Holomorphic mappings into smooth projective varieties

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1. Introduction

Let M and N be two complex manifolds of complex dimensions m and n respectively. A holomorphic map $f: M \to N$ is called *nondegenerate* if in each component of M, there exists a point p, such that the induced linear map of the holomorphic tangent spaces $f_*: T_p(M) \to T_{f(p)}(N)$ is surjective. This can only happen if $m \geq n$.

If L is a holomorphic line bundle on a complex manifold N and R is a positive integer, then we say that *R has property P with respect to L* if there exists a positive holomorphic line bundle T and a positive integer s, such that

$$H^0(N, (L^R \otimes K_N)^s \otimes T^{-1}) \neq 0 \tag{1.1}$$

where K_N is the canonical bundle of N.

The main result of this paper is the following generalization of the big Picard Theorem.

THEOREM 1.2. *Let N be a smooth projective algebraic variety and L a holomorphic line bundle on N. Suppose further that $F^1, \ldots, F^R$ are sections of L in normal position, where R has property P with respect to L, and put $F = F^1 \otimes \ldots \otimes F^R$. Let M be a complex manifold and $f: M \setminus S \to N \setminus |D_F|$ a nondegenerate holomorphic mapping, where S is an analytic subvariety of M. Then f can be extended to a meromorphic map from M into N.*

COROLLARY 1.3. *Let $f: M \setminus S \to P_n$ be a nondegenerate holomorphic mapping, where S is an analytic subvariety of M and P_n is the n-dimensional complex projective space. If f fails to meet R d-dimensional hypersurfaces in normal position and $R \geq (n+1)/d$, then f can be extended to a meromorphic mapping of M into P_n.*

Let us briefly outline the idea of the proof of Theorem 1.2. After a suitable imbedding of N into a P_q has been chosen, f can be represented by certain analytic functions $g_1, \ldots, g_q$. To show that f has a meromorphic extension, it is sufficient to show that the growth of $g_1, \ldots, g_q$ is not too wild near S. We control the growth with potential-theoretic methods, which are outlined in section 3.

It may be pointed out that the Riemann extension theorem implies that if codim $(S) \geq 2$, then any holomorphic mapping $f: M \setminus S \to N$ can be extended to a meromorphic mapping of M into N, whenever N is a smooth projective variety.

For a background to this paper we refer to the survey article by Griffiths [3] and Carlson-Griffiths [2].

The plan of the paper is the following. In section 2 we give the necessary background material. We prove in section 3 some technical results, and in section 4 we prove Theorem 1.2.

2. Notations and terminology

Let M be a complex manifold. Relative to a suitable open covering $\{U_i\}$ of M, a holomorphic line bundle L is given by holomorphic transition functions $f_{ij}: U_i \cap U_j \to \mathbf{C}^* = \mathbf{C} \setminus \{0\}$, satisfying the cocycle condition

$$f_{ik} = f_{ij} f_{jk} \quad \text{in} \quad U_i \cap U_j \cap U_k \,.$$

The tensor product $L_1 \otimes L_2$ of two line bundles L_1 and L_2 is given by multiplication of the corresponding transition functions. A holomorphic section $F = \{F_i\}$ of L is given by holomorphic functions $F_i: U_i \to \mathbf{C}$, satisfying the compatibility relation

$$F_i = f_{ij} F_j \quad \text{in} \quad U_i \cap U_j \,.$$

The vector space of all holomorphic sections of L is denoted by $H^0(M, L)$.

We observe that a section F of L in a natural way gives rise to a divisor D_F on M. We denote by $|D_F|$ the support of D_F, which is the set $\{z \in M: F_i(z) = 0$ for some $i\}$. Suppose $F^1, \ldots, F^p$ are sections of L and put

$$F = F^1 \otimes \ldots \otimes F^p \in H^0(M, L^p).$$

We say that $F^1, \ldots, F^p$ are in *normal position* if for each point $x \in |D_F|$ we can find an i, such that $x \in U_i$ and $\{F_i^s: s \in Z(x)\}$ can be taken as part of a coordinate system around x, where $Z(x) = \{s: F_i^s(x) = 0\}$. It is easy to see that a collection of hyperplanes meets this condition if and only if they are in general position.

A metric in L is given by positive C^∞ functions a_i in U_i, satisfying

$$a_i = |f_{ij}|^2 a_j \text{ in } U_i \cap U_j.$$

Thus for a holomorphic section $F = \{F_i\} \in H^0(M, L)$, the length

$$\|F\|^2 = |F_i|^2 a_i^{-1}$$

is well defined.

