Majorizing sequences and approximation

R. J. Gardner and J. Hawkes

1. Introduction

One approach to approximation theory is the following (see [15, Chapter 10]). If (Ω, ϱ) is a metric space, A is a subset of Ω, and $\varepsilon > 0$ one asks whether there exist points $z_1, z_2, \dots, z_n$ in Ω such that $A \subseteqq \bigcup_{i=1}^n S(z_i, \varepsilon)$. If $N_\varepsilon^\Omega(A)$ is the smallest integer for which the answer is positive the points $z_1, z_2, \dots, z_{N_\varepsilon}$ approximate the set A in the sense that knowing them we can reproduce the set A to within an accuracy ε. The quantity $H_\varepsilon^\Omega(A) = \log_2 N_\varepsilon^\Omega(A)$ is called the ε-entropy of A relative to Ω, and one is then interested in its asymptotic growth as ε tends to zero. This approach has been the subject of much activity (see [13], [14] and [19]).

Since the covering of sets by spheres of equal radii can be quite inefficient, it is, for some purposes, preferable to consider covers by spheres of varying radii. Suppose that (r_i) is a sequence of positive real numbers which are decreasing with limit zero. We say that (r_i) is *majorizing for A in Ω* if there exists a sequence (z_i) of points in Ω such that

$$A \subseteqq \bigcup_{i=n}^{\infty} S(z_i, r_i) \quad \text{for each } n,$$

and we are interested in which sequences are majorizing for A. Again the sequence (z_i) is regarded as approximating A.

The following example illustrates some of the advantages of the second method of approximation. Let Ω be the real line with the usual metric and A the union of the Cantor ternary set and the rationals in $[1, 2]$. If $\varepsilon = \frac{1}{2} \cdot 3^{-n}$ one can show that $N_\varepsilon(A) = 2^n + 3^n$ and that 3^n of the approximating points lie in $[1, 2]$ whereas only 2^n lie in $[0, 1]$ which contains most of the set A. One can also show that (r_i) is majorizing for A in Ω if and only if $\sum_{i=1}^\infty r_i^\alpha$ converges when $\alpha = \log 2/\log 3$. If we recall that $\log 2/\log 3$ is the Hausdorff dimension of A we realise that this method of approximation is more pertinent to the structure of A.

In [3] it was shown that if $\mu^\alpha(\Omega) < \infty$ then $(n^{-1/\alpha})$ is majorizing for Ω in Ω. On the other hand if $(n^{-1/\alpha})$ is majorizing for Ω in Ω and $\int_0^1 h(t) t^{-(1+\alpha)} dt$ is finite,

$\sum_{n=1}^{\infty} h(2n^{-1/\alpha})$ is finite so that $\mu^h(\Omega)=0$. It follows that the Hausdorff dimension of Ω is given by

$$\dim \Omega = \inf \{\alpha\colon (n^{-1/\alpha}) \text{ is majorizing for } \Omega \text{ in } \Omega\}.$$

The reader familiar with Frostman's work on potential theory [2] will realise that the above statements remain true if we replace the phrase "$(n^{-1/\alpha})$ is majorizing for Ω in Ω" by "Ω has zero α-capacity". One of the objects of this paper is to examine the connexion between majorizing sequences and capacities, and to explain such connexions by means of the concept of a random approximating sequence which we introduce in section 4. Majorizing sequences were introduced by Hyllengren in [7]. In an unpublished note he (independently of us) noticed some similarities between the theory of these sequences and potential theory, but all the results presented here are new. We would like to thank Dr. J. M. Anderson for drawing our attention to Hyllengren's work.

The arrangement of the paper is as follows. In section 2 we consider basic definitions. In section 3 we look at potential theory and establish a relationship between capacities and Hausdorff measures which seems finer than those previously known. The concept of a random approximating sequence is introduced in section 4, where we also consider the problem of covering a metric space by randomly placed balls. The existence and non existence of approximating sequences is discussed in section 5, whilst sections 6 and 7 are devoted to the discussion of examples and counter examples. Finally in section 8 we discuss some examples of the application of this circle of ideas in probability, complex variable theory and in the theory of diophantine approximations.

2. Preliminaries

Let $\mathscr{H}$ be the class of all functions h of a non negative real variable such that (i) h is right continuous monotone increasing and (ii) $h(t)=0$ if and only if $t=0$. On $\mathscr{H}$ we can define a partial ordering $\prec$ by $f\prec g$ if and only if $g(t)=o(1)f(t)$ as t tends to zero. The members of $\mathscr{H}$ are called *measure functions.*

If (Ω, ϱ) is a metric space the diameter of a subset C of Ω is defined by $d(C)=\sup\{\varrho(x,y)\colon x,y\in C\}$. If $\delta>0$, $h\in\mathscr{H}$ and A is a subset of Ω we define $\mu_\delta^h(A)=\inf\sum h[d(C_i)]$ where the infimum is taken over all covers of A by sets C_i of diameter less than δ. Then $\mu^h(A)=\lim_{\delta\to 0}\mu_\delta^h(A)$ is called the *Hausdorff h measure.* When $h(t)=t^\alpha$ we write $\mu^\alpha(A)$ so that the Hausdorff dimension of A is given by $\dim A=\inf\{\alpha\colon \mu^\alpha(A)=0\}$.

If $f\in\mathscr{H}$ we define a sequence (r_i) of positive real numbers by

$$r_i = f^{-1}(1/i) = \inf\{t\colon f(t) \geqq 1/i\}. \tag{2.1}$$

Thus we always have $f(r_i-)\leqq 1/i \leqq f(r_i)$. Whenever we speak of a sequence (r_i) and f in the same context we shall assume they are related by (2.1).

