Representations of tensor algebras as quotients of group algebras

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§ 0. Introduction

1. Tensor algebras, or to be precise, projective tensor products of $C(K)$-spaces have important relations both with Hilbert space and with l^1. The relation with Hilbert space was discovered by Grothendieck, and was called by him »the fundamental theorem on the metric theory of tensor products». It certainly is the deepest result in this metric theory [see e.g. 6]. The relations with l^1 were discovered by Varopoulos, and their importance lies in the fact that they relate the algebra structure of tensor algebras to the algebra structure of group algebras [8]. These relations are two-fold, in the first place, a group algebra can in a canonical way be embedded as a closed subalgebra of a tensor algebra. Through this embedding, information on tensor algebras can be obtained from information on group algebras. In the second place a tensor algebra can be represented as a quotient of a group algebra, so that information on tensor algebras can be transferred to group algebras. The main result in the second connection is the following; if $\{K_i\}_{i=1}^n$ are disjoint compact subsets of a compact abelian group, and if $\bigcup K_i$ is a Kronecker set (or a K_p-set), then $A(\sum K_i)$ is a tensor algebra. In this paper we shall consider to what extent the Kronecker condition in the theorem of Varopoulos can be replaced by Helson conditions on the sets. Our main results are the following.

THEOREM A. *To every natural number $n \geq 2$, there corresponds a real number α_n, such that if $\{K_i\}_{i=1}^n$ are disjoint compact subsets of a compact abelian group, and if $\bigcup K_i$ is a Helson-$(1-\alpha)$ set, $\alpha < \alpha_n$, then $A(\sum K_i)$ is a tensor algebra.*

THEOREM B. *To every natural number $q \geq 2$, and every natural number $n \geq 2$, there corresponds a real number $\alpha_{q,n}$ such that if $\{K_i\}_{i=1}^n$ are disjoint compact subsets* *of the group* D_q, *and* $\bigcup K_i$ *is a Helson-*$(\gamma_q - \alpha)$ *set, where* $\alpha < \alpha_{q,n}$ *and* $\gamma_q = \sin(\pi/q)/(\pi/q)$ *is the Helson constant of an infinite* K_q*-set, then* $A(\sum K_q)$ *is a tensor algebra.*

The proofs of these theorems are based on the metric theory of tensor algebras. The main tool in the proof of Theorem A is a rather elementary measure inequality with which we can study the dual space of a tensor algebra.

To prove Theorem B we shall combine the methods used to prove Theorem A, with a theorem of Bohnenblust and Karlin on the geometry of Banach algebras. However, besides this we shall also have to study various possible definitions of Helson-constants for subsets of D_q.

Loosely speaking, the proof of Theorem A is based on a notion of weak approximation, while the proof of Theorem B is based on uniform approximation.

§ 1. Tensor algebras in general groups

1. We shall start by some standard definitions and notations. Let G be a compact abelian group and let E be a closed subset of G. We shall denote by $A(E)$ the restriction of $A(G)$ to E. $A(E)$ can also be represented as the quotient $A(G)/I(E)$, where $I(E)$ is the closed ideal of all functions f in $A(G)$ with $f^{-1}(0) \supset E$. It follows from this representation that $A(E)$ is a Banach algebra with maximal ideal space E, and that every element f in $A(E)$, has an expansion

$$f(x) = \sum a_\chi \chi(x), \quad \chi \in \hat{G}, \quad \sum |a_\chi| < \|f\|_{A(E)} + \varepsilon, \quad x \in E\,. \tag{1.1.1}$$

The dual space of $A(E)$ will be denoted $PM(E)$, and its elements will be called pseudomeasures.

Let X be a compact space (all topological spaces considered in this paper will be assumed to be Hausdorff spaces). We shall denote by

(i) $\mathbf{1}_X$ the identity element of $C(X)$
(ii) $C(X)_1$ the unit ball of $C(X)$
(iii) $S(X)$ the group of all functions $f \in C(X)$ with $|f| = 1$
(iv) $S_q(X)$ the group of all functions $f \in C(X)$ with $f^q = 1$.

With the uniform topology, $S(X)$ is a topological (abelian) group under pointwise multiplication of functions, and $S_q(X)$ is a discrete subgroup of $S(X)$, which separates the points of X if and only if X is totally disconnected.

Let E be a closed subset of a compact abelian group G. We shall call E

a) a *Kronecker set,* if $\hat{G}|_E$ is dense in $S(E)$.

b) a K_q*-set,* if $\hat{G}|_E = S_q(E)$.

c) a *Helson-*(α) *set,* if $A(E) = C(E)$, and if for every f in $C(E)$

$$\|f\|_{C(E)} \geq \alpha \|f\|_{A(E)}\,,$$

or equivalently by duality, if for every measure μ with support on E,

$$\sup_{\chi \in \hat{G}} |\hat{\mu}(\chi)| \geq \alpha \|\mu\|_{M(E)} .$$

In dealing with Helson-(α) sets, $\alpha < 1$, we shall in this paper often make the technical assumption, that for every $\mu \in M(E)$, $\mu \neq 0$, we can in fact find $\chi \in \hat{G}$, such that $|\hat{\mu}(\chi)| > \alpha \|\mu\|_{M(E)}$. Notationally this assumption means that we can avoid certain ε-quantities, and conceptually it means, at most, that we consider a Helson-(α) set as a Helson-(α') set, for some $\alpha' < \alpha$.

Let $\{X_i\}_{i=1}^n$ be compact spaces. We shall denote their cartesian product by $\mathbf{X}$ and their disjoint union by X'. Let further $f_i \in C(X_i)$, we shall define functions $\mathbf{f} \in C(\mathbf{X})$ and $f' \in C(X')$ by resp.

$$\mathbf{f}(x) = \mathbf{f}(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \ldots f_n(x_n) \tag{1.1.2}$$

and

$$f'|_{X_i} = f_i . \tag{1.1.3}$$

We shall denote by

(i) $\prod(\mathbf{X})$ the set of all functions $\mathbf{f} \in C(\mathbf{X})$ defined by (1.1.2), such that $f_i \in C(X_i)_1$ for all i.

(ii) $V(\mathbf{X})$ the projective tensor product $C(X_1) \hat{\otimes} C(X_2) \hat{\otimes} \ldots \hat{\otimes} C(X_n)$ [8].

$V(\mathbf{X})$ is a semi-simple Banach algebra with maximal ideal space $\mathbf{X}$, and every element $F \in V(\mathbf{X})$, has a representation

$$\mathbf{F}(x) = \sum_{k=1}^{\infty} a_k \mathbf{f}_k(x), \quad \mathbf{f}_k \in \prod(\mathbf{X}), \quad \sum_{k=1}^{\infty} |a_k| < \|F\|_{V(\mathbf{X})} + \varepsilon .$$

It is well-known that convex linear combinations of $S(X)$ are dense in $C(X)_1$, so that in the above representation we may in fact assume $f_n \in \prod(\mathbf{X})$, $|f_n| = 1$.

An algebra isomorphic to some $V(\mathbf{X})$ will be called a tensor algebra.

The dual space of $V(\mathbf{X})$ will be denoted $BM(\mathbf{X})$, and is canonically identified with the space of continuous n-linear forms on $\prod_{i=1}^n C(X_i)$. An element of $BM(\mathbf{X})$ will be called a multimeasure of order n, a multimeasure of order 2 is called a bimeasure. For $\Lambda \in BM(\mathbf{X})$, we have $\|\Lambda\|_{BM} = \sup\{|\Lambda(f)| \mid f \in \prod(\mathbf{X})\}$, and we should like to point out, that in contrast to the similar case for measures, we can not turn the sup into a max by going over to Borel functions.

