A p-extremal length and p-capacity equality

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1. Introduction

Let G be a domain in the compactified euclidean n-space $\bar{R}^n = R^n \cup \{\infty\}$, let E and F be disjoint non-empty compact sets in the closure of G. We associate two numbers with this geometric configuration as follows. Let $M_p(E, F, G)$ be the p-modulus (reciprocal of the p-extremal length) of the family of curves connecting E and F in G. Let $\operatorname{cap}_p(E, F, G)$ be the p-capacity of E and F relative to G, defined as the infimum of the numbers $\int_G |\nabla u(x)|^p \, dm(x)$ where u is an ACL function in G with boundary values 0 and 1 on E and F, respectively. We show in this paper that $\operatorname{cap}_p(E, F, G) = M_p(E, F, G)$ whenever E and F do not intersect ∂G. This generalizes Ziemer's [7] result where he makes the assumption that either E or F contains the complement of an open n-ball.

We also obtain a continuity theorem (Theorem 5.9) for the p-modulus and a theorem (Theorem 4.15) on the kinds of densities that can be used in computing the p-modulus.

2. Notation

For $n \geq 2$ we denote by $\bar{R}^n$ the one point compactification of R^n, euclidean n-space: $\bar{R}^n = R^n \cup \{\infty\}$. All topological considerations in this paper refer to the metric space $(\bar{R}^n, q)$ where q is the chordal metric on $\bar{R}^n$ defined by stereographic projection. If $A \subset \bar{R}^n$ then $\bar{A}$ and ∂A denote the closure and boundary of A, respectively. If $b \in \bar{R}^n$ and $B \subset \bar{R}^n$ then $q(b, B)$ denotes the chordal distance of b from B.

If $x \in R^n$ we let $|x|$ denote the usual euclidean norm of x. $B^n(x, r)$ denotes the open n-ball with center x and radius r. We write $B^n(1) = B^n(0, 1)$. If $x \in R^n$ and $A \subset R^n$ we let $d(x, A)$ denote the euclidean distance of x from A.

Lebesgue n-measure on R^n is denoted by m_n or by m if there is no chance for confusion. We let $\Omega_n = m_n(B^n(1))$.

3. The p-modulus and p-capacity

3.1. *Definition.* Let Γ be a collection of curves in $\bar{R}^n$. We let $\mathcal{F}(\Gamma)$ denote the set of Borel functions $\varrho: R^n \to [0, \infty]$ satisfying the condition that for every locally rectifiable $\gamma \in \Gamma$ we have $\int_\gamma \varrho ds \geqq 1$. $\mathcal{F}(\Gamma)$ is called the set of *admissible densities* for Γ. For $p \in (1, \infty)$ the *p-modulus of* Γ, denoted by $M_p(\Gamma)$, is defined as

$$M_p(\Gamma) = \inf \int_{R^n} \varrho^p \, dm_n$$

where the infimum is taken over all $\varrho \in \mathcal{F}(\Gamma)$. For the basic facts about the p-modulus, see [5, Chap. 1]. The p-extremal length of Γ is defined as the reciprocal of the p-modulus of Γ.

3.2. *Definition.* Let G be a domain in $\bar{R}^n$ and let E and F be compact, disjoint, non-empty sets in $\bar{G}$. Let $\Gamma(E, F, G)$ denote the set of curves connecting E and F in G. More precisely, if $\gamma \in \Gamma(E, F, G)$ then $\gamma: I \to G$ is a continuous mapping where I is an open interval and $\overline{\gamma(I)} \cap E$ and $\overline{\gamma(I)} \cap F$ are both non-empty. We write $M_p(E, F, G)$ for the p-modulus of $\Gamma(E, F, G)$. Let $\mathcal{A}(E, F, G)$ denote the set of real valued functions u such that (1) u is continuous on $E \cup F \cup G$, (2) $u(x) = 0$ if $x \in E$ and $u(x) = 1$ if $x \in F$, and (3) u restricted to $G - \{\infty\}$ is ACL. For the definition and basic facts about ACL functions see [5, Chap. 3]. If $p \in (1, \infty)$ we define the *p-capacity of E and F relative to G*, denoted by $\operatorname{cap}_p(E, F, G)$, by

$$\operatorname{cap}_p(E, F, G) = \inf \int_G |\nabla u|^p \, dm_n$$

where the infimum is taken over all $u \in \mathcal{A}(E, F, G)$.

The p-capacity has the following continuity property.

3.3. Theorem. *Let $E_1 \supset E_2 \supset \dots$ and $F_1 \supset F_2 \supset \dots$ be disjoint sequences of non-empty compact sets in the closure of a domain G. Let $E = \bigcap_{i=1}^\infty E_i$, $F = \bigcap_{i=1}^\infty F_i$. Then*

$$\lim_{i \to \infty} \operatorname{cap}_p(E_i, F_i, G) = \operatorname{cap}_p(E, F, G).$$

Proof. Since $\mathcal{A}(E_i, F_i, G) \subset \mathcal{A}(E_{i+1}, F_{i+1}, G) \subset \mathcal{A}(E, F, G)$ for all i, it follows that $\operatorname{cap}_p(E_i, F_i, G)$ is monotone decreasing in i and therefore

$$\lim_{i \to \infty} \operatorname{cap}_p(E_i, F_i, G) \geqq \operatorname{cap}_p(E, F, G).$$

For the reverse inequality, choose $u \in \mathcal{A}(E, F, G)$ and $\varepsilon \in (0, 1/2)$. Define $f: (-\infty, \infty) \to [0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leqq \varepsilon \\ (1-2\varepsilon)^{-1}(x-1+\varepsilon)+1 & \text{if } \varepsilon < x < 1-\varepsilon \\ 1 & \text{if } x \geqq 1-\varepsilon. \end{cases}$$

Let $u'=f\circ u$. Since f is Lipschitz continuous on $(-\infty, \infty)$ with Lipschitz constant $(1-2\varepsilon)^{-1}$, it follows that u' is ACL on $G-\{\infty\}$ and $|\nabla u'|\leqq(1-2\varepsilon)^{-1}|\nabla u|$ a.e. in G.

