Every sequence converging to O weakly in L_2 contains an unconditional convergence sequence[1]

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The aim of this paper is to prove the above statement, which is clearly equivalent to the following:

THEOREM. *For every sequence of measurable functions* f_n *with*

$$\int f_n^2 \leqq K \quad (n = 1, 2 \ldots)$$

there is a subsequence g_n *and a square integrable function* g *such that the sequence* $h_n = g_n - g$ *is an unconditional convergence sequence.*

Recall that a sequence h_n is called a convergence sequence, if the series $\Sigma c_n h_n$ is convergent almost everywhere, whenever the sequence c_n of real numbers satisfies $\Sigma c_n^2 < \infty$. The sequence h_n is called an unconditional convergence sequence, if every rearrangement of h_n is a convergence sequence. (E.g. the sequence r_n (on $[0, 1]$) of Rademacher functions is known to be an unconditional convergence sequence; while the sequence $\sqrt{2/\pi} \cdot \cos(nx)$ (on $[0, \pi]$) is a convergence sequence (Carleson), but — being a complete orthonormal sequence — it is not an unconditional covergence sequence.)

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[1] Throughout the paper all functions are measurable functions on some measure space $\{X, \mathcal{S}, \mu\}$. It is clear that it is sufficient to prove our Theorem in case of finite measure, thus we can take $\mu(X) = 1$.

As a rule, we do not indicate
the arguments of functions: writing φ, f etc. instead of $\varphi(x), f(x)$ etc., and $\mu(f > \lambda)$ instead of $\mu(\{x; f(x) > \lambda\})$,
and the measure: writing $\int \varphi$, $\int \varphi_1\varphi_2$ etc. instead of $\int_X \varphi(x)\mu(dx)$, $\int_X \varphi_1(x)\varphi_2(x)\mu(dx)$ etc.; we also say »almost everywhere» instead of »μ-almost everywhere».

$\alpha_n \underset{L_p}{\rightharpoonup} \alpha$ will stand for weak convergence in L^p.

§ 0. The preliminaries

We summarize shortly the previous results in this direction.

a) Convergence sequence

The following theorem is a classical result of Menchov (see e.g. [1] p. 156 or [2]):

THEOREM A. *Every orthonormal sequence contains a convergence sequence.*

Révész proved that orthogonality is not necessary, i.e.

THEOREM B. [3]. *For every L_2-bounded sequence f_n there is a subsequence g_n and a square integrable g such that the sequence $h_n = g_n - g$ is a convergence sequence.*

This theorem was independently proved also by Gaposhkin ([4] p. 12), and a very simple proof was given by Chatterji ([5] p. 243).

b) Unconditional convergence sequence

The following theorem is due to 0.A. Ziza:

THEOREM C. [6]. *If the orthonormal sequence f_n is pointwise bounded, i.e.*

$$|f_n(x)| \leqq f(x)$$

where $f(x)$ is finite a.e., then it contains an unconditional convergence sequence.

Here, obviously, the strong restriction is not that of the orthogonality, but the boundedness.

The problem of extending this result to arbitrary orthonormal sequences is proposed e.g. in Uljanov's survey on solved and unsolved problems in the theory of trigonometric and orthonormal series [7] p. 54.

For the proof first we established some maximal inequalities for strongly multiplicative sequences, but I. Berkes remarked that the maximal inequalities of Billingsley would do the same.

§ 1. Billingsley's theorem on 4-multiplicative sequences

In his book [8], Billingsley proves some very useful maximal inequalities; here we state one of them in the special case we are going to use.

