can be made small by the argument above. Further,

$$[D,\mathbf{P}_{y_n,t_n}]=[d+d^*,\mathbf{P}_{y_n,t_n}]=[d,\mathbf{P}_{y_n,t_n}]-[d,\mathbf{P}_{y_n,t_n}]^*=e(d\mathbf{P}_{y_n,t_n})-i(d\mathbf{P}_{y_n,t_n})$$

where $\mathbf{P}_{y_n, t_n}$ is considered as an $N \times N$ matrix of functions and $d\mathbf{P}_{y_n, t_n}$ the corresponding $N \times N$ matrix of forms. Further, $d\mathbf{P}_{y_n, t_n} = t_n d\mathbf{P} \cdot C_{y_n, t_n}$. Hence $d\mathbf{P}_{y_n, t_n} \to \to d\mathbf{P}$ uniformly on compacta. It follows that

$$\|[D, \mathbf{P}_{y_n, t_n} - \mathbf{P}_{y, t}](f)\| \to 0 \quad \text{as} \quad n \to 0.$$

If ξ comes from a uniform lattice or $\mathbf{P}$ is constant outside a compact subset of $\Gamma \diagdown G/K$, then $d\mathbf{P}$ is bounded. Hence,

$$\|[D, \mathbf{P}_{y_n, t_n}](f_n - f)\| \leqq C\|f_n - f\|$$

and hence converges to zero.

Lemma 4.4. *The families $\mathscr{D}$ and $\mathscr{P}$ are strongly continuous.*

Proof. A composition of strongly continuous families is strongly continuous. The next step is to check that

$$\mathscr{P}_0 \mathscr{D} = 1 + \mathscr{K}$$

where $\mathscr{K}$ is (locally) a compact family. Consider the families

$$\mathscr{P}_\xi(y, t) = \mathbf{P}_{y,t} P \mathbf{P}_{y,t} \quad \text{and} \quad \mathscr{D}_\xi(y, t) = P_{y,t} \mathscr{D} P_{y,t}.$$

Then

$$\mathscr{P}_\xi \mathscr{D}_\xi = \mathbf{P}_{y,t} P D_y \mathbf{P}_{y,t} + \mathbf{P}_{y,t} P [\mathbf{P}_{y,t}, D_y] \mathbf{P}_{y,t} =$$
$$= \mathbf{P}_{y,t}(1 + K + PK_y') \mathbf{P}_{y,t} + \mathbf{P}_{y,t} P K_y'' \mathbf{P}_{y,t}$$

where $K_y' : \tilde{H}^1(\Omega^*(G/K))^N \to \tilde{H}^1(\Omega^*(G/K))^N$ is given by $1/2(e+i)(d(0, x)^2 - d(y, x)^2)$ and K_y'' by $(e-i)(d\mathbf{P}_{y,t})$. These operators K_y' and K_y'' are bounded on $\tilde{H}^1(\Omega^*(G/K))^N$ and hence compact as operators from $\tilde{H}^1(\Omega^*(G/K))^N$ to $L^2(\Omega^*(G/K))^N$. Therefore

$$\mathscr{P}_\xi \circ \mathscr{D}_\xi(y, t) = \mathbf{P}_{y,t} + K_{y,t}$$

where $K_{y,t} = \mathbf{P}_{y,t} K \mathbf{P}_{y,t} + \mathbf{P}_{y,t} P K_y' \mathbf{P}_{y,t} + \mathbf{P}_{y,t} P K_y'' \mathbf{P}_{y,t}$ is a family of compact operators on $\tilde{H}^1(\Omega^*(G/K))^N$.

Lemma 4.5. *Let $B \subseteqq \tilde{H}^1(\Omega^*(G/K))^N$ be the unit ball. Then*

$$\bigcup_{(y,t) \in G/K \times [0,1]} \mathbf{P}_{y,t}(B)$$

is a bounded subset of $\tilde{H}^1(\Omega^*(G/K))^N$.

Proof. Firstly

$$\|D\mathbf{P}_{y,t} f\| \leqq \|\mathbf{P}_{y,t} Df\| + \|[D, \mathbf{P}_{y,t}]f\| \leqq \|Df\| + \|(e-i)(d\mathbf{P}_{y,t})f\|.$$

The restriction that $\mathbf{P}_{y,t}$ is constant outside a compact subset of $\Gamma \diagdown G/K$ implies that $d\mathbf{P}$ is bounded. It now follows that $\mathbf{P}_{y,t} K \mathbf{P}_{y,t}$ is (locally) a compact family.