Let (A, ϱ) be a metric space and (Ω, ϱ) a metric extension of A. We say that (r_i) is *majorizing for* A *in* Ω if there exists a sequence (z_i) of points in Ω such that

$$A \subseteqq \limsup_{i\to\infty} S(z_i, r_i). \tag{2.2}$$

(We let $S(x, r)$ denote the open ball and $\bar{S}(x, r)$ the closed ball, with centre x and radius r.) The sequence (z_i) is called an *approximating sequence of order* f *for* A *in* Ω. It follows from the definition that, for each z in A, there exists a subsequence (z_{n_i}) of (z_i) such that $\varrho(z, z_{n_i})\to 0$ and

$$f(\varrho(z, z_{n_i})) < 1/n_i \quad \text{for each } i.$$

Thus not only is A contained in the closure of (z_i) but also we have an estimate of the maximum rate at which subsequences can converge to any given point of A.

If (r_i) is strictly decreasing (in which case we may assume that f is continuous and strictly monotone) it makes no difference whether we choose open or closed balls in (2.2). The same is true for general (r_i) if Ω is d-dimensional euclidean space. (A proof of this fact is easily constructed using the fact that the boundaries of $d+1$ generally placed spheres in $\mathbf{R}^d$ can have common intersection consisting of at most one point). For general spaces the choice does make a difference as we show by examples in section 6. Our results indicate that the "correctness" of the definition is related to the convention that the members of $\mathscr{H}$ are right continuous. If we had made them left continuous we should have had to use closed balls in (2.2).

3. Potential theory

In this section we introduce for later reference some of the concepts of potential theory. We also prove a new result relating the polarity of a set to its Hausdorff measure.

Let (Ω, ϱ) be a metric space. If $f\in\mathscr{H}$ we let $\Phi:\Omega\times\Omega\to\mathbf{R}$ be the function defined by

$$\Phi(x, y) = 1/f(\varrho(x, y)). \tag{3.1}$$

The function Φ is called the *kernel* corresponding to f. A subset A of Ω is said to be *Φ-polar* relative to Ω if there is a probability measure m, supported by Ω, such that $\int \Phi(x, y)\,dm(y)=\infty$ whenever $x\in A$. A set A is said to have *zero Φ capacity* if $\int \Phi(x, y)\,dm(y)$ is unbounded for every measure m supported by A, but otherwise it is said to have *positive Φ capacity*. In this case the Φ capacity of A is defined by

$$C^{\Phi}(A) = \sup\left\{m(A) : \int \Phi(x, y)\,dm(y) \leqq 1 \text{ for all } x\right\}.$$

When $f(t)=t^\alpha$ the corresponding capacity is denoted by $C^\alpha(A)$. It is easily seen that every Φ-polar set has zero Φ capacity, but the converse is not always true.

Kametani [11] showed that if $\mu^f(A)<\infty$ then A is Φ-polar. In [18] Taylor showed that this result was best possible in the sense that if Ω is a euclidean space and $f_1 \prec f$ there exists a set A with $\mu^f(A)<\infty$, but A is not Φ_1-polar. Thus there is no uniform improvement of Kametani's result. However, for a fixed set A, we can improve the result as the following theorem shows.

Theorem 3.1. *Let (Ω, ϱ) be a metric space, $h\in\mathscr{H}$, and $A\subseteqq\Omega$ be such that $\mu^h(A)<\infty$. Then there exists $f\in\mathscr{H}$ such that $f\prec h$ and A is Φ-polar where Φ is given by* (3.1).

Proof. If $\mu^h(A)=0$ a result of Besicovitch ([1]) ensures that there exists $f\in\mathscr{H}$ such that $f\prec h$ and $\mu^f(A)=0$. In this case the result follows by applying Kametani's result.

If $0<\mu^h(A)<\infty$ we first need the following lemma.

Lemma 3.1. *If $0<\mu^h(A)<\infty$ and $\delta>0$ there exist a subset A^* of A and $f\in\mathscr{H}$ such that*

(i) $\mu^h(A^*)\leqq\delta$;
(ii) $f\prec h$;
(iii) $\sum_{j=1}^\infty \mu^h[A\cap S(x,r_j)]=\infty$ *whenever* $x\in A\setminus A^*$.

Proof. Kametani shows that $\limsup_{\varepsilon\to 0}\mu^h[A\cap S(x,\varepsilon)]/h(2\varepsilon)\geqq 1$ for μ^h almost all points of Ω. Thus we can define a sequence (a_i), decreasing with limit zero, such that

(i) $h(a_{i+1})\leqq\frac{1}{2}h(a_i)$,
(ii) if $A_i=\{x:\mu^h[A\cap S(x,a)]\geqq\frac{1}{2}h(2a)$ for some $a\in(a_{3i+3},a_{3i+2}]\}$ then $\mu^h(A_i)\geqq \geqq\mu^h(A)-\delta/2^i$.

Next define f by $f/h=i$ on $[a_{3i+3},a_{3i+1}]$ and f/h linear on $[a_{3i+1},a_{3i}]$, and let $A^*=A\cap\left(\bigcap_{i=1}^\infty A_i\right)^c$ so that $\mu^h(A^*)\leqq\delta$. It remains to show that A^* and f have the required properties.

