An application of a general Tauberian remainder theorem

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1. Introduction

Let Φ be a real-valued, measurable and bounded function on $\mathbf{R}$ and let $F \in L^1(\mathbf{R})$. Introduce the Fouriertransform $\hat{F}$ of F

$$\hat{F}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} F(x)\,dx$$

and the convolution

$$\Phi * F(x) = \int_{-\infty}^{\infty} \Phi(x-y) F(y)\,dy.$$

Let us consider a Tauberian relation of the form

$$|\Phi * F(x)| \leqq \varrho(x), \quad x \geqq x_0 \tag{1.1}$$

where $\varrho \searrow$

In an earlier paper [6] a new method was developed and a new set of conditions on $\hat{F}$ were introduced in order to derive an estimate of $|\Phi(x)|$ as $x \to \infty$ from (1.1) and a Tauberian condition for Φ. As an application such results were proved when $1/\hat{F}(\zeta)$, $\zeta = \xi + i\eta$, is analytic in a strip $-\gamma < \eta < \gamma$ around the real axis and the order of magnitude of $1/\hat{F}$ in this strip is known.

In the present paper I use the results in [6] and a lemma for analytic functions proved in Section 2 below to obtain corresponding results when $1/\hat{F}$ is analytic in the strip $0 < \eta < \gamma$ only and the order of magnitude of $1/\hat{F}$ in this strip is known. In this way some new results are obtained. For instance, Theorem 1 in Section 3 below uses no condition on the derivative of $1/\hat{F}$, a condition which is imposed in all earlier theorems of this type (but for the partial result contained in Theorem 1 in [5]). In Theorems 2 and 3 conditions are imposed also on the derivative of $1/\hat{F}$. Theorem 2 extends earlier results of Ganelius and Frennemo and Theorem 3 deals with the case when the 'remainder' $\varrho(x)$ in (1.1) is majorized by $e^{-\alpha x}$ for some $\alpha \geqq \gamma$. In 3.3 I also consider the case when $1/\hat{F}(\zeta)$, $\zeta = \xi + i\eta$, is analytic in a domain $0 < \eta < \gamma(\xi)$ which tapers off at infinity. Theorems 4 and 5 deal with this case.

The theorems are stated for the Tauberian condition $\Phi(x)+Kx\nearrow$, $x>x_0$, for some positive constant K. It is easy to see that corresponding results for the more general Tauberian condition used in 4.2 in [6] can be obtained in an analogous way.

The estimates obtained in Theorems 4 and 5 are best possible and the same holds true for Theorems 1—3 for a wide range of majorants of $1/\hat{F}$ and remainders ϱ.

All functions are supposed to be measurable. I use the notations

$$M_s\{f;\, a, b\} = \Big(\int_a^b |f(x)|^s\, dx\Big)^{1/s}$$

and

$$\|f\|_s = M_s\{f;\, -\infty, \infty\}.$$

2. A result for analytic functions

2.1. *Preliminaries*

Let γ be a positive, even function on $\mathbf{R}$ such that $\gamma(\xi)\searrow$, $\xi\geqq 0$. Let $\zeta=\xi+i\eta$ and let D_γ denote the domain

(2.1.1) $$D_\gamma = \{\zeta;\, 0<\eta<\gamma(\xi)\}.$$

Let $W_0(X)\nearrow$, $X\geqq 0$, and introduce the class $\mathcal{A}_0=\mathcal{A}_0(\gamma;\, W_0)$ of functions g on $\mathbf{R}$ as follows.

Definition. $g\in\mathcal{A}_0(\gamma;\, W_0)$ if $g(\xi)$, $\xi\in\mathbf{R}$, are continuous boundary values of a function g analytic in D_γ and such that

(2.1.2) $$M_2\{g(\xi+i\delta);\, -X, X\}\leqq W_0(X),\quad 0\leqq\delta<\gamma(X),\quad X\geqq X_0.$$

Let $W_1(X)\nearrow$, $X\geqq 0$ and

(2.1.3) $$\varlimsup_{X\to\infty} W_0(X)/XW_1(X)\leqq 1.$$

Introduce the function

(2.1.4) $$W=\sqrt{W_0W_1}.$$

The class $\mathcal{A}_1=\mathcal{A}_1(\gamma;\, W_0, W_1)$ is defined as follows.

Definition. $g\in\mathcal{A}_1(\gamma;\, W_0, W_1)$ if $g\in\mathcal{A}_0(\gamma;\, W_0)$ and

(2.1.5) $$M_2\{g'(\xi+i\delta);\, -X, X\}\leqq W_1(X),\quad 0<\delta<\gamma(X),\quad X\geqq X_0.$$

If $g\in\mathcal{A}_0(\gamma;\, W_0)$ then $g(\xi+i\delta)\to g(\xi)$, $\delta\to 0+$, uniformly on every compact interval. Hence, for every $a>0$,

(2.1.6) $$M_2\{g(\xi+i\delta)-g(\xi);\, -a, a\}\to 0,\quad \delta\to 0+.$$

Let us now prove that if $g\in\mathcal{A}_1(\gamma;\ W_0,\ W_1)$ then $g'(\xi+i\delta)\to g'(\xi)$, $\delta\to 0+$, almost everywhere on the real axis and, for every $a>0$,

$$M_2\{g'(\xi+i\delta)-g'(\xi);\ -a,a\}\to 0,\quad \delta\to 0+. \tag{2.1.7}$$

Choose $b>a$ such that

$$\int_0^{\gamma(b)}\left(|g'(b+i\eta)|^2+|g'(-b+i\eta)|^2\right)d\eta<\infty, \tag{2.1.8}$$

and let ω denote the open interval $(-b,b)$. The assumption (2.1.5) implies that there is a function $h\in L^2(\omega)$ and a sequence $(\delta_n)_1^\infty$ such that $\delta_n\to 0+$, $n\to\infty$, and $g'(\xi+i\delta_n)$ converges weakly in $L^2(\omega)$ to $h(\xi)$ as $n\to\infty$. By using the identity

$$g(\xi+i\delta_n)-g(-b+i\delta_n)=\int_{-b}^{\xi}g'(t+i\delta_n)\,dt,\quad \xi\in\omega,\quad 0<\delta_n<\gamma(b)$$

and letting $n\to\infty$ we thus obtain

$$g(\xi)-g(-b)=\int_{-b}^{\xi}h(t)\,dt,\quad \xi\in\omega.$$

It follows that $g'=h$ a.e. on ω. Thus $g'\in L^2(\omega)$ and

$$\lim_{n\to\infty}\int_{-b}^{b}g'(t+i\delta_n)k(t)\,dt=\int_{-b}^{b}g'(t)k(t)\,dt,\quad k\in L^2(\omega). \tag{2.1.9}$$

Let K_δ, $0\leqq\delta<\gamma(b)$, denote the open rectangle with corners in $\pm b+i\delta$, $\pm b+i\gamma(b)$, let Γ_δ denote its boundary and put $K=K_0$, $\Gamma=\Gamma_0$. If $\zeta\in K$ then by representing $g'(\zeta)$ by its Cauchy integral over Γ_{δ_n}, $n>n_0$ and letting $n\to\infty$ we obtain from (2.1.8) and (2.1.9) that $g'(\zeta)$ may be represented by its Cauchy integral over Γ. Therefore

$$g'(\zeta)=\frac{1}{2\pi i}\int_{\Gamma-\omega}\frac{g'(w)}{w-\zeta}\,dw+\frac{1}{2\pi i}\int_{\omega}\frac{g'(w)}{w-\zeta}\,dw=\varphi_1(\zeta)+\varphi_2(\zeta),\quad \zeta\in K.$$