In particular, if we let $\{U_i\}$ be a covering of M with coordinate neighbourhoods U_i with holomorphic coordinates $Z_i = (z_i^1, \ldots, z_i^m)$, then the canonical bundle K_M of M is given by the transition functions

$$g_{ij} = \det\left(\frac{\partial z_i^k}{\partial z_j^l}\right)^{-1} \quad \text{in} \quad U_i \cap U_j.$$

Thus, if $F = \{F_i\}$ is a section of K_M, then the locally given holomorphic $(m, 0)$-forms $F_i dz_i^1 \wedge \ldots \wedge dz_i^m$ patch together to a holomorphic $(m, 0)$-form on M. In the same way, we observe that the metrics on K_M are in one-to-one correspondence with the positive (m, m)-forms on M.

If h is a C^∞ real form of type $(1, 1)$, then h is given locally by

$$h = \sqrt{-1} \sum h_{ij} dz_i \wedge d\bar{z}_j \quad (h_{ij} = \bar{h}_{ji}).$$

We shall say that h is positive if (h_{ij}) is a positive definite Hermitian matrix.

We are now in a position to define what is meant by a positive line bundle.

Definition 2.1. Let L be a holomorphic line bundle on the complex manifold M. Then L is said to be *positive*, if there exists a metric $\{a_i\}$ on L, such that the real $(1, 1)$-form h on M, given in U_i by

$$h = \sqrt{-1}\, \partial\bar{\partial} \log a_i$$

is positive. This metric is then said to be positive.

3. Some technical results

Since we are going to work in $\mathbf{C}^m$, we start by collecting some notation.

$$\left.\begin{aligned}
&\|z\|^2 = |z_1|^2 + \ldots + |z_m|^2 \quad \text{for} \quad z = (z_1, \ldots, z_m) \in \mathbf{C}^m, \\
&V^1 = \frac{\sqrt{-1}}{2}\, \partial\bar{\partial}\|z\|^2 = \frac{\sqrt{-1}}{2} \left\{ \sum_{j=1}^m dz_j \wedge d\bar{z}_j \right\} \\
&V^k = \underbrace{V^1 \wedge V^1 \wedge \ldots \wedge V^1}_{k}, \quad V^0 = 1, \\
&\Delta = 4\left\{\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \ldots + \frac{\partial^2}{\partial z_m \partial \bar{z}_m}\right\}, \quad \text{the Laplace operator.}
\end{aligned}\right\} \tag{3.1}$$

We shall from now on assume that N is a smooth connected n-dimensional projective variety and that L is a holomorphic line bundle on N and that R has property P with respect to L.

Let $F^1, \ldots, F^R \in H^0(N, L)$ be in normal position and put

$$F = F^1 \otimes \ldots \otimes F^R \in H^0(N, L^R).$$

Hence, we can find a positive bundle T, an integer $s > 0$ and an

$$H \in H^0(N, (L^R \otimes K_N)^s \otimes T^{-1})$$

such that $H \neq 0$. Let $\{a_i\} = a$ be a positive metric on T, and let ε be any positive number. We observe that

$$\alpha = (a|H|^2)^{s^{-1}}|F|^{-2} \prod_{j=1}^{R} [\log (\varepsilon \|F^j\|^2)]^{-2}$$

is a section of $K_N \otimes \bar{K}_N$ in $N \setminus |D_F|$, and therefore gives rise to a non-negative (n, n)-form W in $N \setminus |D_F|$, given locally by

$$W = \left(\frac{\sqrt{-1}}{2}\right)^n (a_i|H_i|^2)^{s^{-1}}|F_i|^{-2} \prod_{j=1}^{R} [\log (\varepsilon\|F^j\|^2)]^{-2} dw_1 \wedge d\bar{w}_1 \wedge \ldots \wedge dw_n \wedge d\bar{w}_n. \tag{3.2}$$

We are now in a position to formulate the main result of this section. For a related result see Carlson-Griffiths [2].

THEOREM 3.3. *There exist two positive numbers ε and c with the following property: For any domain M in $\mathbf{C}^m$ (open, connected set) and any nondegenerate holomorphic function $f\colon M \to N \setminus |D_F|$ the nonnegative function u_f in M defined by*

$$f^* W \wedge V^{m-n} = u_f V^m$$

is such that $\log u_f$ is plurisubharmonic and

$$\Delta \log u_f \geq c(u_f)^{1/n} \quad \text{in} \quad M \setminus \{z \in M\colon u_f(z) = 0\}. \tag{3.4}$$

Proof. Let W_1 be any (n, n)-form on N given locally by

$$W_1 = \left(\frac{\sqrt{-1}}{2}\right)^n h dw_1 \wedge d\bar{w}_1 \wedge \ldots \wedge dw_n \wedge d\bar{w}_n$$

and let Q be the class of subsets of $\{1, \ldots, m\}$ containing exactly n elements. If we write $f = (f_1, \ldots, f_n)$ in local coordinates, then one finds easily that

$$f^* W_1 \wedge V^{m-n} = h \circ f \left\{ \sum_{S \in Q} \left| \det \left(\frac{\partial f_i}{\partial z_j} \right)_{j \in S}^{1 \leq i \leq n} \right|^2 \right\} V^m. \tag{3.5}$$

