Multipliers on homogeneous Banach spaces on compact groups

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0. Introduction

The first section is motivated by a result of K. de Leeuw [9, th. 4.2] and is concerned with Banach modules where the elements of the algebra act as weakly compact operators on the Banach space. In § 2 and § 3 the general properties of such modules are used to characterize the multipliers on homogeneous Banach spaces on compact groups. The results are partly taken from the author's dissertation for the licentiat degree at the Technical University of Norway which was written under the direction of Professor Olav Njåstad.

1. Banach modules with weakly compact action

We start by recalling some basic definitions. If A is a Banach algebra and V is a Banach space, then V is a left Banach A-module if it is a left A-module in the algebraic sense, and $\|av\| \leq \|a\| \|v\|$ for all $a \in A$, $v \in V$. A right Banach module is defined similarly. The closed linear subspace of V spanned by AV is called the essential part of V and is denoted by V_e ([10], p. 454). If $V_e = V$, then V is called an essential A-module. It is well known that if A has a bounded approximate identity, then $V_e = AV_e$, i.e. for al $v \in V_e$, there exists an $a \in A$, $w \in V_e$ such that $v = aw$. (In what follows, we always assume the bound on a bounded appr. id. to be 1.) If V is a left A-module, then V^* is a right A-module under the adjoint action: $(a\mu)(v) \overset{D}{=} \mu(av)$, $a \in A$, $v \in V$, $\mu \in V^*$. The essential part of V^* is called the contragradient of V and is denoted by V^c ([10], p. 455).

If V and W are Banach A-modules, then $\operatorname{Hom}_A(V, W)$ as usual denotes the space of linear, continuous operators T from V to W such that $T(av) = a(Tv)$ for all $a \in A$, $v \in V$.

We say that $a \in A$ is weakly compact if the operator $v \to av$ is weakly compact. Let $\varkappa\colon V \to V^{**}$ be the canonical injection.

LEMMA 1.1. *Let V be an essential A-module. Then $\varkappa(V) = (V^*)^c$ if and only if every $a \in A$ is weakly compact.*

Proof. Let $a \in A$, $v \in V$, $\mu \in V^*$. Then $\varkappa(av)(\mu) = \mu(av) = \varkappa(v)(a\mu) = a\varkappa(v)(\mu)$, that is, $\varkappa(av) \in V^{*c}$. Since the linear span of AV is dense in V, $\varkappa$ is a linear isometry, and V^{*c} is closed, it follows that $\varkappa(V) \subset V^{*c}$. The lemma is now an immediate consequence of the fact that a linear operator on V is weakly compact if and only if its second adjoints maps V^{**} into $\varkappa(V)$ ([2], p. 482).

LEMMA 1.2. *Let A have a bounded appr. id. $\{i_\alpha\}$, and let V be a Banach A-module. Then there exists a natural isometric isomorphism*

$$\varrho\colon V^c \to V^{cc}$$

such that

$$(\varrho\Phi)(\mu) = \Phi(\mu), \; \Phi \in V^{*c}, \mu \in V^c.$$

Proof. One easily checks that ϱ commutes with the action of A which implies that $\varrho(V^{*c}) - V^{cc}$.

Let $f \in V^{cc}$. Then there exists and $a \in A$, $g \in V^{cc}$ such that $f = ag$. If now $\tilde{g} \in V^{**}$ is a continuous extention of g to V^{**}, then

$$\varrho(a\tilde{g})(\mu) = a\tilde{g}(\mu) = \tilde{g}(a\mu) = g(a\mu) = f(\mu) \quad \text{for all} \quad \mu \in V^c.$$

Thus ϱ is onto. Clearly $\|\varrho\Phi\| \leq \|\Phi\|$, and to complete the proof, it remains to prove the opposite inequality:

Let $\Phi \in V^{*c}$. Since $\{i_\alpha\}$ also acts as an approximate identity for V^{*c}, there is an i_α such that $\|i_\alpha\Phi - \Phi\| < \varepsilon/2$.