The function φ_1 is analytic on ω and hence $\varphi_1(\xi+i\delta)\to\varphi_1(\xi)$ as $\delta\to 0+$, uniformly on $(-a,a)$. The function φ_2 is analytic in the upper half-plane. By using well-known results for the Hilbert transform (see [7], Theorems 91 and 93) it is easy to see that there is a function $\varphi_2\in L^2(\mathbf{R})$ such that $\varphi_2(\xi+i\delta)\to\varphi_2(\xi)$, $\delta\to 0+$, almost everywhere on $\mathbf{R}$ and

$$\|\varphi_2(\xi+i\delta)-\varphi_2(\xi)\|_2\to 0,\quad \delta\to 0+.$$

Let $\varphi(\xi)=\varphi_1(\xi)+\varphi_2(\xi)$. From the above results for φ_1 and φ_2 it follows that $g'(\xi+i\delta)\to\varphi(\xi)$, $\delta\to 0+$, almost everywhere on ω and $M_2\{g'(\xi+i\delta)-\varphi(\xi);\ -a,a\}\to 0$, $\delta\to 0+$, the last result by using Minkowski's inequality. Now $g'(\xi+i\delta_n)$ converges weakly in $L^2(\omega)$ to $g'(\xi)$ as $n\to\infty$ and hence $\varphi=g'$ a.e. on ω. Thus the result stated is proved.

2.2. *A fundamental lemma*

The lemma below connects the classes $\mathscr{A}_0$ and $\mathscr{A}_1$ with the classes $\mathscr{B}_2$ and $\mathscr{B}_1$ introduced in 2.6 in [6] and thus makes it possible to apply the Tauberian theorems in [6] if $1/\hat{F}$ belongs to $\mathscr{A}_0$ or $\mathscr{A}_1$.

Lemma. *Let $g\in\mathscr{A}_0(\gamma;\,W_0)$. Then for every $X\geqq\max\big(X_0,\,2\gamma(0)\big)$, there exist functions $f=f_X$ and $k=k_X$ in $L^2(\mathbf{R})$ such that*

$$g(\xi)=f(\xi)+k(\xi). \quad -X\leqq\xi\leqq X, \tag{2.2.1}$$

where k is the Fourier transform in the L^2-sense of a function $K=K_X$ such that $K(x)=0$, $x>0$,

$$\|K\|_2\leqq(2\pi)^{-1/2}W_0(2X) \tag{2.2.2}$$

and

$$\|K\|_\infty\leqq 2X^{1/2}W_0(2X) \tag{2.2.3}$$

and f satisfies

$$M_2\{f^{(n)};\,-X,X\}\leqq 2n!\,W_0(2X)\,\gamma(2X)^{-n}, \quad n=0,1,2,\ldots. \tag{2.2.4}$$

Let us further suppose that $g\in\mathscr{A}_1(\gamma;\,W_0,\,W_1)$ and let W be defined by (2.1.4). *Then there exists X_1 such that if $X\geqq X_1$ then it also holds true that*

$$\|K\|_\infty\leqq 3XW(2X) \tag{2.2.5}$$

$$\|K\|_1\leqq 2W(2X) \tag{2.2.6}$$

and

$$M_2\{f^{(n)};\,-X,X\}\leqq 5(n-1)!\,W_1(2X)\,\gamma(2X)^{1-n}, \quad n=1,2,\ldots. \tag{2.2.7}$$

Proof. Let us suppose that $g\in\mathscr{A}_0(\gamma;\,W_0)$ and let us choose $X\geqq\max\big(X_0,\,2\gamma(0)\big)$ and put $\beta=\gamma(2X)$ and $a=X+\beta$. Then g is analytic in the rectangle $|\xi|<2X$, $0<\eta<\beta$ and

$$M_2\{g(\xi+i\delta);\,-2X,2X\}\leqq W_0(2X), \quad 0\leqq\delta<\beta. \tag{2.2.8}$$

Let $u(\xi)$, $\xi\in\mathbf{R}$ be continuous, $u(\xi)=1$, $|\xi|\leqq a$, $u(\xi)=0$, $|\xi|\geqq 2X$ and u linear over the remaining intervals. Since $a=X+\beta<2X$ we have

$$\|ug\|_2\leqq W_0(2X). \tag{2.2.9}$$

Introduce the inverse Fourier transform oι ug,

$$G=(ug)^{\vee}.$$

Parseval's relation yields

$$\|G\|_2\leqq(2\pi)^{-1/2}W_0(2X) \tag{2.2.10}$$

and using Schwarz' inequality we have

$$\|G\|_\infty \leqq \|ug\|_1 \leqq 2X^{1/2}W_0(2X). \tag{2.2.11}$$

Let H denote the Heaviside function, $H(x)=1$, $x>0$, $H(x)=0$, $x<0$ and let $f=(GH)^\wedge$ and $k=(G(x)H(-x))^\wedge$, the transforms being in the L^2-sense. Then $K(x)=$ $=G(x)H(-x)$ satisfies (2.2.2) and (2.2.3) according to (2.2.10) and (2.2.11). Furthermore, $f+k=ug$, a.e. on the real axis and hence

$$f(\xi)+k(\xi)-g(\xi) = 0, \quad \text{a.e. on} \quad (-a, a). \tag{2.2.12}$$

To prove (2.2.1) and (2.2.4) let $\zeta=\xi+i\eta$ and introduce the functions

$$f(\zeta) = \int_0^\infty e^{-i\zeta x}G(x)\,dx, \quad \eta<0, \quad k(\zeta) = \int_{-\infty}^0 e^{-i\zeta x}G(x)\,dx, \quad \eta>0.$$

These functions are analytic in the domains where they are defined,

$$\lim_{\delta\to 0+} f(\xi-i\delta) = f(\xi) \quad \text{a.e.,} \quad \lim_{\delta\to 0+} g(\xi+i\delta) = g(\xi) \quad \text{a.e. and}$$

$$\|f(\xi-i\delta)-f(\xi)\|_2+\|k(\xi+i\delta)-k(\xi)\|_2 \to 0, \quad \delta\to 0+, \tag{2.2.13}$$

(see [7], Theorems 93 and 95). Furthermore, by Parseval's relation and (2.2.10)

$$\|f(\xi-i\delta)\|_2^2+\|k(\xi+i\delta)\|_2^2 \leqq W_0^2(2X), \quad \delta>0. \tag{2.2.14}$$

Now, by Minkowski's inequality and (2.2.12)

$$M_2\{f(\xi-i\delta)+k(\xi+i\delta)-g(\xi+i\delta);\ -a, a\} \leqq$$
$$\leqq \|f(\xi-i\delta)-f(\xi)\|_2+\|k(\xi+i\delta)-k(\xi)\|_2+M_2\{g(\xi+i\delta)-g(\xi);\ -a, a\},$$
$$0<\delta<\beta.$$

Hence, by (2.1.6) and (2.2.13)

$$M_2\{f(\xi-i\delta)-(g(\xi+i\delta)-k(\xi+i\delta));\ -a, a\} \to 0, \quad \delta\to 0+. \tag{2.2.15}$$

The function f is analytic in the lower half-plane and the function $g-k$ is analytic in the rectangle $-a<\xi<a$, $0<\eta<\beta$. The relation (2.2.15) implies that f can be analytically continued across the interval $(-a, a)$ by $g-k$. Therefore f is continuous on $(-a, a)$. The function g has continuous boundary values on $(-a, a)$ by assumption and hence k has continuous boundary values on $(-a, a)$. Since $X<a$ the identity (2.2.1) thus follows from (2.2.12).