Since f is assumed to be nondegenerate, we have that

$$\sum_{S\in Q}\left|\det\left(\frac{\partial f_i}{\partial z_j}\right)_{j\in S}^{1\leq i\leq n}\right|^2 = J_f$$

is not identically zero. By using Corollary 1.6.8 in Hörmander [5], we find that $\log J_f$ is plurisubharmonic. To show (3.4) it is therefore sufficient to prove that there exists a $c > 0$ such that the inequality

$$c(u_f)^{1/n} \leq \Delta \log \{(a_i \circ f|H_i \circ f|^2)^{s^{-1}}|F_i \circ f|^{-2}\prod_{r=1}^{R} \log (\varepsilon\|F^r \circ f\|^2)^{-2}\}$$

holds in $M_1 = M \setminus \{z \in M: u_f(z) = 0\}$. Since both $\log |H_i \circ f|^2$ and $\log |F_i \circ f|^2$ are harmonic in M_1 it is enough to show that

$$c(u_f)^{1/n} \leq u_0, \tag{3.6}$$

where we have put $u_0 = s^{-1}\{\Delta \log (a_i \circ f)\} - \sum_{r=1}^{R} \Delta \log (\log (\varepsilon\|F^r \circ f\|^2))^2$.

We start by choosing ε so small that $\varepsilon\|F^r\|^2 < 1$ in N for $1 \leq r \leq R$. This is possible by the compactness of N. We localize around $|D_F|$. From the assumptions about F and the compactness of N, it follows that we can cover $|D_F|$ with finitely many coordinate neighbourhoods $\{U\}$, with local coordinates $(w_1, \ldots, w_n)$, such that for some p, $1 \leq p \leq n$, there exist $k_1, \ldots, k_p$, $1 \leq k_1 < k_2 < \ldots < k_p \leq R$, with $w_1 = F_i^{k_1}, \ldots, w_p = F_i^{k_p}$, and if $k \notin \{k_1, \ldots, k_p\}$, then $\inf_{z\in U} \|F^k(z)\| > 0$. Furthermore, there exists a positive $b \in C^\infty(\bar{U})$, such that if $G \in H^0(N, L)$, then $\|G\|^2 = b|G_i|^2$. Without loss of generality we may assume that $w_j = F_i^j$ for $1 \leq j \leq p$. Since we have assumed that T is positive, there exists a number $c > 0$, such that

$$s^{-1}\Delta \log (a_i \circ f) \geq c\sum_{i=1}^{n}\sum_{k=1}^{m}\left|\frac{\partial f_i}{\partial z_k}\right|^2 . \tag{3.7}$$

If $v > 0$, then $\Delta \log v = v^{-1}\Delta v - 4v^{-2}\sum_{k=1}^{m} |\partial v/\partial z_k|^2$. We apply this relation to $v = (\log (\varepsilon\|F^r \circ f\|^2))^2$. Then we have in $f^{-1}(U) \cap M_1$

$$\begin{aligned}\Delta \log (\log (\varepsilon\|F^r \circ f\|^2))^2 &= 2 (\log (\varepsilon\|F^r \circ f\|^2))^{-1}\Delta \log \|F^r \circ f\|^2 - \\ &- 8 (\log (\varepsilon\|F^r \circ f\|^2))^{-2}\sum_{k=1}^{m}\left|\frac{\partial \log \|F^r \circ f\|^2}{\partial z_k}\right|^2 .\end{aligned} \tag{3.8}$$

It is obvious that there exists a number $C > 0$, such that if $1 \leq r \leq R$, then

$$|\Delta \log \|F^r \circ f\|^2| \leq C\sum_{i=1}^{n}\sum_{k=1}^{m}\left|\frac{\partial f_i}{\partial z_k}\right|^2$$

and if $p + 1 \leq r \leq R$, then

$$\sum_{k=1}^{m}\left|\frac{\partial \log \|F^r \circ f\|^2}{\partial z_k}\right|^2 \leq C\sum_{i=1}^{n}\sum_{k=1}^{m}\left|\frac{\partial f_i}{\partial z_k}\right|^2 .$$

Therefore, by combining the estimates above with (3.7) and by choosing ε sufficiently small, we get the following estimate of u_0, with possibly a different $c > 0$

$$u_0 \geq c \sum_{i=1}^{n} \sum_{k=1}^{m} \left| \frac{\partial f_i}{\partial z_k} \right|^2 + 8 \sum_{r=1}^{p} \sum_{k=1}^{m} (\log (\varepsilon \| F^r \circ f \|^2))^{-2} \left| \frac{\partial \log \| F^r \circ f \|^2}{\partial z_k} \right|^2 . \tag{3.9}$$

