An example of a nuclear space in infinite dimensional holomorphy

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Abstract

Let U be an open subset of a complex locally convex space E, and $H(U)$ the space of holomorphic functions from U to $\mathbf{C}$. If the dual E' of E is nuclear with respect to the topology generated by the absolutely convex compact subsets of E, then it is shown that $H(U)$ endowed with the compact open topology is a nuclear space. In particular, if E is the strong dual of a Fréchet nuclear space, then $H(U)$ is a Fréchet nuclear space.

If E is a complex locally convex space and U is an open subset of E, then $H(U)$ is the space of all holomorphic functions from U to $\mathbf{C}$. A function $f\colon U\to\mathbf{C}$ is holomorphic if it is continuous and Gâteaux holomorphic (that is, f restricted to $U\cap F$ is holomorphic or analytic in the usual sense whenever F is a finite dimensional subspace of E). τ_0 will denote the compact open topology on $H(U)$. For each compact $K\subset U$, p_K will denote the semi-norm on $H(U)$ defined by $p_K(f)=$ $=\sup_{x\in K}|f(x)|=|f|_K$. $H(U)$, p_K is the semi-normed space $H(U)$ with respect to the semi-norm p_K.

It is classically known that if $H(U)$ is the space of holomorphic functions on the open subset U of $\mathbf{C}^n$ endowed with the compact open topology, then $H(U)$ is a Fréchet nuclear space. In this article we investigate the nuclearity of $H(U)$ when U is an open subset of a space E whose dual E' is nuclear with respect to the topology generated by the absolutely convex compact subsets of E. Any quasi-complete dual nuclear space E has this property (a space E is quasi-complete if its closed bounded subsets are complete, and E is dual nuclear if its strong topological dual is nuclear). Therefore in particular any Fréchet nuclear or $\mathscr{D}FN$ (strong dual of a Fréchet nuclear space) space has this property.

Proposition 1. *Let E be a locally convex space such that E' is nuclear with respect to the topology generated by the absolutely convex compact subsets of E. If U is an absolutely convex open subset of E, then $H(U)$, τ_0 is a nuclear space.*

Proof. It suffices to show that if K is compact in U, then there exists a compact $\tilde{K}$ such that $U\supset\tilde{K}\supset K$ and the mapping $H(U), p_{\tilde{K}}\to H(U), p_K$ is absolutely summing. By a characterization of Pietsch [3], it suffices to show that there is a positive Radon measure μ on $C(\tilde{K})$ such that $p_K(f)\leqq\int_{\tilde{K}}|f|\,d\mu$ for all $f\in H(U)$.

Let now K be given. Since E' is nuclear with respect to the topology generated by the absolutely convex compact subsets of E, K is contained in the absolutely convex hull of a rapidly decreasing sequence $\{x_n\}$ in E (see [3]), where rapidly decreasing means that $(1+n)^p x_n\to 0$ for all $p>0$. Therefore

$$K\subset X_1=\left\{\sum_{i=1}^{\infty}\lambda_i x_i:\sum_{i=1}^{\infty}|\lambda_i|\leqq 1\right\}.$$

Now choose α such that $0<\alpha<1/2$ and $K\subset(1-2\alpha)U$, and then choose N such that $\left\{\sum_{i=1}^{\infty}\lambda_i(N+i)^4 x_{N+i}:\sum_{i=1}^{\infty}|\lambda_i|\leqq 1\right\}\subset\alpha U$. Define

$$Y=\left\{\sum_{i=1}^{N}\lambda_i x_i:\sum_{i=1}^{\infty}\lambda_i x_i\in K,\ \sum_{i=1}^{\infty}|\lambda_i|\leqq 1\right\},$$

$$X_2=\left\{\sum_{i=1}^{\infty}\lambda_i x_{N+i}:\sum_{i=1}^{\infty}|\lambda_i|\leqq 1\right\},$$

and

$$X_3=\left\{\sum_{i=1}^{\infty}\lambda_i(1+i^2)x_{N+i}:|\lambda_i|\leqq 1\right\}.$$

Note that $\bar{X}_3\subset\left\{\sum_{i=1}^{\infty}\lambda_i(N+i)^4x_{N+i}:\sum_{i=1}^{\infty}|\lambda_i|\leqq 1\right\}\subset\alpha U$, where $\bar{X}_3$ denotes the closure of X_3, and hence $\bar{X}_3$ is compact in αU. Clearly $X_2\subset\bar{X}_3$. Furthermore $Y\subset K+X_2\subset(1-\alpha)U$, and therefore $K\subset Y+X_2\subset Y+\bar{X}_3\subset U$.