Let $\Lambda \in BM(\mathbf{X})$, and let $f_j \in C(X_j)$, $1 \leq j \leq n$, $j \neq i$. The functional on $C(X_i)$, taking $\varphi \in C(X_i)$ into $(f_1 \otimes f_2 \otimes \ldots \otimes f_{i-1} \otimes \varphi \otimes f_{i+1} \otimes \ldots \otimes f_n)$, is then linear and bounded and is by the Riesz representation theorem given by a measure on X_i which we shall denote by $\Lambda_i(f_1 \otimes \ldots f_{i-1} \otimes f_{i+1} \otimes \ldots \otimes f_n)$. It is clear that

$$\|\Lambda_i(f_1 \otimes \ldots \otimes f_{i-1} \otimes f_{i+1} \otimes \ldots \otimes f_n)\|_{M(X_i)} \leq \|\Lambda\|_{BM} \cdot \|f_1\|_\infty \cdots \|f_n\|_\infty .$$

We have thus interpreted Λ as an $(n-1)$-linear operator into $M(X_i)$. In the same way we can consider Λ as a multilinear operator from the products of any of the $C(\mathbf{X})$'s into the multimeasure space over the rest. It is clear that all these operators have the same norm, namely $\|\Lambda\|_{BM}$.

Let now $\Lambda \in BM(\mathbf{X})$, let $f_i \in C(X_i)_1$, and define $\mathbf{f} \in C(\mathbf{X})$ by (1.1.2). We shall denote

$$\lambda_i(\mathbf{f}) = \Lambda_i(f_1 \otimes \ldots \otimes f_{i-1} \otimes f_{i+1} \otimes \ldots \otimes f_n) . \tag{1.1.4}$$

Since we may consider each X_i as a subspace of X', we may also consider the λ_i's as (mutually singular) measures on X', and we can therefore define $\lambda'(\mathbf{f}) \in M(X')$ by

$$\lambda'(\mathbf{f}) = \lambda_1(\mathbf{f}) + \lambda_2(\mathbf{f}) + \ldots + \lambda_n(\mathbf{f}) . \tag{1.1.5}$$

We also observe that since we have for each X_i $\|\lambda_i(\mathbf{f})\|_{M(X_i)} \leq \|\Lambda\|_{BM}$ we have

$$\|\lambda'(\mathbf{f})\|_{M(X')} \leq n \; \|\Lambda\|_{BM(X)} .$$

It is also obvious, but nevertheless important, that $BM(\mathbf{X})$ is a module over each $C(X_i)$ if we define the multiplication as follows: Let $\Lambda \in BM(\mathbf{X})$ and let $e_i \in C(X_i)$, we define $e_i\Lambda$, by

$$(e_i\Lambda)(f_1 \otimes \ldots \otimes f_i \otimes \ldots \otimes f_n) = \Lambda(f_1 \otimes \ldots \otimes e_if_i \otimes \ldots \otimes f_n), \quad f_j \in C(X_j), \quad 1 \leq j \leq n . \tag{1.1.6}$$

Clearly we have then $\|e_i\Lambda\|_{BM} \leq \|e_i\|_\infty \cdot \|\Lambda\|_{BM}$.

The simplest example of a closed subset E of a group G, for which $A(E)$ is a tensor algebra is the following:

Let $E_i \subset G_i$ be Helson-(α_i) sets, and let $E = \prod E_i$, then $A(E) = A(\prod G_i)/I(E)$, is a tensor algebra, and for every f in $A(E)$ we have

$$\|f\|_{V(E)} \geq \prod \alpha_i \cdot \|f\|_{A(E)} .$$

The above example works however only in product groups, and the importance of the theorem of Varopoulos lies in the fact that it does not require the group to be a product. On the other hand it does require a »product set» in an arbitrary group, and such a thing is found in the following way.

Let $\{K_i\}_{i=1}^n$ be a set of closed subsets of a compact abelian group G, denoting their cartesian product by $\mathbf{K}$, we have $\mathbf{K} = \prod K_i \subset \prod G = G^n$. Let further s be the group addition map taking the point $(g_1, g_2, \ldots, g_n) \in G^n$ to the point $g_1 + g_2 + \ldots + g_n \in G$. The image of the set $\mathbf{K}$ under the map s is usually called the sum of the sets K_i, and we shall denote it by $\sum K_i$. The map s is a continuous group homomorphism, and induces a map $\hat{s}$ from $\hat{G}$ into $\widehat{G^n} = \hat{G}^n$. By restriction, the set of functions $\hat{G}|_{\Sigma K_i}$ is mapped into a subset of the set of functions $\hat{G}^n|_{\mathbf{K}}$. Extending $\hat{s}$ by linearity we have a map, which we shall also denote by $\hat{s}$ of $A(\sum K_i)$ into $A(\mathbf{K})$. Identifying $A(\sum K_i)$ with its image under $\hat{s}$ in $A(\mathbf{K})$, we may consider $A(\sum K_i)$ as the space $A_\sigma(\mathbf{K})$ of all functions f in $A(\mathbf{K})$, which have an expansion

$$f(k) = \sum a_i \chi_i(k), \quad \sum |a_i| < \infty, \quad k = (k_1, k_2, \ldots, k_n) , \tag{1.1.7}$$

and with $\chi_i \in \hat{s}(\hat{G})$. $\hat{s}$ is the diagonal imbedding of $\hat{G}$ into $\hat{G}^n$, and is of course the set of all characters χ in $\hat{G}^n$, for which

$$\chi(k) = \chi(k_1 + k_2 + \ldots + k_n), \quad k = (k_1, k_2, \ldots k_n) .$$

The natural embedding of $A_\sigma(\mathbf{K})$ into $A(\mathbf{K})$ is of course normdecreasing. We have further

$$A(\mathbf{K}) = A(K_1 \times K_2 \times \ldots \times K_n) =$$
$$= A(K_1) \hat{\otimes} A(K_2) \hat{\otimes} \ldots \hat{\otimes} A(K_n) \subset C(K_1) \hat{\otimes} C(K_2) \hat{\otimes} \ldots \hat{\otimes} C(K_n) = V(\mathbf{K})$$

and this gives a natural normdecreasing injection of $A(\mathbf{K})$ into $V(\mathbf{K})$. We shall denote by T the composite of $\hat{s}$ and this natural injection. T is then a normdecreasing injective algebra homomorphism of $A(\sum K_i)$ into $V(\mathbf{K})$, or equivalently of $A_\sigma(\mathbf{K})$ into $V(\mathbf{K})$.

2. To prove our main results we shall need the following two theorems from the metric theory of tensor algebras. These theorems will be proved in the next two sections.