The operators $K_y'=1/2(e+i)\big(d(0,x)^2-d(y,x)^2\big)$ form a uniformly continuous family of operators on $L^2(\Omega^*(K/G))^N$, $\cup\mathbf{P}_{y,t}(B)$ is a bounded subset of $\tilde{H}^1(\Omega^*(G/K))^N$ and hence a precompact subset of $L^2(\Omega^*(K/G))^N$. The composition $\mathbf{P}_{y,t}P\,\mathbf{K}_y'$ is strongly continuous and hence for each compact $C\subseteqq G/K$

$$\bigcup_{(y,t)\in C\times[0,1]}\mathbf{P}_{y,t}PK_y'\mathbf{P}_{y,t}(B)$$

is precompact.

Similarly, the family $\mathbf{P}_{y,t}P\,K_y''$ forms a strongly continuous family of operators from $L^2(\Omega^*(G/K))^N$ to $\tilde{H}^1(\Omega^*(G/K))^N$.

Hence finally, the family $K_{y,t}$ is (locally) a compact family as it is the sum of three such families.

We may also show that $\mathscr{D}\cdot\mathscr{P}$ has the form $1+$compact family. The proof is slightly easier than for $\mathscr{P}\cdot\mathscr{D}$ because we are then working with $L^2(\Omega^*(G/K))^N$.

Hörmander's theory now implies that $\mathscr{D}$ is a family of Fredholm operators of constant index and that the kernels and cokernels form locally compact families of spaces.

The proof of Theorem 4.2 is completed by considering the index of the family

$$\mathscr{D}_\xi(y,t)\colon \tilde{H}^1(\Omega^*(G/K)\otimes\xi_{y,t})\to L^2(\Omega^*(G/K)\otimes\xi_{y,t}).$$

We must show the that index of this family is defined as an element of $K(\Gamma\backslash G/K\times[0,1])$.

The family $\mathscr{D}$ is Γ-invariant. Divide out by this action to get a family

$$\mathscr{D}\colon \Gamma\backslash G/K\times[0,1]\to B\big(\tilde{H}^1(\Omega^{\mathrm{ev}}(G/K))^N,\ L^2(\Omega^{\mathrm{odd}}(G/K))^N\big)$$

and a strongly continuous family of projections, $\mathbf{P}_{y,t}$, also parametrized by $\Gamma\backslash G/K\times[0,1]$ such that

$$\mathscr{D}=\mathbf{P}\mathscr{D}\mathbf{P}+(1-\mathbf{P})\mathscr{D}(1-\mathbf{P})=\mathscr{D}_\xi+\mathscr{D}_{\xi\perp}.$$

Let $(y',t')\in\Gamma\backslash G/K\times[0,1]$. Then enlarging the domain we can deform $\mathscr{D}$ into a family surjective in a neighbourhood of (y',t') by taking

$$\mathscr{D}(y,t)\oplus\mathbf{P}_{y,t}\oplus(1-\mathbf{P}_{y,t})\colon \tilde{H}^1(\Omega^{\mathrm{ev}}(G/K))^N+\operatorname{coker}\mathscr{D}_\xi(y',t')+\operatorname{coker}\mathscr{D}_{\xi\perp}(y',t')\to$$
$$\to L^2(\Omega^{\mathrm{odd}}(G/K))^N.$$

On any compact set $C\leqq\Gamma\backslash G/K$ we can deform the family by a finite number of steps of this form so as the the family becomes surjective on $C\times[0,1]$ and finally we have

$$\mathscr{D}\oplus\mathbf{P}^k\oplus(1-\mathbf{P})^k\colon \tilde{H}^1(\Omega^{\mathrm{ev}}(G/K))^N\oplus V_1\oplus V_2\to L^2(\Omega^{\mathrm{odd}}(G/K)^N$$

where V_1 and V_2 are finite dimensional.

Hence the kernel of this deformed family forms a vector bundle over $C\times[0,1]$, call it η, and $\eta-\dim V_1-\dim V_2$ gives the element of $K(C\times[0,1])$ defined by the family $\mathscr{D}$. Notice that the kernel bundle η comes equipped with a continuous family $\mathbf{P}'$ of projections, coming from the projections onto $\tilde{H}^1(\Omega^{ev}(G/K)\otimes\xi_{y,t})\oplus \oplus V_1$. The continuity implies that they are of constant rank. Then $\mathbf{P}'(\eta)-\dim V_1$ gives the element of $K(C\times[0,1])$ represented by the family $\mathscr{D}_\xi$. Finally,

$$\mathbf{P}'(\eta)-\dim V_1|C\times\{0\} = [\xi]$$
$$\mathbf{P}'(\eta)-\dim V_1|C\times\{0\} = [D_\xi]$$

where D_ξ comes from a Fredholm representation. The theorem is proved.

Appendix

We prove that if R denotes the Lie derivative with respect to the radial vector field on G/K and $f\in\Omega^q(G/K)$, then

$$|(Rf,f)+(f,Rf)| \leqq C(\|f\|^2+\|\sqrt{r}f\|^2$$

where C is a positive constant.