Let A and B be open sets in $\bar{R}^n$ such that $\{x\in E\cup F\cup G: u(x)<\varepsilon\}=(E\cup F\cup G)\cap A$ and $\{x\in E\cup F\cup G: u(x)>1-\varepsilon\}=(E\cup F\cup G)\cap B$. For large i we have $E_i\subset A$ and $F_i\subset B$ and, for such i, we can extend u' continuously to $E_i\cup F_i\cup G$ by setting $u'=0$ on $\partial G\cap(E_i-E)$ and $u'=1$ on $\partial G\cap(F_i-F)$. Therefore $u'\in\mathscr{A}(E_i, F_i, G)$ for large i. This implies that for large i we have

$$\operatorname{cap}_p(E_i, F_i, G) \leqq \int_G |\nabla u'|^p\, dm \leqq \frac{1}{(1-2\varepsilon)^p}\int_G |\nabla u|^p\, dm.$$

Hence

$$\lim_{i\to\infty} \operatorname{cap}_p(E_i, F_i, G) \leqq \frac{1}{(1-2\varepsilon)^p}\int_G |\nabla u|^p\, dm.$$

Since $u\in\mathscr{A}(E, F, G)$ and $\varepsilon\in(0, 1/2)$ are arbitrary, we get the reverse inequality, as desired.

4. Complete Families of Densities

4.1. *Definition.* Let Γ be a collection of curves in $\bar{R}^n$. Let $\mathscr{B}\subset\mathscr{F}(\Gamma)$. We say $\mathscr{B}$ is p-complete if

$$M_p(\Gamma) = \inf\int_{R^n} \varrho^p\, dm$$

where the infimum is taken over all $\varrho\in\mathscr{B}$.

4.2. *Example.* Let $\mathscr{B}\subset\mathscr{F}(\Gamma)$ be the collection of $\varrho\in\mathscr{F}(\Gamma)$ such that ϱ is lower semicontinuous. It follows from the Vitali-Caratheodory theorem [4, Thm. 2.24] that $\mathscr{B}$ is p-complete for all $p\in(1, \infty)$.

4.3. Lemma. *Let $\varphi: R^n\to[0, \infty]$ be a Borel function and assume $\varphi\in L^p(R^n)$, $p\in(1, \infty)$. Let $r: R^n\to[0, \infty]$ satisfy $|r(x_2)-r(x_1)|\leqq|x_2-x_1|$ for all $x_1, x_2\in R^n$. Define $T_{\varphi,r}: R^n\to[0, \infty]$ by*

$$T_{\varphi,r}(x) = \frac{1}{\Omega_n}\int_{B^n(1)} \varphi(x+r(x)y)\, dm_n(y).$$

Then $T_{\varphi,r}$ has the following properties.

(1) *If $r(x_0)>0$ then*

$$T_{\varphi,r}(x_0) = \frac{1}{\Omega_n r(x_0)^n}\int_{B^n(x_0, r(x_0))} \varphi(y)\, dm(y) < \infty.$$

(2) *If φ is lower semicontinuous then so is $T_{\varphi,r}$.*

(3) *If $r(x_0)>0$ then $T_{\varphi,r}$ is continuous at x_0.*

(4) *If* φ *is finite and continuous on a domain* G *in* R^n *and if* $0 \leqq r(x) < d(x, R^n - G)$ *then* $T_{\varphi,r}$ *is finite and continuous on* G.

(5) $|T_{\varphi,r}(x) r(x)^{n/p}| \leqq C$ *for some constant* $C \in [0, \infty)$ *and all* $x \in R^n$. *The constant* C *depends on* φ.

(6) *Let* $k = \sup |r(x_2) - r(x_1)| \, |x_2 - x_1|^{-1}$ *where the supremum is taken over all* $x_1, x_2 \in R^n$, $x_1 \neq x_2$. *Then* $\|T_{\varphi,r}\|_p \leqq (1-k)^{-n/p} \|\varphi\|_p$ *where* $\| \ \|_p$ *is the usual* $L^p(R^n)$ *norm and the right hand side of the inequality is infinite in case* $k=1$.

Proof. (1) follows from the change of variables $y' = x_0 + r(x_0) y$ and Hölders inequality. To prove (2), let $x_0 \in R^n$ be arbitrary and suppose $\{x_j\}_{j=1}^{\infty}$ is a sequence in R^n tending to x_0. Fatou's lemma and the lower semicontinuity of φ imply

$$\liminf_{j \to \infty} T_{\varphi,r}(x_j) = \liminf_{j \to \infty} \frac{1}{\Omega_n} \int_{B^n(1)} \varphi(x_j + r(x_j) y) \, dm(y)$$

$$\geqq \frac{1}{\Omega_n} \int_{B^n(1)} \liminf_{j \to \infty} \varphi(x_j + r(x_j) y) \, dm(y)$$

$$\geqq \frac{1}{\Omega_n} \int_{B^n(1)} \varphi(x_0 + r(x_0) y) \, dm(y) = T_{\varphi,r}(x_0).$$

This shows that $T_{\varphi,r}$ is lower semicontinuous. To prove (3), we observe that since r is continuous, $r(x) > 0$ for all x in some neighborhood of x_0 and therefore, by (1),

$$T_{\varphi,r}(x) = \frac{1}{\Omega_n r(x)^n} \int_{B^n(x, r(x))} \varphi(y) \, dm(y)$$

for all x in some neighborhood of x_0. The right hand side of the above formula is continuous in x and therefore, $T_{\varphi,r}$ is continuous at x_0. We proceed to prove (4). We observe that if $x \in G$ then $x + r(x) y \in G$ for any $y \in R^n$ with $|y| \leqq 1$. Fix $x_0 \in G$ and let B be a closed ball with center x_0 and lying in G. Then $B' = \{x' : x' = x + r(x) y, x \in B, |y| \leqq 1\}$ is a compact subset of G. Since φ is uniformly continuous on B', given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\varphi(x_2') - \varphi(x_1')| < \varepsilon$ if $x_1', x_2' \in B'$ and $|x_2' - x_1'| < \delta$. Let $x_1 \in B$ with $|x_1 - x_0| < \delta/2$. Then $|(x_1 + r(x_1) y) - (x_0 + r(x_0) y)| < \delta$ for any $|y| \leqq 1$. Hence,

$$|T_{\varphi,r}(x_1) - T_{\varphi,r}(x_0)| \leqq \frac{1}{\Omega_n} \int_{B^n(1)} |\varphi(x_1 + r(x_1) y) - \varphi(x_0 + r(x_0) y)| \, dm(y) < \varepsilon.$$