Let $\mu \in V^*, \|\mu\| = 1$, so that $|\Phi(\mu)| > \|\Phi\| - \varepsilon/2$. Then $(\varrho\Phi)(i_\alpha\mu) = \Phi(i_\alpha\mu) = \Phi(\mu) - \Phi(\mu) + \Phi(i_\alpha\mu)$, hence

$$\|\varrho\Phi\| \geq |(\varrho\Phi)(i_\alpha\mu)| > \|\Phi\| - \varepsilon/2 - \|(i_\alpha\Phi - \Phi)(\mu)\| > \|\Phi\| - \varepsilon.$$

($\cong$: Read "isometric isomorphic".)

PROPOSITION 1.1. *Let A have a bounded appr. id., and let V be an essential Banach A-module. Then the following are equivalent:*

(i) $V \cong (V^*)^c \cong V^{cc}$.

(ii) *All $a \in A$ are weakly compact.*

(iii) *A have a bounded appr. id. consisting of weakly compact elements.*

Proof. The equivalence of (i) and (ii) is obvious from the lemmas. Clearly (ii) implies (iii), and since the algebra norm majorizes the operator norm, and the composition of a bounded linear operator and a weakly compact operator is weakly compact, (iii) implies that every $a \in A$ is the limit in the uniform operator topology of weakly compact operators, and therefore is weakly compact. Thus (iii) implies (ii).

Let $V \otimes_\gamma W$ be the projective tensor product of the Banach A-modules V and W, and let M be the closed linear span of elements of the form $av \otimes w - v \otimes aw$. The quotient space $V \otimes_\gamma W/M$ is an A-module tensor product and is denoted by $V \otimes_A W$ (See [10], [11]).

Each element φ of $V \otimes_A W$ has an expansion $\varphi = \sum_{i=1}^\infty v_i \otimes w_i$ where $\sum_{i=1}^\infty \|v_i\| \|w_i\| < \infty$. The norm of φ is defined by $\|\varphi\| = \inf \sum_{i=1}^\infty \|v_i\| \|w_i\|$ where the infimum is taken over all possible representations of φ (See [5], [10]). A bilinear operator Ψ from $V \times W$ to a Banach space D is called A-balanced if it is continuous and $\Psi(av, w) = \Psi(v, aw)$ for all $v \in V, w \in W, a \in A$. If $\Psi: V \times W \to D$ is A-balanced, there is a unique linear operator $\overline{\Psi}: V \otimes_A W \to D$ such that

1)
$$
\begin{array}{ccc}
V \times W & \xrightarrow{\;\;\Psi\;\;} & D \\
\Big\downarrow{\scriptstyle \otimes_A} & \circ \quad \nearrow{\scriptstyle \overline{\Psi}} & \\
V \otimes_A W & &
\end{array}
$$

2) $\|\Psi\| = \|\overline{\Psi}\|$.

We refer to [10] for further properties of $V \otimes_A W$.

PROPOSITION 1.2. *Let A be a Banach algebra with a bounded appr. id. Let V and W be essential A-modules and let the action of A on W be weakly compact. Then*

$$\mathrm{Hom}_A (V, W) \cong (V \otimes_A W^c)^* \cong (V \otimes_A W^*)^*.$$

The isomorphism carries $T \in \mathrm{Hom}_A (V, W)$ to $\mu_T \in (V \otimes_A W^c)^*$ (resp. $(V \otimes_A W^*)^*$) defined by

$$\mu_T(v \otimes_A w^*) = w^*(Tv), \; v \in V, \; w^* \in W^c \; (\text{resp. } W^*)$$

Proof. The conditions assert that $W \cong W^{cc}$, and since the natural isomorphism from W to W^{cc} commutes with the action of A, we obviously have $\mathrm{Hom}_A (V, W) \cong \mathrm{Hom}_A (V, W^{cc})$. Thus

$$\mathrm{Hom}_A (V, W) \cong \mathrm{Hom}_A (V, W^{cc})$$

$$\cong \mathrm{Hom}_A (V, W^{c*}) \qquad \text{([10], cor. 3.8)}$$

$$\cong (V \otimes_A W^c)^*. \qquad \text{([10], cor. 3.21)}$$

$\mathrm{Hom}_A\,(V,\,W) \cong (V \otimes_A W^*)^*$ is proved similarly, and it is easily verified that the resulting isomorphism has the form stated. (Actually, if V and W are essential A-modules and A has a bounded appr. id., then $V \otimes_A W^c \cong V \otimes_A W^*$:

Viewing A as both a left and a right A-module one has:

$$V \otimes_A W^* \cong (V \otimes_A A) \otimes_A W^* \cong V \otimes_A (A \otimes_A W^*) \cong V \otimes_A (W^*)_e \cong V \otimes_A W^c.)$$

PROPOSITION 1.3. *If A, V, and W are as in prop. 1.2, and in addition the action of A on V is weakly compact, then:*

$$\mathrm{Hom}_A\,(V,\,W) \cong \mathrm{Hom}_A\,(W^c,\,V^c) \cong \mathrm{Hom}_A\,(W^*,\,V^*).$$

Proof.

$$\mathrm{Hom}_A\,(W^*,\,V^*) \cong (W^* \otimes_A V)^* \cong (V \otimes_A W^*)^*$$
$$\cong \mathrm{Hom}_A\,(V,\,W) \cong (V \otimes_A W^c)^* \cong (W^c \otimes_A V^{cc})^*$$
$$\cong \mathrm{Hom}_A\,(W^c,\,V^c).$$

Remark 1.1. Proposition 1.1 generalizes a result of K. de Leeuw ([9], thm. 4.2). If G is a locally compact group and $\{T_g\}_{g \in G}$ is an isometric, strongly continuous representation of G on a Banach space V, then V becomes an essential Banach $L^1(G)$-module by the composition

$$f \circ v = \int T_g vf(g)dg, f \in L^1(G), v \in V. \tag{1.1}$$

In this case V^c is exactly the closed linear subspace of V^* on which the adjoint representation is strongly continuous. de Leeuw states that if G is compact and Abelian, then $V \cong V^{cc}$ if for each character (x,γ), $y \in \hat{G}$, $(x,\gamma) \circ V$ has a finite dimension, or at least is a reflexive subspace of V. Now the first of these conditions implies that the operator $v \to (x,\gamma) \circ v$ is compact, and the second that it is weakly compact by Corollary 3, p. 483 in [2]. When G is compact, $L^1(G)$ has a bounded apr. id. consisting of trigonometric polynomials, and so condition (iii) in prop. 1.1 is clearly satisfied in de Leeuw's case.

Remark 1.2. For an example where $V \not\cong V^{cc}$, take $A = L^1(\mathbf{R})$, $V = C^0(\mathbf{R})$ with the usual convolution as composition:

Then $V^* = M(\mathbf{R})$, $V^c = L^1(\mathbf{R})$, $V^{c*} = L^\infty(\mathbf{R})$, and $V^{cc} = C^u(\mathbf{R})$, the space of bounded, uniformly continuous functions.

2. Homogeneous Banach spaces

Let V and T be as in Remark 1.1, and let the group G be infinite, compact, and Abelian. If we denote $T_a f$ by f_a, we have

(i) $f_{(a+b)} = (f_a)_b$, $f \in V$, $a, b \in G$.

(ii) $\|f_a\| = \|f\|$ for all $f \in V$, $a \in G$.   (2.1)

(iii) $\|f_a - f\| \xrightarrow[a \to 0]{} 0$.

We also assume the existence of a Fourier transform on V which has the usual properties:

(i) $f \to \hat{f}(\gamma)$ is a continuous linear functional for every $\gamma \in \hat{G}$.

(ii) $f = 0$ if and only if $\hat{f}(\gamma) = 0$ for all $\gamma \in \hat{G}$.   (2.2)

(iii) $\hat{f_a}(\gamma) = (-a, \gamma)\hat{f}(\gamma)$.