Identify p_0 and G/K through the diffeomorphism

$$\exp: \mathbf{p}_0 \to G/K.$$

The Riemannian metric induced on $\mathbf{p}_0$ is then given by

$$(X,Y)_P = B(A(P)X, A(P)Y)$$

where B is the Killing form and $A(P)=\dfrac{\sinh \operatorname{ad}(P)}{\operatorname{ad}(P)}$. Let

$$S = \{X\in\mathbf{p}_0|B(X,X)=1\}.$$

Let $X_2, \dots, X_n$ be an orthonormal set of eigenvectors of the self-adjoint transformation $\operatorname{ad}(P/\|P\|)^2$ on p_0 and hence a basis for the tangent space of S at $P/\|P\|$. Let $\mathcal{O}_2, \dots, \mathcal{O}_n$ denote the corresponding normal coordinates on a neighbourhood N of $P/\|P\|$ in S, with respect to the Riemannian metric induced on S by (,). These functions induce coordinates on the positive cone on N,

$$\mathcal{O}_i(X) = \mathcal{O}_i(X/\|X\|),$$

which together with the radial distance $r(X)=B(X,X)^{1/2}$, form polar coordinates.

Let $f\in\tilde{H}^1(\Omega^*(G/K))$. Then f decomposes as

$$f = w_1+w_2, \quad i(dr)w_1 \neq 0, \quad i(dr)w_2 = 0,$$

so that

$$w_1 = \textstyle\sum_{|I|=q-1} f_{1,I}\, dr\wedge d\mathcal{O}^I, \quad w_2 = \sum_{|J|=q} f_{2,J}\, d\mathcal{O}^J,$$

where $I=(i_1, \dots, i_{q-1})$ is a multi-index and

$$d\mathcal{O}^I = d\mathcal{O}^{i_1} \wedge \dots \wedge d\mathcal{O}_{i_{q-1}}.$$

Then

$$Rf(P) = \sum_I \left(rd/dr(f_{1,I})\, dr \wedge d\mathcal{O}^I + f_{1,I}\, dr \wedge d\mathcal{O}^I\right) + \sum_J rd/dr(f_{2,J})\, d\mathcal{O}^J$$

and

$$\|\mathcal{O}^I(P)\| = r^{-(q-1)} \prod_{j=1}^{q-1} \frac{r\sqrt{\lambda_{i_j}}}{\sinh r\sqrt{\lambda_{i_j}}}.$$

Hence

$$\int_{P_0} rd/dr(f_{1,I})f_{1,I}|\mathcal{O}^I|^2 |\det A(P)|\, dP =$$
$$= -\int |f_{1,I}|^2 |\mathcal{O}^I|^2 |\det A(P)|\, dP -$$
$$-\int (f_{1,I}, rd/drf_{1,I})|\mathcal{O}^I|^2 |\det A(P)|\, dP -$$
$$-\int |f_{1,I}|^2 (rd/dr \ln |\mathcal{O}^I|^2) |\mathcal{O}|^2 |\det A(P)|\, dP -$$
$$-\int |f_{1,I}|^2 |\mathcal{O}^I|^2 (rd/dr \ln \det A(P)) |\det A(P)|\, dP.$$

The derivatives $d/dr \ln |\mathcal{O}^I|^2$ and $d/dr \ln A(P)$ are bounded at ∞.

The term w_2 is similar. Hence the result.

References

1. Atiyah, M. F., *K-theory*. Benjamin, 1968.
2. Helgason, S., *Differential geometry and symmetric spaces*. Academic Press, 1962.
3. Hörmander, L., *Linear partial differential operators*. Springer Verlag, 1963.
4. Hörmander, L., On the index of pseudo-differential operators, Elliptische Differentialgleichungen, Band II, Koll. (1969), Berlin.
5. Janich, K., Vektorraumbündel und der Raum der Fredholm-Operatoren, *Math. Annalen*. **162** (1965), 129—42.
6. Kasparov, G. G., The generalized index of elliptic operators, *Funct. Anal. and its Appl.*, **7** (3) 1973.
7. Matsushima, Y., On the first Betti number of compact quotient spaces of higher dimensional symmetric spaces, *Ann. Math.*, **75** (1962), 312—330.
8. Mishchenko, A. C., Infinite dimensional representations of discrete groups and higher signatures, *Izv. Acad. Nauk S.S.S.R.* **38** (1974), 81—106.
9. Mishchenko, A. C., On Fredholm representations of discrete groups, Funct. Anal. and its Appl., **9** (2) (1975), 36—44.
10. Nelson, E., *Tensor analysis*. Princeton University Press, 1971.
11. De Rham, G., *Variétés differentiables*. Hermann, Paris, 1955.

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