Hence, $T_{\varphi,r}$ is continuous on G. To prove (5) we need only consider $x \in R^n$ such that $r(x) > 0$. For such x we have

$$T_{\varphi,r}(x) = \frac{1}{\Omega_n r(x)^n} \int_{B^n(x, r(x))} \varphi(y) \, dm(y).$$

Applying Hölder's inequality with exponents p and $p/(p-1)$, we get

$$T_{\varphi,r}(x) \leqq \frac{1}{\Omega_n r(x)^n}\left[\int_{B^n(x,r(x))} \varphi^p(y)\,dm(y)\right]^{1/p} [\Omega_n r(x)^n]^{(p-p)/p}.$$

Hence,

$$T_{\varphi,r}(x)r(x)^{n/p} \leqq C = \Omega_n^{-1/p}\left[\int_{R^n} \varphi^p\,dm\right]^{1/p} < \infty,$$

as desired. We proceed to prove (6).

$$\|T_{\varphi,r}\|_p^p = \int_{R^n} T_{\varphi,r}^p(x)\,dm(x) = \int_{R^n}\left[\frac{1}{\Omega_n}\int_{B^n(1)} \varphi(x+r(x)y)\,dm(y)\right]^p dm(x).$$

After applying Hölder's inequality to the inner integral and simplifying, we get

$$\|T_{\varphi,r}\|_p^p \leqq \frac{1}{\Omega_n}\int_{R^n}\int_{B^n(1)} \varphi^p(x+r(x)y)\,dm(y)\,dm(x).$$

Interchanging the order of integration gives

$$\|T_{\varphi,r}\|_p^p \leqq \frac{1}{\Omega_n}\int_{B^n(1)}\int_{R^n} \varphi^p(x+r(x)y)\,dm(x)\,dm(y). \tag{4.4}$$

Define, for $y\in B^n(1)$, $\theta_y: R^n\to R^n$ by $\theta_y(x)=x+r(x)y$. It easily follows that θ_y is injective and hence, by a theorem in topology, $\theta_y(R^n)$ is a domain. Since θ_y is Lipschitz continuous, it follows [6, Thm. 1, Cor. 2] that the change of variables formula for multiple integrals holds with θ_y as the mapping function. Therefore

$$\int_{\theta_y(R^n)} \varphi^p(x)\,dm(x) = \int_{R^n} \varphi^p\circ\theta_y(x)\,\mu_y'(x)\,dm(x) \tag{4.5}$$

where μ_y' is the volume derivative [5, Def. 24. 1] of the homeomorphism θ_y. Since

$$\mu_y'(x) = \lim_{r\to 0} \frac{m(\theta_y(\overline{B^n(x,r)}))}{\Omega_n r^n} \quad \text{a.e.} \quad x,$$

the estimates

$$m(\theta_y(\overline{B^n(x,r)})) \geqq \Omega_n\{\inf_{|x'-x'|=r} |\theta_y(x')-\theta_y(x)|\}^n$$

and

$$|\theta_y(x')-\theta_y(x)| \geqq (1-k)|x'-x|$$

yield $\mu_y'(x)\geqq(1-k)^n$ a.e. x in R^n. This result and (4.4) and (4.5) give

$$\|T_{\varphi,r}\|_p^p \leqq \frac{1}{\Omega_n(1-k)^n}\int_{B^n(1)}\int_{R^n} \varphi^p(x)\,dm(x)\,dm(y) = (1-k)^{-n}\|\varphi\|_p^p,$$

as desired.

For the remainder of this paper, G will denote a domain in $\overline{R}^n$, E and F will be compact, disjoint non-empty sets in $\overline{G}$. We write $\Gamma=\Gamma(E, F, G)$. We let $d: R^n\to[0,\infty)$ be the function defined by $d(x)=d(x, ((\overline{R}^n-G)\cup E\cup F)-\{\infty\})$ and we let l.s.c.(R^n) be the extended real valued lower semicontinuous functions defined on R^n.

4.6. Lemma. *Let $\mathscr{A}\subset\mathscr{J}(\Gamma)$ be the set of $\varrho\in\mathscr{J}(\Gamma)$ satisfying* (1) $\varrho\in$l.s.c.$(R^n)\cap L^p(R^n)$, (2) *ϱ is continuous on $G-(E\cup F\cup\{\infty\})$, and* (3) *$\varrho(x)\cdot d(x)^{n/p}$ is bounded above for $x\in R^n$. Then $\mathscr{A}$ is a p-complete family.*

Proof. It suffices to prove that $M=\inf\int_{R^n}\varrho^p(x)\,dm(x)\leqq M_p(\Gamma)$ where the infimum is taken over all $\varrho\in\mathscr{A}$. Choose $\varrho\in\mathscr{J}(\Gamma)\cap L^p(R^n)\cap$l.s.c.$(R^n)$. Let $\varepsilon\in(0,1)$ and let $g=T_{\varrho,\varepsilon d}$. Suppose $\gamma\in\Gamma$ is locally rectifiable. We may assume, by reparametrizing γ, that $\gamma:(a,b)\to G$ where $a,b\in[-\infty,\infty]$ and that the length of $\gamma|[t_1,t_2]$ is equal to t_2-t_1 for all $t_1,t_2\in(a,b)$. Note that γ restricted to closed subintervals of (a,b) is absolutely continuous.

Let $\gamma_y:(a,b)\to G$, $y\in B^n(1)$, be the curve defined by $\gamma_y(t)=\gamma(t)+\varepsilon d(\gamma(t))y$. Choose $e\in\overline{\gamma(a,b)}\cap E$. Let $t_j\in(a,b)$, $j=1,2,\dots$, be such that $\gamma(t_j)\to e$ as $j\to\infty$. If $e\neq\infty$ then clearly $\gamma_y(t_j)\to e$ as $j\to\infty$. If $e=\infty$ then, for fixed $t'\in(a,b)$, the triangle inequality and the fact that d is Lipschitz continuous with Lipschitz constant 1 imply $|\gamma_y(t_j)-\gamma_y(t')|\geqq(1-\varepsilon)|\gamma(t_j)-\gamma(t')|$ and therefore, $\gamma_y(t_j)\to\infty=e$ as $j\to\infty$. Hence $\overline{\gamma_y(a,b)}\cap E\neq\emptyset$. Similarly, $\overline{\gamma_y(a,b)}\cap F\neq\emptyset$. Therefore $\gamma_y\in\Gamma$. Also, γ_y restricted to closed subintervals of (a,b) is absolutely continuous. An easy estimate shows $|\gamma_y'(t)|\leqq 1+\varepsilon$ a.e. on (a,b).