To prove (2.2.4) let ψ denote the analytic function which equals f in the lower half-plane and equals $g-k$ in D_γ. Then

$$M_2\{\psi(\xi-i\delta);\ -a, a\} \leqq \|f(\xi-i\delta)\|_2, \quad 0\leqq\delta,$$

$$M_2\{\psi(\xi+i\delta);\ -a, a\} \leqq M_2\{g(\xi+i\delta);\ -a, a\}+\|k(\xi+i\delta)\|_2, \quad 0<\delta<\beta,$$

and hence, by (2.2.8) and (2.2.14)

$$\left(\int_{-a}^a |\psi(\xi+i\eta)|^2\,d\xi\right)^{1/2} \leqq 2W_0(2X), \quad \eta<\beta. \tag{2.2.16}$$

The function ψ is analytic in the rectangle $|\xi|<a$, $|\eta|<\beta$. Therefore, Cauchy's formula and an application of Schwarz' inequality yield

$$|\psi^{(n)}(\xi)|^2 = \left|\frac{n!}{2\pi\beta^n}\int_0^{2\pi}\psi(\xi+\beta e^{i\theta})e^{-in\theta}\,d\theta\right|^2 \leqq \frac{1}{2\pi}\left(\frac{n!}{\beta^n}\right)^2\int_0^{2\pi}|\psi(\xi+\beta e^{i\theta})|^2\,d\theta.$$

Integrating over the interval $(-a+\beta, a-\beta)$ and inverting the order of integration we have

$$\int_{-a+\beta}^{a-\beta}|\psi^{(n)}(\xi)|^2\,d\xi \leqq \frac{1}{2\pi}\left(\frac{n!}{\beta^n}\right)^2\int_0^{2\pi}d\theta\int_{-a+\beta}^{a-\beta}|\psi(\xi+\beta e^{i\theta})|^2\,d\xi.$$

The inner integral can be majorized by $4W_0^2(2X)$ according to (2.2.16). Since $X=a-\beta$, $\beta=\gamma(2X)$ and $\psi=f$ on the interval $(-a, a)$ this proves (2.2.4).

Let us now prove the results under the assumption $g\in\mathscr{A}_1(\gamma;\, W_0,\, W_1)$. Choose α, $1<\alpha\leqq 9/8$. According to (2.1.3) there exists X_1, $X_1\geqq\max\,(X_0, \alpha(\alpha-1)^{-1}\gamma(0))$, such that

(2.2.17) $$W_0(2X) \leqq 2\alpha X W_1(2X), \quad X \geqq X_1.$$

Choose X, $X\geqq X_1$, and introduce $\beta=\gamma(2X)$, $a=X+\beta$ and the functions u, G, f, K and ψ as before. Combining (2.2.3) and (2.2.17) we have $\|K\|_\infty\leqq 2^{3/2}\alpha^{1/2}X\,W(2X)$ which proves (2.2.5). Furthermore $|u'(\xi)|=(2X-a)^{-1}\leqq\alpha X^{-1}$, $a<|\xi|<2X$, $u'(\xi)=0$, $|\xi|<a$, and u vanishes outside $(-2X, 2X)$. Thus (2.2.8) and (2.2.17) yield

$$\|u'g\|_2 \leqq \alpha X^{-1}W_0(2X) \leqq 2\alpha^2 W_1(2X)$$

and the assumptions for g' imply that $\|ug'\|_2\leqq W_1(2X)$. Hence

(2.2.18) $$\left\|\frac{d}{d\xi}(u(\xi)g(\xi))\right\|_2 \leqq (1+2\alpha^2)W_1(2X).$$

Now, by an inequality by Carlson and Beurling, $\|G\|_1\leqq\|\hat{G}\|_2\|\hat{G}'\|_2$. By applying this inequality with $\hat{G}=ug$ and using (2.2.9) and (2.2.18) we get

$$\|G\|_1 \leqq (1+2\alpha^2)^{1/2}W(2X),$$

which proves (2.2.6).

To prove (2.2.7) we observe that

(2.2.19) $$\|f'(\xi-i\delta)\|_2^2+\|k'(\xi+i\delta)\|_2^2 \leqq \left\|\frac{d}{d\xi}(u(\xi)g(\xi))\right\|_2^2, \quad \delta>0.$$

By using (2.2.18), (2.2.19), the definition of ψ and the assumptions for g' we obtain

(2.2.20) $$\left(\int_{-a}^{a}|\psi'(\xi+i\eta)|^2\,d\xi\right)^{1/2} \leqq 2(1+\alpha^2)W_1(2X), \quad \eta<\beta.$$

The inequality (2.2.7) then follows from (2.2.20) in the same way as (2.2.4) was derived from (2.2.16). This completes the proof of the lemma.

In some cases when $g'(\zeta)\to 0$ as $|\zeta|\to\infty$, $\zeta\in D_\gamma$, it is better to use L^s-estimates instead of L^2-estimates. In this way the following result is obtained.

Remark. Let s be constant, $1<s\leqq 2$, and $1/s+1/s'=1$. Let g satisfy the conditions in the definition of $\mathscr{A}_1$ but for the fact that M_2 is replaced by M_s in (2.1.2) and (2.1.5). Then there is X_1 such that, for every $X\geqq X_1$, (2.2.1) holds true, where $k=\hat{K}$, $K(x)=0$, $x>0$, and, but for a constant factor depending on s, the inequalities (2.2.4)—(2.2.7) hold true if M_2 is replaced by M_s and W is replaced by $W_0^{1/s'}W_1^{1/s}$.

3. Tauberian theorems

3.1. *Preliminaries*

Let Φ be bounded on $\mathbf{R}$ and $F\in L^1(\mathbf{R})$. Let us consider a Tauberian relation of the type

$$|\Phi * F(x)| \leqq \varrho(x), \quad x \geqq x_0, \tag{3.1.1}$$

where $1/\hat{F}$ belongs to the class $\mathscr{A}_0(\gamma;\, W_0)$ or $\mathscr{A}_1(\gamma;\, W_0,\, W_1)$ introduced in Section 2 and ϱ belongs to the class $\mathscr{E}$ defined in 3.1 in [6]. This means that $\varrho>0$, $\varrho\searrow$ and for every $\varepsilon>0$ there exist x_ε and δ_ε such that

$$\varrho(x-y) \leqq (1+\varepsilon)\varrho(x), \quad x \geqq x_\varepsilon, \quad 0 \leqq y \leqq \delta_\varepsilon.$$

For the sake of simplicity I also introduce the following regularity conditions on ϱ and on the functions W_n, $n=0, 1$. Note that the condition (3.1.3) below makes ϱ regular in the sense introduced in 4.3 in [6].

