Random linear functionals and subspaces of probability one

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1. Introduction

We start with several definitions.

Let E be a *real, locally convex Hausdorff vector space* (l.c.s.) and $(\Omega, \mathscr{F}, R)$ a probability space. Denote by $\mathscr{B}'$ the least σ-algebra of subsets of the topological dual E' of E, which makes every weakly continuous linear functional on E' measurable. A measurable mapping X of $(\Omega, \mathscr{F})$ into $(E', \mathscr{B}')$ will be called a *random continuous linear functional* (r.c.l.f.) over E. The distribution law of X is written P_X or PX^{-1}. It seems convenient to write

$$\langle X(\omega), \varphi\rangle = X_\varphi(\omega), \quad \omega\in\Omega, \quad \varphi\in E.$$

The characteristic function $\mathscr{L}_X$ of X is defined by

$$\mathscr{L}_X(\varphi) = \mathscr{E}(e^{iX_\varphi}), \quad \varphi\in E.$$

Here $\mathscr{E}$ denotes expectation, that is integration with respect to P.

Two r.c.l.f.'s over E are said to be equivalent (abbr. $\equiv$) if they have the same characteristic function (or distribution law).

Suppose E and F are l.c.s.'s and $\Lambda: E\to F$ a linear continuous mapping. Then for every r.c.l.f. Y over F we get an r.c.l.f. X over E by setting $X={}^t\Lambda\circ Y$. For short, we shall write $X={}^t\Lambda Y$.

The class of all (centred) Gaussian r.c.l.f.'s over E is denoted by $\mathscr{G}(E)\big(\mathscr{G}_0(E)\big)$.

An r.c.l.f. X over E is said to belong to the class $\mathscr{M}_s(E)$, if for every $\varphi_1, \ldots, \varphi_n\in E$, and every $n\in\mathbf{Z}_+$, the distribution law P_Y, $Y=(X_{\varphi_1}, \ldots, X_{\varphi_n})$, fulfils the inequality

$$P_Y(\lambda A+(1-\lambda)B) \geqq \big(\lambda P_Y^s(A)+(1-\lambda)P_Y^s(B)\big)^{1/s} \tag{1.1}$$

for every $0<\lambda<1$, and all Borel sets A and B in $\mathbf{R}^n$. Here $s\in[-\infty, 0]$. An r.c.l.f. belonging to the class $\mathscr{M}_{-\infty}(E)$ is called a convex r.c.l.f. over E. Note that $\mathscr{G}(E)\subseteqq\subseteqq\mathscr{M}_0(E)$ [4, Th. 1.1].

Now let E be an l.c.s. such that the weak dual E'_σ of E is a Souslin space [1, p. 114]. Under this assumption it is known that $\mathscr{B}'=\mathscr{B}(E'_\sigma)$, the Borel σ-algebra generated by the weakly open subsets of E' [1, p. 139]. Furthermore, let X be an r.c.l.f. over E. We are interested in two, in general different, classes of affine subspaces of E'. We denote by $I(X)$ the family of all universally $\mathscr{B}(E'_\sigma)$-measurable affine subspaces of E' of P_X-probability one, and by $I_L(X)$ the subfamily of all P_X-Lusin measurable elements of $I(X)$. Thus $G\in I_L(X)$ if and only if $G\in I(X)$ and $\sup\{P_X(K)|K$ weakly compact and convex $\subseteq G\}=1$. The Lusin affine kernel $\mathscr{A}_L(X)$ of X is defined by

$$\mathscr{A}_L(X)=\bigcap[G|G\in I_L(X)].$$

If $X\in\mathscr{G}_0(E)$, the Lusin affine kernel is equal to the reproducing kernel Hilbert space of X and is thus an extremely important object [2, Chap. 9]. It is also known that the Lusin affine kernel plays an important rôle when $E=\varinjlim \mathbf{R}^n$ and P_X reduces to a product measure on $E'=\mathbf{R}^\infty$ [8]. In Section 2 we will give a simple characterization of $\mathscr{A}_L(X)$ when $X\in\mathscr{M}_s(E)$ and $s>-1$. We also show that $\mathscr{A}_L(X)$ is of probability zero when $X\in\mathscr{M}_s(E)$, $s>-1$, and $\dim(\operatorname{supp} P_X)=+\infty$. On the other hand, the Lusin affine kernel is a large set in a topological sense. In fact, we prove that the closure of $\mathscr{A}_L(X)$ is equal to E' if $\operatorname{supp} P_X=E'$ and $X\in\mathscr{M}_s(E)$, $s>-1$. All the results are known in the Gaussian case [2, Chap. 9]. Actually, we here all the time need a mild extra condition on E, condition $C(E)$ below.

Our next task will be to pick out elements of $I(X)$. Suppose G is a subspace of E'. It seems convenient to have the following representation of G; let F be another l.c.s. and $\Lambda\colon E\to F$ a linear continuous mapping. We can, of course, choose F and Λ so that $G={}^t\Lambda(F')$. The problem then is to give necessary and sufficient conditions so that ${}^t\Lambda(F)\in I(X)$. In Section 3 we point out that this question is closely related to the solvability of a certain stochastic linear equation. In Section 4, we give necessary and sufficient conditions so that ${}^t\Lambda(F)\in I(X)$, when F is a separable prehilbert space, and, in Section 5, when F is a nuclear LM space.