We have

$$\int_\gamma g\,ds=\int_a^b g(\gamma(t))\,dt=\frac{1}{\Omega_n}\int_a^b\int_{B^n(1)}\varrho\big(\gamma(t)+\varepsilon d(\gamma(t))y\big)\,dm(y)\,dt$$

$$=\frac{1}{\Omega_n}\int_{B^n(1)}\int_a^b\varrho(\gamma_y(t))|\gamma_y'(t)||\gamma_y'(t)|^{-1}\,dt\,dm(y)$$

$$\geqq\frac{1}{(1+\varepsilon)\Omega_n}\int_{B^n(1)}\int_{\gamma_y}\varrho\,ds\,dm(y)\geqq\frac{1}{1+\varepsilon}.$$

This result and lemma 4.3 show $(1+\varepsilon)g\in\mathscr{A}\subset\mathscr{J}(\Gamma)$. Hence,

$$M\leqq(1+\varepsilon)^p\|g\|_p^p=(1+\varepsilon)^p\|T_{\varrho,\varepsilon d}\|_p^p.$$

From lemma 4.3 (6) we get

$$M\leqq\frac{(1+\varepsilon)^p}{(1-\varepsilon)^n}\int_{R^n}\varrho^p(x)\,dm(x).$$

Since $\varepsilon\in(0,1)$ and $\varrho\in\mathscr{J}(\Gamma)\cap L^p(R^n)\cap$l.s.c.$(R^n)$ are arbitrary, we get $M\leqq M_p(\Gamma)$, as desired.

4.7. *Definition.* For $r\in(0,1)$ we define $E(r)=\{x\in\bar{R}^n:q(x,E)\leqq r\}$ and $F(r)=\{x\in\bar{R}^n:q(x,F)\leqq r\}$. Let $\varrho:R^n\to[0,\infty]$ be a Borel function. We define $L(\varrho,r)$ as the infimum of the integrals $\int_\gamma \varrho ds$ where γ is a locally rectifiable curve in G connecting $E(r)$ and $F(r)$. Since $L(\varrho, r)$ is non-decreasing for decreasing r, we can define

$$L(\varrho) = \lim_{r\to 0} L(\varrho, r).$$

4.8. *Note.* We observe that $L(\varrho)\geqq 1$ if and only if for every $\varepsilon\in(0, 1)$ there exists a $\delta\in(0, 1)$ such that $\int_\gamma \varrho\, ds\geqq 1-\varepsilon$ for every locally rectifiable curve γ in G connecting $E(r)$ and $F(r)$ with $r\leqq\delta$.

4.9. Lemma. *Suppose there exists a p-complete family $\mathscr{B}_0\subset\mathscr{F}(\Gamma)$ such that $L(\varrho)\geqq 1$ for every $\varrho\in\mathscr{B}_0$. Then the family $\mathscr{B}\subset\mathscr{F}(\Gamma)$ consisting of all $\varrho\in\mathscr{F}(\Gamma)$ such that (1) $\varrho\in$ l.s.c.$(R^n)\cap L^p(R^n)$ and (2) ϱ is continuous on $G-\{\infty\}$ is p-complete.*

Proof. Let $\mathscr{B}_1$ be the set of $\varrho\in\mathscr{F}(\Gamma)$ such that $\varrho\in$l.s.c.$(R^n)\cap L^p(R^n)$ and $L(\varrho)\geqq 1$. It follows from the Vitali-Caratheodory theorem [4, Thm. 2.24] that $\mathscr{B}_1$ is p-complete.

Let $\varrho\in\mathscr{B}_1$ and $\varepsilon\in(0, 1)$. Let δ be as in 4.8 and choose $\delta'\in(0, 1)$ such that if $x\in E-\{\infty\}$ (resp., $F-\{\infty\}$) and $y\in R^n$, $|x-y|<\delta'$ then $y\in E(\delta)$ (resp., $F(\delta)$). Let $r\colon R^n\to[0, 1]$ be defined by $r(x)=\varepsilon\delta' \min(1, d(x, R^n-G))$. Let $g=T_{\varrho,r}$. Suppose $\gamma\in\Gamma$ is locally rectifiable and assume that $\gamma\colon(a, b)\to G$ is parametrized as in the proof of 4.6. Let $\gamma_y\colon(a, b)\to G, y\in B^n(1)$, be the curve defined by $\gamma_y(t)=\gamma(t)+ +r(\gamma(t))y$. It follows, using the same method as in the proof of 4.6, that γ_y connects $E(\delta)$ and $F(\delta)$. A computation similar to the one in the proof of 4.6 yields

$$\int_\gamma g\, ds \geqq \frac{1}{(1+\varepsilon)\Omega_n}\int_{B^n(1)}\int_{\gamma_y} \varrho\, ds\, dm(y) \geqq \frac{1-\varepsilon}{1+\varepsilon}.$$

The above and lemma 4.3 show $(1+\varepsilon)(1-\varepsilon)^{-1}g\in\mathscr{B}$. Let $M=\inf\int_{R^n}\varrho^p(x)\, dm(x)$, where the infimum is taken over all $\varrho\in\mathscr{B}$. Then, by lemma 4.3,

$$M \leqq \frac{(1+\varepsilon)^p}{(1-\varepsilon)^p}\|g\|_p^p = \frac{(1+\varepsilon)^p}{(1-\varepsilon)^p}\|T_{\varrho,r}\|_p^p \leqq \frac{(1+\varepsilon)^p}{(1-\varepsilon)^p(1-\varepsilon)^n}\|\varrho\|_p^p.$$

Since $\varrho\in\mathscr{B}_1$ and $\varepsilon\in(0, 1)$ are arbitrary and since $\mathscr{B}_1$ is p-complete, it follows from the above that $M\leqq M_p(\Gamma)$. This completes the proof since the reverse inequality is trivial.