THEOREM D. [8] p. 87—89. *If for all integers $1 \leqq a \leqq b < A \leqq B$ the sequence ψ_n satisfies the inequalities*

$$\int (\sum_{k=a}^{b} \psi_k)^2 \leqq K_1 \sum_{k=a}^{b} c_k^2 \tag{1}$$

$$\int (\sum_{k=a}^{b} \psi_k)^2 (\sum_{l=A}^{B} \psi_l)^2 \leqq K_2 (\sum_{k=a}^{b} c_k^2)(\sum_{l=A}^{B} c_l^2), \tag{2}$$

(K_1 and K_2 are absolute constants, c_k are real numbers), then for all $1 \leqq a \leqq b$ and $\lambda > 0$

$$\mu(\max_{a \leqq t \leqq b} |\sum_{k=a}^{t} \psi_k| \geqq \lambda) \leqq 10^6 K_2 \frac{(\sum_{k=a}^{b} c_k^2)^2}{\lambda^4} + 4K_1 \frac{\sum_{k=a}^{b} c_k^2}{\lambda^2}. \tag{3}$$

As an immediate consequence of Theorem D, one has the following corollary of independent interest[2]):

COROLLARY 1. *If the sequence φ_n satisfies the following four conditions for all different indices k, l, m, n:*

$$\int \varphi_k^2 \leqq K \tag{4}$$

$$\int \varphi_k^2 \varphi_l^2 \leqq K^2 \tag{5}$$

$$\int \varphi_k^2 \varphi_l \varphi_m = 0 \tag{6}$$

$$\int \varphi_k \varphi_l \varphi_m \varphi_n = 0, \tag{7}$$

then it is an unconditional convergence sequence.

Indeed, (5), (6) and (7) imply (2) for the sequence $\psi_n = c_n \varphi_n$ with $K_2 = K^2$. Using (4) we get

$$\int (\sum_{k=a}^{b} \psi_k)^2 = \int \sum_{k=a}^{b} \psi_k^2 + 2 \int \sum_{a \leqq k < l \leqq b} \psi_k \psi_l \leqq K \sum_{k=a}^{b} c_k^2 + 2 \sqrt{\int (\sum_{a \leqq k < l \leqq b} \psi_k \psi_l)^2} \leqq$$
$$\leqq K \sum_{k=a}^{b} c_k^2 + 2K \sum_{k=a}^{b} c_k^2 = 3K \sum_{k=a}^{b} c_k^2,$$

i.e. (1) holds with $K_1 = 3K$. Thus, ψ_k satisfies (3), that is known to be sufficient in order that the series $\Sigma c_n \varphi_n$ converge almost everywhere, if only $\Sigma c_n^2 < \infty$ (see e.g. [11] pp. 1—4).

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[2]) Corollary 1 is similar to the Theorem in [9] (see also Theorem C in [10]); but it does not require the existence of the fourth moments.

Now φ_n is clearly an unconditional convergence sequence, since conditions (4), (5), (6) and (7) are invariant under rearrangements of φ_n.

Corollary 1 can be slightly sharpened as follows:

COROLLARY 2. *If the sequence* φ_n *satisfies the following two conditions:*

$$\text{(A)} \qquad \int \varphi_k^2 \leqq K, \quad \int \varphi_k^2 \varphi_l^2 \leqq K^2 \quad (1 \leqq k < l)$$

$$\text{(B)} \qquad \sum \left| \int \varphi_k \varphi_l \varphi_m \varphi_n \right| < \infty,$$

where the summation runs over all indices k, l, m, n *such that at least three of them are different, then it is an unconditional convergence sequence.*

Conditions (A) and (B) are invariant under rearrangements of φ_n thus it is sufficient to check that they imply that the sequence $\psi_n = c_n \varphi_n$ satisfies conditions (1) and (2).

Using the inequality

$$|c_k| \leqq \sqrt{\sum_{k=a}^{b} c_k^2} \quad (a \leqq k \leqq b)$$

one easily gets that (2) holds with $K_2 = K^2 + 4C$, where

$$C = \sum \left| \int \varphi_k \varphi_l \varphi_m \varphi_n \right|,$$

and using the inequality

$$\int \left(\sum_{k=a}^{b} c_k \varphi_k\right)^2 \leqq \sum_{k=a}^{b} c_k^2 \int \varphi_k^2 + 2 \sqrt{\int \left(\sum_{a \leqq k < l \leqq b} c_k c_l \varphi_k \varphi_l\right)^2}$$

we get that (1) holds with $K_1 = K + 2\sqrt{K^2 + 6C}$.