We include three simple examples.

2. The affine kernel of an r.c.l.f.

Let E be an l.c.s. and X an r.c.l.f. over E. We define the vector subspace $\mathscr{H}(X)$ of E' as the set of all $a\in E'$ such that

$$\lim_{j\to\infty}\langle a,\varphi_j\rangle=0 \tag{2.1}$$

for every denumerable sequence $\{\varphi_j\}$ in E such that

$$\lim_{j\to\infty}\langle u,\varphi_j\rangle=0 \quad \text{a.s.} \quad [P_X]. \tag{2.2}$$

The purpose of this section is to find relations among the affine subspaces $\mathscr{A}_L(X)$, $\mathscr{H}(X)$, and supp P_X. To this end we must assume that E fulfils the following condition;

$C(E)$: *there exists a locally convex topology* $\mathscr{T}'$ *on* E', *compatible with the duality* (E', E) *such that* $E'(\mathscr{T}')$ *is a complete Souslin space.*

This condition is, in particular, satisfied if E is the strict inductive limit of an increasing denumerable sequence of separable Fréchet subspaces [7, Th. I.5.1].

We shall first prove

Theorem 2.1. *Let* E *be an l.c.s. which fulfils the condition* $C(E)$. *Suppose that* X *is an r.c.l.f. over* E *such that* $0\in\mathscr{A}_L(X)$. *Then*

$$\mathscr{A}_L(X) = \mathscr{H}(X).$$

Remark 2.1. There exists an r.c.l.f. X over $C([0, 1])$, the vector space of all continuous functions on the unit interval, equipped with the sup-norm topology such that $\mathscr{A}_L(X)=\emptyset$. (An example due to E. Alfsen; private communication.) Below we will see that this pathology cannot occur when $X\in\mathscr{M}_s(E)$, $s>-1$, and E fulfils the condition $C(E)$.

Recently, J. Hoffmann–Jørgensen has given a better characterization of $\mathscr{A}_L(X)$ in the special case when $0\in\mathscr{A}_L(X)$, $E=\varinjlim \mathbf{R}^n$, and P_X is a product measure with non-degenerated factors [8, Th. 4.4]. Our method of proof is similar to that in [8].

Proof. Suppose $G\in\mathscr{A}_L(X)$ and $a\notin G$. Since G is a P_X-Lusin affine space and $0\in G$, there are weakly compact, convex, and symmetric sets K_j, $j\in\mathbf{N}$, such that

$$K_j \subseteqq G, \quad 2K_j \subseteqq K_{j+1}, \quad P_X(K_j) > 1-2^{-j}. \tag{2.3}$$

Now choose $\varphi_j\in E$ such that $\langle a, \varphi_j\rangle=1$ and $|\langle u, \varphi_j\rangle|\leqq 1$ when $u\in K_j$. It is readily seen that $\langle u, \varphi_j\rangle\to 0$, as $j\to\infty$, for every $u\in\bigcup K_j$. In particular, (2.2) is true. Since (2.1) is not fulfilled, we have $a\notin\mathscr{H}(X)$. Hence $\mathscr{A}_L(X)\subseteqq\mathscr{H}(X)$.

Conversely, assume that $a\notin\mathscr{H}(X)$. Then there is a sequence $\{\varphi_j\}$ in E so that $\langle a, \varphi_j\rangle=1$ for all j, and (2.2) is valid. We can thus find a subsequence $\{\psi_k\}=\{\varphi_{j_k}\}$ such that

$$P[|X_{\psi_k}| > 2^{-k}] < 2^{-k}.$$

Set

$$N(u) = \sum |\langle u, \psi_k\rangle|, \quad u\in E',$$

and let $G=\{N<+\infty\}$. Clearly, $G\in I(X)$. We shall prove that $G\in I_L(X)$. Therefore, let $\varepsilon>0$ be given and choose $\lambda\in\mathbf{R}_+$ such that

$$P_X[N \leqq \lambda] > 1-\varepsilon$$

Now observe that $\mathscr{B}(E'(\mathscr{T}'))=\mathscr{B}(E'_\sigma)$ [1, p. 121]. Since, by assumption, $E'(\mathscr{T}')$ is a Souslin space there is a $\mathscr{T}'$-compact subset K of $\{N\leqq\lambda\}$ such that

$$P_X(K) > 1-\varepsilon.$$

(See e.g. [1, p. 132].) Let $\hat{K}$ be the $\mathscr{T}'$-closed, convex hull of K. Then $\hat{K}$ is $\mathscr{T}'$-compact and, of course, also weakly compact. We also have that $K\subseteqq\hat{K}\subseteqq\{N\leqq\lambda\}\subseteqq G$. Hence $G\in I_L(X)$ and the theorem is proved.

The definition of the space $\mathscr{H}(X)$ can be simplified if $X\in\mathscr{M}_s(E)$, $s>-\infty$.

Theorem 2.2. *Suppose* $X\in\mathscr{M}_s(E)$, $s>-\infty$, *and let* $p\in]0, -1/s[$.