If $S(x)\nearrow$, $x>x_0$, let χ_S denote the function

$$\chi_S(x) = \frac{xD^+S(x)}{S(x)}. \tag{3.1.2}$$

Let $\lim_{X\to\infty}(\log W_n(X))^{-1}(\log\log W_n(X))^{-1}\chi_{W_n}(X)$ exist, finite or infinite and let $r=1/\varrho$ satisfy

$$\lim_{x\to\infty} (\log x)^{-1}\log\chi_r(x) = \omega. \tag{3.1.3}$$

These assumptions are maintained throughout the present paper.

Let $v(x)\nearrow$, $x\geqq 0$, $v(x)>1$, $x>0$ and introduce the class $R[v]$ as in 2.3 in [6] by the following definition.

Definition. $\varrho\in R[v]$ if $\varrho>0$, $\varrho\searrow$, $\varrho(x)\to 0$, $x\to\infty$ and $\varrho(x-y)\leqq\varrho(x)v(y)$, $y\geqq 0$, $x\in\mathbf{R}$.

In the theorems below I impose a condition of the type $\varrho\in R[v]$, where v is a function determined by the class $\mathscr{A}_0(\gamma;\, W_0)$ or $\mathscr{A}_1(\gamma;\, W_0,\, W_1)$ respectively. This condition can always be replaced by $\varrho\in R[bv]$ for some constant $b>1$ and the same result will hold true but for the fact that the constants will depend also on b.

For the sake of simplicity I use the following Tauberian condition. Let, for some positive constant K,

$$\Phi(x)+Kx\nearrow,\quad x\geqq x_0. \tag{3.1.4}$$

This condition may be weakened in the following way. If the result of the theorem is $\Phi(x)=O(\sigma(x))$, $x\to\infty$, then (3.1.4) may be replaced by

$$\Phi(x)-\Phi(x+y)\leqq K\sigma(x),\quad 0<y\leqq\sigma(x),\quad x\geqq x_0 \tag{3.1.5}$$

and the same result will be valid.

3.2. *The function $1/\hat{F}$ analytic in a strip above the real axis*

Three theorems will be proved in which the domain D_γ introduced in (2.1.1) is a strip, i.e. $\gamma\equiv$constant.

Introduce the functions t_0 and t in the following way. Let $W=\sqrt{W_0W_1}$, let

$$U_0(X)=XW_0(X),\quad U(X)=XW(X) \tag{3.2.1}$$

and let U_0^{-1} and U^{-1} denote the inverse functions of U_0 and U respectively. Let

$$t_0(x)=1/U_0^{-1}\left(\frac{1}{\varrho(x)}\right) \tag{3.2.2}$$

and

$$t(x)=1/U^{-1}\left(\frac{1}{\varrho(x)}\right). \tag{3.2.3}$$

With these notations the theorems may be stated as follows.

Theorem 1. (1) *Let Φ be bounded on* **R** *and $\Phi(x)+Kx\nearrow$, $x>x_0$ for some $K>0$. Let $F\in L^1(\mathbf{R})$ and $|\Phi*F(x)|\leqq\varrho(x)$, $x\geqq x_0$.*

(2) *Let $1/\hat{F}\in\mathscr{A}_0(\gamma;W_0)$, $\gamma\equiv$constant, and let $\psi=\Phi*F$ satisfy*

$$M_2\{\psi;x,\infty\}\leqq\varrho(x),\quad x\geqq x_0. \tag{3.2.4}$$

(3) *Let θ be constant, $0<\theta<1$, and*

$$\varrho\in R[e^{\theta\gamma x}]. \tag{3.2.5}$$

Then

$$\varlimsup_{x\to\infty}\frac{|\Phi(x)|}{t_0(x)}\leqq C_1K+C_2, \tag{3.2.6}$$

where $C_1=C_1(\gamma)$, $C_2=C_2(\gamma,\theta)$ and t_0 is defined by (3.2.2).

Theorem 2. *Let Conditions* (1) *and* (3) *of Theorem 1 hold true and let* $1/\hat{F}\in \in\mathcal{A}_1(\gamma;\, W_0,\, W_1)$, $\gamma\equiv$constant. *Then*

$$\overline{\lim_{x\to\infty}}\ \frac{|\Phi(x)|}{t(x)} \leqq C_1K+C_2, \tag{3.2.7}$$

where $C_1=C_1(\gamma)$, $C_2=C_2(\gamma,\theta)$ *and* t *is defined by* (3.2.3).

Theorem 3. *Let Condition* (1) *of Theorem 1 hold true and let* $1/\hat{F}\in\mathcal{A}_1(\gamma;\, W_0,\, W_1)$, $\gamma\equiv$*constant. Let* t *be defined by* (3.2.3) *and suppose that for some constants* θ, c *and* β, $\theta\geqq 1$, $c\geqq 0$, $\beta>\gamma$,

$$\varrho\in R[1+(\theta x)^{c+1}e^{\theta\gamma x}] \tag{3.2.8}$$

and

$$W_1\left(\frac{1}{t(x)}\right)x^{c+3/2}\exp(x\beta\theta^2\log\theta)\leqq W\left(\frac{1}{t(x)}\right),\quad x\geqq x_1. \tag{3.2.9}$$

Then (3.2.7) *holds true with* $C_1=C_1(\gamma, c)$ *and* $C_2=C_2(\gamma, c, \theta)$.

Before proving the theorems I shall make a number of comments. Introduce χ_W according to (3.1.2) and let $\underline{W}$ be defined by

$$\underline{W}=\min(W_0, W_1). \tag{3.2.10}$$

If $\underline{W}(X)=o(W(X))$, $X\to\infty$, then C_2 is independent of θ in Theorem 2. If $\chi_{W_0}(X)\to\infty$, $X\to\infty$, then we may choose $C_2=0$ in the result of Theorems 1 and 2 and also in Theorem 3 provided that (3.2.9) holds true with t replaced by rt, $1\leqq r\leqq B$, for some $B>1$.

Let λ denote a positive constant and replace the assumption $\Phi(x)+Kx\nearrow$ by (3.1.5) where $\sigma=\lambda t_0$ in Theorem 1 and $\sigma=\lambda t$ in Theorems 2 and 3. Then the results (3.2.6) and (3.2.7) respectively hold true with $C_1=(1+\lambda)C(\gamma)$ and $C_2=C_2(\gamma,\theta)$.

The condition $0<\theta<1$ can be replaced by $\theta\geqq 1$ in Theorem 1 provided that, for some $A>0$,

$$\log W_0(X)\leqq A\chi_{W_0}(X),\quad X\geqq X_1, \tag{3.2.11}$$

and in Theorem 2 provided that

$$\log W(X)\leqq A\chi_W(X),\quad X\geqq X_1, \tag{3.2.12}$$

and then the results (3.2.6) and (3.2.7) of these theorems hold true with $C_1=C_1(\gamma,\theta,A)$ and $C_2=0$.