The proof of our Theorem will be based on Corollary 2.

§ 2. Proof of the Theorem

Two sequences H_n and Φ_n are said to be equivalent (this will be indicated by $H_n \sim \Phi_n$), if for an arbitrary sequence c_n of real numbers the two sets

$$\{x; \sum_{n=1}^{\infty} c_n H_n(x) \ \text{ is convergent}\}$$

and

$$\{x; \sum_{n=1}^{\infty} c_n \Phi_n(x) \quad \text{is convergent}\}$$

differ in only a set of zero measure, and this holds also for an arbitrary rearrangement $H_{p(k)}, \Phi_{p(k)}$ $(p(1), p(2), \ldots$ is a permutation of the numbers $1, 2, \ldots)$. It is clear that if $H_n \sim \Phi_n$ and $n_1 < n_2 < \ldots$ are positive integers, then the subsequences $H_{n_k} \sim \Phi_{n_k}$.

Now the basic step before applying Corollary 2, can be formulated as follows:

PROPOSITION. *For every sequence* H_n *with* $H_n \underset{L_2}{\rightharpoonup} 0$ *one can construct a sequence* Φ_n, *which is equivalent to a subsequence* $\bar{H}_n$ *of* H_n, *and has the following properties:*

(i) $\Phi_n \underset{L_2}{\rightharpoonup} 0$

(ii) $\Phi_n^2 \underset{L_1}{\rightharpoonup} \Phi^2$

with $0 \leqq \Phi < 1$ *a.e.*

(iii) $|\Phi_n| \leqq K_n$ *a.e.,*

where K_n *is some sequence of positive numbers.*

The Proposition will be proved in § 3. After having this proposition, we can complete the proof of the theorem as follows: Since f_n is L_2-bounded, it contains a subsequence f_n' such that $f_n' \underset{L_2}{\rightharpoonup} g$ with some square integrable g. Taking $H_n = f_n' - g$ we have $H_n \underset{L_2}{\rightharpoonup} 0$. Applying the Proposition, we get a subsequence $\bar{H}_n$ of H_n, and a sequence Φ_n with $\Phi_n \sim \bar{H}_n$ and Φ_n satisfies (i), (ii) and (iii).

Now we construct a subsequence φ_n of Φ_n satisfying (A) and (B). Define the sequence n_k of positive integers by induction as follows:

Since $0 \leqq \Phi < 1$,

$$\int \Phi_n^2 \to \int \Phi^2 < 1 \quad \text{and} \quad \int \Phi_n^2 \Phi^2 \to \int \Phi^4 < 1,$$

one can choose n_1 so large that for all $n \geqq n_1$

$$\int \Phi_n^2 < 1 \quad \text{and} \quad \int \Phi_n^2 \Phi^2 < 1.$$

Assume that $n_1 < n_2 < \ldots < n_k$ have already been defined in such a way that we have

$$\int \Phi_{n_j}^2 \Phi_{n_i}^2 < 1 \qquad 1 \leqq j < i \leqq k$$

$$\left| \int \Phi_{n_l}^2 \Phi_{n_j} \Phi_{n_i} \right| < \frac{1}{2^i} \qquad 1 \leqq l < j < i \leqq k$$

$$\left|\int \Phi_{n_l}\Phi^2_{n_j}\Phi_{n_i}\right| < \frac{1}{2^i} \qquad 1 \leqq l < j < i \leqq k$$

$$\left|\int \Phi_{n_l}\Phi_{n_j}\Phi^2_{n_i}\right| < \frac{1}{2^j} \qquad 1 \leqq l < j < i \leqq k$$

$$\left|\int \Phi_{n_j}\Phi_{n_i}\Phi^2\right| < \frac{1}{2^i} \qquad 1 \leqq j < i \leqq k$$

$$\left|\int \Phi_{n_m}\Phi_{n_l}\Phi_{n_j}\Phi_{n_i}\right| < \frac{1}{2^i} \quad 1 \leqq m < l < j < i \leqq k,$$

and try to define the number $n_{k+1} > n_k$ in such a way that the above six conditions hold if k is replaced by $k+1$, i.e. also for $i = k+1$.