THEOREM 1.1. *For every natural number $n \geq 1$, there exists a continuous real-valued function $\varepsilon_n(x, y)$, $0 \leq x \leq 1$, $0 \leq y \leq 2n$, such that $\varepsilon_n(0, 0) = 0$, $\varepsilon_n(x, y)$ is concave and increasing in each variable, and having also the following property:*

Let $\{X_i\}_{i=1}^n$ be compact spaces, and let $\mathbf{X}$ and X' be resp. their cartesian product and disjoint union. Let $A \in BM(\mathbf{X})$, $\|A\| = 1$, let $f_i, g_i \in C(X_i)_1$, be such that

$$\operatorname{Re}\{A(\mathbf{f})\} \geq 1 - x \tag{1.2.1}$$

$$\operatorname{Re}\left\{\int_{X'} g' d\lambda'(\mathbf{f})\right\} \geq n - y , \tag{1.2.2}$$

where $\mathbf{f}$, g' *and* $\lambda'(\mathbf{f})$ *are defined by resp.* (1.1.2), (1.1.3) *and* (1.1.5). *Defining now also* $\mathbf{g}$ *by* (1.1.2), *we have*

$$|1 - A(\mathbf{g})| \leq \varepsilon_n(x, y) . \tag{1.2.3}$$

THEOREM 1.2. *For every natural number $n \geq 2$, there exists a continuous real-valued function $\eta_n(y)$, $0 \leq y$, with $\eta_n(0) = 0$, and having the following property:*

Let $\{X_i\}_{i=1}^n$ be compact spaces, let T be a locally compact space and let u be a positive measure on T with

$$\|u\|_{M(T)} \leq 1 + y . \tag{1.2.4}$$

Let further $f_i(x_i, t) \in C(X_i \times T)_1$, *define* $\mathbf{f} \in C(\mathbf{X} \times T)$ *and* $f' \in C(X' \times T)$ *by resp.* (1.1.2) *and* (1.1.3), *and let*

$$\int_T f'(x', t)du(t) = \mathbf{1}_{X'} . \tag{1.2.5}$$

Writing now

$$F(x) = \int_{\hat{T}} \mathbf{f}(x, t)du(t) \tag{1.2.6}$$

we have $F \in V(\mathbf{X})$, $\|F\|_V \leq 1 + y$, *and*

$$\|1 - F\|_V \leq \eta_n(y) .$$

3. Using Theorem 1.1 we shall now state and prove Theorem A in a slightly more precise form as follows.

THEOREM 1.3. *Let* δ, $0 < \delta < 1$, *be a real number. For every natural number* $n \geq 2$, *there exists a real number* $\beta_n > 0$, *such that if* $\{K_i\}_{i=1}^n$ *are disjoint compact subsets of a compact abelian group* G, *and if* $\bigcup K_i$ *is a Helson*-$(1 - \beta)$ *set*, $\beta < \beta_n$, *then the map*

$$T : A(\textstyle\sum K_i) \to V(\mathbf{K})$$

is a topological isomorphism, and $\|T^{-1}\| \leq \delta^{-1}$.

Proof. We first define β_n in terms of the function $\varepsilon_n(\cdot,\cdot)$, by the relation $1 - \varepsilon_n(0, n\beta_n) = \delta$. Since we further assume $\beta < \beta_n$, we can then choose $\varepsilon > 0$, such that

$$1 - \varepsilon_n(\varepsilon, n(\beta + \varepsilon)) = \delta . \tag{1.3.1}$$

Let now $\Lambda \in BM(\mathbf{K})$, $\|\Lambda\|_{BM} = 1$. To prove the theorem it suffices by well-known duality arguments to prove that

$$\|T'(\Lambda)\|_{PM(\Sigma K_i)} \geq \delta , \tag{1.3.2}$$

and this means that we must find a character χ in $\hat{s}(\hat{G})$, such that

$$|\Lambda(\chi)| \geq \delta . \tag{1.3.3}$$

Let $\varepsilon > 0$ be the ε defined by (1.3.1). By the definition of norm in $BM(\mathbf{K})$ we can then find functions $f_i \in C(K_i)$, such that

$$\Lambda(\mathbf{f}) = \mathrm{Re}\,\{\Lambda(\mathbf{f})\} > 1 - \varepsilon ,$$

where we have defined $\mathbf{f}$ by (1.1.2). Let now $\lambda'(\mathbf{f}) \in M(\bigcup K_i) = M(K')$ be the measure defined by (1.1.5), and let $f' \in C(\bigcup K_i) = C(K')$ be defined by (1.1.3). Since we now have by definition

$$\int_{\cup K_i} f' d\lambda'(\mathbf{f}) = nA(\mathbf{f}) > n(1-\varepsilon)\,, \tag{1.3.5}$$

and since $\|f'\|_\infty \leq 1$, we have $\|\lambda'(\mathbf{f})\|_{M(\cup K_i)} \geq n(1-\varepsilon)$. By the assumption on the set $\bigcup K_i$, we can now find a character $\chi \in G$, such that

$$\left|\int_{\cup K_i} \chi d\lambda'(\mathbf{f})\right| > (1-\beta)\|\lambda'(\mathbf{f})\| \geq n - n(\beta+\varepsilon)\,. \tag{1.3.6}$$

Let $e^{-i\theta}$ be the argument of the integral in (1.3.6), and put $g_i = e^{i\theta}\chi|_{K_i}$. We have then $g_i \in C(K_i)$ and we define $\mathbf{g}$ and g' by (1.1.2) and (1.1.3). But this implies then that

$$\operatorname{Re}\left\{\int g' d\lambda'(\mathbf{f})\right\} = \operatorname{Re}\left\{e^{i\theta}\int \chi d\lambda'(\mathbf{f})\right\} = \left|\int \chi d\lambda'(\mathbf{f})\right| > n - n(\beta+\varepsilon)\,. \tag{1.3.7}$$

By (1.3.4), (1.3.7) and Theorem 1.1 we have then

$$|1 - A(\mathbf{g})| \leq \varepsilon_n(\varepsilon, n(\beta+\varepsilon)) = 1 - \delta\,,$$

and therefore $|A(\mathbf{g})| \geq \delta$. But we have also $|A(\mathbf{g})| = |e^{in\theta}A(\chi)| = |A(\chi)|$, and therefore $|A(\chi)| \geq \delta$, and this proves the theorem.

§ 2. Two metric lemmas

1. To prove Theorem 1.1 we shall need two lemmas. The first lemma is the key lemma of the proof. It is an inequality for bimeasures, which follows from a more general inequality valid in complex L^p-spaces, $1 \leq p \leq 2$. The second lemma is simply a convenient quantitative version of the qualitative statement, that for every μ in $M(K)$, the function g in the unit ball of $L^\infty(|\mu|)$, for which $\int g d\mu = \|\mu\|$, is unique as an element of $L^\infty(|\mu|)$.

LEMMA 2.1. *Let X_1 and X_2 be compact Hausdorff spaces, let $A \in BM(X_1 \times X_2)$, and let $e_i \in C(X_1)$, $i = 1, 2$, be such that $\sup_{x\in X_1}\{|e_1(x)| + |e_2(x)|\} \leq 1$. Let further $e_iA \in BM(\mathbf{X})$, be the bimeasure defined by $(e_iA)(f \otimes g) = A(e_if \otimes g)$, $f \in C(X_1)$, $g \in C(X_2)$.*

Then

$$\|A\|^2_{BM} \geq \|e_1A\|^2_{BM} + \frac{4}{\pi^2}\|e_2A\|^2_{BM}$$

The proof of Lemma 2.1 is based on the following general inequality.

PROPOSITION 2.2. *Let μ be a positive measure on a locally compact Hausdorff space X, and let f and g belong to $L^p(\mu)$, then the following inequalities hold: if $p = 1$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta}g\|_1 d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} \left| \|f\|_1 + e^{i\theta}\|g\|_1 \right| d\theta \tag{2.1.1}$$

If $1 \leq p \leq 2$, then

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta}g\|_p^p \, d\theta\right)^{2/p} \geq \|f\|_p^2 + \frac{4}{\pi^2}\|g\|_p^2 . \tag{2.1.2}$$

Note 1. In the case $p = 2$, a stronger inequality with $4/\pi^2$ replaced by 1 is true, and is of course elementary.

Note 2. For all $p \geq 1$, we have

$$\sup_{0 \leq \theta \leq 2\pi} \|f + e^{i\theta}g\|_p \geq \left(\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta}g\|_p^p \, d\theta\right)^{1/p} \tag{2.1.3}$$

In application we shall often need to combine (2.1.3) with the inequalities (2.1.1) or (2.1.2) above.