4.10. Lemma. *Suppose $(E\cup F)\cap\partial G=\emptyset$. Let $\varrho\colon R^n\to[0, \infty]$ be a Borel function and assume $\varrho|G-(E\cup F\cup\{\infty\})$ is finite valued and continuous. Let $\varepsilon\in(0, \infty)$. Then there exists a locally rectifiable curve $\gamma\in\Gamma$ such that*

$$\int_\gamma \varrho\, ds \leqq L(\varrho)+\varepsilon.$$

Proof. We may assume that $L(\varrho)<\infty$. Let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of positive numbers such that $\sum_{k=1}^\infty \varepsilon_k<\varepsilon/8$. Let $\{r_k\}_{k=1}^\infty$ be a strictly monotone decreasing sequence of positive numbers such that (1) $\operatorname{Lim}_{k\to\infty} r_k=0$ and (2) $E(r_k)\cap F(r_k)=\emptyset$,

$E(r_k)$, $F(r_k)\subset G$, and $\infty\notin\partial E(r_k)$, $\partial F(r_k)$ for $k=1,2,\ldots$. It follows that $\partial E(r_k)\cap E=\emptyset$, $\partial F(r_k)\cap F=\emptyset$ for $k=1,2,\ldots$. Let Γ_k be the curves in G connecting $E(r_k)$ and $F(r_k)$, $k=1,2,\ldots$. Choose $\gamma_k\in\Gamma_k$ such that γ_k is locally rectifiable and

$$\int_{\gamma_k}\varrho\,ds\leqq L(\varrho,r_k)+\frac{\varepsilon}{2}\leqq L(\varrho)+\frac{\varepsilon}{2}. \tag{4.11}$$

Let x_{kj} (resp., y_{kj}), defined for $j<k$, the be last (resp., first) point of γ_k in $E(r_j)$ (resp., $F(r_j)$). We have $x_{kj}\in\partial E(r_j)$ and $y_{kj}\in\partial F(r_j)$. By considering successive subsequences and then a diagonal sequence and then relabeling the sequences, we may assume $x_{kj}\to x_j\in\partial E(r_j)$ and $y_{kj}\to y_j\in\partial F(r_j)$ as $k\to\infty$. Let $V_j\subset G-(E\cup F\cup\{\infty\})$ (resp., $W_j\subset G-(E\cup F\cup\{\infty\})$) be an open euclidean ball with center x_j (resp., y_j) such that $\int\varrho\,ds<\varepsilon_j$ where the integral is taken over any line segment lying in V_j (resp., W_j), $j=1,2,\ldots$. This can be done since ϱ is continuous on $G-(E\cup F\cup\{\infty\})$ and hence, locally bounded there.

Let Ψ_j (resp., Φ_j) be the set of rectifiable curves $\alpha:[a,b]\to G$ such that $\alpha(a)\in V_j$ (resp., $\alpha(a)\in W_j$) and $\alpha(b)\in V_{j-1}$ (resp., $\alpha(b)\in W_{j-1}$), $j=2,3,\ldots$. Let Λ be the set of rectifiable curves $\alpha:[a,b]\to G$ such that $\alpha(a)\in V_1$ and $\alpha(b)\in W_1$. For any positive integer k there exists a curve in the sequence $\{\gamma_i\}_{i=1}^{\infty}$, say $\gamma_{i(k)}$, such that $x_{i(k),j}\in V_j$ and $y_{i(k),j}\in W_j$ for $j=1,2,\ldots,k$. This implies that $\gamma_{i(k)}$ has distinct subcurves in $\Psi_2,\Psi_3,\ldots,\Psi_k$, $\Phi_2,\Phi_3,\ldots,\Phi_k$, Λ. Hence, for every positive integer k we have, using (4.11),

$$\inf_{\gamma\in\Lambda}\int_\gamma\varrho\,ds+\sum_{j=2}^{k}\inf_{\gamma\in\Psi_j}\int_\gamma\varrho\,ds+\sum_{j=2}^{k}\inf_{\gamma\in\Phi_j}\int_\gamma\varrho\,ds\leqq\int_{\gamma_{i(k)}}\varrho\,ds\leqq L(\varrho)+\frac{\varepsilon}{2}.$$

Since k is arbitrary, we get

$$\inf_{\gamma\in\Lambda}\int_\gamma\varrho\,ds+\sum_{j=2}^{\infty}\inf_{\gamma\in\Psi_j}\int_\gamma\varrho\,ds+\sum_{j=2}^{\infty}\inf_{\gamma\in\Phi_j}\int_\gamma\varrho\,ds\leqq L(\varrho)+\frac{\varepsilon}{2}. \tag{4.12}$$

Choose $\theta\in\Lambda$ such that

$$\int_\theta\varrho\,ds<\inf_{\gamma\in\Lambda}\int_\gamma\varrho\,ds+\varepsilon_1. \tag{4.13a}$$

Choose $\tau_j\in\Psi_j$, $\sigma_j\in\Phi_j$, $j=2,3,\ldots$, such that

$$\int_{\tau_j}\varrho\,ds<\inf_{\gamma\in\psi_j}\int_\gamma\varrho\,ds+\varepsilon_j \tag{4.13b}$$

and

$$\int_{\sigma_j}\varrho\,ds<\inf_{\gamma\in\Phi_j}\int_\gamma\varrho\,ds+\varepsilon_j. \tag{4.13c}$$

Let α_j (resp., β_j) be the line segment in V_j (resp., W_j) connecting the endpoints of τ_j and τ_{j+1} (resp., σ_j and σ_{j+1}), $j=2, 3, \ldots$. Let α_1 (resp., β_1) be the line segment in V_1 (resp., W_1) connecting the endpoints of τ_2 and θ (resp., σ_2 and θ). We have

$$\int_{\alpha_j} \varrho\, ds < \varepsilon_j, \quad \int_{\beta_j} \varrho\, ds < \varepsilon_j, \quad j = 1, 2, \ldots. \tag{4.13d}$$

Let $\gamma \in \Gamma$ be the locally rectifiable curve $\gamma = \ldots \tau_3 \alpha_2 \tau_2 \alpha_1 \theta \beta_1 \sigma_2 \beta_2 \sigma_3 \ldots$. We have, by (4.12) and (4.13)

$$\int_\gamma \varrho\, ds = \sum_{j=1}^\infty \int_{\alpha_j} \varrho\, ds + \sum_{j=1}^\infty \int_{\beta_j} \varrho\, ds + \int_\theta \varrho\, ds + \sum_{j=2}^\infty \int_{\tau_j} \varrho\, ds + \sum_{j=2}^\infty \int_{\sigma_j} \varrho\, ds$$

$$\leqq \sum_{j=1}^\infty \varepsilon_j + \sum_{j=1}^\infty \varepsilon_j + \varepsilon_1 + \sum_{j=2}^\infty \varepsilon_j + \sum_{j=2}^\infty \varepsilon_j + L(\varrho) + \frac{\varepsilon}{2} \leqq L(\varrho) + \varepsilon,$$

as desired.