Now we have that

$$\frac{\partial \log \| F^r \circ f \|^2}{\partial z_k} = \frac{\partial \log b \circ f}{\partial z_k} + (w_r \circ f)^{-1} \frac{\partial f_r}{\partial z_k} .$$

We recall the following elementary inequality. If ξ_1 and ξ_2 are two complex numbers, then $|\xi_1 + \xi_2|^2 \geq 1/2 |\xi_1|^2 - |\xi_2|^2$. By using this inequality with $\xi_1 = \partial f_r / \partial z_k$ and $\xi_2 = \partial \log b \circ f / \partial z_k$ we find from (3.9) that

$$u_0 \geq c \sum_{i=1}^{n} \sum_{k=1}^{m} \left| \frac{\partial f_i}{\partial z_k} \right|^2 + 4 \sum_{r=1}^{p} \sum_{k=1}^{m} |w_r \circ f|^{-2} (\log (\varepsilon \| F^r \circ f \|^2))^{-2} \left| \frac{\partial f_r}{\partial z_k} \right|^2 - \\ - 8 \sum_{r=1}^{p} \sum_{k=1}^{m} (\log (\varepsilon \| F^r \circ f \|^2))^{-2} \left| \frac{\partial \log b \circ f}{\partial z_k} \right|^2 .$$

We may as before absorb the third term of the right hand side of the inequality above into the first term by choosing ε sufficiently small, and we are left with

$$u_0 \geq c \sum_{i=1}^{n} \sum_{k=1}^{m} \left| \frac{\partial f_i}{\partial z_k} \right|^2 + 4 \sum_{r=1}^{p} \sum_{k=1}^{m} |w_r \circ f|^{-2} (\log (\varepsilon \| F^r \circ f \|^2))^{-2} \left| \frac{\partial f_r}{\partial z_k} \right|^2 \geq c \sum_{i=1}^{n} \sum_{k=1}^{m} \alpha(i) \left| \frac{\partial f_i}{\partial z_k} \right|^2 ,$$

where we have put

$$\alpha(i) = \begin{cases} 1 + |w_i \circ f|^{-2} (\log (\varepsilon \| F^i \circ f \|^2))^{-2} & \text{if } 1 \leq i \leq p \\ 1 \quad \text{if } p + 1 \leq i \leq n. \end{cases}$$

Hence we can find a $c > 0$, such that

$$u_0 \geq c \sum_{S \in Q} \sum_{i=1}^{n} \sum_{k \in S} \alpha(i) \left| \frac{\partial f_i}{\partial z_k} \right|^2 . \tag{3.10}$$

From the inequality between the arithmetical and the geometrical mean we have

$$\sum_{i=1}^{n} \sum_{k \in S} \alpha(i) \left| \frac{\partial f_i}{\partial z_k} \right|^2 \geq \left(\prod_{i=1}^{n} \sum_{k \in S} \alpha(i) \left| \frac{\partial f_i}{\partial z_k} \right|^2 \right)^{1/n} = \left(\prod_{i=1}^{n} \alpha(i) \right)^{1/n} \left(\prod_{i=1}^{n} \sum_{k \in S} \left| \frac{\partial f_i}{\partial z_k} \right|^2 \right)^{1/n} .$$

Now we have from Hadamard's inequality that

$$\prod_{i=1}^{n} \sum_{k \in S} \left| \frac{\partial f_i}{\partial z_k} \right|^2 \geq \left| \det \left(\frac{\partial f_i}{\partial z_k} \right)_{k \in S}^{1 \leq i \leq n} \right|^2 .$$

Since $\inf \{ \| F^r(z) \| : z \in U \} > 0$ for $p + 1 \leq r \leq R$, it follows that

$$\prod_{i=1}^{n} \alpha(i) \geq c\|F \circ f\|^{-2} \prod_{r=1}^{R} (\log (\varepsilon\|F^r \circ f\|^2))^{-2},$$

for some $c > 0$ in $f^{-1}(U)$. By combining this with (3.10) we get, with some $c > 0$

$$u_0 \geq c(\|F \circ f\|^{-2} \prod_{r=1}^{R} (\log (\varepsilon\|F^r \circ f\|^2))^{-2})^{1/n} \sum_{S \in Q} \left| \det \left(\frac{\partial f_i}{\partial z_k}\right)_{k \in S}^{1 \leq i \leq n} \right|^{2/n}.$$

From this inequality (3.6) follows by observing that the terms a_i and $|H_i|$ can be taken to be bounded and by applying the inequality $\sum x_i^{1/n} \geq (\sum x_i)^{1/n}$ to the last factor of the inequality above. To complete the proof we have to show that we can choose ε such that $\log u_f$ is also plurisubharmonic, but that is done in the same way, and is therefore omitted.

