A Banach space with basis constant > 1.

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If a Banach space has a Schauder basis β, then $\beta_K = \sup_{x,n} \|\sum_{i=1}^{n} a_i e^i\|/\|x\|$ exists, where $x = \sum_{i=1}^{\infty} a_i e^i$. Inf β_K taken over all β is called the basis constant of the Banach space. It is obvious that if the Banach space B has the basis constant p, then every finite-dimensional subspace C of B can be approximated by subspaces D_n of B – by approximating a set of basis vectors of C with vectors of finite expansions in some basis – such that each D_n can be embedded into a finite-dimensional subspace E_n of B, onto which there is a projection from B of norm arbitrarily close to p.

In this paper we construct a separable infinite-dimensional Banach space B with a two-dimensional subspace C_1 with the following properties: There is a $p > 1$ such that, if D is a two-dimensional subspace of B sufficiently close to C_1 and E is a finite-dimensional subspace of B containing D, then there is no projection from B onto E of norm $\leq p$. Thus the basis constant of this Banach space is $\geq p$. This seems to be by now the strongest result in negative direction on the well-known basis problem. The previously strongest result seems to be Gurarii's example of a Banach space where $\beta_K > 1$ for every β. (See Singer [1] pp. 218–42.)

We now start by giving a general and somewhat unprecise description of the ideas behind the construction and of the problems we meet. We consider a two-dimensional subspace C_1 of $l_\infty(\Gamma)$, where Γ is the set of pairs of positive integers. We assume that the projection constant of C_1 is > 1. Now our first ambition will be to embed C_1 in a larger space E_1, such that there is no projection of norm close to 1 from E_1 onto spaces close to C_1 and such that no subspace C_2 of E_1 containing a subspace of E_1 sufficiently close to C_1 has a projection constant near to 1. However, if we try to do this we have to get control of quite many linear spaces. In order to describe how we obtain the necessary simplifications, we give now a description of the way we estimate norms of projections.

Let us assume that C_1 is a two-dimensional subspace of a Banach space B and that $0, a_1, a_2, \ldots, a_n$ are $n+1$ points in C_1 such that $\|a_j\| = 1$; $j = 1, 2, \ldots, n$, $\|a_i - a_j\| \leq 1$; $i, j = 1, 2, \ldots, n$. We say that $b \in B$ is a *central point* for the $n+1$ points, if $\|b\| = \frac{1}{2}$ and $\|b - a_j\| = \frac{1}{2}$; $j = 1, 2, \ldots, n$. It is obvious that if b is the only central point in B for the $n+1$ points $0, a_1, a_2, \ldots, a_n$ and points far away from b are far from being central points, then a subspace of B which contains a subspace close to C_1 and onto which there is a projection from B of norm close to 1 must contain a vector near to b. For otherwise by such a projection b would be mapped onto a point whose distance to some of the points $0, a_1, a_2, \ldots, a_n$ would be essentially larger than $\frac{1}{2}$.

This description suggests how we can get control of the projection constants of subspaces of B containing a subspace close to C_1. It namely gives that we can restrict our attention to subspaces of B containing a subspace close to C_1 and containing a vector close to b. In our example we have in C_1 two sets $0, a_{11}, a_{12}, \ldots, a_{1n}$ and $0, a_{21}, a_{22}, \ldots, a_{2n}$ with unique central points b_1 and b_2 and so we can restrict our attention to subspaces of B containing vectors close to b_1 and b_2.

The way in which we will build up our Banach space is as follows. We start with a subspace C_1 of $l_\infty(\Gamma)$, generated by two vectors e^1 and e^2. In C_1 we have two point sets with unique central points $\frac{1}{2}e^3$ and $\frac{1}{2}e^4$ in B. The construction is made so that for every two-dimensional subspace D of $l_\infty(\Gamma)$ sufficiently close to C_1 there are point sets of D such that $\frac{1}{2}e^3$ and $\frac{1}{2}e^4$ are the unique central points in B for these point sets and points far from $\frac{1}{2}e^3$ and $\frac{1}{2}e^4$ are far from being central points for these point sets. This is important since it gives that there will be no accumulation of approximations in our construction. Now in every subspace of B sufficiently close to that generated by e^3 and e^4 there are point sets with the unique central points $\frac{1}{2}e^5$ and $\frac{1}{2}e^6$. The facts just described are part of Lemma 2 below and give an idea of how the discussion goes on. We will finally conclude that if a subspace E of B contains a subspace close to C_1 and there is a projection from B onto E of norm close to 1, then E can not be finite-dimensional.

We now define the vectors e^j, $e^j \in l_\infty(\Gamma)$, which generate B. We let $e^j_{r,k}$ denote the component of e^j which corresponds to the pair (r, k).