4.14. Lemma. *Suppose $(E \cup F) \cap \partial G = \emptyset$. Let $\mathscr{B} \subset \mathscr{J}(\Gamma)$ be the set of $\varrho \in \mathscr{J}(\Gamma)$ such that (1) $\varrho \in$ l.s.c.$(R^n) \cap L^p(R^n)$ and (2) ϱ is continuous on $G - \{\infty\}$. Then $\mathscr{B}$ is p-complete.*

Proof. Lemma 4.10 shows that $L(\varrho) \geqq 1$ for every ϱ in the p-complete family $\mathscr{A}$ defined in lemma 4.6. Hence, this family $\mathscr{A}$ satisfies the hypotheses of lemma 4.9. Therefore, $\mathscr{B}$ is p-complete.

4.15. Theorem. *Suppose $(E \cup F) \cap \partial G = \emptyset$. Let $\mathscr{C} \subset \mathscr{J}(\Gamma)$ be the set of $\varrho \in \mathscr{J}(\Gamma)$ such that (1) $\varrho \in$ l.s.c.$(R^n) \cap L^p(R^n)$, (2) ϱ is continuous on $G - \{\infty\}$, (3) $\varrho(x) \cdot d(x)^{n/p}$ is bounded above for $x \in R^n$, and (4) $L(\varrho) \geqq 1$. Then $\mathscr{C}$ is a p-complete family.*

Proof. Choose ϱ in the p-complete family $\mathscr{B}$ of lemma 4.14 and let $\varepsilon \in (0, 1)$. Let $g = T_{\varrho, \varepsilon d}$. It follows exactly as in the proof of lemma 4.6 that $\int_\gamma g\, ds \geqq (1+\varepsilon)^{-1}$ for every locally rectifiable curve $\gamma \in \Gamma$. An application of lemma 4.3 and lemma 4.10 shows $(1+\varepsilon) g \in \mathscr{C}$. Let $M = \inf \int_{R^n} \varrho^p(x)\, dm(x)$ where the infimum is taken over all $\varrho \in \mathscr{C}$. We have, by lemma 4.3,

$$M \leqq (1+\varepsilon)^p \|g\|_p^p \leqq \frac{(1+\varepsilon)^p}{(1-\varepsilon)^n} \|\varrho\|_p^p = \frac{(1+\varepsilon)^p}{(1-\varepsilon)^n} \int_{R^n} \varrho^p(x)\, dm(x).$$

Since $\varrho \in \mathscr{B}$ and $\varepsilon \in (0, 1)$ are arbitrary and $\mathscr{B}$ is p-complete, it follows that $M \leqq M_p(\Gamma)$. Since the reverse inequality is trivial, we are done.

4.16. *Comments.* (1) Part 2 of lemma 4.6 was proved independently by Aseev [1], Ohtsuka [3, Thm. 2.8], and the author. Lemma 4.10 is modeled after [3, lemma 2.9].

5. Relations between the p-modulus and p-capacity

5.1 *Definition.* Let $\gamma:[a, b]\to R^n$ be a rectifiable curve in R^n and let $\gamma_0:[0, L]\to R^n$ be the arc length parametrization of γ. Let f be an ACL function defined in a neighborhood of $\gamma([a, b])=\gamma_0([0, L])$. We say f is absolutely continuous on γ if

$$\int_0^t \nabla f\cdot\frac{d\gamma_0}{dt}\,dt = f\circ\gamma_0(t)-f\circ\gamma_0(0)$$

for all $t\in[0, L]$. The integrand is the inner product of $d\gamma_0/dt$ and $\nabla f=$the gradient of f. We use the convention that $\partial f/\partial x_i=0$ at points x where $\partial f/\partial x_i$ is not defined. The above definition differs slightly from [5, Def. 5.2] in that we require a little more than the absolute continuity of $f\circ\gamma_0$.

5.2. Lemma. $M_p(\Gamma)\leqq \operatorname{cap}_p(E, F, G)$.

Proof. Let $u\in\mathcal{A}(E, F, G)\cap L^p(G)$. Let Γ_0 be the locally rectifiable curves $\gamma\in\Gamma$ for which u is absolutely continuous on every rectifiable subcurve of γ. Define $\varrho: R^n\to[0, \infty]$ by

$$\varrho(x) = \begin{cases} |\nabla u(x)| & \text{if } x\in G-\{\infty\} \\ 0 & \text{if } x\in R^n-G.\end{cases}$$

Suppose $\gamma\in\Gamma_0$ and $\gamma:(a, b)\to G$ is parametrized as in the proof of lemma 4.6. If $a<t_1<t_2<b$ then

$$\int_\gamma \varrho\,ds = \int_a^b \varrho\circ\gamma(t)\,dt \geqq \int_{t_1}^{t_2} |\nabla u(\gamma(t))|\,dt \geqq \left|\int_{t_1}^{t_2} \nabla u(\gamma(t))\cdot\frac{d\gamma}{dt}\,dt\right|$$
$$= |u\circ\gamma(t_2)-u\circ\gamma(t_1)|.$$

Since t_1 and t_2 are arbitrary, the above implies $\int_\gamma \varrho\,ds\geqq 1$. Hence, $\varrho\in\mathcal{F}(\Gamma_0)$. Therefore

$$M_p(\Gamma_0) \leqq \int_{R^n} \varrho^p(x)\,dm(x) = \int_G |\nabla u(x)|^p\,dm(x).$$

By a theorem of Fuglede [5, Thm. 28.2] we have $M_p(\Gamma)=M_p(\Gamma_0)$. Therefore,

$$M_p(\Gamma) \leqq \int_G |\nabla u(x)|^p\,dm(x).$$

Since $u\in\mathcal{A}(E, F, G)\cap L^p(G)$ is arbitrary, we get the desired result.