Proof of the proposition. We start by observing that for arbitrary complex numbers a, b, we always have

$$\frac{1}{2\pi}\int_0^{2\pi} |a + e^{i\theta}b| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left||a| + e^{i\theta}|b|\right| d\theta \tag{2.1.4}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \left||a| + e^{i\theta}|b|\right| d\theta = \frac{1}{\pi}\int_0^{\pi} \left||a| + e^{i\theta}|b|\right| d\theta \tag{2.1.5}$$

$$\geq \frac{1}{\pi}\left|\int_0^{\pi} (|a| + e^{i\theta}|b|) d\theta\right| = \left||a| + \frac{2i}{\pi}|b|\right| = \left(|a|^2 + \frac{4}{\pi^2}|b|^2\right)^{1/2} .$$

We now prove (2.1.1).

Using the definition of norm, Fubini's theorem and (2.1.4) we have

$$\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta}g\|_1 \, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_X |f + e^{i\theta}g| d\mu \, d\theta =$$

$$\int_X \frac{1}{2\pi}\int_0^{2\pi} |f + e^{i\theta}g| d\theta \, d\mu = \int_X \frac{1}{2\pi}\int_0^{2\pi} \big||f| + e^{i\theta}|g|\big| d\theta \, d\mu \, .$$

Now using Fubini's theorem again, then the triangle inequality, and then again the definition of norm, we have

$$\int_X \frac{1}{2\pi}\int_0^{2\pi} \big||f| + e^{i\theta}|g|\big| d\theta \, d\mu = \frac{1}{2\pi}\int_0^{2\pi}\int_X \big||f| + e^{i\theta}|g|\big| d\mu \, d\theta \geq$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} \left| \int_X (|f| + e^{i\theta}|g|) d\mu \right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \big|\|f\|_1 + e^{i\theta}\|g\|_1\big| d\theta \, .$$

This proves (2.1.1).

Before proving (2.1.2), we observe, that if $p \geq 1$, then by Hölder's inequality, (2.1.4) and (2.1.5), we have

$$\frac{1}{2\pi}\int_0^{2\pi} |a + e^{i\theta}b|^p d\theta \geq \left(\frac{1}{2\pi}\int_0^{2\pi} |a + e^{i\theta}b| d\theta\right)^p \geq \left(|a|^2 + \frac{4}{\pi^2}|b|^2\right)^{p/2} \tag{2.1.6}$$

We now prove (2.1.2). Using the definition of norm, and Fubini's theorem, we have

$$\begin{aligned}\left(\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta}g\|_p^p \, d\theta\right)^{2/p} &= \left(\frac{1}{2\pi}\int_0^{2\pi}\int_X |f + e^{i\theta}g|^p \, d\mu d\theta\right)^{2/p} = \\ &= \left(\int_X \frac{1}{2\pi}\int_0^{2\pi} |f + e^{i\theta}g|^p d\theta \, d\mu\right)^{2/p} .\end{aligned} \tag{2.1.7}$$

Applying (2.1.6) and Minkowski's inequality (observe that $p/2 \leq 1$), we have

$$\left(\int_X \frac{1}{2\pi}\int_0^{2\pi} |f + e^{i\theta}g|^p d\theta \, d\mu\right)^{2/p} \geq \left(\int_X \left(|f|^2 + \frac{4}{\pi^2}|g|^2\right)^{p/2} d\mu\right)^{2/p} \geq$$

$$\geq \left(\int_X |f|^p d\mu\right)^{2/p} + \frac{4}{\pi^2}\left(\int_X |g|^p \, d\mu\right)^{2/p}$$

which is by definition $\|f\|_p^2 + (4/\pi^2)\|g\|_p^2$, and so (2.1.2) is proved.

Proof of Lemma 2.1. We first observe that it follows immediately from Pro position 2.2, that if $\mu, \nu \in M(X)$, then

$$\max_{0\le\theta\le 2\pi} \|\mu + e^{i\theta}\nu\| \ge \left(\|\mu\|^2 + \frac{4}{\pi^2}\|\nu\|^2\right)^{1/2}. \qquad (2.1.8)$$

We consider A as an operator from $C(X_1)$ into $M(X_2)$. Using our standard notations we have then obviously

$$\left.\begin{aligned} \|A\|_{BM} &= \sup \|A_2(f)\|_{M(X_2)} \\ \|e_1A\|_{BM} &= \sup \|A_2(e_1f)\|_{M(X_2)} \\ \|e_2A\|_{BM} &= \sup \|A_2(e_2f)\|_{M(X_2)} \end{aligned}\right\} f \in C(X_1) \qquad (2.1.9)$$

Now let ε be a positive number. We can find $f_i \in C(X_1)$, such that $\|A_2(e_if_i)\|_{M(X_2)} \ge \|e_iA\|_{BM} - \varepsilon$. By linearity, and by (2.1.8) we have now

$$\max_{0\le\theta\le 2\pi} \|A_2(e_1f_1 + e^{i\theta}e_2f_2)\|^2_{M(X_2)} = \max \|A_2(e_1f_1) + e^{i\theta}A_2(e_2f_2)\|^2_{M(X_2)} \ge$$

$$\ge \|A_2(e_1f_1)\|^2 + \frac{4}{\pi^2}\|A_2(e_2f_2)\|^2 \ge (\|e_1A\|_{BM} - \varepsilon)^2 + \frac{4}{\pi^2}(\|e_2A\|_{BM} - \varepsilon)^2 .$$

Since ε is arbitrary, and since $\|e_1f_1 + e^{i\theta}e_2f_2\|_\infty \le 1$, the lemma follows.

2. In the proof of the theorem, we shall also need the following lemma.

LEMMA 2.3. *Let a, b be positive numbers, and let X be a compact Hausdorff space. Let further $\mu \in M(X)$, $f, g \in L^\infty(|\mu|)$, be such that*

$$\|\mu\| \le 1, \ \|f\| \le 1, \ \|g\| \le 1 \qquad (2.2.1)$$

and

$$\operatorname{Re}\int f d\mu \ge 1 - a, \ \operatorname{Re}\int g d\mu \ge 1 - b , \qquad (2.2.2)$$

then

$$\left(\int |f - g| d|\mu|\right)^2 \le \int |f - g|^2 d|\mu| \le 2(\sqrt{a} + \sqrt{b})^2 .$$

Proof. Write $\mu = \varphi \cdot |\mu|$, $|\varphi| = 1$ a.e. $(|\mu|)$.

Replacing then f and g by $f\varphi$, $g\varphi$, and μ by $|\mu|$, we see that we may assume μ to be positive.

We now write

$$f = f_1 + if_2, \ g = g_1 + ig_2 ; f_i, \ g_i \text{ real}. \qquad (2.2.3)$$

We have then

$$\int f_1\, d\mu \geq 1 - a, \quad \int g_1\, d\mu \geq 1 - b \tag{2.2.4}$$

and therefore

$$\int f_1^2\, d\mu \geq \left(\int f_1\, d\mu\right)^2 \geq (1-a)^2 \geq 1 - 2a, \quad \int g_1^2\, d\mu \geq 1 - 2b \tag{2.2.5}$$

but then

$$\int f_2^2\, d\mu \leq \int (1 - f_1^2)\, d\mu \leq 1 - (1 - 2a) = 2a, \quad \int g_2^2\, d\mu \leq 2b \tag{2.2.6}$$

Now

$$\begin{aligned}\left(\int |f-g| d\mu\right)^2 &\leq \int |f-g|^2 d\mu = \int (|f|^2 + |g|^2 - f\bar{g} - \bar{f}g) d\mu \\ &\leq \int (2 - f\bar{g} - \bar{f}g) d\mu = A\,.\end{aligned} \tag{2.2.7}$$

Writing this in terms of real and imaginary parts, we have

$$A = \int (2 - 2(f_1g_1 + f_2g_2))d\mu = 2\int (1 - f_1 + f_1(1 - g_1) - f_2g_2)d\mu\,.$$

Using (2.2.4) we have now

$$A \leq 2\left(a + b + \left|\int f_2g_2\, d\mu\right|\right),$$

and finally by Schwarz's inequality and (2.2.6),

$$A \leq 2(a + b + (2a \cdot 2b)^{1/2}) = 2(\sqrt{a} + \sqrt{b})^2\,.$$

This proves the lemma.