We denote a Banach space which satisfies 2.1 and 2.2 a *homogeneous Banach space* (HBS). This notion has been used by various authors for certain subspaces of $L^1(G)$ where the Fourier transform has been inherited from $L^1(G)$, and T has been the regular representation, i.e. $f_a(x) = f(x - a)$. The spaces $C(G)$ and $L^p(G)$, $1 \leq p < \infty$, are well known examples. But it is convenient not to restrict oneself to spaces of functions on G: $L^p(\hat{G})$, $1 \leq p < \infty$, becomes a HBS if we take the "Fourier transform" to be the identity mapping and the action of G to be the multiplication with a character: $(T_a f)(\gamma) = (-a, \gamma)f(\gamma)$.

A *HBS* is an essential $L^1(G)$-module if the action is defined as in eq. 1.1. Obviously, $\widehat{g \circ f}(\gamma) = \hat{g}(\gamma)\hat{f}(\gamma)$ for all $\gamma \in \hat{G}, f \in V, g \in L^1(G)$, and by 2.2 (ii), $(x, \gamma) \circ V$ is either $\{0\}$ or a one dimensional subspace of V. Thus the action of $L^1(G)$ on V is compact, and the results of § 1 apply. Define $\{e_\gamma\}_{\gamma \in \hat{G}}$ by $e_\gamma(\tau) = \delta_{\tau\gamma}$ if $(x, \gamma) \circ V \neq \{0\}$, 0 if $(x, \gamma) \circ V = \{0\}$. Then clearly $\{e_\gamma\}$ spans V.

The adjoint representation of G on V^* is defined by $\mu_a(f) = \mu(f_a)$, $\mu \in V^*$, $f \in V$. It will be isometric but not necessarily strongly continuous. We define the Fourier transform of a continuous linear functional by

$$\hat{\mu}(\gamma) = \mu(e_\gamma), \, u \in V^*, \gamma \in \hat{G}.$$

The mapping $\mu \to \hat{\mu}$ satisfies 2.2, and therefore, if V^e inherits the Fourier transform from V^*, it becomes a *HBS*.

PROPOSITION 2.1. *Let V and W be HBS's which we also consider as $L^1(G)$ modules. Then $V \otimes_{L^1(G)} W$ is a HBS if we define*

(i) $(f \otimes g)_a = f \otimes g_a$, $f \in V, g \in W, a \in G$.

(ii) $\widehat{f \otimes g}(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma)$, $f \in V, g \in W, \gamma \in \hat{G}$.

Proof. It is well known and easily proved that (i) defines an isometric, strongly continuous representation on $V \otimes_{L^1} W$ (See [10], cor. 3.10). The Fourier transform defined by (ii) clearly satisfies eq. 2.2 (i) and (iii). Let

$$\varphi \in V \otimes_{L^1} W, \quad \varphi = \sum_{i=1}^{\infty} f_i \otimes g_i, \quad \sum_i \|f_i\| \|g_i\| < \infty.$$

If we define the action of $L^1(G)$ on $V \otimes_{L^1} W$ by eq. (1.1), we see that $\varphi = 0$ if and only if $(x, \gamma) \circ \varphi = 0$ for all $\gamma \in \hat{G}$. But

$$(x, \gamma) \circ \varphi = \int \left(\sum_i (f_i \otimes g_i)_x\right)(x, \gamma) dx$$

$$= \sum_i f_i \otimes ((x, \gamma) \circ g_i)$$

$$= \sum_i f_i \otimes ((x, \gamma) \circ (x, \gamma) \circ g_i)$$

$$= \sum_i ((x, \gamma) \circ f_i) \otimes ((x, \gamma) \circ g_i)$$

$$= \sum_i \hat{f_i}(\gamma)\hat{g_i}(\gamma)e_\gamma^V \otimes e_\gamma^W,$$

which shows that $\varphi = 0$ if and only if

$$\sum_i \hat{f_i}(\gamma)\hat{g_i}(\gamma) = \hat{\varphi}(\gamma) = 0 \quad \text{for all} \quad y \in G.$$