It follows from the last remark that Theorem 3 is of any interest only if (3.2.12) is not satisfied. In fact, Theorem 3 can be applied only if W and ϱ are sufficiently small. This is due to the fact that (3.2.9) and (2.1.3) imply that

$$t(x)=O\big(x^{-2c-3}\exp(-2x\beta\theta^2\log\theta)\big),\quad x\to\infty \tag{3.2.13}$$

and (3.2.8) implies that $1/\varrho(x)=O(x^{c+1}e^{\theta\gamma x})$, $x\to\infty$. If the conditions of Theorem 3 are satisfied for some $\theta>1$ then, by the above inequalities and the definition of t, W is dominated by a polynomial and ϱ is exponentially decreasing.

The condition (3.2.4) of Theorem 1 is irrelevant if $M_2\{\varrho; x, \infty\}\leqq A\varrho(x)$, $x\geqq x_0$ for some constant A. The same holds true for a larger set of functions ϱ if W_0 does not increase too slowly. For instance, if $\varrho\in L^s(x_0, \infty)$ for some s, $0<s<2$ and (3.2.11) is satisfied or if $1/\log(1/\varrho)\in L^s(x_0, \infty)$ for some $s>0$ and $\log\log W_0(X)\leqq \leqq A\chi_{W_0}(X)$, $X\geqq X_1$, then the condition (3.2.4) of Theorem 1 may be omitted and the same result holds true but for the fact that C_1 will depend also on s and A.

Let the conditions of Theorem 2 and any of the above-mentioned conditions on ϱ and W_0 be satisfied. Then either Theorem 1 or Theorem 2 may be applied, and if $W_1(X)=o(W_0(X))$, $X\to\infty$, Theorem 2 seems to yield the best estimate. This is so, however, only when W_0 does not increase too fast. Let us suppose for instance that, for some $A>0$,

$$\log X \leqq A\chi_{W_0}(X), \quad X\geqq X_1. \tag{3.2.14}$$

Then $\log X\leqq 2A\chi_W(X)$, $X\geqq X_1$. By combining this inequality with (2.1.3) we get $W_0(X)\leqq W(e^{2A}X)$, $X\geqq X_2$ and hence $t_0(x)\leqq e^{2A}t(x)$, $x\geqq x_1$. Thus Theorem 2 does not, except possibly for the value of the constants, yield a better estimate than Theorem 1. If $W_0(X)=o(W_1(CX))$, $X\to\infty$ for every $C>0$ then $t_0(x)=o(t(x))$, $x\to\infty$, and Theorem 1 yields a better estimate than Theorem 2.

It follows from a theorem of Ganelius ([4], Th. 4.2.1, p. 34) that the estimates obtained in Theorems 2 and 3 are best possible in the sense that (3.2.7) cannot be replaced by $\Phi(x)=O(\delta(x)t(x))$, $x\to\infty$, for any function δ such that $\delta(x)\to 0$, $x\to\infty$, if

$$\log W_0(X) = O(X^2), \quad X\to\infty \tag{3.2.15}$$

and if either (3.2.14) is satisfied or $XW_1(X)=O(W_0(X))$, $X\to\infty$. Therefore, by the above argument, Theorem 1 is best possible in the sense that (3.2.6) cannot be replaced by $\Phi(x)=O(\delta(x)t_0(x))$, $x\to\infty$, for any function δ such that $\delta(x)\to 0$, $x\to\infty$, if (3.2.14) and (3.2.15) hold true and ϱ and W_0 satisfy any of the conditions which yield that the assumption (3.2.4) may be omitted in Theorem 1. The above statements hold, in fact, true if (3.2.15) is replaced by

$$\varlimsup_{X\to\infty} X^{-1}\log\log W_0(X) < \frac{\pi}{2\gamma}. \tag{3.2.16}$$

This follows by applying Ganelius' method with the auxiliary function e^{-x^2}, used by him, replaced by $\exp(-e^{\alpha x}-e^{-\alpha x})$ where

$$\varlimsup_{X\to\infty} X^{-1}\log\log W_0(X) < \alpha < \frac{\pi}{2\gamma}.$$

Proceeding to the proof of the theorems I shall first introduce some notations. These are the same as the ones used in [6] but for the function S and the classes

$\mathscr{B}_1$ and $\mathscr{B}_2$. For the sake of convenience S and hence $\mathscr{B}_1$ and $\mathscr{B}_2$ are introduced here in a way slightly different from their definitions in [6].

The sequence $P=(P_n)_0^\infty$ and the function h_P are introduced as in [6]. Thus

$$h_P(x) = \sum_{n=0}^{\infty} \frac{x^n}{P_n}, \quad x \geqq 0. \tag{3.2.17}$$

For the conditions on (P_n) the reader is referred to 2.1 in [6]. For the present purpose it suffices to know that the sequences $P_n=n!\,\gamma^{-n}$, $n=0, 1, 2, \ldots$ and $P_0=1, P_n=(n-1)!\,\gamma^{1-n}$, $n=1, 2, \ldots$, satisfy these conditions. Note that these sequences are also regular in the sense introduced in 4.3 in [6].

The functions S_n, $n=0, 1, 2, \ldots$ and $\bar{S}_1$ are introduced as in 2.1 in [6]. Thus $S_n(X)\nearrow$, $X\geqq X_0$, $n=0, 1, 2, \ldots$, $S_0(X)\leqq\frac{3}{2}XS_1(X)$, $X\geqq X_1$, and

$$\bar{S}_1 = \sup_{n\geqq 1} S_n. \tag{3.2.18}$$

Let $S(X)\nearrow$, $X\geqq X_0$, and

$$S \geqq \sqrt{S_0 S_1}. \tag{3.2.19}$$

When S and ϱ are given, the function $\tau=\tau_{S,\varrho}$ is introduced as in 4.3 in [6] by the following definition. Let $T(X)=XS(X)$, let T^{-1} denote the inverse function of T and

$$\tau(x) = 1/T^{-1}\left(\frac{1}{\varrho(x)}\right). \tag{3.2.20}$$

The classes $\mathscr{B}_1\big((P_n), (S_n), S\big)$ and $\mathscr{B}_2\big((P_n), (S_n), S\big)$ of functions $g(\xi)$, $\xi\in\mathbf{R}$ are introduced literally in the same way as the classes $\mathscr{B}_1$ and $\mathscr{B}_2$ are introduced in 2.6 in [6] by the following definition.

Definition. $g\in\mathscr{B}_1\big((P_n), (S_n), S\big)$ if for every $X\geqq X_0$ there exist functions $f=f_X$ and $k=k_X$ satisfying

$$f(\xi)+k(\xi) = g(\xi), \quad -X \leqq \xi \leqq X,$$

and such that

$$M_2\{f^{(n)}; -X, X\} \leqq P_n S_n(X), \quad n = 0, 1, 2, \ldots$$

and $k=\hat{K}$ where $K(x)=0$, $x>0$, $\|K\|_\infty\leqq XS(X)$ and

$$\|K\|_1 \leqq S(X). \tag{3.2.21}$$

$\mathscr{B}_2\big((P_n), (S_n), S\big)$ denotes the class of functions g satisfying the above conditions but for the fact that k is Fourier transform in the L^2-sense of K and (3.2.21) is replaced by

$$\|K\|_2 \leqq S(X).$$

The condition (3.2.19) thus replaces the definition $S=\sqrt{S_0 S_1}$ used in [6]. It is easy to see that the theorems in [6] hold true also for functions S satisfying (3.2.19) if τ and $\mathscr{B}_1$ and $\mathscr{B}_2$ are defined as above.