§ 3. Two metric theorems

1. We shall now first prove Theorem 1.1. To do this we shall first prove the special case $n = 2$, and we shall then prove the general case by induction on n.

Proof of special case. We shall prove that the function

$$\varepsilon_2(x, y) = \sqrt{2y} + \pi(2x + \sqrt{2x} + \sqrt{2y})^{1/2}$$

satisfies the conditions of the theorem, and we start by observing that it is clearly increasing and concave.

We next observe that since $\|\lambda_i(\mathbf{f})\|_{M(X_i)} \leq 1$, we clearly have both

$$\mathrm{Re}\left\{\int_{X_1} g_1 d\lambda_1(\mathbf{f})\right\} \geq 1 - y \quad \text{and} \quad \mathrm{Re}\left\{\int_{X_2} g_2 d\lambda_2(\mathbf{f})\right\} \geq 1 - y\,. \tag{3.1.1}$$

By linearity we have now

$$\Lambda(\mathbf{g}) = \Lambda(g_1 \otimes g_2) = \Lambda((f_1 + g_1 - f_1) \otimes g_2) = \Lambda(f_1 \otimes g_2) - \Lambda((f_1 - g_1) \otimes g_2) \quad (3.1.2)$$

and therefore

$$|1 - \Lambda(\mathbf{g})| \leq |1 - \Lambda(f_1 \otimes g_2)| + |\Lambda((f_1 - g_1) \otimes g_2)| = A + B\,. \quad (3.1.3)$$

By definition of norm we have now $|\Lambda(f_1 \otimes g_2)| \leq 1$, and by (3.1.1) we have $\mathrm{Re}\,\{\Lambda(f_1 \otimes g_2)\} \geq 1 - y$, and this implies that $A \leq \sqrt{2y}$.

It remains to prove that $B \leq \pi(2x + \sqrt{2x} + \sqrt{2y})^{1/2}$, and towards this we first observe that

$$\mathrm{Re}\left\{\int_{X_1} f_1 d\lambda_1(\mathbf{f})\right\} = \mathrm{Re}\,\{\Lambda(\mathbf{f})\} \geq 1 - x\,. \quad (3.1.4)$$

By (3.1.1), (3.1.4) and Lemma 2.3 we have therefore

$$\int_{X_1} |f_1 - g_1| d|\lambda_1(\mathbf{f})| \leq \sqrt{2x} + \sqrt{2y}\,. \quad (3.1.5)$$

Let us write $e_2 = |f_1 - g_1|/2$, and $e_1 = 1 - e_2$.

We have then

$$\mathrm{Re}\,\{(e_1\Lambda)(f_1 \otimes f_2)\} = \mathrm{Re}\left\{\int_{X_1} (1 - e_2) f_1 d\lambda_1(\mathbf{f})\right\} \geq 1 - x - \frac{1}{2}\int_{X_1} |f_1 - g_1| d|\lambda_1(\mathbf{f})| \geq$$

$$\geq 1 - x - \frac{1}{2}\,(\sqrt{2x} + \sqrt{2y}) \geq \|e_1\Lambda\|_{BM}\,. \quad (3.1.6)$$

By Lemma 2.1 we have further $\|e_1\Lambda\|^2 + \frac{4}{\pi^2}\,\|e_2\Lambda\|^2 \leq 1$, and therefore

$$\|e_2\Lambda\|_{BM} \leq \frac{\pi}{2}\left\{1 - \left(1 - x - \frac{1}{2}\left(\sqrt{2x} + \sqrt{2y}\right)\right)^2\right\}^{1/2} \leq$$

$$\leq \frac{\pi}{2}\,(2x + \sqrt{2x} + \sqrt{2y})^{1/2} \quad (3.1.7)$$

Finally we have now, denoting $h = \mathrm{sign}\,(f_1 - g_1)$,

$$B = |\Lambda((f_1 - g_1) \otimes g_2)| = |2\Lambda(e_2 h \otimes g_2)| = 2|(e_2\Lambda)(h \otimes g_2)|$$

$$\leq 2\|e_2\Lambda\| \leq \pi(2x + \sqrt{2x} + \sqrt{2y})^{1/2}\,, \quad (3.1.8)$$

and this proves the special case $n = 2$.

2. *The general case.*

The proof is by induction. We define $\varepsilon_1(x, y) = y$, and we take for $\varepsilon_2(x, y)$ the function defined above. We now assume $\varepsilon_k(x, y)$, $1 \leq k \leq n-1$ constructed, and we shall construct $\varepsilon_n(x, y)$. Let therefore $A \in BM(X)$ be a multimeasure of order n, and let $f, g \in \prod(X)$, satisfy the assumptions of the theorem. We write $f = f_1 \otimes f_2 \otimes f_3 \otimes \ldots \otimes f_n$, $g = g_1 \otimes g_2 \otimes g_3 \otimes \ldots \otimes g_n$, and we use the fact that a multimeasure may be considered as an operator from a product of some of the $C(X)$ spaces into the multimeasure space over the rest, which is a multimeasure space of lower order. This means that we may use the induction hypothesis to replace any k, $k \leq n-1$, of the f's by g's. Doing this we have e.g.

$$\begin{aligned} &\text{Re}\{A(f_1 \otimes f_2 \otimes g_3 \otimes g_4 \otimes \ldots \otimes g_n)\} > 1 - \varepsilon_{n-2}(x, y) \\ &\text{Re}\{A(f_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes \ldots \otimes g_n)\} > 1 - \varepsilon_{n-1}(x, y) \\ &\text{Re}\{A(g_1 \otimes f_2 \otimes g_3 \otimes g_4 \otimes \ldots \otimes g_n)\} > 1 - \varepsilon_{n-1}(x, y) . \end{aligned} \tag{3.2.1}$$

But now $A(\varphi \otimes \psi \otimes g_3 \otimes g_4 \otimes \ldots \otimes g_n)$, $\varphi \in C(X_1)$, $\psi \in C(X_2)$, is a bimeasure in $BM(X_1 \times X_2)$, and we use the special case proved above to conclude that

$$|A(G)| > 1 - \varepsilon_2(\varepsilon_{n-2}(x, y),\ 2\varepsilon_{n-1}(x, y)) . \tag{3.2.2}$$

We define thus $\varepsilon_n(x, y) = \varepsilon_2(\varepsilon_{n-2}(x, y),\ 2\varepsilon_{n-1}(x, y))$, and this completes the proof of Theorem 1.1. It is easy to verify that $\varepsilon_n(x,y)$ is concave and increasing.

3. In the proof of Theorem 1.2 an essential role is played by a result of Bohnenblust and Karlin (Proposition 3.1 below). To state this result we shall need some notations.