We choose $e^1_{1,k} = 0.611 - (k-1) \cdot 0.001$, $k = 1, 2, \ldots, 12$. We then choose $e^2_{1,1} = -0.600$ and choose $e^2_{1,k}$, $k = 2, 3, \ldots, 12$ inductively by the equations $0.611 - (k-1) \cdot 0.001 - (0.9995 + 0.01(k-1))e^2_{1,k} = 0.611 - (k-2) \cdot 0.001 - (0.9995 + 0.01 \cdot (k-1))e^2_{1,k-1}$. By this choice we obtain that $e^2_{1,k}$ $k = 1, 2, \ldots, 12$ lies between -0.600 and -0.612 and that the component $(1, k)$ is the largest of the 12 components $(1, 1)$, $(1, 2), \ldots, (1, 12)$ of the vector $e^1 - (1 + 0.01 \cdot (k-1))e^2$, $k = 1, 2, \ldots, 12$. We also see that if v^1 and v^2 are sufficiently close to e^1 and e^2 in $l_\infty(\Gamma)$ then the component $(1, k)$ will be the largest of the 12 components $(1, 1)$, $(1, 2), \ldots, (1, 12)$ of $v^1 - (1 + 0.01 \cdot (k-1))v^2$, $k = 1, 2, \ldots, 12$. We put $e^1_{1,k} = e^2_{1,k} = 0$ if $k > 12$. Components (r, k) with $r > 1$ will be defined later. We will have need for such components to save the uniqueness of central points.

We will now define e^3 and e^4 so that $\frac{1}{2}e^3$ is the unique central point in B for the 7 points $0, (e^1 - (1 + 0.01(k-1))e^2)/\|e^1 - (1 + 0.01(k-1))e^2\|, \ k = 1, 2, \ldots, 6$ and so that $\frac{1}{2}e^4$ is the unique central point in B for the 7 points 0, $(e^1 - (1 + 0.01(k-1))e^2)/\|e^1 - (1 + 0.01(k-1))e^2\|, \ k = 7, 8, \ldots, 12$ and so that this holds even if e^1 and e^2 are replaced by approximating vectors v^1 and v^2 in B. To this end we put $e^3_{1,1} = e^3_{1,2} = \ldots = e^3_{1,6} = 1$ and we choose $e^3_{1,7}, e^3_{1,8}, \ldots, e^3_{1,12}$ in such a way that:

(a) the distance in l_∞-norm between the 6-vector $\frac{1}{2} \cdot (e^3_{1,7}, e^3_{1,8}, \ldots, e^3_{1,12})$ and each of the seven 6-vectors which consist of the components $(1, 7), (1, 8), \ldots, (1, 12)$ of the vectors $0, (e^1 - (1 + 0.01(k-1))e^2)/\|e^1 - (1 + 0.01(k-1))e^2\|, k = 1, 2, \ldots, 6$, is strictly less than $\frac{1}{2}$.

(b) the four 6-vectors $(e^1_{1,7}, e^1_{1,8}, \ldots, e^1_{1,12})$, $(e^2_{1,7}, e^2_{1,8}, \ldots, e^2_{1,12})$, $(e^3_{1,7}, e^3_{1,8}, \ldots, e^3_{1,12})$ and $(1, 1, 1, 1, 1, 1)$ are linearly independent.

We need (a) since $\frac{1}{2}e^3$ shall be a central point even if we approximate e^1 and e^2 by v^1 and v^2 and (b) since $\frac{1}{2}e^4$ defined below shall be a unique central point. We put $e^4_{1,7} = e^4_{1,8} = \ldots = e^4_{1,12}$ and choose $e^4_{1,1}, e^4_{1,2}, \ldots, e^4_{1,6}$ in such a way that:

(a_1) the distance in l_∞-norm between the 6-vector $\frac{1}{2} \cdot (e^4_{1,1}, e^4_{1,2}, \ldots, e^4_{1,6})$ and each of the seven 6-vectors which consist of the components $(1, 1), (1, 2), \ldots, (1, 6)$ of the vectors $0, (e^1 - (1 + 0.01(k-1))e^2)/\|e^1 - (1 + 0.01(k-1))e^2\|, k = 7, 8, \ldots, 12$, is strictly less than $\frac{1}{2}$.

(b_1) the four 6-vectors $(e^j_{1,1}, e^j_{1,2}, \ldots, e^j_{1,6})$, $j = 1, 2, 4$ and $(1, 1, 1, 1, 1, 1)$ are linearly independent.

As above we need (a_1) since $\frac{1}{2}e^4$ shall be a central point even if we approximate e^1 and e^2 by v^1 and v^2 and (b_1) since $\frac{1}{2}e^3$ shall be a unique central point.