If $S=\sqrt{S_0S_1}$ I use the same notation as in [6]. Thus, for $k=1, 2$, let

$$\mathscr{B}_k((P_n), (S_n)) = \mathscr{B}_k((P_n), (S_n), \sqrt{S_0 S_1}).$$

When all the functions S_n equal S, I write, for $k=1, 2$,

$$\mathscr{B}_k((P_n), S) = \mathscr{B}_k((P_n), (S_n)), \quad S_n = S, \quad n = 0, 1, 2, \ldots.$$

Proof of Theorem 1. Let $P_n=n!\ \gamma^{-n}$, $n=0, 1, 2, \ldots$, and let h_P be defined by (3.2.17). Then $h_P(x)=e^{\gamma x}$ and $\varrho\in R[h_P(\theta x)]$ according to the assumption (3.2.5). Let $S(X)=2W_0(2X)$. By applying the lemma in 2.2 we find that the assumption $1/\hat{F}\in\mathscr{A}_0(\gamma;\ W_0)$ implies that $1/\hat{F}\in\mathscr{B}_2((P_n), S)$. The function S is regular in the sense introduced in 4.3 in [6] according to the regularity conditions imposed on W_0, and the sequence (P_n) satisfies

$$\text{(3.2.22)} \qquad \log (P_{n+1}/P_n) = o(n), \quad n \to \infty.$$

The assumption $1/\hat{F}\in\mathscr{A}_0(\gamma;\ W_0)$ further yields that $1/\hat{F}$ is continuous on $\mathbf{R}$ and hence $\hat{F}$ cannot vanish on $\mathbf{R}$. Thus the conditions of Theorem 3 in [6], modified according to the remark in 4.4 in [6], are satisfied. By applying this theorem we get

$$\text{(3.2.23)} \qquad \varlimsup_{x\to\infty} |\Phi(x)|/\tau(x) \leqq C_0K+C,$$

where τ is defined by (3.2.20), $C_0=C_0(\gamma)$ and $C=C(\gamma, \theta)$. Since $\tau=2t_0$ this proves (3.2.6).

Proof of Theorem 2. Let us choose $P_n=n!\ \gamma^{-n}$ as in the previous proof. Then $\varrho\in R[h_P(\theta x)]$ and (3.2.22) is satisfied. Let $\varkappa=\max(1, \gamma)$, $\underline{W}=\min(W_0, W_1)$ and let us choose $S_0(X)=2W_0(2X)$, $S_n(X)=5\varkappa\underline{W}(2X)$, $n=1, 2, \ldots$, and $S(X)=5\varkappa W(2X)$. Then $\sqrt{S_0S_1}<S$ and $S_0(X)\leqq XS_1(X)$, $X\geqq X_1$, according to (2.1.3). From the lemma in Section 2 and the assumption $1/\hat{F}\in\mathscr{A}_1(\gamma;\ W_0, W_1)$ it follows that $1/\hat{F}\in\mathscr{B}_1((P_n), (S_n), S)$. Furthermore, $\bar{S}_1\leqq S$ and S is regular in the sense introduced in 4.3 in [6] according to the regularity conditions imposed on W_0 and W_1. Thus, the conditions of Theorem 3 in [6] are satisfied. By applying this theorem we obtain (3.2.23). Since $\tau\leqq 5\varkappa t$ this proves (3.2.7).

Proof of Theorem 3. Let $P_0=1$, $P_n=(n-1)!\ \gamma^{1-n}$, $n=1, 2, \ldots$. Then (3.2.22) is satisfied and $h_P(x)=1+xe^{\gamma x}$. Thus $\varrho\in R[(1+\theta x)^c h_P(\theta x)]$ according to the assumption (3.2.8). Let $S_0(X)=2W_0(2X)$, $S_n(X)=5W_1(2X)$, $n=1, 2, \ldots$, and $S(X)=5W(5X)$. Then $\sqrt{S_0S_1}<S$ and $S_0(X)\leqq XS_1(X)$, $X\geqq X_1$. The assumption (3.2.9) implies that $W_1(X)=o(W(X))$, $X\to\infty$, and therefore $\bar{S}_1(X)\leqq S(X)$, $X\geqq X_2$. From the lemma in Section 2.2 and the assumption $1/\hat{F}\in\mathscr{A}_1(\gamma;\ W_0, W_1)$ it follows that $1/\hat{F}\in\mathscr{B}_1((P_n), (S_n), S)$. Since $\tau=5t$ the assumption (3.2.9) yields

$$\text{(3.2.24)} \qquad \bar{S}_1\left(\frac{1}{\tau(x)}\right) x^{c+3/2} \exp(x\beta\theta^2 \log\theta) \leqq S\left(\frac{1}{\tau(x)}\right), \quad x \geqq x_0.$$

If the inequality (3.2.24) were satisfied with $x^{c+3/2}$ replaced by x^{c+2} then Theorem 4 in [6] could be applied and (3.2.23) would follow with $C_0=C_0(\gamma, c)$ and $C=C(\gamma, c, \theta)$. To obtain (3.2.23) under the assumption (3.2.24) we proceed as follows. Let

$$p(x) = \sup_n \frac{x^n}{P_n}, \quad x \geqq 0.$$

Theorem 4 in [6] was proved for a large class of sequences (P_n) satisfying the inequality $h_P(x) \leqq C(P)(1+x)p(x)$, $x \geqq 0$. For the sequence chosen in this proof it is easy to verify the stronger inequality

$$h_P(x) \leqq C(\gamma)\big(1+\sqrt{x}\big)p(x), \quad x \geqq 0. \tag{3.2.25}$$

By taking into account the improvements obtained in Lemma 3 in [6] and hence in Theorem 4 in [6] by using the inequality (3.2.25) the result (3.2.23) follows. Since $\tau=5t$ this proves Theorem 3.

3.3. $1/\hat{F}$ *analytic in a domain above the real axis which tapers off at infinity*

Let us now consider the case when $1/\hat{F}$ is analytic in a domain D_γ of the type (2.1.1) and $\gamma(\xi)\to 0$, $\xi\to\infty$. Proceeding as in 5.4 in [6] we introduce an auxiliary sequence $(M_n)_0^\infty$ such that the sequence $P_n=n!\,M_n$, $n=0, 1, 2, \ldots$ satisfies the conditions introduced in 2.1 in [6] and is regular in the sense introduced in 4.3 in [6]. To this end it suffices to choose (M_n) such that $M_0=1$, $M_n^{1/n}\nearrow$, $n\geqq 1$, $M_n^{1/n}\to\infty$, $n\to\infty$, $(n!\,M_n)_0^\infty$ is logarithmically convex and $\lim_{n\to\infty}(\log n)^{-1}\log(M_{n+1}/M_n)$ exists, finite or infinite.

