Composite integral operators and nuclearity

James A. Cochran

1. Introduction

Let the integrable two-variable function $K(x, y)$, $a \leqq x, y \leqq b$, serve as the kernel of the integral operator T given by

$$(T\varphi)(x) = \int_a^b K(x, y)\varphi(y)\,dy. \tag{1}$$

If $K(x, y)$ is square-integrable, then T is a *Hilbert—Schmidt* operator from $L^2[a, b]$ into itself, i.e.

$$\sum_{n=1}^{\infty} [s_n(T)]^2 < \infty.$$

Here the $s_n(T)$ are the characteristic numbers ([9], [16]; so-called s-numbers [12]) of T, defined to be the eigenvalues of the related compact nonnegative selfadjoint operator $(T^* T)^{1/2}$, arranged in decreasing order and repeated according to multiplicity.

Some compact operators between Hilbert spaces have characteristic numbers which are γ-summable for exponents γ smaller than 2. In the case $\gamma = 1$ the operator is said to be *nuclear* or of *trace class*. Since the pioneering work of Grothendieck [13] on nuclear spaces (see also Gohberg and Krein [11], [12]; Gel'fand and Vilenkin [10]) it has been of interest to enquire under what various sorts of conditions compact operators are nuclear.

For integral operators the following are amongst the known results (see [1], [2], [4], [5], [6], [9], [12], [21], [22], for example):

Theorem A (Mercer): *T is nuclear if $K(x, y)$ is continuous, Hermitian, and nonnegative (nonpositive) definite.*

Theorem B (Chang): *T is nuclear if and only if*

$$K(x, y) = \int_a^b K_1(x, z) K_2(z, y)\,dz$$

where K_1 and K_2 are in $L^2[a, b]$.

Theorem C (Smithies, Stinespring): *T is nuclear if $K(x, y)$, as a function of one of its two variables, is in* Lip α *for some* $\alpha > 1/2$.

More generally,

Theorem D (Smithies): *T is nuclear if $K(x, y)$, as a function of one of its two variables, is in* Lip (α, p) *with* $p \geqq 1$ *for some* $\alpha > 1/\min(2, p)$.

Theorem E (Cochran): *T is nuclear if $K(x, y)$, as a function of one of its two variables, belongs both to* Lip (α, p) *and to* Lip (β, q) *for some* $1 \leqq p < q$ *with*

i) *for* $\alpha \leqq \beta$,

$$\beta > 1/\min(2, q);$$

ii) *for* $\beta < \alpha \leqq \beta + \frac{1}{p} - \frac{1}{q}$,

$$\begin{cases} \beta > 1/q & q \leqq 2 \\ \alpha p(q-2) + \beta q(2-p) > q - p & p \leqq 2 < q \\ \alpha > 1/2 & 2 < p; \end{cases}$$

iii) *and for* $\beta + \frac{1}{p} - \frac{1}{q} < \alpha$,

$$\alpha > 1/\min(2, p).$$

As a special case of this last result we have

Corollary 1: *T is nuclear if $K(x, y)$, as a function of one of its two variables, is relatively uniformly of bounded p-variation for some* $1 \leqq p < 2$ *and also in* Lip (β, q) *for some* $q \geqq 1$, $\beta q > 1$.

Readers will note that each of the above sets of sufficient conditions for nuclearity has a counterpart in the theory of absolutely convergent Fourier series. Indeed, it has been largely an exploitation of precise analogies with the classical Fourier series results of Bernstein, Zygmund, Hardy and Littlewood, Szász, and others which has engendered most of these theorems (see the survey article [7]).

Recently, amongst the continuing stream of papers concerning convergence questions for Fourier series there have appeared several in which the functions of interest are convolutions ([15], [19], [23]; also the earlier [3], [24]). Using the Fourier series analogue of Theorem B above as the starting point, the various authors show how the series for the convolution of the two given functions remains absolutely convergent when the conditions on one of the functions are relaxed, if simultaneously the conditions on the other are suitably strengthened. In this paper we show that many of these recent Fourier series results for convolution functions, in a sense, are special cases of comparable results for composite integral operators. In view of our earlier remarks, moreover, the operator results are also of some independent interest.