Since $\Phi_{n_1}, \ldots, \Phi_{n_k}$ are bounded (by (iii)), for $n \to \infty$

$$\int \Phi^2_{n_j}\Phi^2_n \to \int \Phi^2_{n_j}\Phi^2 < 1, \quad j = 1, \ldots, k$$

thus

$$\int \Phi^2_{n_j}\Phi^2_{n_{k+1}} < 1, \quad j = 1, \ldots, k$$

if n_{k+1} is chosen sufficiently large. Since $\Phi_n \underset{L_2}{\rightharpoonup} 0$, the second, third, fifth and sixth conditions will be satisfied for $i = k+1$, if n_{k+1} is chosen large enough.

Now for $1 \leqq l < j \leqq k$

$$\left|\int \Phi_{n_l}\Phi_{n_j}\Phi^2_n\right| \leqq \left|\int \Phi_{n_l}\Phi_{n_j}\Phi^2\right| + \left|\int \Phi_{n_l}\Phi_{n_j}(\Phi^2_n - \Phi^2)\right| <$$
$$< \frac{1}{2^j} + \left|\int \Phi_{n_l}\Phi_{n_j}(\Phi^2_n - \Phi^2)\right|,$$

and since the second term in the right-hand side of this inequality tends to zero as $n \to \infty$, the fourth condition also holds, if n_{k+1} is chosen sufficiently large.

Choosing $\varphi_k = \Phi_{n_k}$, we obtained a sequence φ_k satisfying (A) and (B) with $K = 1$, thus φ_k is an unconditional convergence sequence, and (since $\bar{H}_n \sim \Phi_n$) so is $\bar{H}_{n_k}$, that will be the sequence h_k in our Theorem. Qu.e.d.

§ 3. Proof of the Proposition

We will use a lemma of Gaposhkin [4], p. 14 (which is also contained implicitely in [12], however it is not explicitely stated there):

LEMMA. *If α_n is an L_1-bounded sequence:*

$$\int |\alpha_n| \leqq K,$$

then it contains a subsequence β_n *such that* β_n *can be written as*

$$\beta_n = \beta_n^{(1)} + \beta_n^{(2)},$$

where $\beta_n^{(1)}\beta_n^{(2)} \equiv 0$, $n = 1, 2, \ldots$, $\beta_n^{(1)}$ *is weakly convergent* $\beta_n^{(1)} \underset{L_1}{\rightharpoonup} \beta$, *and*

$$\sum_{n=1}^{\infty} \mu(\beta_n^{(2)} \neq 0) < \infty.$$

We will also use the following simple remark: if $\gamma_n \underset{L_p}{\rightharpoonup} \gamma$ and ψ is a bounded function, then

$$\psi\gamma_n \underset{L_p}{\rightharpoonup} \psi\gamma \quad (p \text{ is arbitary, } 1 \leqq p < \infty).$$

Further, if $|\gamma_n'| \leqq |\gamma_n|$, $\mu(\gamma_n' \neq \gamma_n) \to 0$, then $\gamma_n' \underset{L_p}{\rightharpoonup} \gamma$.

Now applying the Lemma (H_n^2 is L_1-bounded), one can choose a subsequence $\bar{H}_n$ of H_n such that

$$\bar{H}_n^2 = \beta_n^{(1)} + \beta_n^{(2)},$$

where

$$\beta_n^{(1)}\beta_n^{(2)} \equiv 0, \quad \beta_n^{(1)} \underset{L_1}{\rightharpoonup} \beta,$$

and

$$\sum_{n=1}^{\infty} \mu(\beta_n^{(2)} \neq 0) < \infty.$$

Define the sequence γ_n as follows

$$\gamma_n = \begin{cases} \bar{H}_n & \text{if } \beta_n^{(1)} \neq 0 \\ 0 & \text{if } \beta_n^{(1)} = 0. \end{cases}$$

Since $\gamma_n^2 = \beta_n^{(1)}$, we have

(*) $\gamma_n^2 \underset{L_1}{\rightharpoonup} \beta$.