Let m be the function defined by

$$m(x) = \sup_n \frac{x^n}{M_n}, \quad x \geqq 0. \tag{3.3.1}$$

Then, for $\xi \geqq 0$,

$$\gamma(\xi)^{-n} \leqq M_n m\big(\gamma(\xi)^{-1}\big), \quad n = 0, 1, 2, \ldots. \tag{3.3.2}$$

Let us first suppose that $g=1/\hat{F}\in\mathscr{A}_0(\gamma; W_0)$. Choose X, $X\geqq\max\big(X_0, 2\gamma(0)\big)$, and introduce $f=f_X$ and $k=k_X$ as in the lemma in Section 2. By combining (2.2.4) and (3.3.2) we get

$$M_2\{f^{(n)}; -X, X\} \leqq 2n!\,M_n W_0(2X)m\big(\gamma(2X)^{-1}\big), \quad n=0, 1, 2, \ldots. \tag{3.3.3}$$

Let

$$S^*(X) = 2W_0(2X)m\big(\gamma(2X)^{-1}\big) \tag{3.3.4}$$

and

$$P_n = n!\,M_n, \quad n = 0, 1, 2, \ldots \tag{3.3.5}$$

and let h_P be defined by (3.2.17). It follows from the lemma and (3.3.3) that

$$1/\hat{F} \in \mathscr{B}_2((P_n), S^*) \tag{3.3.6}$$

and the theorems in 4.3 in [6] may be applied.

In certain cases it is possible to obtain sharp results also with this method. Let us suppose that (M_n) can be chosen so that, for some $\mu > 2$,

$$2W_0(2X)m(\gamma(2X)^{-1}) \leqq W_0(\mu X), \quad X \geqq X_1. \tag{3.3.7}$$

Let

$$S(X) = W_0(\mu X).$$

Then $S^*(X) \leqq S(X)$, $X \geqq X_1$ and (3.3.6) yields that $1/\hat{F} \in \mathscr{B}_2((P_n), S)$.

If $\Phi(x) + Kx \nearrow$, $x \geqq x_0$ for some $K > 0$, $|\Phi * F(x)| \leqq \varrho(x)$, $x \geqq x_0$ and $\psi = \Phi * F$ satisfies $M_2\{\psi; x, \infty\} \leqq \varrho(x)$, $x \geqq x_0$, then Theorems 3' or 3 in [6], modified according to the remark in 4.4 in [6], may be applied and yield that $\Phi(x) = O(\tau(x))$, $x \to \infty$, where τ is defined by (3.2.20). Since $\tau < \mu t_0$ we thus obtain a result of the same form as in Theorem 1, namely

$$\Phi(x) = O\left(1/U_0^{-1}\left(\frac{1}{\varrho(x)}\right)\right), \quad x \to \infty \tag{3.3.8}$$

where U_0 is defined by (3.2.1).

In Theorem 1 the function γ was constant, and if $\varkappa$ is constant, $0 < \varkappa < \gamma$, then (3.3.8) holds true for $\varrho \in R[e^{\varkappa x}]$. In the present case when $\gamma(\xi) \to 0$, $\xi \to \infty$, the result (3.3.8) is obtained only for a smaller class of functions ϱ. For instance, if θ is constant, $0 < \theta < 1$, and $P_n = n!\, M_n$ satisfies (3.2.22), then (3.3.8) holds true for $\varrho \in R[h_P(\theta x)]$.

To illustrate the method we shall prove the following theorem in which W_0 is chosen as the exponential function. The result of the theorem is best possible in the sense that (3.3.9) cannot be replaced by $\Phi(x) = O(\delta(x)/\log(1/\varrho(x)))$, $x \to \infty$, for any function δ such that $\delta(x) \to 0$, $x \to \infty$. This follows from Ganelius' theorem in the same way as the corresponding result for Theorem 1 since $W_0(X) = \exp(\beta X)$ satisfies (3.2.11), (3.2.14) and (3.2.15).

Theorem 4. *Let K, α, β and s, $s < 2$, denote positive constants. Let Φ be a bounded function on $\mathbf{R}$ such that $\Phi(x) + Kx \nearrow$, $x \geqq x_0$. Let $F \in L^1(\mathbf{R})$ and $|\Phi * F(x)| \leqq \varrho(x)$, $x \geqq x_0$, where $\varrho \in L^s(x_0, \infty)$. Suppose further that $1/\hat{F} \in \mathscr{A}_0(\gamma; W_0)$ where $\gamma(\xi) = (\log \xi)^{-\alpha}$, $\xi \geqq \xi_0$, and $W_0(X) = \exp(\beta X)$. If $\varrho \in R[\exp(Bx/(\log(x+e))^\alpha)]$ for some $B > 0$ then*

$$\Phi(x) = O\left(1/\log\left(\frac{1}{\varrho(x)}\right)\right), \quad x \to \infty. \tag{3.3.9}$$

Proof of Theorem 4. Let $M_n = (\log(n+e))^{\alpha n}$, $n = 0, 1, 2, \ldots$, and let m be defined by (3.3.1). It is easy to see that $\log m(x) = o(\exp(x^{1/\alpha}))$, $x \to \infty$. Therefore (3.3.7) holds true for $\mu=3$ and it follows that $1/\hat{F}\in\mathscr{B}_2((P_n), S)$ where $S(X)=\exp(3\beta X)$ and $P_n=n!\,M_n$, $n=0, 1, 2, \ldots$. The sequence (P_n) satisfies (3.2.22). Let h_P be defined by (3.2.17). It is easy to verify that $h_P(x)>\exp\left(x(\log(x+e))^{-\alpha}\right)$, $x\geqq x_1$. Let $\varkappa=\frac{1}{2}\min(1, B^{-1})$ and $\varrho^*=\varrho^{\varkappa(1-s/2)}$. Then $\varrho^*\in R[bh_P(x/2)]$ for some $b>1$ and $\psi=\Phi*F$ satisfies $M_2\{\psi; x, \infty\}\leqq M_2\{\varrho; x, \infty\}\leqq\varrho^*(x)$, $x\geqq x_1$. Theorem 3 in [6], modified according to the remark in 4.4 in [6], may be applied with ϱ replaced by ϱ^*. Thus we get

$$\Phi(x) = O\left(1/\log\left(\frac{1}{\varrho^*(x)}\right)\right), \quad x\to\infty.$$

Since $\varrho^*=\varrho^{\varkappa(1-s/2)}$ this proves (3.3.9).

Let us now suppose that $g=1/\hat{F}\in\mathscr{A}_1(\gamma; W_0, W_1)$ and $\gamma(\xi)\to 0$, $\xi\to\infty$. In some cases when $W_1(X)/W_0(X)$ tends to zero in an appropriate way as $X\to\infty$ and $\gamma(\xi)$ does not tend to zero too fast as $\xi\to\infty$, we may obtain sharp results even when W_0 is a polynomial and thus (3.3.7) cannot be satisfied. Let us suppose that (M_n) may be chosen so that, for some $\lambda\geqq 1$

$$\varlimsup_{X\to\infty} W_1(X)m\left(\gamma(X)^{-1}\right)W(X)^{-1} < \lambda. \tag{3.3.10}$$

If X is large enough and $f=f_X$ denotes the function introduced in the lemma in Section 2 then, by (2.2.7), (3.3.2) and (3.3.10)

$$M_2\{f^{(n)}; -X, X\} \leqq 5\lambda(n-1)!\,M_{n-1}W(2X), \quad n=2, 3, \ldots. \tag{3.3.11}$$