Then $a\in\mathscr{H}(X)$ *if and only if there is a constant* $C=C(a)>0$ *such that*

$$|\langle a, \varphi\rangle|^p \leqq C\mathscr{E}(|X_\varphi|^p), \quad \varphi\in E. \tag{2.4}$$

Here $-1/0=+\infty$.

Proof. Suppose $a\notin\mathscr{H}(X)$. Then there is a sequence $\{\varphi_j\}$ in E such that $|\langle a, \varphi_j\rangle|\geqq 1$ and (2.2) is valid.

Set

$$N(u) = \sup_j |\langle u, \varphi_j\rangle|. \tag{2.5}$$

Then N is an $\overline{\mathbf{R}}_+$-valued seminorm which is finite a.s. $[P_X]$. Hence $N^p\in L^1(P_X)$ [4, Th. 3.1]. From the Lebesgue dominated convergence theorem we now deduce that the inequality (2.4) cannot be valid for any $C>0$.

Conversely, if the inequality (2.4) cannot be valid for any C, it is trivial to show that $a\notin\mathscr{H}(X)$. This proves the theorem.

We shall now try to give a better description of the affine kernel $\mathscr{A}_L(X)$ when $X\in\mathscr{M}_s(E)$ and $s>-1$.

We first need a preliminary result.

Theorem 2.3. *Suppose* $X\in\mathscr{M}_s(E)$, $s>-1$, *and assume that* E *fulfils the condition* $C(E)$.

Then for any $h\in L^\infty(\Omega, \mathscr{F}, P)$ *the linear mapping*

$$\Phi_X(h): E\ni\varphi\to\mathscr{E}(hX_\varphi)\in\mathbf{R}$$

belongs to E'.

Proof. First note that every sequentially continuous linear functional on E is continuous. In fact, $E'(\mathscr{T}')$ is both complete and separable and the statement follows from [13, p. 150]. Using [4, Th. 3.1] again it is readily seen that $\Phi_X(h)$ is sequentially continuous, which proves the theorem.

Under the same assumptions as in Theorem 2.3, let us write $\Phi_X(1)=\mathscr{E}(X)$ and define

$$\tilde{X}_\varphi(\omega) = X_\varphi(\omega) - \langle \mathscr{E}(X), \varphi \rangle, \quad \varphi \in E, \quad \omega \in \Omega.$$

Note that $X \in \mathscr{M}_s(E)$. Theorem 2.2 also gives that

(2.6) $$\Phi_{\tilde{X}}\big(L^\infty(\Omega, \mathscr{F}, P)\big) = \mathscr{H}(\tilde{X}).$$

We now have

Theorem 2.4. *Suppose* $X \in \mathscr{M}_s(E)$, $s > -1$, *and assume that* E *fulfils the condition* $C(E)$.

Then

a) $\mathscr{A}_L(X) = \mathscr{E}(X) + \mathscr{H}(\tilde{X})$.

b) $\overline{\mathscr{A}_L(X)} = \operatorname{supp} P_X$, *if* $\operatorname{supp} P_{X_\varphi} =$ *singleton set or* **R** *for all* $\varphi \in E$.

c) $P_X(\mathscr{A}_L(X)) = 0$ *or* 1 *according as* $\dim(\mathscr{H}(\tilde{X})) = +\infty$ *or* $< +\infty$.

Theorem 2.4 is well known in the Gaussian case. (See e.g. [10], [11], and [2, Chap. 9].) Our methods of proof seem to be rather different from those in the quoted papers.

Proof of a). First note that $\mathscr{A}_L(X) = \mathscr{A}_L(\tilde{X}) + \mathscr{E}(X)$. In view of Theorem 2.1, we thus only have to prove that $0 \in \mathscr{A}_L(\tilde{X})$. Now choose $H \in \mathscr{A}_L(\tilde{X})$ arbitrarily, and write $H = a + G$, where G is a $P_{\tilde{X}-a}$-Lusin linear space. Suppose $a \notin G$, and let us choose the K_j as in (2.3) with X replaced by $\tilde{X} - a$. Furthermore, we choose the φ_j exactly as in the proof of Theorem 2.1. Defining N as in (2.5), we have $N \in L^1(P_{\tilde{X}-a})$. Hence

$$\lim_{j \to +\infty} \int |\langle u, \varphi_j \rangle| \, dP_{\tilde{X}-a}(u) = 0.$$

On the other hand

$$\int |\langle u, \varphi_j \rangle| \, dP_{\tilde{X}-a}(u) = \int |\langle u, \varphi_j \rangle - \langle a, \varphi_j \rangle| \, dP_{\tilde{X}}(u) \geqq \Big| \int (\langle u, \varphi_j \rangle - \langle a, \varphi_j \rangle) \, dP_{\tilde{X}}(u) \Big| = 1.$$

This contradiction shows that $a \in G$. Hence $0 \in H$ and part a) is proved.

Proof of b). We know that $\operatorname{supp} P_X$ is equal to the intersection of all closed affine subspaces of E'_σ of probability one [4, Th. 5.1]. In particular,

$$\overline{\mathscr{A}_L(X)} \subseteqq \operatorname{supp} P_X.$$

To prove the opposite inclusion choose $a \notin \overline{\mathscr{A}(X)}$ arbitrarily. By part a) we have that $a - \mathscr{E}(X) \notin \overline{\mathscr{H}(\tilde{X})}$. Now choose $\varphi_0 \in E$ such that $\langle a - \mathscr{E}(X), \varphi_0 \rangle = 1$ and $\langle u, \varphi_0 \rangle = 0$ for all $u \in \mathscr{H}(\tilde{X})$.