3. Multipliers on HBS's

If X and Y are HBS's or possibly the duals of such spaces, we say that $\phi = \{\phi(\gamma)\}_{\gamma \in \hat{G}}$ is a (X, Y)-multiplier if for all $f \in X$, $\phi\hat{f}$ is the Fourier transform of an element in Y. The set of all (X, Y)-multipliers is denoted by (X, Y). The connection between multipliers and "translation invariant" operators is well known:

PROPOSITION 3.1. *Let V and W be HBS's and let $U: V \to W$ be a linear operator. Then the following are equivalent:*
   (i) *There is a $\phi \in (V, W)$ such that $\widehat{Uf}(\gamma) = \phi(\gamma)\hat{f}(\gamma)$ for all $f \in V, \gamma \in \hat{G}$.*
   (ii) *U is continuous and $(Uf)_a = U(f_a)$ for all $f \in V, a \in G$.*
   (iii) *U is continuous and $U(h \circ f) = h \circ Uf$ for all $f \in V, h \in L^1(G)$.*

We omit the proof, but remark that while (iii) $\Leftrightarrow$ (i) $\Rightarrow$ (ii) holds even if we substitute V and W by V^*, W^* resp., (ii) does not necessarily imply (i) in this case. (See [12], p. 220 for a counter example).

The next proposition is also simple to prove:

PROPOSITION 3.2. *If V and W are HBS's, then the following identities obtain:*
   (i) $(V, W) = (W^*, V^*) = (W^c, V^c)$,
   (ii) $(V, W^*) = (V, W^c)$,
   (iii) $(V, W^*) = (W, V^*)$.

We assign to each $\phi \in (V, W)$ a corresponding multiplier operator U_ϕ defined by $\widehat{U_\phi f}(\gamma) = \phi(\gamma)\hat{f}(\gamma)$. We identify multipliers which generate the same operators, and we norm (V, W) by setting $\|\phi\| = \|U_\phi\|$. This gives us by prop. 3.1 an isometric isomorphism of (V, W) onto $\operatorname{Hom}_{L^1(G)}(V, W)$.

PROPOSITION 3.3. *Let* V *and* W *be* HBS's. *Then there exists an isometric isomorphism*

$$i: (V, W) \to (V \otimes_{L^1(G)} W^c)^*$$

such that

$$\widehat{i\phi}(\gamma) = \phi(\gamma) \text{ for all } \phi \in (V, W), y \in \hat{G}.$$

Proof. The existence and the natural definition of i follows immediately from prop. 1.2 and the remarks above. It remains to prove that $\widehat{i\phi}(\gamma) = \phi(\gamma)$:

$$\begin{aligned}
\widehat{i\phi}(\gamma) &= i\phi(e_\gamma^V \otimes_{L^1} e_\gamma^{W^c}) \\
&= e_\gamma^{W^c}(U_\phi e_\gamma^V) \qquad\qquad\qquad \text{(Prop. 1.2)} \\
&= (U_\phi e_\gamma^V)^\wedge(\gamma) \\
&= (\phi(\gamma)e_\gamma^V)^\wedge(\gamma) = \phi(\gamma).
\end{aligned}$$

We remark that by prop. 2.1, $V \otimes_{L^1} W^c$ is a *HBS*, and thus the multipliers between two arbitrary *HBS*'s can always be identified with the dual space of a *HBS*.

COROLLARY 3.1. *If* V *is a* HBS, *then* $(V, V) \cong (V \otimes_{L_1(G)} V^c)^*$.

As in the case of the L^p-spaces, $1 \le p < \infty$, (See [4], [11]), we can identify $V \otimes_{L^1(G)} V^c$ with a Banach space of continuous functions:

Let $\overline{\Psi}: V \otimes_{L^1(G)} V^c \to C(G)$ be the lifting of the $L^1(G)$-balanced operator $\Psi: V \times V^c \to C(G)$, $\Psi(f, \mu)(t) = \mu(f_{-t})$. We first prove that $\overline{\Psi}$ is injective:

Let $\varphi = \sum_i f_i \otimes \mu_i \in \ker \overline{\Psi}$, $\sum_i \|f_i\|\|\mu_i\| < \infty$.