Let A be a Banach algebra with unit element $\mathbf{1}$, and dual space A'. We shall define a convex set $D_A \subset A'$, by

$$D_A = \{f \in A' \mid f(\mathbf{1}) = 1,\ \|f\|_{A'} = 1\} . \tag{3.3.1}$$

Let further $a \in A$, we shall then write

$$V(a) = \{z \in C \mid z = f(a),\ f \in D_A\} , \tag{3.3.2}$$

and

$$v(a) = \sup |z|,\ z \in V(a) . \tag{3.3.3}$$

$V(a)$ is called the numerical range and $v(a)$ the numerical radius of the element $a \in A$. The result that we shall need is the following

PROPOSITION 3.1. [2] *Let A be a Banach algebra with unit $\mathbf{1}$, and let $a \in A$, then $\|a\|_A \leq e \cdot v(a)$.*

Proof of Theorem 1.2. To see that $F \in V$ we first observe that for all $t \in T$ we have trivially $f \in V$, $\|f\|_V \leq 1$. We next use the fact that the function $t \to f(x, t)$ is a continuous function from T to V. Therefore the integral is well-defined and takes its value in V. Finally we have

$$\|F\|_V \leq \int_T \|f(x, t)\|_V du(t) \leq 1 + y . \tag{3.3.4}$$

For the second part of the theorem we shall prove that for any function $\varepsilon_n(x, y)$ satisfying the conditions of Theorem 1.1, the function $\eta_n(y)$, defined by

$$\eta_n(y) = e \cdot \left\{y + (1 + y)\varepsilon_n\left(0, \frac{ny}{1 + y}\right)\right\} \tag{3.3.5}$$

satisfies the conditions of Theorem 2. To do this it suffices by proposition 3.1 to prove that

$$v(\mathbf{1} - F) \leq \left\{y + (1 + y)\varepsilon_n\left(0, \frac{ny}{1 + y}\right)\right\} , \tag{3.3.6}$$

where $v(\mathbf{1} - F)$ denotes the numerical radius.

Towards this, let $\Lambda \in D_V$ and let the (probability) measures $\lambda_i = \lambda_i(\mathbf{1})$ and λ' be defined by (1.1.4) and (1.1.5). We have now

$$\begin{aligned} |\Lambda(\mathbf{1} - F)| &= |1 - \Lambda(F)| = |- y + (1 + y - \Lambda(F))| \\ &= \left|- y + \int_T (1 - \Lambda(f(x, t))du(t)\right| \leq y + \int_T |1 - \Lambda(f(x, t))|du(t) . \end{aligned} \tag{3.3.7}$$

Comparing the final term in (3.3.7) with the right hand term in (3.3.6) we see that it suffices to prove that

$$\frac{1}{1 + y} \int_T |1 - \Lambda(f(x, t))|du(t) \leq \varepsilon_n\left(0, \frac{ny}{1 + y}\right) . \tag{3.3.8}$$

Let now $t \in T$. We shall write

$$n - y(t) = \operatorname{Re} \left\{\int_{X'} f'(x', t)d\lambda(x')\right\} , \tag{3.3.9}$$

and we have then

$$|1 - \Lambda(f(x, t))| \leq \varepsilon_n(0, y(t)) . \tag{3.3.10}$$

Now by (1.2.5) and Fubini's theorem we have also

$$\begin{aligned}\int_T y(t)du(t) &= \int_T ndu(t) - \operatorname{Re}\left\{\int_T\int_{X'} f'(x',t)d\lambda(x')du(t)\right\} = \\ &= n(1+y) - \int_{X'} 1\cdot d\lambda(x') = n+ny-n = ny\,.\end{aligned} \tag{3.3.11}$$

Since the function $\varepsilon_n(0, y)$ is concave we have by Jensen's inequality (turned upside down)

$$\begin{aligned}\frac{1}{1+y}\int_T |1-A(f(x,t))|du(t) &\le \frac{1}{1+y}\int_T \varepsilon_n(0, y(t))du(t) \\ &\le \varepsilon_n\left(0, \frac{1}{1+y}\int_T y(t)du(t)\right) = \varepsilon_n\left(0, \frac{ny}{1+y}\right),\end{aligned} \tag{3.3.12}$$

and this proves the theorem.

Remark: In the case $n = 2$, one can using Grothendieck's »fundamental theorem» prove that the function $\eta_2(y) = 15y$ satisfies the conditions of the theorem.

As an immediate consequence of Theorem 2, we have the following

COROLLARY 3.1. *Let* $\{X_i\}_{i=1}^n$ *be compact spaces and let* $\mathbf{X}$ *and* X' *be as above. Let for each* i, $g_i \in S(X_i)$, *i.e.* $g_i \in C(X_i)$ *and* $|g_i| = 1$, *and define the functions* $g \in C(\mathbf{X})$ *and* $g' \in C(X')$ *by* (1.1.2) *and* (1.1.3). *Let* T *be a locally compact space and let* u *be a positive measure on* T *with* $\|u\| \le 1 + y$. *Let further for each* i, $f_i \in C(X_i \times T)_1$ *and let* f *and* f' *be as above, and let*

$$\int_T f'(x', t)du(t) = g'(x')\,.$$

Writing now $F(x) = \int_T f(x, t)du(t)$
we have $F \in V$, $\|F\|_V \le 1 + y$ *and* $\|g - F\| \le \eta_n(y)$.

Proof. We shall define functions $h_i \in C(X_i \times T)$ by $h_i(x_i, t) = \overline{g(x_i)}f_i(x_i, t)$. Defining then h, h', and H by resp. (1.1.2), (1.1.3) and (1.2.6) we have by Theorem 1.2.

$$H \in V,\ \|H\|_V \le 1 + y \text{ and } \|\mathbf{1} - H\| \le \eta_n(y)\,. \tag{3.3.13}$$

But we have also $g = g\cdot\mathbf{1}$, $F = g\cdot H$, and $\|g\|_V \le 1$, and therefore $\|F\| \le \|g\|\cdot\|H\| \le 1 + y$ and $\|g - F\| \le \|g\|\cdot\|\mathbf{1} - H\| \le \eta_n(y)$.

COROLLARY 3.2. *Let* $\{X_i\}_{i=1}^n$ *be compact spaces, and let* $F \in V(X)$ *be such that*

$$\|F\|_V \leq 1\,, \quad \|\mathbf{1} - F\|_\infty \leq y\,, \tag{3.3.14}$$

then

$$\|\mathbf{1} - F\|_V \leq 10\eta_n(y)\,.$$

§ 4. Helson sets in $\mathbf{D}_q$

1. The group D_q is the product, as a group and as a topological space, of a denumerably infinite number of compact abelian groups isomorphic to $Z(q)$, the cyclic group of order q. The dual group $\widehat{D_q}$, is the sum of the corresponding dual groups. D_q is compact metrizable and totally disconnected, and $\widehat{D_q}$ is discrete and denumerably infinite [7].

In § 1 a closed subset E of a compact abelian group G was called a K_q-set if $\hat{G}|_E = S_q(E)$. It is easy to see that the group D_q contains K_q-sets, and it was observed by Varopoulos, that if $\{K_i\}_{i=1}^n$ are closed subsets of D_q, and if $\bigcup K_i$ is a K_q-set, then $A(\sum K_i)$ is a tensor algebra. On the other hand the group D_q does not contain Helson-(α)-sets, for α arbitrarily close to 1, so Theorem 1.1 can not in general be used to provide weak conditions on a set $\{K_i\}_{i=1}^n$ of subsets, ensuring $A(\sum K_i)$ to be a tensor algebra. Nevertheless it seems reasonable, in the light of Theorem 1.1 to conjecture that if $\bigcup K_i$ is almost a K_q-set then $A(\sum K_i)$ is a tensor algebra. In the next paragraph we shall prove this in the case $n = 2$, i.e. $A(K_1 + K_2)$ is a tensor algebra if $K_1 \cup K_2$ is almost a K_q-set. In the proof of this result an important role is played by functions with $f^q = 1$, in fact we shall have to assume that the characters are sufficiently dense in the appropriate metric sense in $S_q(K_1 \cup K_2)$. This assumption is however not used in the definition of the Helson constant, and we shall therefore consider also another type of Helson-constant, which is more suitable for our purposes, and is conceptually more natural for subsets of D_q. We then study the relations, between the two concepts, and we shall prove that they are essentially equivalent.