Using (2.6), we get

$$\int |\langle u, \varphi_0\rangle|\, dP_{\tilde{X}}(u) = \langle \Phi_{\tilde{X}}(\operatorname{sign} \tilde{X}_{\varphi_0}), \varphi_0\rangle = 0.$$

From this it follows that $\langle u-\mathscr{E}(X), \varphi_0\rangle = 0$ for every $u \in \operatorname{supp} P_X$. In particular, we have that $a \notin \operatorname{supp} P_X$, which concludes the proof of part b).

Proof of c). Suppose first that $\dim(\mathscr{H}(\tilde{X})) = +\infty$. It is then obvious that the vector subspace $\{\tilde{X}_\varphi | \varphi \in E\}$ of $L^1(\Omega, \mathscr{F}, P)$ is of infinite dimension. From the Dvoretsky—Rogers theorem [6, Th. 3] we now deduce that there exists a sequence $\{\varphi_j\}$ in E such that

$$\sum \mathscr{E}(|\tilde{X}_{\varphi_j}|) = +\infty$$

and

$$\sum |\mathscr{E}(h\tilde{X}_{\varphi_j})| < +\infty$$

for every $h \in L^\infty(\Omega, \mathscr{F}, P)$.

Set

$$N(u) = \sum |\langle u, \varphi_j\rangle|, \quad u \in E'.$$

From the definition of $\mathscr{H}(\tilde{X})$ we have

$$\mathscr{H}(\tilde{X}) \subseteqq \{u \in E' | N(u) < +\infty\}. \tag{2.7}$$

The function N is an $\overline{\mathbf{R}}_+$-valued seminorm and

$$\int N\, dP_{\tilde{X}} = \sum_1^\infty 1 = +\infty.$$

We know from the zero-one law [4, Th. 4,1] that $P_{\tilde{X}}[N < +\infty] = 0$ or 1, and this probability is equal to one only if $N \in L^1(P_{\tilde{X}})$ [4, Th. 3.1]. The inclusion (2.7) and part a) of Theorem 2.4 thus prove that $P_X(\mathscr{A}_L(X)) = 0$. On the other hand, if $\dim(\mathscr{H}(\tilde{X})) < +\infty$, then $\mathscr{H}(\tilde{X})$ is closed. The proof of part b) above then shows that $\mathscr{A}_L(X) \supseteqq \operatorname{supp} P_X$. Hence $P_X(\mathscr{A}(X)) = 1$. This proves part c) and concludes the proof of Theorem 2.4.

Corollary 2.1. *Let E be an l.c.s. satisfying the condition $C(E)$ and let $X, Y \in \mathscr{M}_s(E)$, $s > -1$.*

Then P_X and P_Y are singular if

$$\mathscr{E}(X) + \mathscr{H}(\tilde{X}) \neq \mathscr{E}(Y) + \mathscr{H}(\tilde{Y}).$$

Corollary 2.1 follows at once from Theorem 2.4, a) and the zero-one law.

We shall conclude this section by giving a few examples.

Example 2.1. Let $X \in \mathscr{M}_s(\mathbf{R}_0^\infty)$, $s > -1$, where $\mathbf{R}_0^\infty = \varinjlim \mathbf{R}^n$, and suppose that $\mathscr{E}(X) = 0$. In view of the Kolmogorov zero-one law it can be interesting to know when

$$\mathbf{R}_0^\infty \subseteqq \mathscr{A}_L(X). \tag{2.8}$$

Note that a set $G \subseteqq \mathbf{R}^\infty = (\mathbf{R}_0^\infty)'$ is a tail event if and only if $\mathbf{R}_0^\infty + G \subseteqq G$. Let us write

$$X_\varphi = \sum \varphi_j X_j, \quad \varphi = \{\varphi_j\} \in \mathbf{R}_0^\infty,$$

where the X_j are real-valued random variables. Denote by M_k the closure in $L^1(\Omega, \mathcal{F}, P)$ of the vector space spanned by the X_j, $j \neq k$. Let $e_1 = (1, 0, 0, \ldots) \in \mathbf{R}^\infty$. From Theorems 2.2 and 2.4 we now deduce that $e_1 \in \mathcal{A}_L(X)$ if and only if there is a constant $C > 0$ such that

$$1 \leqq C\mathcal{E}\left(\left|X_1 + \textstyle\sum_2^\infty \varphi_j X_j\right|\right)$$

for all $\{\varphi_j\} \in \mathbf{R}_0^\infty$. Equivalently, this means that $X_1 \notin M_1$. Hence (2.8) is valid if and only if $X_j \notin M_j$ for all j. Note that this condition is fulfilled if the X_j are independent and non-zero.