2. Composite Integral Operators

Consider integrable two-variable functions $K(x, y)$, $a \leqq x, y \leqq b$ generating integral operators T according to (1). (Additional conditions, such as Zaanen's Property P [25], are generally needed so that T is well-defined in some appropriate sense.) We shall say that K, periodic in x with period $b-a$, is in Lip α if

$$\Delta K \equiv |K(x+h, y) - K(x-h, y)| < |h|^\alpha A(y) \quad (0 < \alpha \leqq 1)$$

where $A(y)$ is nonnegative and square-integrable. More generally, for $p \geqq 1$, K is said to be in Lip (α, p) if its L^p modulus of continuity satisfies

$$\omega_p(K, \delta) \equiv \sup_{0<h\leqq\delta} \left[\int_a^b (\Delta K)^p dx\right]^{1/p} < \delta^\alpha A(y) \quad (0 < \alpha \leqq 1)$$

with comparable A. Analogous definitions are valid if the roles of x and y are reversed.

Let $K(x, y)$ be such that its induced integral operator T is compact on $L^2[a, b]$ and has the Schmidt expansion [12] (polar representation [20])

$$T = \sum_{n=1}^\infty s_n(T)(\cdot, \psi_n)\varphi_n.$$

Here $s_n(T) \to 0$ as $n \to \infty$ and $\{\varphi_n\}$, $\{\psi_n\}$ are orthonormal systems in $L^2[a, b]$ satisfying the coupled equations

$$\left.\begin{aligned} s_n \varphi_n(x) &= \int_a^b K(x, y)\psi_n(y)\,dy, \\ s_n \psi_n(x) &= \int_a^b K^*(x, y)\varphi_n(y)\,dy. \end{aligned}\right. \qquad n = 1, 2, \dots .$$

If

$$\sum_{n=1}^\infty [s_n(T)]^p < \infty$$

for some $0 < p < \infty$, T is a member of the operator class C_p. The classes C_1 and C_2 are the collections of all nuclear and Hilbert—Schmidt operators, respectively.

In the sequel we will need the following results concerning the characteristic numbers of our integral operators:

Property 1 ([9], [12], [16], [20]): $s_n(T^*) = s_n(T)$ $n = 1, 2, \dots$.

Property 2 ([8]): If $0 < r \leqq 2$, then for arbitrary $m \geqq 1$,

$$\sum_{n=m}^\infty [s_n(T)]^r = \inf \sum_{n=m}^\infty \|T\Psi_n\|^r,$$

where the inf is taken over all orthonormal bases $\{\Psi_n\}$ of the underlying Hilbert space.

Property 3 ([18]): Let $K(x, y)$ be such that $K \in \mathscr{L}_{p,\infty}$, $K^* \in \mathscr{L}_{q,\infty}$ for some $1 \leqq p, q \leqq 2$, with $\max(p, q) > 1$, where $K \in \mathscr{L}_{p,s}$ implies

$$\|K\|_{p,s} \equiv \left\{\int_a^b \left[\int_a^b |K(x, y)|^p dx\right]^{s/p} dy\right\}^{1/s} < \infty.$$

Then the integral operator T generated by K is compact and belongs to C_r with $r=2\min(p, q)/(p+q-2)$.

Property 4 ([12], [14]): For any compact operators S and T,

$$\sum_{n=1}^{\infty}[s_n(ST)]^r \leqq \sum_{n=1}^{\infty}[s_n(S)s_n(T)]^r \quad (r>0).$$

We remark that, as far as the inducing kernels K, L, and M of S, T, and ST, respectively, are concerned, the composition of two integral operators involves nothing more than the usual kernel composition

$$M(x, y) = \int_a^b K(x, z)L(z, y)\,dz.$$

Employing Hölder's inequality, we can derive a useful extension of this last result, namely

Property 5. Let $p>1$ and $q=p/(p-1)$ be its conjugate. For any compact operators S and T, and any positive monotone (increasing or decreasing) function λ

$$\sum_{n=1}^{\infty}[s_n(ST)]^r \leqq \left\{\sum_{n=1}^{\infty}[s_n(S)\lambda(n)]^{pr}\right\}^{1/p}\left\{\sum_{n=1}^{\infty}[s_n(T)/\lambda(n)]^{qr}\right\}^{1/q} \quad (r>0).$$

In the case $\lambda\equiv 1$, we have as an immediate corollary

Property 6 ([9], [14], [16]): If $S\in C_p$ and $T\in C_q$ for some $0<p, q<\infty$, then the composite operators ST and TS both belong to C_r with $1/r=1/p+1/q$.