Let

$$S_0(X) = 2W_0(2X), \quad S_1(X) = 5W_1(2X), \quad S_n(X) = S(X) = 5\lambda W(2X), \quad n=2, 3, \ldots, \tag{3.3.12}$$

and

$$P_0 = 1, \quad P_n = (n-1)!\,M_{n-1}, \quad n=1, 2, \ldots. \tag{3.3.13}$$

Then $\sqrt{S_0S_1}<S$, $S_0(X)\leqq XS_1(X)$, $X\geqq X_1$ and $\bar{S}_1(X)=S(X)$, $X\geqq X_1$. From the lemma and (3.3.11) it follows that $1/\hat{F}\in\mathscr{B}_1((P_n), (S_n), S)$. If $|\Phi*F(x)|\leqq\varrho(x)$, $x\geqq x_0$ and $\Phi(x)+Kx\nearrow$, $x\geqq x_0$ for some $K>0$, then Theorems 3' or 3 in [6] may be applied and yield a result of the same form as in Theorem 2, namely

$$\Phi(x) = O\left(1/U^{-1}\left(\frac{1}{\varrho(x)}\right)\right), \quad x\to\infty, \tag{3.3.14}$$

where U is defined by (3.2.1). In Theorem 2, $\gamma\equiv\gamma_0$ and the result (3.3.14) holds true for $\varrho\in R[e^{\varkappa x}]$ if $\varkappa$ is constant, $0<\varkappa<\gamma_0$. In the present case where $\gamma(\xi)\to 0$, $\xi\to\infty$, (3.3.14) is obtained only for a smaller class of functions ϱ which cannot contain the class $R[e^{\varkappa x}]$ for any $\varkappa>0$. If W is majorized by a polynomial then such a restriction on the class of functions ϱ for which (3.3.14) holds true is necessary. This is a consequence of the following theorem ([5], Th. 6, p. 347).

Theorem. *Let δ, $\varkappa$ and α, $\alpha<1$, be positive constants and let* $(1+|x|)^{\delta+1/2}F(x) \in L^1(\mathbf{R})$. *If* $|\Phi * F(x)| \leqq \varrho(x)$, $x \geqq x_0$, *implies that* $\Phi(x)=O(\varrho(x)^\alpha)$, $x\to\infty$, *for every bounded function Φ satisfying the Tauberian condition* (3.1.4) *and for every* $\varrho\in R[e^{\varkappa x}]$ *then* $1/\hat{F}(\xi)$, $\xi\in\mathbf{R}$, *are continuous boundary values of a function* $g(\zeta)$, $\zeta=\xi+i\eta$, *analytic in the strip* $0<\eta<\alpha\varkappa$.

If W is dominated by a polynomial then $1/U^{-1}(1/\varrho(x))=O(\varrho(x)^\alpha)$, $x\to\infty$ for some α, $0<\alpha<1$, and from the above theorem it follows that it is impossible to obtain the estimate (3.3.14) for $\varrho\in R[e^{\varkappa x}]$ for any $\varkappa>0$ if the conditions for $1/\hat{F}$ are imposed only in a domain D_γ which tapers off at infinity.

The method described above will be used to prove Theorem 5 below. The L^2-conditions for $1/\hat{F}$ and its derivative are now replaced by an O-condition in order to include the case when the assumption (3.3.15) holds true with $0<a\leqq 1/2$. If $a>1/2$ then the assumption (3.3.15) may be replaced by the corresponding L^2-condition and if $0<a\leqq 1/2$ it may be replaced by a corresponding L^s-condition, $0<s<$ $<1/(1-a)$.

The result of the theorem is best possible in the sense that (3.3.16) cannot be replaced by $\Phi(x)=O(\delta(x)\varrho(x)^{1/(a+1)})$, $x\to\infty$, for any function δ such that $\delta(x)\to 0$, $x\to\infty$. This follows from Ganelius' theorem in the same way as the corresponding result for Theorem 2.

Theorem 5. *Let K, a and α denote positive constants. Let Φ be a bounded function on* $\mathbf{R}$ *such that* $\Phi(x)+Kx\nearrow$, $x\geqq x_0$, *let* $F\in L^1(\mathbf{R})$ *and* $|\Phi * F(x)|\leqq\varrho(x)$, $x\geqq x_0$. *Suppose that* $1/\hat{F}(\xi)$, $\xi\in\mathbf{R}$ *are boundary values of a function* $g(\zeta)$, $\zeta=\xi+i\eta$, *analytic in the domain* $D_\gamma=\{\zeta\,|\,0<\eta<\gamma(\xi)\}$ *and such that*

$$(1+|\zeta|)^{1-a}g'(\zeta) \text{ is bounded in } D_\gamma. \tag{3.3.15}$$

If $\gamma(\xi)=(2\alpha/\log\xi)^\alpha$, $\xi\geqq\xi_0$, *and* $\varrho\in R[\exp(Bx^{1/(\alpha+1)})]$ *for some* B, $0<B<\alpha+1$, *then*

$$\Phi(x) = O(\varrho(x)^{1/(a+1)}), \quad x\to\infty. \tag{3.3.16}$$

Proof of Theorem 5. Let us first consider the case $a>1/2$. The assumptions on $\hat{F}$ imply that $1/\hat{F}\in\mathscr{A}_1(\gamma;W_0,W_1)$ where, apart from constant factors, $W_0(X)=$ $=X^{a+1/2}$, $W_1(X)=X^{a-1/2}$ and $W(X)=X^a$. Let $M_0=1$, $M_n=n^{n\alpha}e^{-n\alpha}$, $n=1,2,\ldots$ and let m be defined by (3.3.1). It is easy to see that $\log m(x)\leqq\alpha x^{1/\alpha}$, $x\geqq 0$. Therefore $m(\gamma(X)^{-1})\leqq X^{1/2}$, $X\geqq X_1$. Since $W_1(X)=O(X^{-1/2}W(X))$, $X\to\infty$, (3.3.10) is satisfied for some $\lambda\geqq 1$ and (3.3.11) follows. Therefore $1/\hat{F}\in\mathscr{B}_1((P_n),(S_n),S)$, where (S_n) and S are defined by (3.3.12) and (P_n) by (3.3.13). The sequence (P_n) satisfies (3.2.22) and $\tau(x)=\text{const}\ \varrho(x)^{1/(a+1)}$. It is easy to see that

$$h_P(x)\geqq\exp((1+\alpha)x^{1/(1+\alpha)}),\quad x\geqq x_1.$$

Let $\theta=(B/(1+\alpha))^{1+\alpha}$. Then $0<\theta<1$ and $\varrho\in R[bh_P(\theta x)]$ for some $b\geqq 1$. The result thus follows from Theorem 3 in [6].

The case $0<a\leqq 1/2$ is treated similarly by using L^s-estimates instead of L^2-estimates and by the aid of the remarks to the lemma in Section 2 and to Lemma 1 in 2.2 in [6]. The details are omitted.

In Section 5 of the paper [6] some examples were given under the assumption that $1/\hat{F}$ is analytic in a domain including the real axis. Corresponding results when $1/\hat{F}$ is analytic in a domain D_γ of the type (2.1.1) were stated without proof. These results now either follow directly from the above theorems or are easily proved by using the same methods.

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Received September 23, 1974

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