Example 2.2. Let E be a separable Hilbert space and suppose that $X \in \mathcal{M}_s(E)$, $s > -1/2$. Then, since the norm in E belongs to $L^2(P_{\tilde{X}})$ [4, Th. 3.1], there is a symmetric non-negative Hilbert—Schmidt operator S on E such that

$$\|S\varphi\|^2 = \mathcal{E}(\tilde{X}_\varphi^2), \quad \varphi \in E.$$

Hence

$$\mathcal{A}_L(X) = \mathcal{E}(X) + \text{range}\,(S).$$

3. A connection between $I(X)$ and a certain linear stochastic equation

We now turn to the problem of picking out elements of $I(X)$. The following theorem, which is an immediate consequence of a measurable selection theorem, will play an important rôle later on.

Theorem 3.1. *Let E and F be l.c.s.'s and $\Lambda: E \to F$ a linear continuous mapping. Furthermore, assume that the weak duals of E and F, respectively, are Souslin spaces. Then,*

a) *if X is an r.c.l.f. over E, it is true that ${}^t\Lambda(F) \in I(X)$ if and only if there exists an r.c.l.f. Y over F such that*

$$X \equiv {}^t\Lambda Y. \tag{3.1}$$

b) *the equation* (3.1) *has an r.c.l.f. solution Y over F for every r.c.l.f. X over E if and only if ${}^t\Lambda$ is surjective.*

Before the proof we introduce a new notation. If $(\Omega, \mathcal{F})$ is a measurable space, we denote by $\tilde{\mathcal{F}}$ the σ-algebra of all $\mathcal{F}$-universally measurable subsets of Ω.

Proof. a). Note first that (3.1) is equivalent to the identity

$$P_X = P_Y({}^t\Lambda)^{-1}.$$

Note also that ${}^t\Lambda(F')$, under the given assumptions, is universally measurable [1, p. 123, p. 129, p. 132]. Therefore, if (3.1) is valid it follows at once that ${}^t\Lambda(F') \in I(X)$. We now prove the "only if " part of part a). There is no loss of generality to assume that X is the identity mapping on E'. Note that the transpose mapping ${}^t\Lambda$ is a continuous mapping of F'_σ onto $G = {}^t\Lambda(F')$ equipped with the relative $\sigma(E', E)$-topology, here denoted by $\mathcal{T}'$. In particular, the surjective mapping

$${}^t\Lambda : (F', \mathcal{B}(F'_\sigma)) \to (G, \mathcal{B}(G(\mathcal{T}')))$$

is measurable. We also know that the σ-algebra $\mathcal{B}(G(\mathcal{T}'))$ is countably generated since $G(\mathcal{T}')$ is a Souslin space [1, p. 138, p. 124]. We recall that F'_σ is a Souslin space. Under these circumstances it is known that there exists a measurable mapping

$$y : (G, \widetilde{\mathcal{B}(G(\mathcal{T}'))}) \to (F', \mathcal{B}(F'_\sigma))$$

so that

$$u = {}^t\Lambda y(u), \quad u \in G.$$

(See [12, Cor. 2, p. 121] or [9, Cor. 7, p. 150].) Let us now define $Y(u) = y(u)$, $u \in G$, and $Y(u) = 0$, $u \in E' \setminus G$. This gives us an r.c.l.f.

$$Y : (E', \widetilde{\mathcal{B}(E'_\sigma)}, P_X) \to (F', \mathcal{B}(F'_\sigma))$$

so that (3.1) is valid. This proves part a) of Theorem 3.1.

It only remains to be proved the "only if" part of part b). To this end choose $u \in E'$ arbitrarily. Suppose X is an r.c.l.f. over E which equals u with probability one and choose Y so that (3.1) is valid. It is obvious that there exists a $v \in F$ so that $u = {}^t\Lambda v$. The mapping ${}^t\Lambda$ is thus surjective. This proves part b) and concludes the proof of Theorem 3.1.

In applications it is, of course, in general, very hard to decide whether the equation (3.1) has a solution Y or not. In the following sections we shall see that this question is closely related to continuity of the characteristic function $\mathcal{L}_X$ with respect to a suitable topology. In general, however, it is easier to decide whether a certain moment

$$m_X^p(\varphi) = \mathcal{E}(|X_\varphi|^p), \quad \varphi \in E, \quad (p > 0)$$

is continuous or not. Before proceeding the following result can therefore be worth pointing out.

Theorem 3.2. *Let $X \in \mathcal{M}_s(E)$, $s > -\infty$, and let $\mathcal{T}$ be a locally convex topology on E.*

Then the following assertions are equivalent;

a) *$\mathcal{L}_X$ is $\mathcal{T}$-continuous.*

b) *there exists a $p \in]0, -1/s[$ so that m_X^p is $\mathcal{T}$-continuous.*

c) *m_X^p is $\mathcal{T}$-continuous for all $p \in]0, -1/s[$.*

Proof, a)⇒c). Choose $\varepsilon > 0$. It can be assumed that $-\infty < s < 0$. Let $p \in]0, -1/s[$ be fixed. Since X is continuous in probability [7, Th. II. 2.3, p. 37] there is a convex $\mathcal{T}$-neighborhood V of the origin so that

$$P[|X_\varphi| > 1/4] < 1/4, \quad \varphi \in V.$$

Set $\theta = 1 - P[|X_\varphi| > 1/4]$. From [4, Lemma 3.1] we then have, for all $\varphi \in V$,