If $x\in A\setminus A^*$ we choose $a=a(x)=a(x,i)$ such that $a\in(a_{3i+3},a_{3i+2}]$ and $\mu^h[A\cap S(x,a)]\geqq\frac{1}{2}h(2a)$, and define

$$n(a)=\max\{j:f(a)\leqq 1/j\}=\max\{j:ih(a)\leqq 1/j\}.$$

Thus $1/[1+n(a)]<ih(a)$ and $2/[1+n(a)]<2ih(a)\leqq 2ih(a_{3i+2})\leqq ih(a_{3i+1})=f(a_{3i+1})$. Thus $a\leqq f^{-1}(1/j)\leqq a_{3i+1}$ if $n(a)/2<j\leqq n(a)$. Now

$$\sum_{j=1}^\infty \mu^h[A\cap S(x,f^{-1}(1/j))]\geqq\sum_{i=1}^\infty\sum_{a_{3i+3}\leqq f^{-1}(1/j)\leqq a_{3i+1}}\mu^h[A\cap S(x,f^{-1}(1/j))]\geqq$$

$$\geqq\sum_{i=1}^\infty\left\{n(a)-\frac{n(a)}{2}-1\right\}\frac{h(2a)}{2}\geqq\sum_{i=1}^\infty\frac{n(a)-2}{2}\frac{1}{2\{n(a)+1\}i}.$$

As $a\to 0$, $n(a)\to\infty$ and the summand is asymptotically $1/4i$. Thus if $x\in A\setminus A^*$, $\sum_{j=1}^{\infty}\mu^h[A\cap S(x,f^{-1}(1/j))]=\infty$. Since $f\prec h$ the lemma is proved.

We now return to the proof of the theorem. Let (δ_n) be a sequence decreasing with limit zero. Using Lemma 3.1 and induction we can find a sequence (A_i) of subsets of A and a sequence (f_i) of members of $\mathscr{H}$ such that

(i) $A_i\subset A_{i+1}$;
(ii) $\mu^h(A\setminus A_i)\leqq\delta_i$;
(iii) $f_i\prec h$;
(iv) $\sum_{j=1}^{\infty}\mu^h[A\cap S(x,f_i^{-1}(1/j))]=\infty$ if $x\in A_i$;
(v) $f_{i+1}(t)\leqq f_i(t)$.

(The last inequality is arranged by choosing appropriate subsequences of (a_i) at each stage of the argument.)

Next we choose $g\in\mathscr{H}$ such that $g\prec h$ and $f_i\prec g$ for each i. It follows that

$$\sum_{j=1}^{\infty}\mu^h[A\cap S(x,g^{-1}(1/j))]=\infty \tag{3.2}$$

for μ^h almost all x in A. Let $\tilde{A}$ be the set of points in A where this fails to hold. As already remarked at the start of the proof there exists $\tilde{f}\prec h$ such that $\tilde{A}$ is $\tilde{\Phi}$-polar. Let $f(t)=\min[\tilde{f}(t),g(t)]$. Then $f\prec h$ and $\tilde{A}$ is Φ-polar. On the other hand (3.2) implies that

$$\int_A\Phi(x,y)\,d\mu^h(y)=\infty \quad\text{if}\quad x\in A\setminus\tilde{A},$$

so that $A\setminus\tilde{A}$ is Φ-polar. Since the union of two Φ-polar sets is again Φ-polar the proof of the theorem is complete.

4. Random approximating sequences

Let (Ω,ϱ) be a metric space, P a Borel probability measure on Ω, and (Z_n) a sequence of independent (Ω valued) random variables, each distributed according to the law $P(Z\in B)=P(B)$. We say that Ω *admits a random f-sequence for A* if for some P

$$P\{x\in\limsup_{i\to\infty}S(Z_i,r_i)\}=1$$

whenever $x\in A$. We say that Ω *admits a random uniform f-sequence for A* if there is a measure P such that

$$P\{A\subseteqq\limsup_{i\to\infty}S(Z_i,r_i)\}=1.$$

Whilst the existence of a random f-sequence may (or may not) imply the existence of a random uniform f-sequence, the same random sequence will not always work for both cases, as the following example shows. Let $\Omega=\mathbf{R}$, the real line, $A=\mathbf{I}$, the unit interval, and P be the uniform distribution on $\mathbf{I}$. Then $P\{x\in\limsup_{i\to\infty} S(Z_i, r_i)\}=1$ for each x in $\mathbf{I}$ if and only if $\sum_{i=1}^{\infty} r_i$ is infinite. On the other hand Shepp has shown ([17]) that $P\{\mathbf{I}\subseteq\limsup_{i\to\infty} S(Z_i, r_i)\}=1$ if and only if $\sum_{j=1}^{\infty} j^{-2}\exp(2\sum_{i=1}^{j} r_i)$ is infinite.

In this section we obtain a necessary and sufficient condition for the existence of a random f-sequence, and also a sufficient condition for the existence of a random uniform f-sequence.

Theorem 4.1. *Let Ω, A and f be as above. Then Ω admits a random f-sequence for A if and only if A is Φ-polar for A in Ω, when $\Phi(x, y)=1/f[\varrho(x, y)]$.*

Proof. Let $A(x, i)$ be the annulus $\{z: r_{i+1}\leqq\varrho(x, z)<r_i\}$. Then

$$\int_{\Omega\setminus\{x\}}\Phi(x, y)\,dP(y) = \sum_{i=1}^{\infty}\int_{A(x,i)}\Phi(x, y)\,dP(y)+\int_{\{y:\varrho(x,y)\geqq r_1\}}\Phi(x, y)\,dP(y).$$

Now since $f(r_{i-1})\leqq 1/i\leqq f(r_i)$,

$$iP\{A(x, i)\}\leqq\int_{A(x,i)}\Phi(x, y)\,dP(y)\leqq(i+1)P\{A(x, i)\}. \tag{4.1}$$

Thus

$$\int_{\Omega\setminus\{x\}}\Phi(x, y)\,dP(y),\quad \sum_{i=1}^{\infty} iP\{A(x, i)\},\quad\text{and}\quad \sum_{i=1}^{\infty} P\{S(x, r_i)\setminus\{x\}\}$$

converge or diverge together, so that

$$\int_{\Omega}\Phi(x, y)\,dP(y)=\infty \quad\text{if and only if}\quad \sum_{i=1}^{\infty} P\{S(x, r_i)\}=\infty. \tag{4.2}$$

Now suppose that there is a probability measure P with $\int\Phi(x, y)dP(y)=\infty$ if $x\in A$, and let Z_n be a sequence of independent (Ω-valued) random variables each distributed according to P. We will then have $\sum_{i=1}^{\infty} P\{0\leqq\varrho(x, Z_i)<r_i\}=\infty$ whenever $x\in A$, and so by the Borel—Cantelli lemma $P\{0\leqq\varrho(x, Z_i)<r_i$ infinitely often$\}=1$ and $P\{x\in\limsup_{i\to\infty} S(Z_i, r_i)\}=1$ whenever $x\in A$. Thus there exists a random approximating sequence. To complete the proof of the theorem we just reverse the argument.