We shall now turn to some consequences of Theorem 3.3. Suppose $p_0 \in N \setminus |D_F|$ and that there exists an $H \in H^0(N, (L^R \otimes K_N)^s \otimes T^{-1})$ with $H(p_0) \neq 0$. Fix a coordinate neighbourhood U around p_0 with local coordinates $(w_1, \ldots, w_n)$. If $f: B^m(r) \to N$ is a holomorphic mapping, where $B^m(r) = \{z \in \mathbf{C}^m: \|z\| < r\}$, with $f(0) = p_0$, then we put

$$J_f(z) = \sum_{S \in Q} \left| \det \left(\frac{\partial f_i}{\partial z_k}\right)_{k \in S}^{1 \leq i \leq n} \right|^2, \quad \text{for } z \in f^{-1}(U).$$

Here Q is the set of all subsets of $\{1, \ldots, m\}$ containing n elements and we have written $f = (f_1, \ldots, f_n)$ in local coordinates $(w_1, \ldots, w_n)$.

As an application of Theorem 3.3 we shall prove the following generalization of Landau's theorem (cf. Kodaira [7] and Carlson-Griffiths [2]).

THEOREM 3.11. *To each $m \geq n$ there exists a constant $r_0 > 0$ with the following properties: For any holomorphic mapping $f: B^m(\varrho) \to N \setminus |D_F|$ with $f(0) = p_0$ and $J_f(0) \geq 1$, the intequality $\varrho \leq r_0$ holds.*

From this result we deduce the following variant of the (small) Picard theorem.

COROLLARY 3.12. *Any holomorphic mapping $f: \mathbf{C}^m \to N \setminus |D_F|$ is degenerate.*

Proof of Corollary 3.12. Pick a $G \in H^0(N, (L^R \otimes K_N)^s \otimes T^{-1})$ with $G \neq 0$. Let M be the set of all points $z \in \mathbf{C}^m$ such that $f_*: T_z(\mathbf{C}^m) \to T_{f(z)}(N)$ is surjective. Suppose that M is not empty. Then M is open and hence $f(M)$ is open in N. Therefore $f(M)$ contains a point p_0 such that $G(p_0) \neq 0$. We may assume $f(0) = p_0$ and $0 \in M$. We can choose a coordinate system around p_0 such that $J_f(0) = 1$. Theorem 3.11 implies that f can be defined only in a bounded domain and this contradiction finishes the proof.

Proof of Theorem 3.11. We wish to each $t > 0$ construct a positive infinitely differentiable function V_t in $B^m(t)$ with the following properties:

(a) $\Delta \log V_t \leq V_t^{1/n}$,

(b) $\lim_{z \to z_0} V_t(z) = \infty$ for all $z_0 \in \partial B^m(t)$,

(c) $\lim_{r \to t} V_r(z) = V_t(z)$ for each fixed $z \in B^m(r)$,

(d) $\lim_{t \to \infty} V_t(0) = 0$.

If $t > 0$ is given, then we put $g_t(r) = n \log (8n(m+1)) + \log (t^{2n}(t^2 - r^2)^{-2n})$, $0 \leq r < t$. We construct the family $\{V_t\}$ by putting

$$V_t(z) = \exp g_t(r), \quad \|z\| = r.$$

We now have

$$\Delta \log V_t(z) = r^{-2m+1} \frac{d}{dr}\left(r^{2m-1} \frac{d}{dr}(g_t(r))\right) =$$

$$= 4n \frac{2m(t^2 - r^2) + 2r^2}{(t^2 - r^2)^2} < 8n(m+1)t^2(t^2 - r^2)^{-2} = (V_t(z))^{1/n},$$

and hence the family $\{V_t\}$ satisfies (a). The rest is straightforward verification. Next we consider the function u_f as constructed in Theorem 3.3. From the assumptions about f, it follows that there exists a number $B > 0$, such that

$$u_f(0) \geq B, \quad \text{for all } f \in \mathcal{F}_\varrho \tag{3.13}$$

where $\mathcal{F}_\varrho$ is the set of all holomorphic mappings $f\colon B^m(\varrho) \to N \setminus |D_F|$, satisfying $f(0) = p_0$ and $J_f(0) \geq 1$. Let $\mathcal{H}_\varrho$ be the set of all nondegenerate holomorphic mappings $f\colon B^m(\varrho) \to N \setminus |D_F|$. We want to show, that there exists a number $A > 0$, such that for all $\varrho > 0$ and all $f \in \mathcal{H}_\varrho$ we have

$$u_f \leq A V_\varrho. \tag{3.14}$$

We prove relation (3.14) with the aid of a classical trick due to Ahlfors [1]. From property (c) of V_t and the continuity of u_f, it follows that to each $t < \varrho$, there exists a point $\xi \in B^m(t)$, such that