$$P[|X_\varphi| > t/4] \leqq \left\{\frac{t+1}{2}[(1-\theta)^s - \theta^s] + \theta^s\right\}^{1/s}, \quad t \geqq 1,$$

where the right-hand side decreases in θ. Hence, for all $\varphi \in V$,

$$m_X^p(\varphi) = p \int_0^\infty t^{p-1} P[|X_\varphi| \geqq t]\, dt \leqq$$

$$\leqq 4^{-p} + p 4^{-p} \int_1^\infty t^{p-1} \left\{\frac{t+1}{2}[(1/4)^s - (3/4)^s] + (1/4)^s\right\}^{1/s} dt = C,$$

where $C < +\infty$. From this it follows that

$$m_X^p(\varphi) < \varepsilon \quad \text{if} \quad \varphi \in (\varepsilon/(1+C))^{1/p} V.$$

Since m_X^p is continuous at the origin, it is easy to show the continuity at each point of E.

The implications c)⇒b) and b)⇒a) are both trivial.

4. F a separable prehilbert space

Let F be a separable prehilbert space. A positive semidefinite quadratic form B on F is said to be of finite trace class if there exists a $C \in \mathbf{R}_+$ so that

$$\sum B(\psi_k, \psi_k) \leqq C$$

for every orthonormal sequence $\{\psi_k\}$ in F. The seminorms $F \ni \psi \to \sqrt{B(\psi, \psi)} \in \mathbf{R}$, where B varies over all positive semidefinite quadratic forms on F of finite trace class, determine a locally convex topology $\mathcal{HS}(F)$ on F. By Sazonov's theorem

[7, Th. II. 3.4, p. 46], a positive semi-definite function f on F is the characteristic function of an r.c.l.f. over F if and only if $f(0)=1$ and f is $\mathscr{HS}(F)$-continuous. Observe here that F'_σ is a Souslin space.

Theorem 4.1. *Let E be an l.c.s. such that E'_σ is a Souslin space and let F be a separable prehilbert space. Furthermore, assume that $\Lambda\colon E\to F$ is a linear continuous mapping and denote by $\mathscr{T}$ the weakest topology on E which makes the mapping $\Lambda\colon E\to F(\mathscr{HS}(F))$ continuous.*

Then,

a) *if X is an r.c.l.f. over E, it is true that ${}^t\Lambda(F')\in I(X)$ if and only if $\mathscr{L}_X$ is $\mathscr{T}$-continuous.*

b) *if $X\in\mathscr{M}_s(E)$, $s\geqq-\infty$, it is true that ${}^t\Lambda(F')\in I(X)$ if and only if the equation* (3.1) *has a solution $Y\in\mathscr{M}_s(F)$.*

c) *if $X\in\mathscr{G}(E)$, it is true that ${}^t\Lambda(F')\in I(X)$ if and only if the equation* (3.1) *has a solution $Y\in\mathscr{G}(F)$.*

Proof of a). Suppose first that ${}^t\Lambda(F')\in I(X)$. By Theorem 3.1, a) there is an r.c.l.f. Y over F such that (3.1) holds. Hence $\mathscr{L}_X=\mathscr{L}_Y\circ\Lambda$ and Sazonov's theorem implies that $\mathscr{L}_X$ is $\mathscr{T}$-continuous. Conversely, let us assume that $\mathscr{L}_X$ is $\mathscr{T}$-continuous. Since $\mathscr{L}_X$ is a positive semi-definite function and $\mathscr{L}_X(0)=1$, we have the inequality

$$|\mathscr{L}_X(\varphi_0)-\mathscr{L}_X(\varphi_1)|^2\leqq 2|1-\operatorname{Re}\mathscr{L}_X(\varphi_0-\varphi_1)|, \tag{4.1}$$

valid for all $\varphi_0,\varphi_1\in E$. It is therefore possible to define a positive semi-definite function f on the vector space $\Lambda(E)$ by setting $f(\psi)=\mathscr{L}_X(\varphi)$, when $\psi=\Lambda\varphi$ and $\varphi\in E$. Since the topology $\mathscr{HS}(F)$ induces a weaker topology on $\Lambda(E)$ than the $\mathscr{HS}(\Lambda(E))$-topology (these topologies are in fact identical) we deduce that f is $\mathscr{HS}(\Lambda(E))$-continuous. By Sazonov's theorem there is an r.c.l.f. Z over $\Lambda(E)$ so that $f=\mathscr{L}_Z$. Let $\hat{F}$ be the completion of F and denote by $\overline{\Lambda(E)}$ the closure of $\Lambda(E)$ in $\hat{F}$. It is obvious that Z can be considered an r.c.l.f. over $\overline{\Lambda(E)}$. Let $p\colon\hat{F}\to\overline{\Lambda(E)}$ be the canonical projection. By setting $Y={}^tpZ$ we have an r.c.l.f. over $\hat{F}$ such that $\mathscr{L}_X=\mathscr{L}_Y\circ\Lambda$, that is $X\equiv{}^t\Lambda Y$. Since Y can be regarded as an r.c.l.f. over F, part a) is proved.