Let Ω, A, f and (r_i) be as above and let P be a probability measure on Ω such that $\int\Phi(x, y)dP(y)=\infty$ if $x\in A$. Define $\psi(x, b)=\int\min\{\Phi(x, y),\ 1/f(b)\}dP(y)$, $\psi(b)=\inf_{x\in A}\psi(x, b)$, and $h(b)=\exp[-\psi(b)]$, so that $h\in\mathscr{H}$.

Theorem 4.2. *If $\mu^h(A)=0$ we have $P\{A\subseteq\limsup_{i\to\infty} S(Z_i, 2r_i)\}=1$.*

Proof. Given $b>0$ we define $n(b)$ by $r_{n+1}<b\leqq r_n$. If

$$\pi_m(b)=\inf_{x\in A}\sum_{j=m}^{n(b)} P\{S(x, r_j)\}\quad\text{and}\quad h_m(b)=\exp[-\pi_m(b)]$$

an application of (4.1) shows that $h_m(b)=O(1)h(b)$ as $b\to 0$.

Now suppose $x \in A$. Then

$$P\left\{S(x, b) \nsubseteq \bigcup_{j=m}^{\infty} S(Z_j, 2r_j)\right\} \leqq P\left\{S(x, b) \nsubseteq \bigcup_{j=m}^{n(b)} S(Z_j, 2r_j)\right\} \leqq$$

$$\leqq P\{Z_j \notin S(x, r_j), j = m, \ldots, n(b)\} = \prod_{j=m}^{n(b)} [1 - P\{Z_j \in S(x, r_j)\}] \leqq$$

$$\leqq \exp[-\sum_{j=m}^{n(b)} P\{Z_j \in S(x, r_j)\}] \leqq h_m(b),$$

where the equality follows by independence. When $\mu^h(A)=0$ we can, given $\varepsilon>0$, cover A by spheres $S_i = S_i(x_i, b_i)$ such that $\sum_i h[d(S_i)] < \varepsilon$. Hence

$$P\left\{A \nsubseteq \bigcup_{j=m}^{\infty} S(Z_j, 2r_j)\right\} \leqq \sum_i P\left\{S_i \nsubseteq \bigcup_{j=m}^{\infty} S(Z_j, 2r_j)\right\} \leqq \sum_i h_m[d(S_i)] = O(1)\varepsilon.$$

Letting ε tend to zero we have $P\{A \nsubseteq \bigcup_{j=m}^{\infty} S(Z_j, 2r_j)\}=0$ for each m. Thus $P\{A \nsubseteq \limsup_{i\to\infty} S(Z_j, 2r_j)\}=0$ and the theorem is proved.

In [6] Hoffmann-Jørgensen obtains results related to those presented above. Although his objectives are different to ours his methods could be applied to our situation.

5. Existence of approximating sequences

Our main result on the existence of approximating sequences is the following:

Theorem 5.1. *Let (Ω, ϱ) be a metric space, and $A \subseteqq \Omega$ be compact and Φ-polar. Then $(2r_n)$ is majorizing for A in A.*

Proof. Since A is Φ-polar there exists a probability measure P such that

$$\sum_{j=1}^{\infty} P[S(x, r_j)] = \infty, \quad x \in A.$$

Since A is compact we can use the lower semicontinuity of the summand to define an increasing sequence (M_i) of integers such that $M_1=0$ and

$$\{x : \sum_{M_i+1}^{M_{i+1}} P[S(x, r_j)] > 2\} \supseteqq A.$$

We will show that there exist points x_j, $j=M_i+1, \ldots, M_{i+1}$ such that $x_j \in A$ and

$$(*) \qquad \bigcup_{M_i+1}^{M_{i+1}} S(x_j, 2r_j) \supseteqq A.$$

To this end it can be assumed that $A \neq \emptyset$. We consider the typical case $i=1$. First choose $x_1 \in A$ so that the inequality

$$(**) \qquad 2P[S(x_j, r_j)] \geqq P[S(x, r_j)]$$

is valid for $j=1$ and every $x \in A$. Define $S_1 = S(x_1, 2r_1)$. If $A \setminus S_1 \neq \emptyset$ choose x_2 so that $(**)$ is valid for $j=2$ and every $x \in A \setminus S_1$. Set $S_2 = S(x_2, 2r_2)$ and repeat the process. The process stops if at some stage $A \subset \bigcup_1^p S_j$, $p < M_2$. In this case $(*)$ is clear. On the other hand if $A \setminus \bigcup_1^p S_j \neq \emptyset$, $p = 1, \ldots, M_2 - 1$, then

$$S(x_k, r_k) \cap S(x_j, r_j) = \emptyset, \quad 1 \leqq j < k \leqq M_2 - 1,$$

and $(**)$ is valid with $x = x_{M_2}$. Hence

$$2 \geqq 2P\left[\bigcup_1^{M_2} S(x_j, r_j)\right] = \sum_1^{M_2} 2P[S(x_j, r_j)] \geqq \sum_1^{M_2} P[S(x_{M_2}, r_j)]$$

which is a contradiction. This proves the result.