$$u_f(\xi)/V_t(\xi) = \sup \{u_f(z)/V_t(z) \colon z \in B^m(t)\}.$$

Since f is nondegenerate, we must have that $u_f(\xi) > 0$, and hence u_f is infinitely differentiable in a neighbourhood of ξ. Moreover, since ξ is a local maximum for $\log (u_f/V_t)$, we have

$$\Delta \log (u_f/V_t)(\xi) \leq 0.$$

Now we have from Theorem 3.3 that

$$(u_f(\xi))^{1/n} \leq C\Delta \log u_f(\xi) \leq C\Delta \log V_t(\xi) \leq C(V_t(\xi))^{1/n},$$

and hence $u_f(\xi) \leq A V_t(\xi)$. Therefore, we have for all $z \in B^m(\varrho)$

$$u_f(z) \leq A \lim_{t \uparrow \varrho} V_t(z) = A V_\varrho(z).$$

This proves relation (3.14). In particular, we have from (3.13) that if $f \in \mathcal{F}_\varrho$, then

$$B \leq u_f(0) \leq A V(0) = A\{8n(m+1)\}^n \varrho^{-2n},$$

and this gives

$$\varrho \leq (A/B)^{1/2n} 8n(m+1),$$

which completes the proof of Theorem 3.11.

From the proof of Theorem 3.11 we have the following corollary.

COROLLARY 3.15. *To each* $m \geq n$ *there exists a constant* $K > 0$ *with the following property: For all domains* M *in* $\mathbf{C}^m$ *and all holomorphic mappings* $f: M \to N \setminus |D_F|$ *the inequality*

$$u_f(z) \leq K(d(z, M))^{-2n} \tag{3.15}$$

holds, where $d(z, M) = \operatorname{dist}\{z, \partial M\}$ *and* f *is nondegenerate.*

Proof. We continue using the notation of the proof of Theorem 3.11. It is sufficient to treat the case $z = 0$. For each $\varrho < d(0, M)$ we have $f \in \mathcal{H}_\varrho$ and from (3.14) we have

$$u_f(0) \leq A V_\varrho(0) = A\{8n(m+1)\}^n \varrho^{-2n}.$$

Letting $\varrho \to d(0, M)$ we have (3.16), and this finishes the proof.

It is possible to prove Theorem 3.11 along different lines.

Second proof of Theorem 3.11. We continue using the notation of the previous proof of Theorem 3.11. Let G be the Green function of $B^m(r)$ with pole at 0. This function is given by

$$G(z) = g(\|z\|) = \begin{cases} \log \dfrac{r}{\|z\|} & \text{if } m = 1 \\ \|z\|^{2-2m} - r^{2-2m} & \text{if } m \geq 2. \end{cases}$$

We have from the Riesz representation formula applied to the subharmonic function $\log u_f$ that there exists a positive measure μ_f on $B^m(\varrho)$, such that for all r, $0 < r < \varrho$,

$$\log u_f(0) = \sigma_m^{-1} \int_{\partial B^m(1)} \log u_f(r\xi) d\sigma(\xi) - \beta_m \int_{B^m(r)} G(z) d\mu_f(z). \tag{3.16}$$

Here $d\sigma$ is the area measure on $\partial B^m(1)$, σ_m the area of $\partial B^m(1)$, $\beta_1 = (2\pi)^{-1}$ and $\beta_m = \{\sigma_m(2m-2)\}^{-1}$ if $m \geq 2$. From Theorem 3.3 we have $d\mu_f \geq c(u_f)^{1/n} V^m$. Put $P_f(r) = \int_{B^m(r)} (u_f)^{1/n} V^m$. Hence we have

$$\beta_m \int_{B^m(r)} G(z) d\mu_f(z) \geq c\beta_m \int_0^r P_f'(t) g(t) dt.$$

A partial integration gives that

$$\beta_m \int_0^r P_f'(t) g(t) dt = \alpha_m \int_0^r P_f(t) t^{1-2m} dt,$$

where α_m is a constant. Recalling (3.13) and (3.16) we have

$$\log B + c\alpha_m \int_0^r P_f(t) t^{1-2m} dt \leq \sigma_m^{-1} \int_{\partial B^m(1)} \log u_f(r\xi) d\sigma(\xi). \tag{3.17}$$

Arguing as in Kodaira [7], one can from inequality (3.17) finish the proof. We omit the argument.

4. The generalized Picard Theorem

Let $B = \{z \in \mathbf{C}: |z| < 3\}$, $B^* = B \setminus \{0\}$, $D^* = B^* \times \underbrace{B \times \ldots \times B}_{m-1}$ and $D = B \times \ldots \times B$. We next prove Theorem 1.2.