Proof of b). The "if" part is clear. To prove the other direction assume that ${}^t\Lambda(F')\in I(X)$. By part a) $\mathscr{L}_X$ is $\mathscr{T}$-continuous. We can thus define Z as in the proof of part a) above and observe that $X\equiv{}^t\Lambda Z$. Since the map $\Lambda\colon E\to\Lambda(E)$ is surjective, it follows that $Z\in\mathscr{M}_s(\Lambda(E))$ [4, Sect. 2]. Using the same convention as above we also have $Z\in\mathscr{M}_s(\overline{\Lambda(E)})$. Defining Y as above and using [4, Th. 2.1] again, it is readily seen that $Y\in\mathscr{M}_s(F)$, thus proving part c).

Proof of c). The proof is "exactly" the same as the proof of part b).

This concludes the proof of Theorem 4.1.

5. F a nuclear LM space

An l.c.s. F is said to be an LM space if F is the strict inductive limit of an increasing denumerable sequence of metrizable subspaces. If, in addition, F is nuclear it follows that F is separable and the weak dual of F is a Souslin space ([13, Cor. 2, p. 101], [7, Section 5]).

The main tool in this section is Minlos' theorem. The following variant seems convenient to us.

Theorem 5.1. ([7, Th. II. 3.3, p. 43].) *Let F be a separable nuclear space. Then*

(i) *every continuous positive semi-definite function f on F such that $f(0)=1$ is the characteristic function of an r.c.l.f. over F.*

(ii) *if, in addition, F is an LM space, the characteristic function of every r.c.l.f. over F is continuous.*

Theorem 3.1 now gives us the following extension theorem for positive semi-definite functions. Actually, we have no need for it here, but it can be worth pointing out since it seems to be of independent interest.

Theorem 5.2. *Let F be a nuclear LM space and E a subspace of F. Then every continuous positive semi-definite function on E can be extended to a continuous positive semi-definite function on F.*

Proof. Suppose f is a continuous positive semi-definite function on E and $f(0)=1$. Then, by Theorem 5.1(i), there is an r.c.l.f. X over E such that $\mathscr{L}_X=f$. Here it shall be observed that E, equipped with the relative topology, is separable. In fact, there is an obvious stronger inductive limit topology on E, which makes E into a nuclear LM space. Let $\Lambda: E\to F$ be the canonical injection and note that ${}^t\Lambda(F')=E'$. Hence ${}^t\Lambda(F')\in I(X)$ and Theorem 3.1 implies that there exists an r.c.l.f. Y over F such that $\mathscr{L}_X=\mathscr{L}_Y\circ\Lambda$. In virtue of Theorem 5.1(ii), $\mathscr{L}_Y$ is a continuous positive semi-definite function on F, which extends f.

We shall now prove.

Theorem 5.3. *Let E be an l.c.s. such that E'_σ is a Souslin space and let F be a nuclear LM space. Suppose $\Lambda: E\to F$ is a continuous linear mapping and denote by $\mathscr{T}$ the weakest topology on E which makes Λ continuous.*

Then,

a) *if X is an r.c.l.f. over E, it is true that ${}^t\Lambda(F')\in I(X)$ if and only if $\mathscr{L}_X$ is $\mathscr{T}$-continuous.*

b) *if $X\in\mathscr{G}_0(E)$, it is true that ${}^t\Lambda(F')\in I(X)$ if and only if the equation* (3.1) *has a solution $Y\in\mathscr{G}_0(F)$.*

Proof of a). Suppose first that ${}^t\Lambda(F')\in I(X)$. Then Theorem 3.1 gives us an r.c.l.f. Y over F so that $\mathcal{L}_X=\mathcal{L}_Y\circ\Lambda$. By Minlos' theorem (ii), $\mathcal{L}_Y$ is continuous. This proves the "only if" part. Conversely, assume that $\mathcal{L}_X$ is $\mathcal{T}$-continuous. The inequality (4.1) then makes it possible to define a continuous positive semi-definite function f on $\Lambda(E)$ such that $f\circ\Lambda=\mathcal{L}_X$. By Minlos' theorem (i) there is an r.c.l.f. Y over $\Lambda(E)$ (or F) so that $\mathcal{L}_Y\circ\Lambda=\mathcal{L}_X$. Hence $P_X=P_Y({}^t\Lambda)^{-1}$ and it follows that ${}^t\Lambda(F')\in I(X)$.

Proof of b). The "if" part is clear. Conversely, assume that ${}^t\Lambda(F')\in I(X)$. From Theorem 3.1 we have that there exists an r.c.l.f. Y over F such that (3.1) is valid. In particular,

$$\int\langle u,\varphi\rangle^2\,dP_X(u)=\int\langle v,\Lambda\varphi\rangle^2\,dP_Y(v),\quad \varphi\in E.$$

Since $X\in\mathcal{G}_0(E)$, we also have

$$\mathcal{L}_X(\varphi)=\exp\Big(-1/2\int\langle u,\varphi\rangle^2\,dP_X(u)\Big),\quad \varphi\in E.$$

Part a) of Theorem 5.3 implies that $\mathcal{L}_X$ is $\mathcal{T}$-continuous. We can therefore find a continuous seminorm q on F such that

$$\int\langle v,\psi\rangle^2\,dP_Y(v)\leqq q^2(\psi),\quad \psi\in\Lambda(E).\tag{5.1}$$

Since F is nuclear, the positive semi-definite continuous quadratic form on the left-hand side of (5.1) can be extended to a positive semi-definite continuous quadratic form B on F [13, Cor. 2, p. 102]. Using the Minlos theorem (i) again, we conclude that there exists a $Y_0\in\mathcal{G}_0(F)$ such that

$$\mathcal{L}_{Y_0}(\psi)=\exp(-1/2B(\psi,\psi)),\quad \psi\in F.$$

Hence $X\equiv{}^t\Lambda Y_0$, which was to be proved. This concludes the proof of Theorem 5.3.