The theorem can be applied to give existence theorems when only the Hausdorff measure properties of a set are known. We have

Corollary 5.1. *Let (Ω, ϱ) be a metric space, $h \in \mathscr{H}$, and $A \subseteqq \Omega$ be compact with $\mu^h(A) < \infty$. Then there exists $f \in \mathscr{H}$ such that $f \prec h$ and $(2r_n)$ is majorizing for A in A.*

This follows by applying Theorem 3.1 to the above theorem. The need for compactness can be avoided by modifying the proof. (See [3] for another proof of this corollary.)

When $0 < \mu^h(A) < \infty$ and $\liminf_{\varepsilon \to 0} \mu^h[A \cap S(x, \varepsilon)]/h(2\varepsilon) > 0$ we say that A *has positive lower h density*. In this case we obtain a necessary and sufficient condition for (r_n) to be majorizing.

Corollary 5.2. *Let (Ω, ϱ) be a metric space, $h \in \mathscr{H}$ with $h(2t) = O(1)h(t)$ as $t \to 0$, and $A \subseteqq \Omega$ be compact and have positive lower h density. Then*

$$\sum_{n=1}^{\infty} h(2r_n) = \infty \tag{5.1}$$

is a necessary and sufficient condition for (r_n) to be majorizing.

Proof. If $\sum_{n=1}^{\infty} h(2r_n) < \infty$, (r_n) is not majorizing since otherwise we would have $\mu^h(A) = 0$. If $\sum_{n=1}^{\infty} h(2r_n) = \infty$, $\sum_{n=1}^{\infty} h(r_n) = \infty$ and A has positive lower $h(t/2)$ density. In the present situation Taylor's result ([18]) implies that A is $\Phi(2t)$-polar and, by Theorem 5.1, (r_n) is majorizing for A in A.

Note. The corollary remains true without the assumption that A be compact. This can be seen by modifying the proof of Theorem 5.1.

Under the hypotheses of the corollary, (5.1) is a necessary and sufficient condition for A to be Φ-polar. Thus in this situation majorizing and polarity are equivalent concepts. It seems that where the space admits a simple test for polarity this equivalence always holds (see § 6 for further examples). Unfortunately it does not hold in all cases (see § 7).

6. Ohtsuka's theorem in Cantor spaces

In this section we consider a class of Cantor like spaces, and by establishing an Ohtsuka type test for polarity show that the concepts of majorizing, polarity and capacity zero coincide in these spaces. We also show that it is vital to take open spheres in the definition of majorizing.

Let (n_j), $j=1, 2, \ldots$ be a sequence of integers such that $n_j \geqq 2$ for each j. The Cantor space corresponding to (n_j) is the metric space (X, ϱ) defined by taking sequences $\mathbf{i}=(i_j)$, $j=1, 2, \ldots$ of integers and letting $X=\{\mathbf{i}: 0 \leqq i_j < n_j\}$. Define $\mathbf{i}|k=(i_1, i_2, \ldots, i_k, 0, \ldots)$ and let $\varrho(\mathbf{i}, \mathbf{j})=2^{-n}$ where $n=\max\{k: \mathbf{i}|k=\mathbf{j}|k\}$. It is easy to check that (X, ϱ) is a compact metric space.

The space (X, ϱ) can be described by $X=\bigcap_{j=1}^{\infty} F_j$ where F_j consists of $n_1 n_2 \ldots n_j$ closed spheres of radius 2^{-j-1} (these spheres also have diameter 2^{-j-1}), whose mutual distances are greater than or equal to 2^{-j}. Each of these closed spheres contains n_{j+1} closed spheres of radius 2^{-j-2} whose mutual distances are 2^{-j-1}. It follows that the ordinary Cantor type sets, when suitably metrized, are Cantor spaces. In what follows m is the probability measure on X induced by the set function $m(S)=(n_1 n_2 \ldots n_j)^{-1}$ when S is a closed sphere in F_j.

Theorem 6.1. *Let (X, ϱ) be as above. Then the following are equivalent:*

(i) *X has zero Φ capacity*;

(ii) *X is Φ-polar*;

(iii) *(r_j) is majorizing for X*;

(iv) $\sum_{j=1}^{\infty} \frac{1}{n_1 n_2 \ldots n_j} \left\{\frac{1}{f(2^{-j+1})} - \frac{1}{f(2^{-j})}\right\} = \infty.$

Proof. We define $M_j = \#\{n: 2^{-j-1} < r_n \leqq 2^{-j}\}$ and $N_j = n_1 n_2 \ldots n_j$ so that N_j is the number of spheres in F_j. We shall show that (i)—(iv) are all equivalent to

$$\sum_{j=1}^{\infty} M_j/N_j = \infty. \tag{6.1}$$

We may suppose, without loss of generality, that $r_1 < \frac{1}{2}$ so that for $x \in X$ $\sum_{n=1}^{\infty} m\{S(x, r_n)\} = \sum_{j=1}^{\infty} M_j/N_j$. It follows from (4.2) that (i) and (ii) are equivalent to (6.1).

To show that (iv) is equivalent to (6.1) we have to estimate M_j. Clearly $M_j = \#\{n: r_n > 2^{-j-1}\} - \#\{n: r_n > 2^{-j}\}$. Now $r_n > 2^{-j}$ implies that $f(r_n -) \geqq f(2^{-j})$ which implies that $n \leqq 1/f(2^{-j})$. On the other hand $n < 1/f(2^{-j})$ implies that $f(2^{-j}) < 1/n \leqq f(r_n)$ and that $2^{-j} < r_n$. Thus we have

$$\#\{n: r_n > 2^{-j}\} = 1/f(2^{-j}) + O(1).$$

It follows that (iv) is also equivalent to (6.1).