$(\overline{\Psi}\varphi)(t) = \sum_i \mu_i((f_i)_{-t})$, and, as the series converges uniformly, we calculate its Fourier transform:

$$\begin{aligned}
0 &= \widehat{\overline{\Psi}\varphi}(\gamma) \\
&= \sum_i \mu_i((f_i)_{-t})^\wedge(\gamma) \\
&= \sum_i \int (-t, \gamma)\mu_i((f_i))_{-t}dt \qquad\qquad (3.1) \\
&= \sum_i \mu_i\left(\int (-t, \mu)(f_i)_{-t}dt\right)
\end{aligned}$$

$$= \sum_i \mu_i(\hat{f}_i(\gamma)e_\gamma)$$

$$= \sum_i \hat{\mu}_i(\gamma)\hat{f}_i(\gamma).$$

By prop. 2.1, $\varphi = 0$.

Let A_V denote the range of $\overline{\Psi}$. To make A_V a Banach space, it now suffices to define $\|\overline{\Psi}\varphi\|_{A_V} = \|\varphi\|$. Observe also that the induced representation and Fourier transform on A_V coincide with the canonical.

Remark 3.1. *G non Abelian.* Let G be a non-Abelian compact group with dual object Σ, that is, the set of equivalence classes of continuous irreducible representations of G. For each $\sigma \in \Sigma$, we choose a fixed representation U^σ on the Hilbert space H^σ, and the set $\prod_{\sigma \in \Sigma} \mathrm{Hom}\,(H^\sigma)$ is denoted $\mathfrak{E}(\Sigma)$ (See [6], ch. 7).

It is now convenient to define a *HBS* V as a Banach space on which G acts isometrically and strongly continuous in a way resembling both the left and the right regular representations. Moreover, we assume to have a mapping from V to $\mathfrak{E}(\Sigma), f \to \hat{f}$, similar to (2.2):

(i) $f = 0$ if and only if $\hat{f}(\sigma) = 0$ for all $\sigma \in \Sigma$,

(ii) $\widehat{L_a f}(\sigma) = \overline{U_a^\sigma}\hat{f}(\sigma)$, for $a \in G, \sigma \in \Sigma$,

(iii) $\widehat{R_a f}(\sigma) = \hat{f}(\sigma)\overline{U_a^\sigma}$, for $a \in G, \sigma \in \Sigma$.

Familiear spaces like $L^p(G), 1 \leq p < \infty, C(G)$, and also $\mathfrak{E}^p(\Sigma), 1 \leq p < \infty$, defined in [6] p. 77, have all these properties. The results in § 2 and 3 can now be generalized with appropriate modifications. The tensor product $V \otimes_{L^1} W$ equals $V \otimes_\gamma W/K$ where K is the closed subspace of $V \otimes_\gamma W$ spanned by elements of the form $(\int_G R_x f h(x)dx) \otimes g - f \otimes (\int_G L_x g h(x)dx), f \in V, g \in W, h \in L^1(G)$.

$V \otimes_{L^1} W$ becomes a *HBS* if we define:

(i) $L_a(f \otimes g) = (L_a f) \otimes g$,

(ii) $R_a(f \otimes g) = f \otimes (R_a g)$,

(iii) $\widehat{f \otimes g}(\sigma) = \hat{f}(\sigma)\hat{g}(\sigma)$.

Proposition 3.1 has one left and one right handed version, and in proposition 3.2, the class of left multipliers from V to W, defined in the obvious way, is equal to the class of right multipliers from W^* to V^*.