Since $|\gamma_n| \leqq |\bar{H}_n|$, $\mu(\gamma_n \neq \bar{H}_n) \to 0$ and $\bar{H}_n \underset{L_2}{\rightharpoonup} 0$, we have

(**) $\gamma_n \underset{L_2}{\rightharpoonup} 0$.

Since

$$\sum_{n=1}^{\infty} \mu(\gamma_n \neq \bar{H}_n) < \infty,$$

thus $\gamma_n \sim \bar{H}_n$ (actually on almost all x, $\gamma_n = \bar{H}_n$ for $n > n_c = n_c(x)$).

Define

$$\varphi_n = \frac{\gamma_n}{\sqrt{1+\beta}}.$$

Since $0 \leqq 1/\sqrt{1+\beta} \leqq 1$ and $0 \leqq 1/(1+\beta) \leqq 1$, (*) and (**) imply $\varphi_n \underset{L_2}{\rightarrow} 0$ and $\varphi_n^2 \underset{L_1}{\rightarrow} \beta/1+\beta$. Now put

$$\Phi_n = \begin{cases} \varphi_n & \text{if} \quad |\varphi_n| \leqq 2^n \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mu(\Phi_n \neq \varphi_n) \leqq \int |\varphi_n|/2^n$, thus

$$\sum_{n=1}^{\infty} \mu(\Phi_n \neq \varphi_n) < \infty,$$

and hence

$$\Phi_n \sim \varphi_n,$$

$$\Phi_n \underset{L_2}{\rightarrow} 0$$

and

$$\Phi_n^2 \underset{L_1}{\rightarrow} \frac{\beta}{1+\beta}.$$

Since $|\Phi_n| \leqq 2^n$, Φ_n satisfies all conditions of the Proposition.

Let us mention that the above transformation (dividing by $\sqrt{1+\beta}$) makes it possible to choose a subsequence of Φ_n with $\int \Phi_{n_k}^2 \Phi_{n_l}^2 < K$ (that was done in the proof of the Theorem). But (though $\int \Phi^4 < 1$) it cannot ensure the existence of higher moments, e.g. $\int \Phi_{n_k}^4 \rightarrow \infty$ as $k \rightarrow \infty$ can still happen. Thus, the above used Billingsley's theorem cannot be replaced by any theorem using higher moments than second.

References

1. Alexits, G., *Convergence problems of orthogonal series.* Publishing House of the Hungarian Academy of Sciences, Budapest, 1961.
2. Menchov, D., Sur la convergence et la sommation des séries de fonctions orthogonales. *Bull. Soc. Math. France* 64 (1936), 147–170.
3. Révész, P.: On a problem of Steinhaus. *Acta Math. Acad. Sci. Hungar* 16 (1965), 310–318.
4. Gaposhkin, V. F., Lacunary series and independent functions. *Uspehi Mat. Nauk* 6/21 (1966), 3–82. (In Russian).
5. Chatterji, S. D., A general strong law. *Invent. Math.* 9 (1970), 235–245.
6. Ziza, O. A.: On a property of orthonormal sequences (in Russian). *Mat. Sbornik,* 55 (100) (1962), 3–16.

7. ULJANOV, P. L., Solved an unsolved problems in the theory of trigonometric and orthonormal series. *Uspehi Mat. Nauk* 19/1 (1964), 3–69.. (In Russian.)
8. BILLINGSLEY, P., *Convergence of probability measures.* John Wiley and Sons Inc., 1968.
9. KOMLÓS, J. & RÉVÉSZ, P., Remark to a paper of Gaposhkin. *Acta Sci. Math. Szeged* 33 (1972), 237–241.
10. —»— On the series $\Sigma c_k \varphi_k$. *Studia Sci. Math. Hungar* 7 (1972), 451–458.
11. GARSISA, A., *Topics in almost everywhere convergence.* Chicago, 1970.
12. KOMLÓS, J., A generalization of a problem of Steinhaus. *Acta Math. Acad. Sci. Hungar.* 18 (1967), 217–229.

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