It remains to show that (iii) is equivalent to (6.1). First suppose that

$\sum_{j=1}^{\infty} M_j/N_j < \infty$ and choose J such that $\sum_{j=J}^{\infty} M_j/N_j < \frac{1}{4}$. Then we let (z_i) be any sequence of points in X and estimate $m\{\bigcup_{r_i<2^{-J}} S(z_i, r_i)\}$. If $2^{-(p+1)} < r_i \leqq 2^{-p}$ and $z \in X$, $m\{S(z_i, r_i)\} = 1/N_p$ and so $m\{\bigcup_{r_i<2^{-J}} S(z_i, r_i)\} \leqq \sum_{j=J}^{\infty} M_j/N_j < \frac{1}{4}$. Since $m(X)=1$ we can never have $X = \bigcup_{r_i<2^{-J}} S(z_i, r_i)$ with the result that (r_i) is not majorizing, and (iii) implies (6.1).

Now suppose that $\sum_{j=1}^{\infty} M_j/N_j = \infty$, that $r_1 \leqq \frac{1}{2}$, and that J is the least integer such that $\sum_{j=1}^{J} M_j/N_j \geqq 1$. We first cover M_1 of the spheres of F_1 by open spheres with radii in $(\frac{1}{4}, \frac{1}{2}]$. Then $N_2 - N_2 N_1^{-1} M_1$ of the spheres of F_2 will be uncovered. Next we cover M_2 of these by open spheres with radii in $(\frac{1}{8}, \frac{1}{4}]$. Then $N_3 - N_3 N_1^{-1} M_1 - N_3 N_2^{-1} M_2$ of the spheres of F_3 will be uncovered. Proceed in this way until after the $J-1$th choice when $N_J\{1 - \sum_{j=1}^{J-1} M_j N_j^{-1}\}$ of the spheres of F_J will be uncovered. This integer is at most M_J, so we can cover these by open spheres whose radii belong to the interval $(2^{-J+1}, 2^{-J}]$. Now let J_1 be the least integer such that $\sum_{j=J+1}^{J_1} M_j N_j^{-1} \geqq 1$ and repeat the above process. In this way we obtain a sequence of spheres $S(z_i, r_i)$ such that $X = \limsup_{i\to\infty} S(z_i, r_i)$ which completes the proof.

Now define $\overline{M}_j = \#\{n: 2^{-j-1} \leqq r_n < 2^{-j}\}$, $\bar{f}(t) = \sup\{f(s): s<t\}$ and $\bar{\Phi}(x, y) = 1/\bar{f}[\varrho(x, y)]$. The methods of the above theorem also yield the following:

Theorem 6.2. *If (X, ϱ) is as above, the following statements are equivalent:*

(i) *X has zero $\bar{\Phi}$ capacity;*

(ii) *X is $\bar{\Phi}$-polar;*

(iii) *there exists a sequence (z_i) of points of X such that $X = \limsup_{i\to\infty} \bar{S}(z_i, r_i)$;*

(iv) $\sum_{j=1}^{\infty} \overline{M}_j N_j^{-1} = \infty$.

Remark 1. The last two results enable us to construct a class of examples for which covering by closed spheres is possible, but covering by open spheres is not possible. All one has to do is to arrange that

$$\sum_{j=1}^{\infty} \overline{M}_j N_j^{-1} = \infty \quad \text{but} \quad \sum_{j=1}^{\infty} M_j N_j^{-1} < \infty.$$

To do this we let $n_j = 2^j$ and $r_i = 2^{-(k+1)}$ when

$$\sum_{j=1}^{k-1} N_j < i \leqq \sum_{j=1}^{k} N_j.$$

In this way $\overline{M}_j = N_j$ and $M_j = N_{j-1} = N_j 2^{-j}$.

Remark 2. The methods of these theorems apply to generalized Cantor sets in euclidean spaces but the conclusions must sometimes be modified.

Remark 3. If we know that $\liminf_{t\to 0} f(2t)/f(t) > 1$ then each of the statements of Theorem 6.1 is equivalent to

$$\sum_{j=1}^{\infty} [N_j f(2^{-j})]^{-1} = \infty.$$

This provides a generalization of Ohtsuka's result ([16]).

7. A counterexample

In this section we construct a random subset A of $[1, 2]$ such that A has positive Φ capacity for $\Phi(x, y)=|x-y|^{-\alpha}$ (i.e. has positive α-capacity) with probability one, whilst $(j^{-1/\alpha})$ is always majorizing for A in A. Thus the converse to Theorem 5.1 does not hold.

The proof relies on the potential theory of the stable subordinator (see [4] for full details).

Suppose $0<\alpha<1$, let $T_t(\omega)$ be a stable subordinator of index $1-\alpha$ and $R(\omega)$ be its range. Then if $B\subset[1, 2]$, $P\{R(\omega)\cap B\neq\emptyset\}=0$ if and only if B has zero α-capacity (Theorem 3 of [4]). Let

$$N_p(\omega) = \#\{j\colon R(\omega)\cap[j2^{-p}, (j+1)2^{-p}] \neq \emptyset,\ 2^p \leqq j < 2^{p+1}\}.$$

Then we prove the following:

Lemma 7.1. *Let*

$$k(x) = x^{-\alpha}/\Gamma(1-\alpha) \quad \textit{if} \quad x > 0$$
$$= 0 \qquad\qquad\qquad \textit{if} \quad x \leqq 0.$$

Then for a certain positive constant c

$$EN_p \sim c2^{p(1-\alpha)}\int_1^2 k(x)\,dx \tag{7.1}$$

and

$$EN_p^2 \sim c^2 2^{2p(1-\alpha)}\int_1^2\int_1^2 k(x)k(y-x)\,dx\,dy. \tag{7.2}$$

Proof. For some positive constant d the capacity of an interval of length I is dI^α. Since

$$EN_p = \sum_{j=2^p}^{2^{p+1}-1} P\{R(\omega)\cap[j2^{-p}, (j+1)2^{-p}] \neq \emptyset\},$$

(7.1) follows from the formula for the hitting probability for an interval ([4, Lemma 1]). A similar idea gives (7.2).