2. We shall presently consider subsets E of the group D_q that are either K_q-sets or Helson-sets. In these considerations an essential role is played by the groups $S_q(E)$ and its subgroup $\widehat{D_q}|_E$. To see the nature of this role we shall however first consider a more general problem. We shall need some definitions and notations. Let K be a compact metrizable Hausdorff space, and let P be a compact convex set, containing 0, in the complex plane. We shall denote by $P(K)$ the set of all functions f in $C(K)$ with $f(K) \subset P$, and by $P'(K)$ the set of Borel functions with $f(K) \subset P$. $P(K)$ is a bounded convex subset of $C(K)$. To avoid trivialities we assume that P contains some point outside 0, and it is easy to see that then, every function in $C(K)$ can be represented by a finite linear combination of functions

from $P(K)$. We can therefore define an equivalent norm in $C(K)$, which we shall call the *P-norm,* as follows

$$\|f\|_P = \inf \sum |a_k| \quad \text{over all representations} \quad f = \sum a_k f_k, \ f_k \in P(K) . \tag{4.2.1}$$

We shall denote the space $C(K)$, when given the P-norm by $C_P(K)$. The norm-dual of $C_P(K)$ will be denoted $M_P(K)$, and is the usual space $M(K)$, with norm given by

$$\|\mu\|_P = \sup \left| \int f \, d\mu \right| , \quad f \in P(K) . \tag{4.2.2}$$

We shall later consider mainly the cases, when P is a set $P_q =$ the convex hull of the q^{th} roots of unity, and when P is the unit interval. We also observe that the P-norm obtained, when P is the unit disc, is the usual sup norm in $C(K)$ resp. the usual total mass norm in $M(K)$.

For a convex set E of complex numbers we shall denote the radius of the set by $r(E)$, i.e. $r(E) = \sup |z|$, $z \in E$, and we shall denote the perimeter by $p(E)$. For a bounded convex set containing 0, we have $2r(E) \leq p(E) \leq 2\pi \cdot r(E)$. For normalization of the base set P, we shall assume that $1 \in P$, and that $r(P) = 1$. Denoting the unit interval by I, and the unit disc by D, we assume thus $I \subset P \subset D$. These assumptions imply that the P-norm of a measure is at most the total mass, and that the P-norm of a positive measure is the total mass.

A natural starting-point for investigations on P-norms is the following fact, well-known in the theory of convex sets.

Let C and D be convex sets in the plane, and let E be the sum-set $E = C + D$. Then E is a convex set, and

$$p(E) = p(C) + p(D) . \tag{4.2.3}$$

The identity (4.2.3) is implicitly contained in some more general formulas in e.g. Bonnesen—Fenchel [1], and is obvious on inspection when C and D are polygons. Finally an analytic proof of (4.2.3) can be based on a formula of Cauchy, showing the perimeter of a convex set to be a linear function of the width of the set. This formula can be written e.g. as follows [see 1, p. 48]

$$p(C) = \frac{1}{2} \int_0^{2\pi} [\sup_{x \in C} (x \cdot u) - \inf_{y \in C} (y \cdot u)] \, d\theta , \tag{4.2.4}$$

$u = (\cos \theta, \ \sin \theta)$, $(x \cdot u)$ the usual inner product in $\mathbf{R}^2$.

Formula (4.2.4) clearly shows the additivity of the perimeter. We observe also that the radius is subadditive under addition of convex sets, so that if $E = C + D$, then

$$r(E) \leq r(C) + r(D) . \tag{4.2.5}$$

We now introduce some notions, that will be used throughout this §. Let P be a compact convex set containing 0 in the complex plane. Let a be a complex number, we shall write

$$aP = \{w \mid w = az, \quad z \in P\} . \tag{4.2.6}$$

Let now $\mathbf{a} = \{a_k\}_{k=1}^n$ be a set of complex numbers. The P-range of $\mathbf{a}$ is the set

$$Q_P(\mathbf{a}) = \sum_{k=1}^{n} (a_k P) = \{z \mid z = \sum_{k=1}^{n} a_k z_k, \quad z_k \in P\} . \tag{4.2.7}$$

It is obvious that $p(aP) = |a| \cdot p(P)$, and it then follows from (4.2.3), that

$$p(Q_P(\mathbf{a})) = (\sum_{k=1}^{n} |a_k|) \cdot p(P) . \tag{4.2.8}$$

Let now K be a compact metrizable Hausdorff space, and let $\mu \in M(K)$. The *P-range of* μ is the set

$$Q_P(\mu) = \mathrm{cl} \left\{ z \mid z = \int_K f d\mu \, , \ f \in P(K) \right\} , \tag{4.2.9}$$

or equivalently

$$Q_P(\mu) = \left\{ z \mid z = \int f d\mu \, , \ f \in P'(K) \right\} . \tag{4.2.9'}$$

$Q_P(\mu)$ is the continuous linear image under μ, of the bounded convex set $P(K)$, and is therefore bounded and convex. It follows moreover e.g. from a partition of unity argument, that

$$p(Q_P(\mu)) = p(P) \cdot \|\mu\|_{M(K)} . \tag{4.2.10}$$

By definition we have further $\|\mu\|_P = r(Q_P(\mu))$. The fact that both the P-norm and the usual norm can be read off from the set $Q_P(\mu)$, indicates the importance of this set for problems on relations between P-norms and usual norms.

Now the perimeter of a convex set is a monotone function on the set of all convex sets, so a set of perimeter L cannot be contained in a circle of radius r if $r < L/2\pi$, and we have therefore

$$\|\mu\|_P \geq \frac{p(P)}{2\pi} \|\mu\|_{M(K)} . \tag{4.2.11}$$

Denoting by I_P the identity map of $C(K)$, considered as a map from $C(K)$ into $C_P(K)$, we see thus that $\|I_P\| \leq 2\pi/p(P)$.

Let now P be one of the sets P_q (resp. the unit interval I), and let us speak of q-norms instead of P_q-norms, and of spaces $C_q(K)$, and $M_q(K)$, and of the q-range $Q_q(\mu)$. We have in this case $p(P_q) = 2q \cdot \sin(\pi/q)$ (resp. $p(I) = 2$), and from (4.2.11) we have then

$$\|\mu\|_q \geq \frac{2q \sin(\pi/q)}{2\pi} \|\mu\|_{M(K)} \tag{4.2.12}$$

resp.,
$$\|\mu\| \geq \frac{1}{\pi} \|\mu\|_{M(K)} . \tag{4.2.13}$$

If the set K is totally disconnected, then convex combinations of functions from $S_q(K)$ are dense in $P_q(K)$, and therefore the set $Q_q(\mu)$ is in fact the closed convex hull of the set of all z in C, such that $z = \int f d\mu$, with f in $S_q(K)$. In particular we have therefore $\|\mu\|_q = r(Q_q(\mu)) = \sup |\int f\, d\mu|$, $f \in S_q(K)$. Combining this with the definition of a K_q-set, we see that the Helson constant of a K_q-set is $\gamma_q = \sin(\pi/q)/(\pi/q)$.