For $k \geq 1$ we now choose $e^3_{1,12+k} = e^1_{1,k}$ and $e^4_{1,12+k} = e^2_{1,k}$. We see that this does not destroy that $\frac{1}{2}e^3$ and $\frac{1}{2}e^4$ are central points for the sets described above. We also obtain that the components $(1, 1), (1, 2), \ldots, (1, 12)$ of $e^3 - (1 + 0.01(k-1)) \cdot e^4$, $k = 1, 2, \ldots, 12$ are all < 0.25 and that the component $(1, 12 + k)$ is the largest of the components $(1, 13), (1, 14), \ldots, (1, 24)$ of $e^3 - (1 + 0.01(k-1)) \cdot e^4$, $k = 1, 2, \ldots, 12$, and is > 1. The last sentence is obviously true even if e^3 and e^4 are replaced by approximating vectors v^3 and v^4 in $l_\infty(\Gamma)$.

We now choose $e^5_{1,k}, e^6_{1,k}$ $k = 1, 2, \ldots, 12$, to be all in the interval $(0.05, 0.10)$ and choose them in such a way that the six 6-vectors $(e^j_{1,1}, e^j_{1,2}, \ldots, e^j_{1,6})$, $j = 1, 2, \ldots, 6$, are linearly independent and such that the six 6-vectors $(e^j_{1,7}, e^j_{1,8}, \ldots, e^j_{1,12})$ are linearly independent. This is done in order to make $\frac{1}{2}e^3$ and $\frac{1}{2}e^4$ unique central points. For all $k \geq 1$ we put $e^5_{1,12+k} = e^3_{1,k}$ and $e^6_{1,12+k} = e^4_{1,k}$.

Now for all $k \leq 12j$, $j \geq 1$, we put $e^{6+(2j-1)}_{1,k} = 0$ and for all $k \geq 1$ and $j \geq 1$ we put $e^{6+(2j-1)}_{1,12j+k} = e^5_{1,k}$. For all $k \leq 12j$, $j \geq 1$, we put $e^{6+2j}_{1,k} = 0$ and for all $k \geq 1$ and $j \geq 1$ we put $e^{6+2j}_{1,12j+k} = e^6_{1,k}$.

Now we have to introduce components (r, k) where $r > 1$. For if we did not do that, we would only have obtained the following: $\frac{1}{2}e^{2j+1}$ and $\frac{1}{2}e^{2j+2}$ are both central points for sets consisting of 0 and linear combinations of e^{2j-1} and e^{2j}. This holds even if e^{2j-1} and e^{2j} are approximated by vectors v^{2j-1} and v^{2j} in B. But $\frac{1}{2}e^{2j+1}$ and $\frac{1}{2}e^{2j+2}$ will not be unique central points. For certain linear combinations of e^i: s, $i \leq 2j-2$ or $i \geq 2j+5$ could be added to $\frac{1}{2}e^{2j+1}$ and $\frac{1}{2}e^{2j+2}$ without destroying their property of being central points.

For all $j \geq 1$ and $k \geq 1$, $e^j_{[(j+3)/2],k} = 1$, $e^{2j-1}_{j+2,k} = 0.1$ and $e^{2j}_{j+2,k} = -0.1$. The definition of $e^{2j-1}_{j+2,k}$ and $e^{2j}_{j+2,k}$ is made so that $\frac{1}{2}e^{2j+1}$ and $\frac{1}{2}e^{2j+2}$ are central points even if e^{2j-1} and e^{2j} are approximated by vectors v^{2j-1} and v^{2j} in B. For those triplets (j, r, k) where $e^j_{r,k}$ is not yet defined, we put $e^j_{r,k} = 0.1 \cdot (-1)^{[(k-1/2)]}$. This last definition will save the uniqueness of $\frac{1}{2}e^{2j+1}$ and $\frac{1}{2}e^{2j+2}$ with respect to linear combinations of e^i: s, $i \leq 2j-2$ or $i \geq 2j+5$. For if we add such a linear combination to $\frac{1}{2}e^{2j+1}$ or $\frac{1}{2}e^{2j+2}$ the distance to 0 will be $> \frac{1}{2}$.

We now resume in three lemmas the immediate consequences of our definitions of the vectors e^j, $j \geq 1$.