Next let $T=$ vector span of $\{x_1,\dots,x_N\}$ and choose $y_1,\dots,y_t$ from $\{x_1,\dots,x_N\}$ to form a basis for T. Let $U_T=U\cap T$. Since $\bar{Y}$ is compact in $(1-\alpha)U_T$, we may find $2r$ polydiscs of dimension t, $L_1,\dots,L_r$, $M_1,\dots,M_r$, such that $\bar{Y}\subset L_1\cup\dots\cup L_r\subset M_1\cup\dots\cup M_r\subset(1-\alpha)U_T$, $L_i\subset M_i$ for each $i=1,\dots,r$, and if L_i has polycenter $(z_1^i y_1, z_2^i y_2,\dots,z_t^i y_t)$ and polyradius $(r_1^i, r_2^i,\dots,r_t^i)$, then M_i has the same polycenter with polyradius $(r_1^i+\varepsilon, r_2^i+\varepsilon,\dots,r_t^i+\varepsilon)$ where $\varepsilon>0$. For each $i=1,\dots,r$, let $K_i=L_i+X_2$ and $\tilde{K}_i=M_i+\bar{X}_3$. Note that $K\subset\bigcup_{i=1}^{r}K_i$, and that $K_i\subset\tilde{K}_i\subset U$ for each $i=1,\dots,r$. If $\tilde{K}=\bigcup_{i=1}^{r}\tilde{K}_i$, then $\tilde{K}$ is compact in U and contains K. To show that $H(U), p_{\tilde{K}}\to H(U), p_K$ is absolutely summing, it clearly suffices to show that $H(U), p_{\tilde{K}_i}\to H(U), p_{K_i}$ is absolutely summing for each $i=1,\dots,r$. We do this for $i=1$.

To show that $H(U), p_{\tilde{K}_1}\to H(U), p_{K_1}$ is absolutely summing, we show the existence of a positive Radon measure μ on $C(\tilde{K}_1)$ such that

$$|f|_{K_1}\leqq\int_{\tilde{K}_1}|f|\,d\mu\quad\text{for all}\quad f\in H(U).$$

For notation's sake, we let $(z_1y_1,\dots,z_ty_t)$ be the polycenter of L_1 and $(r_1,\dots,r_t)$ the polyradius. Let $C=\prod_{j=1}^{\infty}(1+j^{-2})$ and $D=\prod_{j=1}^{t}(r_j+\varepsilon)\varepsilon^{-t}$. For each $n=1,2,\dots,$

define the positive Radon measure μ_n on $C(\tilde{K}_1)$ by

$$\mu_n(\varphi) = D(2\pi)^{-t-n} \prod_{j=1}^n (1+j^{-2}) \int_0^{2\pi} \dots \int_0^{2\pi} \varphi\big(\sum_{k=1}^t (z_k y_k + e^{i\alpha_k}(r_k+\varepsilon) y_k) + \\ + \sum_{k=1}^n (e^{i\beta_k}(1+k^2)x_{N+k})\big)\, d\alpha_1 \dots d\alpha_t\, d\beta_1 \dots d\beta_n.$$

Note that $\mu_n(\varphi) \leqq DC|\varphi|_{\tilde{K}_1}$ for all $\varphi \in C(\tilde{K}_1)$. Also if $f \in H(U)$, it follows by the Cauchy integral formula that

$$|f|_{L_1+X_{2,n}} \leqq \mu_n(|f|),$$

where $X_{2,n} = \{\sum_{i=1}^n \lambda_i x_{N+i} : \sum_{i=1}^n |\lambda_i| \leqq 1\}$.

To see this, let $w \in L_1 + X_{2,n}$. Then $w = \sum_{k=1}^t (z_k + w_k) y_k + \sum_{k=1}^n (w_{N+k} x_{N+k})$, and by the Cauchy integral formula

$$f(w) = (2\pi i)^{-t-n} \int \dots \int f\big(\sum_{k=1}^t (z_k + \xi_k) y_k + \\ + \sum_{k=1}^n \xi_{N+k} x_{N+k}\big) \prod_{k=1}^t (\xi_k - w_k)^{-1} \prod_{k=1}^n (\xi_{N+k} - w_{N+k})^{-1} d\xi_1 \dots d\xi_t\, d\xi_{N+1} \dots d\xi_{N+n}$$

where

$$\xi_k = (r_k+\varepsilon) e^{i\alpha_k} \quad k = 1, \dots, t,$$

$$\xi_{N+k} = (1+k^2) e^{i\beta_k} \quad k = 1, \dots, n.$$

Taking absolute values, it follows that $|f(w)| \leqq \mu_n(|f|)$, and hence

$$|f|_{L_1+X_{2,n}} \leqq \mu_n(|f|).$$

Suppose we show that $\lim_n \mu_n(\varphi)$ exists for all $\varphi \in C(\tilde{K}_1)$. Defining $\mu(\varphi) = \lim_n \mu_n(\varphi)$, it follows that μ is a positive Radon measure on $C(\tilde{K}_1)$. Moreover, since $|f|_{L_1+X_{2,n}} \leqq \mu_n(|f|)$, it follows that

$$|f|_{K_1} = \lim_n |f|_{L_1+X_{2,n}} \leqq \mu(|f|) \quad \text{for all} \quad f \in H(U).$$

Hence it remains to show that $\lim_n \mu_n(\varphi)$ exists for all $\varphi \in C(\tilde{K}_1)$.