One direction of the earlier Chang result (Theorem B) is a consequence of Property 6; the other follows by explicit construction. A similar construction leads to the converse of Property 6.

Theorem 1: *Every operator R in C_r, $0<r<\infty$, admits of decompositions into operators $S\in C_p$, $T\in C_q$ with $R=ST$, $1/r=1/p+1/q$, $r<p, q<\infty$.*
Proof: Let R have the Schmidt expansion (polar representation)

$$R = \sum_n s_n(R)(\cdot, \psi_n)\varphi_n.$$

For arbitrary $p, q>r$ satisfying $1/p+1/q=1/r$ form

$$S \equiv \sum_n [s_n(R)]^{r/p}(\cdot, \psi_n)\varphi_n \quad \text{and} \quad T \equiv \sum_n [s_n(R)]^{r/q}(\cdot, \psi_n)\psi_n.$$

Clearly $R=ST$. Moreover, since $R\in C_r$, it follows readily that $S\in C_p$ and $T\in C_q$.

3. Nuclear Operators

In the remainder of the paper we take $a=0$, $b=2\pi$, for convenience, and assume that all functions are extended periodically for argument values outside the fundamental interval. We also designate the classical orthonormal complex Fourier functions by $\Phi_n(x)$, $n=0, \pm 1, \ldots$.

We now note that our earlier Theorem D is actually a special case of the following result, due essentially to Smithies [21], for which we provide a new and simplified proof:

Lemma 1: *If the generating kernel K is in* $\mathrm{Lip}\,(\alpha, p)$ *for some* $p \geqq 1$, $0 < \alpha \leqq 1$, *with* $\alpha + 1/2 > 1/p$, *then the induced integral operator T belongs to C_r for all* $r > [\alpha + 1 - 1/\min(2, p)]^{-1}$.

Proof: Expand $K(x,y)$, as a function of its first variable, in a classical complex Fourier series, viz.

$$K(x, y) \sim \sum_n \Phi_n(x) \bar{C}_n(y).$$

For $1 \leqq p \leqq 2$, it then follows (see McLaughlin [17], for instance) that for all $0 < \beta \leqq q$ with q conjugate to p, and arbitrary real γ,

$$\sum_n |C_n(y)|^\beta |n|^\gamma \leqq \text{const.} \sum_n |n|^{\gamma - \beta/q} [\omega_p(K, |n|^{-1})]^\beta. \tag{2}$$

Using the fact that K is in $\mathrm{Lip}\,(\alpha, p)$, the special case $\beta = 2$ leads upon integration to

$$\sum_n \|C_n\|^2 |n|^\gamma \leqq \text{const.} \sum_n |n|^{\gamma - 2(\alpha + 1 - 1/p)}.$$

Now $\alpha + 1/2 > 1/p$, so that $\varrho \equiv [\alpha + 1 - 1/p]^{-1} < 2$, and thus, using Hölder's inequality, the convergence of the RHS of this last expression for $\gamma < 2\alpha + 1 - 2/p$ implies the convergence of

$$\sum_n \|C_n\|^r$$

for all $\varrho < r \leqq 2$. Since $\|C_n\| = \|T^* \Phi_n\|$, Properties 1 and 2, and the obvious nesting relationship of the operator classes C_p, then yield the desired result in this case. To complete the proof we merely need to note that $\mathrm{Lip}\,(\alpha, p) \subset \mathrm{Lip}\,(\alpha, 2)$ for all $p > 2$.

Combining this lemma with our earlier Property 6 leads to

Theorem 2: *The composition of two operators generated by kernels in* $\mathrm{Lip}\,(\alpha, p)$ *and* $\mathrm{Lip}\,(\beta, q)$, *respectively, each of which satisfies the hypotheses of Lemma 1, is nuclear.*

Proof: Obvious.

Since, as is well-known, periodic difference kernels $K(x, y) = k(x - y)$ lead to normal operators with Schmidt functions (eigenfunctions, in this situation) which are the classical Fourier functions, i.e. $\{\varphi_n\} = \{\psi_n\} = \{\Phi_n\}$, Theorem 2 contains as special cases earlier Fourier series results of Cheng [3] $(\alpha > 1/2p = \beta)$ and Yadav [24] (p, q conjugate).