We now describe the construction of the set. Let $k_i=[2^{p_i\alpha}]$ where (p_i) is an increasing sequence of integers such that $k_1+k_2+\ldots+k_n\sim k_n$. At the ith stage we choose independently and at random k_i integers $J(k; i)$, $1\leqq k\leqq k_i$ in such a way that $P\{J(k; i)=l\}=2^{-p_i}$ for $2^{p_i}\leqq l<2^{p_i+1}$, and let $I(k; i)=\{x\colon J(k; i)2^{-p_i}< <x<[J(k; i)+1]2^{-p_i}\}$.

Define $A_i=\bigcup_{k=1}^{k_i} I(k; i)$ and $A=\limsup_{i\to\infty} A_i$. We show that with probability one A has positive α-capacity. First we let R be the range of a stable subordinator of index $1-\alpha$, which is independent of the above construction. Then we define $M_i=\#\{k\colon R\cap I(k; i)\neq\emptyset\}$. Now

$$EM_i = \sum_{k=1}^{k_i}\sum_{l=2^{p_i}}^{2^{p_i+1}-1} P\{R\cap S(l, p_i) \neq \emptyset | J(k; i) = l\}P\{J(k; i) = l\}.$$

(Here $S(l, p) = \{x : l2^{-p} < x < (l+1)2^{-p}\}$.) Now by the independence of R and the $\{J\}$

$$P\{R \cap S(l, p_i) \neq \emptyset | J(k; i) = l\} = P\{R \cap S(l, p_i) \neq \emptyset\}$$

so that

$$EM_i = k_i 2^{-p_i} \sum_{l=2^{p_i}}^{2^{p_i+1}-1} P\{R \cap S(l, p_i) \neq \emptyset\}$$

and

$$EM_i \sim 2^{p_i(\alpha-1)} EN_{p_i}.$$

Similarly $EM_i^2 \sim 2^{2p_i(\alpha-1)} EN_{p_i}^2$ and so by Lemma 7.1 there is a positive constant C such that $(EM_i)^2/EM_i^2 \to C$ as $i \to \infty$. Now if $0 < \lambda < 1$, $P\{X \geqq \lambda E(X)\} \geqq (1-\lambda)^2 [E(X)]^2/EX^2$, see [10, p. 6]. Thus for some positive constant c, independent of i, $P\{M_i > \lambda EM_i\} \geqq \geqq c > 0$, $P\{M_i \geqq 1$ infinitely often$\} \geqq c > 0$, and $P\{R \cap \bigcup_{k=i}^{\infty} A_k \neq \emptyset\} \geqq c > 0$ for all i.

Letting i tend to infinity we have $P\{R \cap A \neq \emptyset\} \geqq c > 0$ and hence, by the property of the stable subordinator we have already quoted, $P\{A$ has positive α-capacity$\} > 0$. Now by a zero one law $P\{A$ has positive α-capacity$\} = 1$.

It remains to show that A admits an α-sequence. Let $r_j = 2^{-(p_i+1)}$ if $\sum_{n=1}^{i-1} k_n < < j \leqq \sum_{n=1}^{i} k_n$. Then, by construction, (r_j) is majorizing for A. Since $2r_j \leqq j^{-1/\alpha}$ for all j, $(j^{-1/\alpha})$ is majorizing for A in A. Thus with probability one A has the property asserted.

8. Examples

In this section we consider a number of examples related to the circle of ideas we have been discussing.

First we observe that if (x_i) is any sequence of points in a metric space Ω and if we define

$$A = \limsup_{i \to \infty} S(x_i, r_i)$$

then (r_i) is majorizing for A in Ω. Kaufman ([12]) considers the problem of making A as large as possible by making a suitable random choice of (x_i). In our notation his result reads as follows.

Theorem 8.1. *Let (r_i) be a sequence of positive real numbers, decreasing with limit zero, such that $\sum_{i=1}^{\infty} r_i < \infty$. Suppose that $h \in \mathscr{H}$ is concave with $\sum_{n=1}^{\infty} h(2r_n) = \infty$. Then, by choosing a suitable sequence of independent identically distributed real random variables (x_i), we can obtain a random set A such that*

(i) *(r_n) is majorizing for A;*

(ii) *A is Φ-polar;*

(iii) *$\mu^h(A) > 0$,*

each statement holding with probability one.

Remark. Part (ii) is not stated in [12] but it is implicit in the proof. It is inserted here to give a further example of the connexion with potential theory.

Let f be a meromorphic function, a be a complex number, and $\Delta(a,f)$ be the Valiron deficiency (see [8]) of f at a. Hyllengren shows [8] that for a function of finite order and any set U we have

$$U \subset \{a : \Delta(a,f) > 0\} \quad \text{for some } f$$

if and only if, for some $c>0$, $r_n = \exp\{-\exp(cn)\}$ is majorizing for U. The following (an unpublished result of Hyllengren) is an immediate consequence of this result and the above theorem.

Theorem 8.2. *Let $h \in \mathscr{H}$ be concave. Then there exists a plane set U and a meromorphic function f of finite order such that*

$$\mu^h(U) > 0 \quad \textit{and} \quad U \subseteqq \{a : \Delta(a,f) > 0\}$$

if and only if

$$\int_0^1 \frac{h(t)\,dt}{t \log 1/t} = \infty.$$

Proof. The last equation is equivalent to

$$\sum_{n=1}^{\infty} h[2\exp(-\exp cn)] = \infty$$

for some $c>0$.