Let $\varphi \in C(\tilde{K}_1)$ and $\delta > 0$ be given. There exists a neighborhood W of 0 in E, such that if $x - y \in W$, and $x, y \in \tilde{K}_1$, then $|\varphi(x) - \varphi(y)| < \delta$. Pick s such that if $n > s$, then $\{\sum_{j=s}^n \lambda_j (1+j^2) x_{N+j} : |\lambda_j| \leqq 1\} \subset W$ and $|1 - \Pi_{j=s+1}^n (1 - (1+j^2)^{-1})| < \delta$. Then

$$|\mu_n(\varphi) - \mu_s(\varphi)| \leqq \\ D(2\pi)^{-t-n} \prod_{j=1}^n (1+j^{-2}) \int \dots \int \big| \varphi\big(\sum_{k=1}^t (z_k y_k + e^{i\alpha_k}(r_k+\varepsilon) y_k) + \\ + \sum_{k=1}^n (e^{i\beta_k}(1+k^2) x_{N+k})\big) - \prod_{j=s+1}^n (1 - (1+j^2)^{-1}) \varphi\big(\sum_{k=1}^t (z_k y_k + e^{i\alpha_k}(r_k+\varepsilon) y_k) + \\ + \sum_{k=1}^s (e^{i\beta_k}(1+k^2) x_{N+k})\big)\big|\, d\alpha_1 \dots d\alpha_t\, d\beta_1 \dots d\beta_n \leqq DC\delta + DC\delta |\varphi|_{\tilde{K}_1}.$$

Hence $\mu_n(\varphi)$ is Cauchy, and $\lim_n \mu_n(\varphi)$ exists. This completes the proof of the proposition.

It is now clear that if V is any translate of an absolutely convex open set in E, where E' is nuclear with respect to the topology generated by the absolutely convex compact subsets of E, then $H(V), \tau_0$ is nuclear. Since every point in an arbitrary open subset U of E has a fundamental system of neighborhoods V of this type, we are able to prove the following theorem.

Theorem 1. *Let E be a locally convex space such that E' is nuclear with respect to the topology generated by the absolutely convex compact subsets of E. Then if U is an open subset of E, $H(U), \tau_0$ is nuclear.*

Proof. Let K be compact in U. Then there exist compact sets $K_1, \dots, K_m$ and open translates of absolutely convex sets $V_1, \dots, V_m$ such that $K = K_1 \cup \dots \cup K_m \subset V_1 \cup \dots \cup V_m \subset U$. From the remark following Proposition 1, we can find compact sets $\tilde{K}_1, \dots, \tilde{K}_m$ where $K_i \subset \tilde{K}_i \subset V_i$ and positive Radon measures μ_i such that $|f|_{K_i} \leqq \int_{\tilde{K}_i} |f| \, d\mu_i$ for all $f \in H(U)$ and $i = 1, \dots, m$. Let $\tilde{K} = \tilde{K}_1 \cup \dots \cup \tilde{K}_m$ and define the positive Radon measure μ on $C(\tilde{K})$ by

$$\mu(\varphi) = \sum_{i=1}^m \int_{\tilde{K}_1} \varphi \, d\mu_i \quad \text{for} \quad \varphi \in C(\tilde{K}).$$

Clearly $|f|_K \leqq \sum_{i=1}^m |f|_{K_i} \leqq \mu(|f|)$ for all $f \in H(U)$. It follows that $H(U), p_{\tilde{K}} \to H(U), p_K$ is absolutely summing, and this completes the proof.

Corollary 1. *If U is an open subset of the strong dual of a Fréchet nuclear space E, then $H(U), \tau_0$ is a Fréchet nuclear space.*

In general, if U is an open subset of the locally convex space E, we may consider another topology τ_d on $H(U)$ defined as follows. A compact set K in U will be said to be rapidly decreasing if it is contained in the absolutely convex hull of a rapidly decreasing sequence in E. τ_d will denote the topology on $H(U)$ of uniform convergence on rapidly decreasing compact sets of U. In general, $\tau_d \leqq \tau_0$, and τ_d is finer than the topology of uniform convergence on compact sets of finite dimension. It is clear from the results in this paper that $H(U), \tau_d$ is a nuclear space.

Author's Note: Many of the results in this paper have been proved independently by Prof. Lucien Waelbroeck. His proofs will be published in the Proceedings of the Conference on Infinite Dimensional Holomorphy, State University of Campinas, 1975.

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