Remark 3.2. Applications of prop. 3.3. Rieffel [10] has shown that

$$L^1(G) \otimes_{L^1(G)} V \cong V \otimes_{L^1(G)} L^1(G) \cong V$$

if V is an essential $L^1(G)$-module and $L^1(G)$ is considered as an essential $L^1(G)$-module over itself. This gives us the identifications:

$$(L^1(G),\, V) \cong (L^1(G) \otimes_{L^1(G)} V^c)^* \cong (V^c)^*,$$

$$(V,\, C(G)) \cong (V \otimes_{L^1(G)} L^1(G))^* \cong V^*.$$

Then $(L^1(G), L^1(G)) \cong C(G)^* \cong M(G)$, $(L^1(G), L^p(G)) \cong L^q(G)^* \cong L^p(G)$, $(L^1(G),$ $C(G)) \cong L^1(G)^* \cong L^\infty(G)$ etc.

If we look at the table p. 410—411 in [6], proposition 3.3 provides a tensor product characterization of all pairs (X, Y) in the table where X is a *HBS* (Recall that $(V, W^*) = (V, W^c)$). That is, all rows except the ones for $\mathcal{E}^\infty$, $M(G)$, $L^\infty(G)$. But if X and Y both are dual spaces, we may use prop. 3.2 (i).

Multipliers between L^p-spaces have been characterized by means of tensor products by Rieffel [11]. If $1 \leq p, q < \infty$, one identifies the tensor products with Banach spaces of integrable (or even continuous) functions by examining the lifting of the operator $\Psi\colon L^p(G) \times L^q(G) \to L^1(G)$, $\Psi(f, g) = f * g$. One can prove that Ψ is injective by a similar calculation to eq. 3.1.

R. Larsen has considered the multipliers on the spaces

$$A_p(G) = \{f; \|f\|_{L^1(G)} + \|\hat{f}\|_{L^p(\hat{G})} < \infty\}.$$

If $1 \leq p < \infty$, these are easily seen to be *HBS*'s, in particular, if G is compact, $(A_p(G), A_p(G))$ can be identified with the dual space of a Banach space of continuous functions as proved by Larsen [8], pp. 207.

Remark 3.3. Further results. Here we mention some other simple results which might be of interest:
  a) In prop. 3.3, $i(\{\phi; U_\phi \text{ is compact}\}) = (V \otimes_{L^1} W^c)^c$. The proof is essentially the same as the proof of thm. 3.1 (ii) in [1].
  b) The representation of multipliers as Fourier transforms of continuous linear functionals on certain Banach spaces is essentially unique: If V, M, X, Y are *HBS*'s and $X^* \cong (V, W) \cong Y^*$, then there exists a continuous isomorphism j from X onto Y such that $\widehat{jx}(\gamma) = \hat{y}(\gamma)$. This can be used to identify the tensor products if the multiplier classes are known.
  c) If V is a *HBS* such that $(x, \gamma) \circ V \neq \{0\}$ for all $\gamma \in \hat{G}$, then

$$A(G) = \{f; \hat{f} \in L^1(\hat{G})\} \subseteq A_V \subseteq C(G).$$

  d) $(A_V, A_V) \cong (V, V)$.

Remark 3.4. G non-compact. If G is non-compact, much of the above theory breaks down. First, to assume the existence of a mapping like (2.2) is now a more serious restiction than it was in the compact case. Secondly, as the example in remark 1.2 shows, it is easy to find cases where the action of $L^1(G)$ is not weakly compact. In this particular example, the conclusion of proposition 1.3 is nevertheless true:
  $\mathrm{Hom}_{L^1}(C^0, C^0) \cong M$ (We omit the elementary argument)
  $\mathrm{Hom}_{L^1}(L^1, L^1) \cong M$ (See e.g. [8], p. 3)
  $\mathrm{Hom}_{L^1}(M, M) \cong (M \otimes_{L^1} C^0)^* \cong M$.

However, if we compute $\mathrm{Hom}_{L^1}\,(L^\infty, L^\infty)$, we get

$$\mathrm{Hom}_{L^1}\,(L^\infty, L^\infty) \cong (L^\infty \otimes_{L^1} L^1)^* \cong (C^u)^*,$$

that is,

$$\mathrm{Hom}_{L^1}\,(L^1, L^1) \ncong \mathrm{Hom}_{L^1}\,(L^\infty, L^\infty).$$

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