A somewhat more interesting result is

Theorem 3: *Let the integral operator S be induced by a kernel K where* $K \in \mathscr{L}_{p,\infty}$, $K^* \in \mathscr{L}_{q,\infty}$ *with* $1 < q \leqq p \leqq 2$. *Assume T is generated by a kernel* $L \in \mathrm{Lip}\,(\alpha, \varrho)$ *with*

$\varrho \geqq 1$, $0<\alpha\leqq 1$, and $\alpha+1/2>1/\varrho$. Then the composite integral operators ST and TS both belong to C_r for all

$$r > [3/2-(2-p)/2q+\alpha-1/\min(2, \varrho)]^{-1}.$$

In particular, whenever $2\alpha+1-2/\min(2, \varrho)>(2-p)/q$ these composite operators are nuclear.

Proof: Immediate by Properties 3,6 and Lemma 1.

In view of the celebrated Hausdorff—Young inequality, as special cases Theorem 3 contains Fourier series results both of the Izumis [15] $(\varrho=q=p<2)$ and of Onneweer [19] $(q=p,\ \varrho=\infty)$.

For our final theorem we will need the following very useful result closely related to Property 2:

Lemma 2: *If T is a Hilbert—Schmidt operator, then for all $r>0$ and arbitrary $m\geqq 1$,*

$$\sum_{n=2m-1}^{\infty} n^r [s_n(T)]^2 \leqq \text{const.} \sum_{n=m}^{\infty} n^r \|T^* \varphi_n\|^2$$

where $\{\varphi_n\}$ is any orthonormal basis for the underlying Hilbert space.

Proof: Given a Hilbert—Schmidt operator T we have, from Properties 1, 2 that

$$\sum_{n=m}^{\infty} [s_n(T)]^2 \leqq \sum_{n=m}^{\infty} \|T^* \varphi_n\|^2$$

for any orthonormal basis $\{\varphi_n\}$. Since the $s_n(T)$ are nonincreasing as $n\to\infty$, it follows that

$$m[s_{2m-1}(T)]^2 \leqq \sum_{n=m}^{\infty} \|T^* \varphi_n\|^2,$$

and hence, for positive r and $m\geqq 1$,

$$\sum_{n=2m-1}^{\infty} n^r [s_n(T)]^2 \leqq \text{const.} \sum_{k=m}^{\infty} k^{r-1} \sum_{n=k}^{\infty} \|T^* \varphi_n\|^2$$

$$\leqq \text{const.} \sum_{n=m}^{\infty} n^r \|T^* \varphi_n\|^2.$$

Theorem 4: *Let the integral operator S be such that*

$$\sum_{n=1}^{\infty} [s_n(S)]^2 n^{1-2/p} < \infty \tag{3}$$

for some $1<p\leqq 2$, and assume T is generated by a kernel K whose L^p modulus of continuity satisfies

$$\int_0^1 t^{1-4/p} \int_0^{2\pi} [\omega_p(K, t)]^2\, dy\, dt < \infty \tag{4}$$

for the same p. Then the composite operators ST and TS are both nuclear.

Proof: We again expand $K(x, y)$, as a function of its first variable, in a classical complex Fourier series. For $\beta=2$, $\gamma=2/p-1$, the relevant equation (2) now becomes, upon integration,

$$\sum_n \|C_n\|^2 |n|^{2/p-1} \leqq \text{const.} \sum_n |n|^{4/p-3} \int_0^{2\pi} [\omega_p(K, |n|^{-1})]^2 dy.$$

In view of the monotonicity of $\omega_p(K, t)$ as a function of t, however, the RHS of this last relation is bounded from above by a constant multiple of the finite expression (4).

The case $p=2$ is trivial. For $1<p<2$, we note that $T\in C_2$ and employ Lemma 2 with $r=2/p-1$ and $\{\varphi_n\}=\{\Phi_n\}$. The desired conclusion is then an obvious consequence of Property 5 with $\lambda(n)=n^{(p-2)/2p}$.