In connection with Theorem 5.3 we have not been able to prove a complete analogue to Theorem 4.1, b) but the following can be said; assume E is a nuclear LM space and let $X\in\mathcal{M}_s(E)$, $s>-1/2$. Since the second order moment m_X^2 is continuous there is a linear functional H on $E\otimes E$, equipped with the projective topology $\mathcal{T}_P(E)$, so that

$$\mathcal{E}(X_{\varphi_0}\cdot X_{\varphi_1})=\langle H,\varphi_0\otimes\varphi_1\rangle,\quad \varphi_0,\varphi_1\in E.$$

As we see H is symmetric and positive semi-definite. A continuous linear functional on $(E\otimes E)(\mathcal{T}_P(E))$ with these properties is said to be a covariance.

We now have

Theorem 5.4. *Suppose E and F are nuclear LM spaces and let $\Lambda\colon E\to F$ be a continuous linear mapping. Assume $X\in\mathcal{M}_s(E)$, $s>-1/2$, and denote by H the covariance of X.*

Then ${}^t\Lambda(F')\in I(X)$ *if and only if there exists a covariance* K *on* $(F\otimes F)(\mathcal{T}_p(F))$ *such that* $H=({}^t\Lambda\otimes{}^t\Lambda)K$.

Proof. From Theorems 3.2 and 5.3, a) we have that ${}^t\Lambda(F')\in I(X)$ if and only if H is continuous on $E\otimes E$, equipped with that projective topology $\mathcal{U}$ as we get by giving E the $\mathcal{T}$-topology defined in Theorem 5.3. Let us now define $X'\in\mathcal{G}_0(E)$ by setting

$$\mathcal{L}_{X'}(\varphi) = \exp(-1/2\langle H, \varphi\otimes\varphi\rangle), \quad \varphi\in E,$$

which is possible in view of Minlos' theorem (i). From Theorem 5.3 we deduce that H is $\mathcal{U}$-continuous if and only if there exists a $Y\in\mathcal{G}_0(F)$ such that $X'\equiv{}^t\Lambda Y$. Hence H is $\mathcal{U}$-continuous if and only if there exists a covariance K on $(F\otimes F)(\mathcal{T}_p(F))$ such that $\langle H, \varphi\otimes\varphi\rangle=\langle K, \Lambda\varphi\otimes\Lambda\varphi\rangle$, $\varphi\in E$. This proves the theorem.

Example 5.1. Let M be an open subset of $\mathbf{R}^n$ and $Q(x, D)$ a linear partial differential operator in M with real $C^\infty(M)$-coefficients. Furthermore, assume that μ is a given Borel probability measure on $(\mathcal{D}'(M))_\sigma$ and denote by $\hat{\mu}$ the Fourier transform of μ, that is

$$\hat{\mu}(\varphi) = \int e^{i\langle u, \varphi\rangle}\, d\mu(u), \quad \varphi\in\mathcal{D}(M).$$

Then, in particular, Theorem 5.3, a) gives a necessary and sufficient condition so that the equation

$$u = Q(x, D)v \tag{5.2}$$

has a distribution solution $v\in\mathcal{D}'(M)$ for μ-almost all $u\in\mathcal{D}'(M)$. The condition is as follows;

for every $\varepsilon>0$ *there exists a continuous seminorm* p *on* $\mathcal{D}(M)$ *such that*

$$p({}^tQ(x, D)\varphi) < 1 \Rightarrow |1-\hat{\mu}(\varphi)| < \varepsilon.$$

In view of Theorem 5.4 this condition can be much simplified if $\mu\in\mathcal{M}_s((\mathcal{D}'(M))_\sigma)$, that is if the identity mapping

$$j: (\mathcal{D}'(M), \mathcal{B}((\mathcal{D}'(M))_\sigma), \mu) \to (\mathcal{D}'(M), \mathcal{B}((\mathcal{D}'(M))_\sigma))$$

belongs to $\mathcal{M}_s(\mathcal{D}(M))$, and $s>-1/2$. In fact, let H be the covariance of j and note that $H\in\mathcal{D}'(M\times M)$ by the kernel theorem [14, Th. 51.7]. We thus have that the equation (5.2) has a distribution solution $v\in\mathcal{D}'(M)$ for μ-almost all $u\in\mathcal{D}'(M)$ if and only if there exists a covariance $K\in\mathcal{D}'(M\times M)$ such that

$$H = Q(x, D)Q(y, D)K.$$

Furthermore, if this condition violates, the set of all $u\in\mathcal{D}'(M)$ such that the equation (5.2) has a solution $v\in\mathcal{D}'(M)$, is of μ-measure zero.

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