Proof of Theorem 1.2. Since the set of singular points of S are of codimension ≥ 2, we may assume that S is nonsingular. Localizing, we can assume that f is defined in D^*. We have to show that the pull back of the rational functions on N can be extended to meromorphic functions on D. There exists by Kodaira [6] a positive line bundle E on N, such that if $\{s^0, \ldots, s^q\}$ is a basis of $H^0(N, E)$, then the mapping $F: N \to P_q$, defined by

$$F: z \to [s^0(z): \ldots : s^q(z)] \tag{4.1.}$$

is an imbedding. Let $s \in H^0(N, E)$, $s \neq 0$, and put $G = f^{-1}(|D_s|)$. We start by demonstrating that $\bar{G}$ is an analytic subset of D. We may assume that $\|s\| \leq 1$. It follows from Bishop's Theorem (Theorem F of Stolzenberg [8]), that it is sufficient to show that there exists a neighbourhood U of $\{z \in D: z_1 = 0\}$, such that

$$\int_{*s \cap U} V^{m-1} < \infty. \tag{4.2}$$

It follows from the proof of Theorem 3.3, that there exists a $\delta > 0$, such that the function $v_f = \log \|s \circ f\|^\delta u_f$ is plurisubharmonic in D^*. Without loss of generality, we may assume that there exists a z_1, such that $v_f(z_1, 0 \ldots 0) > -\infty$, otherwise we change our coordinate system. We can also assume, possibly after a change of the coordinate system, that if $|z_1| = 1$, $|z_2| \leq 1, \ldots, |z_m| \leq 1$, then $v_f(z_1, z_2, \ldots, z_m) > -\infty$. Let $B_{m-1} = \{\xi \in \mathbf{C}^{m-1}: \|\xi\| < 1\}$ and $n(r, \xi)$ be the number of roots, counted with multiplicity, of the equation $s(f(z_1, \xi)) = 0$ in $\{z_1: r < |z_1| < 1\}$. Fix $\xi \in B_{m-1}$ and put $v_\xi(z) = v_f(z, \xi)$. We can find a sequence $\{v_j\}_{j=1}^\infty$ of twice continuously differentiable subharmonic functions in B^*, such that $v_j \searrow v_\xi$. For each j, an application of Green's formula gives

$$\int_0^{2\pi} \frac{\partial}{\partial r} v_j(e^{\sqrt{-1}\vartheta})d\vartheta - r \int_0^{2\pi} \frac{\partial}{\partial r} v_j(re^{\sqrt{-1}\vartheta})d\vartheta = 4 \int_{r<|z_1|<1} \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} v_j(z_1) dV(z_1) .$$

Here dV is the planar Lebesgue measure on $\mathbf{C}$. Dividing by r and then integrating, we have

$$\begin{aligned} &\int_0^{2\pi} v_j(re^{\sqrt{-1}\vartheta})d\vartheta - \int_0^{2\pi} v_j(e^{\sqrt{-1}\vartheta})d\vartheta + \\ &+ (\log r^{-1}) \int_0^{2\pi} \frac{\partial}{\partial r} v_j(e^{\sqrt{-1}\vartheta})d\vartheta = \int_r^1 4t^{-1} \int_{t<|z_1|<1} \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} v_j(z_1) dV(z_1). \end{aligned} \tag{4.3}$$

It is clear that there exists a constant K, such that if $|z_1| = 1$, then $|v_j(z_1)| \leq K$ for all j. Moreover, this constant can be chosen independent of $\xi \in B_{m-1}$. The functions v_j can be taken as iterated mean values. From the explicit representation of the derivatives of such functions, see Helms [4, p. 20], it is clear that there exists a constant K, also independent of $\xi \in B_{m-1}$, such that if $|z_1| = 1$, then $|\partial v_j(z_1)/\partial r| \leq K$. Letting j tend to ∞ in such a way that the measures $(\partial^2/\partial z_1 \partial \bar{z}_1) v_j dV$ tend weakly to the Riez measure of v_ξ, we find that

$$\int_r^1 t^{-1} n(t, \xi) dt - \int_0^{2\pi} \log \|s \circ f(re^{\sqrt{-1}\vartheta}, \xi)\| d\vartheta \leq \int_0^{2\pi} \log u_f(re^{\sqrt{-1}\vartheta}, \xi) d\vartheta + K \log r^{-1}. \tag{4.4}$$

One consequence of (4.4) should be noted here. If we take $\xi = 0$, then an integration gives

$$\int_{r<|z_1|<1} v_f(z_1, 0) dV(z_1) \geq -K \int_r^1 t \log t^{-1} dt \geq -K. \tag{4.5}$$