LEMMA 1. *There exists an* $\varepsilon > 0$ *such that if* $\|v^{2j-1} - e^{2j-1}\| \leq \varepsilon$ *and* $\|v^{2j} - e^{2j}\| \leq \varepsilon$ *holds for two vectors* v^{2j-1} *and* v^{2j} *in* B, $j \geq 1$, *then* $v^{2j-1} - (1 + 0.01(k-1))v^{2j}$ *has its largest component on the same place as* $e^{2j-1} - (1 + 0.01(k-1))e^{2j}$, $k = 1, 2, \ldots, 12$, *namely some component* (r, m) *with* $r = 1$, *and the distance between* $\frac{1}{2}e^{2j+1}$ *and each of the elements* 0, $(v^{2j-1} - (1 + 0.01(k-1))v^{2j})/\|v^{2j-1} - (1 + 0.01(k-1))v^{2j}\|$, $k = 1, 2, \ldots, 6$, *is* $\frac{1}{2}$. *Obviously a similar result holds for* $\frac{1}{2}e^{2j+2}$ *for* $k = 7, 8, \ldots, 12$.

LEMMA 2. *Let* v^{2j-1} *and* v^{2j} *be as in Lemma 1. Then* $\frac{1}{2}e^{2j+1}$ *is the only vector of* B *such that the distance between the vector and each of the vectors* 0, $(v^{2j-1} - (1 + 0.01(k-1))v^{2j})/\|v^{2j-1} - (1 + 0.01(k-1))v^{2j}\|$, $k = 1, 2, \ldots, 6$, *is* $\frac{1}{2}$ *and there are positive constants* K *and* ε_1 *such that if* $g \in B$ *and* $\|g - e^{2j+1}\| = \delta$, *then the distance between* $\frac{1}{2}g$ *and some of the*

$$0,\ (v^{2j-1} - (1 + 0.01(k-1))v^{2j})/\|v - (1 + 0.01(k-1))v^{2j}\|,\quad k = 1, 2, \ldots, 6,$$

is at least $\frac{1}{2} + \frac{1}{2}K \min(\delta, \varepsilon_1)$. *A similar result holds for* $\frac{1}{2}e^{2j+2}$ *for* $k = 7, 8, \ldots, 12$.

Proof. The lemma holds since the six 6-vectors $(e^{2j-2+t}_{1,12j+1}, e^{2j-2+t}_{1,12j+2}, \ldots, e^{2j-2+t}_{1,12j+6})$, $t = 1, 2, \ldots, 6$, are linearly independent and since $(j+2, k)$, $k \geq 1$, are pairs where $e^{2j+1}_{j+2,k} = 1$. The last property gives that $\|e^{2j+1} + h\| \geq \|e^{2j+1}\| + 0,1 \cdot \|h\|$ if h is a linear combination of vectors e^i, $i \leq 2j-2$ or $i \geq 2j+5$.

We can assume that ε, ε_1 and K in Lemma 1 and 2 are the same for e^{2j+1} and e^{2j+2}.

LEMMA 3. *Let* v^{2j-1} *and* v^{2j} *be as in Lemma* 1. *If* C *is a subspace of* B, $v^{2j-1} \in C$ *and* $v^{2j} \in C$, *and* $\inf_{g \in C} \|g - e^{2j+1}\| = \delta$, *then there is no projection from the space* D *generated by* C *and* e^{2j+1} *onto* C *of norm* $< 1 + K \cdot \min(\delta, \varepsilon_1)$ *where* K *and* ε_1 *are defined as in Lemma* 2. *A similar result holds for* e^{2j+2}.

Proof. The lemma holds since by any projection from D onto C the vector $\frac{1}{2}e^{2j+1}$ is mapped onto a vector whose distance to some of the elements $0, (v^{2j-1} - (1 + 0.01(k-1))v^{2j})/\|v^{2j-1} - (1 + 0.01(k-1))v^{2j}\|$, $k = 1, 2, \ldots, 6$, is $\geq \frac{1}{2} + \frac{1}{2}K \min(\delta, \varepsilon_1)$.

We now prove that B has the basis constant $\geq p > 1$. We choose $p = 1 + \frac{1}{2}K \cdot \min(\varepsilon, \varepsilon_1)$ where ε is that of Lemma 1 and K and ε_1 are the same as in Lemma 2. Let D be a two-dimensional subspace of B generated by v^1 and v^2, v^1 and v^2 as in Lemma 1. If E is a subspace of B, $D \subset E$, such that there exists a projection from B onto E of norm $\leq p$, then by Lemma 3 E must contain vectors whose distances to e^3 and e^4 are $< \varepsilon$. Thus by Lemma 1 and Lemma 3 E must contain vectors whose distances to e^5 and e^6 are $< \varepsilon$. By induction we see that E must contain vectors whose distances to e^{2j-1} and e^{2j} are $< \varepsilon$ for every $j \geq 1$. Thus B has the basis constant $\geq p$.

Remark. It has been shown by J. Lindenstrauss that the construction above can be modified so that we get a uniformly convex Banach space isomorphic to Hilbert space with basis constant > 1.

References

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