5.3. Lemma. *Let U be a domain in R^n, let $g: U\to[0, \infty)$ be continuous and suppose K is a non-empty bounded compact set with $K\subset U$. Define $f:U\to[0, \infty)$ by $f(x)= =\inf\int_\beta g\,ds$ where the infimum is taken over all rectifiable curves $\beta:[a, b]\to U$ with $\beta(a)\in K$ and $\beta(b)=x$. Then,* (1) *if the closed line segment $[x_1, x_2]$ lies in U then*

$$|f(x_2)-f(x_1)| \leqq \max_{x\in[x_1, x_2]} g(x)\,|x_2-x_1| \tag{5.4}$$

and (2) *if* $f:U\to[0,\infty)$ *satisfies* (5.4) *then* f *is differentiable a.e. in* U *and* $|\nabla f(x)|\leqq g(x)$ *a.e. in* U.

Proof. Let β be a rectifiable curve connecting K and x_1. Then

$$f(x_2)\leqq\int_\beta g\,ds+\int_{[x_1,x_2]}g\,ds\leqq\int_\beta g\,ds+\max_{x\in[x_1,x_2]}g(x)|x_2-x_1|.$$

Since β is arbitrary, we get

$$f(x_2)\leqq f(x_1)+\max_{x\in[x_1,x_2]}g(x)|x_2-x_1|.$$

In a similar way, we get

$$f(x_1)\leqq f(x_2)+\max_{x\in[x_1,x_2]}g(x)|x_2-x_1|.$$

This proves (5.4).

If f satisfies (5.4) then f is locally Lipschitz continuous in U and therefore, by the theorem of Rademacher and Stepanov [5, Thm. 29.1], f is differentiable a.e. in U. Suppose now that $x_0\in U$ is a point of differentiability of f. Then $f(x_0+h)-f(x_0)=\nabla f(x_0)\cdot h+|h|\varepsilon(h)$ where $h\in R^n$ and $\lim\varepsilon(h)=0$ as $h\to 0$. For small $t\in(0,1)$ let $h=t\nabla f(x_0)/|\nabla f(x_0)|$. Substituting in the above formula gives $\big||\nabla f(x_0)|+\varepsilon(h)\big|\leqq\max_{x\in[x_0,x_0+h]}g(x)$. If we let $t\to 0$ we get $|\nabla f(x_0)|\leqq g(x_0)$, as desired.

5.5. Theorem. *Suppose* $(E\cup F)\cap\partial G=\emptyset$. *Then* $M_p(\Gamma)=\operatorname{cap}_p(E,F,G)$.

Proof. It suffices, by lemma 5.2, to prove

$$\operatorname{cap}_p(E,F,G)\leqq M_p(\Gamma). \tag{5.6}$$

We assume, without any loss of generality, that E is bounded and we let $\mathscr{C}\subset\mathscr{F}(\Gamma)$ be as in theorem 4.15. The proof is divided into four cases.

Case 1. Suppose $\infty\notin G$. Let $\varrho\in\mathscr{C}$ and define $u:G\to[0,\infty)$ by $u(x)=\min(1,\inf\int_\beta\varrho\,ds)$ where the infimum is taken over all rectifiable curves β in G connecting E and x. It follows, using lemma 5.3, that $u\in\mathscr{A}(E,F,G)$ and $|\nabla u|\leqq\varrho$ a.e. in G. Therefore

$$\operatorname{cap}_p(E,F,G)\leqq\int_G|\nabla u|^p\,dm\leqq\int_{R^n}\varrho^p.$$

Since $\varrho\in\mathscr{C}$ is arbitrary and $\mathscr{C}$ is p-complete, we get (5.6).

Case 2. Suppose $\infty\in G$ and $\infty\in F$. Choose $\varrho\in\mathscr{C}$ and $\varepsilon\in(0,1)$. Since $L(\varrho)\geqq 1$ we can choose a small $r\in(0,1)$ so $\int_\gamma\varrho\,ds\geqq 1-\varepsilon$ for every locally rectifiable curve γ in G connecting $E(r)$ and $F(r)$. Define $u:G-\{\infty\}\to[0,\infty)$ by $u(x)=\min\big(1,(1-\varepsilon)^{-1}\inf\int_\beta\varrho\,ds\big)$ where the infimum is taken over all rectifiable curves β in G connecting $E(r)$ and x. Since u is identically 1 in a deleted neighborhood of ∞, we see that u extends continuously to all of G. It follows, using lemma 5.3, that $u\in\mathscr{A}(E, F, G)$ and $|\nabla u|\leqq(1-\varepsilon)^{-1}\varrho$ a.e. in G. Therefore,

$$\mathrm{cap}_p(E, F, G) \leqq \int_G |\nabla u|^p\, dm \leqq (1-\varepsilon)^{-p}\int_{R^n} \varrho^p\, dm.$$

Since $\varrho\in\mathscr{C}$ and $\varepsilon\in(0, 1)$ are arbitrary and $\mathscr{C}$ is p-complete, we get (5.6).