It is interesting to note that, whilst Hyllengren's solution of this problem for functions of finite order involves majorizing sequences, for functions of infinite order the solution involves logarithmic capacity. Recently Hayman has shown ([5]) that for any F_σ-set U

$$U \subseteqq \{a : \Delta(a,f) > 0\}$$

for some f of infinite order if and only if U has zero logarithmic capacity.

Our final example involves some problems in diophantine approximation. We give the simplest case. If x is a real number and q a positive integer $\|qx\|$ denotes the fractional part of qx. Let $\omega(t)$ be a positive function of t which decreases to zero as t tends to infinity. Define

$$A = \{x \in [0,1] : \|qx\| < q\omega(q) \text{ infinitely often}\}.$$

In [9] Jarnik shows that (provided ω and h satisfy certain natural conditions of monotonicity) A has zero or non σ-finite μ^h measure depending on whether or not $\int_1^\infty th[\omega(t)]\,dt < \infty$. We now show how this result can be reinterpreted in terms of approximating sequences.

Let (x_n) be an enumeration of the rationals in $[0, 1)$ (with possible repetitions) which is such that the vulgar fraction p/q always occurs in unreduced form and that rationals with smaller denominators always precede those with larger denominators. Now choose any $f \in \mathscr{H}$ and ask what is the size of the set for which the rationals with this enumeration form an approximating sequence of order f, that is what is the size of the set $B = \limsup_{n\to\infty} (x_n - r_n, x_n + r_n)$? Jarnik's result allows us to answer this question for sufficiently regular f. Let

$$\omega_1(k) = r_{k(k-1)/2}, \quad \omega_2(k) = r_{k(k+1)/2}$$

and

$$A_i = \{x \in [0, 1) : \|qx\| < q\omega_i(q) \text{ infinitely often}\}.$$

Then $A_2 \subset B \subset A_1$ and if f and h are smooth enough $\int_1^\infty th[\omega_1(t)]\,dt$ and $\int_1^\infty th[\omega_2(t)]\,dt$ converge or diverge together. In these circumstances the convergence of either integral is equivalent to the convergence of $\sum_{n=1}^{\infty} h(2r_n)$. Thus we see that B has zero or non σ-finite μ^h measure depending on whether or not $\sum_{n=1}^{\infty} h(2r_n)$ converges.

These observations give us an alternative method of constructing sets of positive μ^h measure which admit approximating sequences of order f.

References

1. Besicovitch, A. S., On the definition of tangents to sets of infinite linear measure, *Proc. Cambridge Philos. Soc.*, **52** (1956), 20—29.
2. Frostman, O., *Potentiel d'equilibre et capacité des ensembles avec quelques applications à la théorie des fonctions,* Lund Univ. Thesis, 1935.
3. Gardner, R. J., Approximating sequences and Hausdorff measure, *Proc. Cambridge Philos. Soc.*, **76** (1974), 161—172.
4. Hawkes, J., Polar sets, regular points and recurrent sets for the symmetric and increasing stable processes, *Bull. London Math. Soc.*, **2** (1970), 53—59.
5. Hayman, W. K., On the Valiron deficiencies of integral functions of infinite order, *Ark. Mat.* **10** (1972), 163—172.
6. Hoffmann-Jørgensen, J., Coverings of metric spaces with randomly placed balls, *Math. Scand.*, **32** (1973), 169—186.
7. Hyllengren, A., Über die untere Ordnung der ganzen Funktion $f(z)e^{az}$, *Festschrift zur Gedächtnisfeier für Karl Weierstrass* 1815—1965, Hsg. von Behnke, Kopfermann, West-deutscher Verlag (1966), 555—577.
8. Hyllengren, A., Valiron deficient values for meromorphic functions in the plane, *Acta Math.*, **124** (1970), 1—8.
9. Jarnik, V., Über die simultanen diophantische Approximationen, *Math. Z.*, **33** (1931), 503—543.
10. Kahane, J.-P., *Some random series of functions,* D. C. Heath and Co., Lexington, 1968.
11. Kametani, S., On Hausdorff's measures and generalized capacities with some of their applications to the theory of functions, *Japan. J. Math.*, **19** (1945), 217—257.
12. Kaufman, R., Fractional dimension and the Borel—Cantelli lemma, *Acta Math. Acad. Sci. Hungar.*, **21** (1970), 235—238.
13. Kolmogorov, A. N. & Tihomirov, V. M., ε-entropy and ε-capacity of sets in functional spaces, *Amer. Math. Soc. Trans.*, **17** (1961), 277—364.

14. Lorentz, G. G., Metric entropy and approximation, *Bull. Amer. Math. Soc.*, **72** (1966), 903—937.
15. Lorentz, G. G., *Approximation of functions*, Holt, Rinehart and Winston, New York, 1966.
16. Ohtsuka, M., Capacité d'ensembles de Cantor généralisés, *Nagoya Math. J.*, **11** (1957), 151—160.
17. Shepp, L. A., Covering the circle with random arcs, *Israel J. Math.*, **11** (1972), 328—345.
18. Taylor, S. J., On the connexion between Hausdorff measure and generalized capacity, *Proc. Cambridge Philos. Soc.*, **57** (1961), 524—531.
19. Vitushkin, A. G., *Transmission and processing of information*, Pergamon Press, London, 1961.

Received May 5, 1975.

R. J. Gardner
Department of Mathematics
Westfield College
Kidderpore Avenue
London NW3 7ST.

J. Hawkes
Statistics Department
University College Swansea
Singleton Park
Swansea SA2 8PP.