We remark that this last result is in the style of Theorem 1 of the Izumis [15]. The condition (3) is slightly stronger than $S\in C_q$ with q conjugate to p and (4) is slightly weaker than $K\in \text{Lip}\,(\alpha, p)$ with $\alpha q+1>q/p$. Theorem 4 is typical, therefore, of the further extensions of Theorem 3 which may be obtained.

4. Acknowledgment

It is a pleasure to express our gratitude to our colleague Professor Charles Oehring for a number of stimulating discussions relative to this research.

References

1. Bljumin S. L., Kotljar B. D., Hilbert—Schmidt operators and the absolute convergence of Fourier series, *Izv. Akad. Nauk SSSR Ser. Mat.* **34** (1970), 209—217, translated in *Math. USSR—Izv.* **4** (1970), 215—223.
2. Chang S. H., A generalization of a theorem of Lalesco, *J. London Math. Soc.* **22** (1947), 185—189.
3. Cheng M.—T. The absolute convergence of Fourier series, *Duke Math. J.* **9** (1942), 803—810.
4. Cochran J. A., *The analysis of linear integral equations*, McGraw—Hill, New York, (1972).
5. Cochran J. A., The nuclearity of operators generated by Hölder continuous kernels, *Proc. Cambridge Philos. Soc.* **75** (1974), 351—356.
6. Cochran J. A., Growth estimates for the singular values of squareintegrable kernels, *Pacific J. Math.* **56** (1975), 51—58.
7. Cochran J. A., Summability of singular values of L^2 kernels — analogies with Fourier series, *Enseignement Math.* (2), **22** (1976), 141—157.
8. Cochran J. A., An extension of an operator inequality for s-numbers, *Proc. Amer. Math. Soc.*, to appear.
9. Dunford N., Schwartz J. T., *Linear operators, Part II*, Interscience, New York, (1963).
10. Gel'fand I. M., Vilenkin N. Ya., *Generalized functions*, vol. 4 (translated by A. Feinstein from a revision of the 1961 Russian ed.), Academic, New York, (1964).

11. GOHBERG I. C., KREIN M. G., The basic propositions on defect numbers, root numbers, and indices of linear operators, *Uspehi Mat. Nauk* (N. S.) **12** (2) (74) (1957), 43—118, translated in *Amer. Math. Soc. Transl.* (2) **13** (1960), 185—264.
12. GOHBERG I. C., KREIN M. G., *Introduction to the theory of linear nonself-adjoint operators*, Translations of Math. Mono., vol. 18, Amer. Math. Soc., Providence, R. I., (1969).
13. GROTHENDIECK A., Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **16** (1955), 1—140.
14. HORN A., On the singular values of a product of completely continuous operators, *Proc. Nat. Acad. Sci. U.S.A.* **36** (1950), 374—375.
15. IZUMI M., IZUMI S.-I., Absolute convergence of Fourier series of convolution functions, *J. Approximation Theory* **1** (1968), 103—109.
16. MCCARTHY C. A., C_p, *Israel J. Math.* **5** (1967), 249—271.
17. MCLAUGHLIN J. R., Absolute convergence of series of Fourier coefficients, *Trans. Amer. Math. Soc.* **184** (1973), 291—316.
18. OEHRING C., COCHRAN J. A., Integral operators and an analogue of the Hausdorff—Young theorem, *J. London Math. Soc.*, (2), **15** (1977).
19. ONNEWEER C. W., On absolutely convergent Fourier series, *Ark. Mat.* **12** (1974), 51—58.
20. SCHATTEN R., *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin, (1960).
21. SMITHIES F., The eigen-values and singular values of integral equations, *Proc. London Math. Soc.* (2) **43** (1937), 255—279.
22. STINESPRING W. F., A sufficient condition for an integral operator to have a trace, *J. Reine Angew. Math.* **200** (1958), 200—207.
23. YADAV B. S., On the class of Young's continuous functions, *Amer, Math. Monthly* **73** (1966), 1097—1099.
24. YADAV B. S., On the class of Young's continuous functions, II, *Mat. Vesnik* **1** (16) (1964), 299—302.
25. ZAANEN A. C., *Linear Analysis*, Interscience, New York, (1953).

Received June 14, 1976

James A. Cochran
Department of Mathematics
Virginia Polytechnic Institute
and State University
Blacksburg, Virginia 24061
USA