Case 3. Suppose $\infty\in G$, $\infty\notin F$ and $1<p<n$. Choose $\varrho\in\mathscr{C}$. Since $((\overline{R}^n-G)\cup E\cup F)-\{\infty\}$ lies inside some ball, it follows that $|x|\leqq\text{constant}\cdot d(x)$ for large $|x|$. Therefore,

$$\varrho(x) \leqq C|x|^{-n/p} \tag{5.7}$$

for some constant $C\in(0, \infty)$ and all large $|x|$, say $|x|>r_0$. Define $v: G-\{\infty\}\to[0, \infty)$ by $v(x)=\inf\int_\beta \varrho\, ds$ where the infimum is taken over all rectifiable curves β connecting E and x. We proceed to show that $v(\infty)$ can be defined continuously. Set $v(\infty)=\inf\int_\beta \varrho ds$ where the infimum is taken over all continuous β such that $\beta:[a, b]\to G$ with $\beta(a)\in E$, $\beta(b)=\infty$ and $\beta|[a, t]$ is rectifiable for all $t\in[a, b)$. Choose any $x_0\in R^n$ so that the curve $[x_0, \infty]$ lies in G, where $[x_0, \infty](t)=tx_0$, $t\in[1, \infty]$. Let γ by any rectifiable curve in G connecting E and x_0. Let β the curve obtained by connecting the curves γ and $[x_0, \infty]$. Then

$$v(\infty) \leqq \int_\beta \varrho\, ds = \int_\gamma \varrho\, ds + \int_{[x_0, \infty]} \varrho\, ds.$$

Clearly $\int_\gamma \varrho\, ds$ is finite and $\int_{[x_0, \infty]} \varrho\, ds$ is finite by the estimate (5.7) and the fact that $1<n/p$. Hence $v(\infty)$ is finite. Choose $r\in(r_0, \infty)$ large enough so that the complement in $\overline{R}^n$ of $\overline{B^n(0, r)}$ lies in G and $E\subset B^n(0, r)$. Let $x_0\in G-\{\infty\}$ and $|x_0|>r$.

Suppose β is a rectifiable curve in G connecting E and x_0. We have

$$v(\infty) \leqq \int_\beta \varrho\, ds + \int_{[x_0, \infty]} \varrho\, ds \leqq \int_\beta \varrho\, ds + C\int_r^\infty t^{-n/p}\, dt.$$

Since the above is true for all such β, we get

$$v(\infty)-v(x_0) \leqq c\int_r^\infty t^{-n/p}\, dt. \tag{5.8a}$$

Suppose now that β is a curve connecting E and ∞ and is of the type used in defining $v(\infty)$. Let τ be a curve which is part of a great circle on the sphere $\{x\in R^n: |x|=|x_0|\}$ and which connects x_0 and y_0 where y_0 is some point on the curve β. Let β_1 be a subcurve of β connecting E and y_0. We have

$$v(x_0) \leqq \int_{\beta_1} \varrho\, ds + \int_\tau \varrho\, ds \leqq \int_\beta \varrho\, ds + \int_\tau \varrho\, ds.$$

Also,

$$\int_\tau \varrho\, ds \leqq \frac{C}{|x_0|^{n/p}}\cdot \text{length}(\tau) \leqq 2\pi C|x_0|^{1-n/p}.$$

Hence

$$v(x_0) \leqq \int_\beta \varrho\, ds + 2\pi C|x_0|^{1-n/p} \leqq \int_\beta \varrho\, ds + 2\pi C r^{1-n/p}.$$

Since the above is true for all β connecting E and ∞, we have

$$v(x_0)-v(\infty) \leqq 2\pi C r^{1-n/p}. \tag{5.8b}$$

Relations (5.8) show v is continuous at ∞.

Define $u: G \to [0, \infty)$ by $u(x)=\min(1, v(x))$. Then it follows, using lemma 5.3, that $u \in \mathscr{A}(E, F, G)$ and $|\nabla u| \leqq \varrho$ a.e. in G. Therefore

$$\operatorname{cap}_p(E, F, G) \leqq \int_G |\nabla u|^p\, dm \leqq \int_{R^n} \varrho^p\, dm.$$

Since $\varrho \in \mathscr{C}$ is arbitrary and $\mathscr{C}$ is p-complete, we get (5.6).

Case 4. Suppose $\infty \in G$, $\infty \notin F$ and $p \geqq n$. Define $\theta: R^n \to [0, 1]$ by

$$\theta(x) = \begin{cases} 1/e & \text{if } |x| \leqq e \\ 1/(|x| \log |x|) & \text{if } |x| > e. \end{cases}$$

It is straightforward to verify that $\theta \in L^p(R^n)$ and $\int_0^\infty \theta(|x|)\, d|x| = \infty$. Choose $\varrho \in \mathscr{C}$ and $\varepsilon \in (0, 1)$. Let $\varrho' = \varrho + \varepsilon\theta$. Define $u: G - \{\infty\} \to [0, \infty)$ by $u(x) = \min(1, \inf \int_\beta \varrho'\, ds)$ where the infimum is taken over all rectifiable β in G connecting E and x. Choose $r \in (0, \infty)$ so that $E \subset B^n(0, r)$. If $|x_0| > r$ and if β connects E and x_0 then

$$\int_\beta \varrho'\, ds \geqq \varepsilon \int_\beta \theta\, ds \geqq \varepsilon \int_r^{|x_0|} \theta(|x|)\, d|x|.$$

It follows that if $|x_0|$ is large then $\int_\beta \varrho'\, ds \geqq 1$. Therefore, u extends continuously to $u: G \to [0, \infty)$. We get, using lemma 5.3, that $u \in \mathscr{A}(E, F, G)$ and $|\nabla u| \leqq \varrho'$ a.e. in G. Hence,

$$\operatorname{cap}_p(E, F, G) \leqq \int_G |\nabla u|^p\, dm \leqq \int_{R^n} (\varrho + \varepsilon\theta)^p\, dm.$$

Since $\varrho \in \mathscr{C}$ and $\varepsilon \in (0, 1)$ are arbitrary and $\mathscr{C}$ is p-complete, we get (5.6).

We use the previous theorem to prove a continuity theorem for the modulus.

5.9. Theorem. *Suppose $E_1 \supset E_2 \supset \dots$ and $F_1 \supset F_2 \supset \dots$ are disjoint sequences of non-empty compact sets in a domain G. Then*

$$\lim_{i \to \infty} M_p(E_i, F_i, G) = M_p\left(\bigcap_{i=1}^{\infty} E_i, \bigcap_{i=1}^{\infty} F_i, G\right).$$

Proof. The theorem follows immediately from theorems 5.5 and 3.3.

5.10. *Comment.* The reader may wish to compare the proof of 5.6 with Ziemer's proof [7]. Ziemer defines a function u derived from a density ϱ in a way that is similar to the one in this paper. Ziemer's technique will not work for the situation considered in this paper since the "limiting curve" of [7, lemma 3.3] need not necessarily lie in G. The present proof "works" because there is a p-complete family of densities ϱ with $L(\varrho) \geqq 1$.

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