It follows from Corollary 3.15, that $\log u_f(re^{\sqrt{-1}\vartheta}, \xi) \leq K \log r^{-1}$, with K independent of $\xi \in B_{m-1}$, and hence we have from (4.4)

$$\int_r^1 t^{-1}n(t, \xi)dt - \delta \int_0^{2\pi} \log \|s \circ f(re^{\sqrt{-1}\vartheta}, \xi)\| d\vartheta \leq K \log r^{-1}. \tag{4.6}$$

We get from (4.6) that

$$n(t, \xi) \leq K \quad \text{for all} \quad \xi \in B_{m-1}. \tag{4.7}$$

We may therefore assume that $v_f(z_1, 0) > -\infty$ for $0 < |z_1| \leq 1$. If $z \in B^*$, then we define $v_z(\xi) = v_f(z, \xi)$, $\xi \in B_{m-1}$. Let Δ_{m-1} be the Laplace operator on $\mathbf{C}^{m-1}$. Since v_z is subharmonic in B_{m-1}, we have by the Riesz representation formula

$$v_z(0) = \int_{\partial B_{m-1}} v_z(t) d\sigma_{m-1}(t) - \alpha_m^{-1} \int_{B_{m-1}} \Delta_{m-1} v_z(\xi) g_m(\xi) dV_{m-1}(\xi), \tag{4.8}$$

where $d\sigma_{m-1}$ is the surface measure on ∂B^{m-1}, normalized so $\int_{\partial B_{m-1}} d\sigma_{m-1} = 1$, dV_{m-1} is the Lebesgue measure on $\mathbf{C}^{m-1}$, $g_m(\xi) = |\xi|^{4-2m} - 1$, $m \geq 3$, $g_2(\xi) = \log |\xi|^{-1}$ and $\alpha_m = (2m-4)(2m-2) \int_{B_{m-1}} dV_{m-1}$ for $m \geq 3$ and $\alpha_2 = (2\pi)^{-1}$. Let $\mu(z)$ be the $2(m-2)$-volume of $G \cap \{z\} \times \{\xi \in \mathbf{C}^{m-1}: \|\xi\| < 1/2\}$. Since $g_m(\xi) \geq \lambda_m^{-1} > 0$ if $\|\xi\| < 1/2$, we have from (4.8) that

$$\mu(z) \leq K \log |z|^{-1} - \alpha_m \lambda_m v(z, 0).$$

If we now make use of (4.5), then an integration gives

$$\int_{r < |z| < 1} \mu(z) dV(z) \leq K \quad \text{for all} \quad r, \quad 0 < r < 1. \tag{4.9}$$

Now the relations (4.7) and (4.9) establish Bishop's condition (4.2).

Pick any two sections s_1 and s_2 of E, $s_2 \neq 0$. To complete the proof, it is sufficient to show that the meromorphic function $h = (s_1 \circ f_1)/(s_2 \circ f)$ has a meromorphic extension to D. In view of (4.2), we may assume that

$$f^{-1}(|D_{s_2}|) \subset \{z \in D: z_1 = 0\}.$$

Hence we may assume that h is analytic in D^*. Let g have the Laurent expansion

$$g(z_1, z_2, \ldots, z_m) = \sum_{j=-\infty}^{\infty} z_1^j A_j(z_2, \ldots, z_m).$$

Fix $\xi \in B_{m-1}$. We note that $|h| = \|s_1 \circ f\|/\|s_2 \circ f\|$, and since we may assume that $\|s_i\| \leq 1$, $i = 1, 2$, we have that $(\log^+ = \max(0, \log))$

$$\int_0^{2\pi} \log^+ |g(re^{\sqrt{-1}\vartheta}, \xi)| d\vartheta \leq - \int_0^{2\pi} \log \|s_1 \circ f(re^{\sqrt{-1}\vartheta}, \xi)\| d\vartheta \leq K \log r^{-1}, \tag{4.10}$$

where the last inequality follows from inequality (4.6). From the Nevanlinna theory it now follows that there exists a number $k \geq 0$ such that $A_j(\xi) = 0$ for all $j \leq -k$. If we for $r \in Z^+$ put $H(r) = \{\xi \in B_{m-1}: A_j(\xi) = 0$ for all $j \leq -r\}$, then we have just shown that $\bigcup_r H(r) = B_{m-1}$. Since the sets $H(r)$ are closed, at least one, $H(r_2)$ say, contains an open subset of B_{m-1}. Since the functions A_j are analytic, we have that $A_j = 0$ for $j \leq r_0$ and this proves Theorem 1.2.

Proof of Corollary 1.3. Let H be the hyperplane bundle of P_n and K the canonical bundle of P_n. Since $K = H^{-n-1}$, it is easily seen that if $R \in Z^+$ and $R > (n+1)/d$, then R has property P with respect to H^d.

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Received November 10, 1972;
in revised form June 30, 1973

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