When P is the unit interval, we use the general principle, that what can almost be done by continuous functions, can be done by Borel functions, to conclude the existence of a Borel function φ, which we may in fact assume to be idempotent, with $|\int \varphi\, d\mu| \geq \|\mu\|_{M(K)}/\pi$.

These results are all well-known, [3, p. 565; 5]. The standard proofs are different, but also the more natural approach given here is known, see e.g. [5, p. 674].

3. By very general arguments we have proved that no infinite subset of D_q can have a Helson constant greater than γ_q, and that on the other hand a K_q-set has the Helson constant γ_q. If K is a K_q-set we have in fact $A(K) = C_q(K)$ canonically, which means that for every function f in $C(K)$ resp. every measure μ in $M(K)$, we have

$$\|f\|_{A(K)} = \|f\|_q \leq \gamma_q^{-1} \|f\|_\infty \tag{4.3.1}$$

$$\text{resp.} \quad \|\mu\|_{PM(K)} = \|\mu\|_q \geq \gamma_q \|\mu\|_{M(K)} . \tag{4.3.1$'$}$$

Let now E be a Helson set in D_q. The Helson constant $\alpha(E)$ of E is defined by comparing norms in $A(E)$ and norms in $C(E)$. By the above we may write $\alpha(E) = \gamma_q(1 - \eta)$, and we have then

$$\|f\|_\infty \geq \gamma_q(1 - \eta) \cdot \|f\|_{A(E)}, \quad \text{all } f \in C(E) , \tag{4.3.2}$$

$$\text{resp.} \quad \|\mu\|_{PM(E)} \geq \gamma_q(1 - \eta)\|\mu\|_{M(E)}, \quad \text{all } \mu \in M(E) . \tag{4.3.2$'$}$$

If η is small, then condition (4.3.2′) is, in terms of a fixed μ, a strong condition if $\|\mu\|_q/\|\mu\|_M$ is close to γ_q, while it requires considerably less if $\|\mu\|_q/\|\mu\|_M$ is close to 1. It is therefore natural to consider instead of (4.3.2) and (4.3.2′) conditions of the following type

$$\|f\|_q \geq (1 - \delta)\|f\|_A, \quad \text{all } f \in C(E) , \tag{4.3.3}$$

$$\text{resp.} \quad \|\mu\|_{PM} \geq (1 - \delta)\|\mu\|_q, \quad \text{all } \mu \in M(E) , \tag{4.3.3$'$}$$

and we shall make the following definition:

Definition 4.1. Let K be a Helson set in D_q. We shall say that K is a *$H(q, \beta)$ set* if for every $f \in C(K)$, we have

$$\|f\|_{A(K)} < \beta^{-1}\|f\|_q$$

or equivalently, if for every $\mu \in M(K)$ we have

$$\|\mu\|_{PM} > \beta\|\mu\|_q .$$

It is clear that a $H(q, \beta)$ set with β close to 1, is a Helson set with Helson constant close to γ_q. We shall however also prove, and this is the main result of this section, that a Helson set, with Helson constant close to γ_q is a $H(q, \beta)$ set, with β close to 1.

The main step towards this is to prove the following theorem.

THEOREM 4.1. *Let α be a positive number, $0 < \alpha < \min(\pi^2/8q^2, 1/15)$, and let K be a totally disconnected metrizable compact Hausdorff space. Let further $\mathcal{F} \subset S_q(K)$ be a family of functions with the following property:*

For every measure $\mu \in M(K)$, there exists $g \in \mathcal{F}$, with

$$\left|\int g\, d\mu\right| > \frac{\sin(\pi/q)}{\pi/q}(1-\alpha)\|\mu\|_{M(K)} . \tag{4.3.4}$$

Then: If $\mu = \bar{f} \cdot |\mu|$, f a Borel function with $f^q = 1$, there exists $g \in \mathcal{F}$, such that

$$\left|\int g\, d\mu\right| > (1 - 14(\alpha/q)^{2/3})\|\mu\|_{M(K)} \quad \textit{if} \quad q \geq 2 \tag{4.3.5}$$

and

$$\left|\int g\, d\mu\right| > (1 - 5\alpha^{2/3}) \cdot \|\mu\|_{M(K)} \qquad \textit{if} \quad q = 2 . \tag{4.3.6}$$

Remark 1. Without loss of generality we assume that the family $\mathcal{F}$ is closed under multiplication by q^{th} rooths of unity (otherwise simply adjoin all multiples by q^{th} roots of unity), and we shall under this condition prove that we can in fact find $g \in \mathcal{F}$, such that

$$\operatorname{Re}\left(\int g\, d\mu\right) > (1 - 14(\alpha/q)^{2/3})\|\mu\|_{M(K)} .$$

Remark 2. Theorem 4.1 is stated and shall be proved without using the concepts of q-norms. In the following proof, the norm of a measure will always be the total mass norm.

Remark 3. If $\alpha = \pi^2/8q^2$, then $(1-\alpha)\gamma_q > (1 - 14(\alpha/q)^{2/3})$, and this shows that the given range of α's contains the range of interest.

Remark 4. The observant reader has already noticed that we have followed our own advice from § 1, and used strict inequalities in both assumptions and conclusions of the theorem.

4. *Outline of proof:* Any function g in $P_q(K)$ with

$$|I_\mu(g)| = \left| \int g \bar{f} d|\mu| \right| > (1 - \varepsilon)\|\mu\| ,$$

(and with $-(\pi/q) < \arg(I_\mu(g)) \leq (\pi/q)$) is close in $L^1(|\mu|)$ to the function f. The proof consists therefore essentially in finding a function g in $\mathcal{F}$ sufficiently close to f. This we can not do only in terms of the measure μ, and we shall therefore construct another measure ν, which is more adequate for our purposes. In the case $q = 2$, the measure μ is assumed to be real, and the problem is to find a suitable imaginary part to add to μ. The constructions in the proof are mainly geometric, and a crucial step in the proof is a purely geometric lemma on the perimeter of certain convex sets.

Proof of theorem. From general facts about sets $Q_P(\mu)$, we know that $Q_q(\mu)$ is for every $\mu \in M(K)$ a compact convex set containing 0. We also know that $p(Q_q(\mu)) = p(P_q) \cdot \|\mu\| = 2\pi\gamma_q \cdot \|\mu\|$.

We next observe that if $f \in P_q(K)$ then $f \cdot P_q(K) \subset P_q(K)$, and therefore $Q_q(f \cdot \mu) \subset Q_q(\mu)$. If we have moreover $f^q = 1$, then $f \cdot P_q(K) = P_q(K)$ and hence

$$Q_q(f \cdot \mu) = Q_q(\mu) . \tag{4.4.1}$$

In fact, (4.4.1) holds also if $f^q = 1$, and if f is a Borel function. In particular it follows that $Q_q(\mu)$ is invariant under multiplication by q^{th} roots of unity. Another consequence of (4.4.1) is that if $\mu = \bar{f} \cdot |\mu|$, $f^q = 1$ then

$$Q_q(\mu) = \|\mu\| \cdot P_q . \tag{4.4.2}$$

To see this we assume, by (4.4.1), that μ is positive. We take then $g = 1 \in P_q(K)$, and we see that the complex number $\|\mu\|$ belongs to $Q_q(\mu)$. Since $Q_q(\mu)$ is convex and invariant under multiplication by q^{th} roots of unity, we have then $\|\mu\| \cdot P_q \subset Q_q(\mu)$. Now we also know that the sets have equal perimeters, and therefore they are equal.

To illustrate the geometric notions involved, we shall construct explicitly the set $Q_q(\mu)$, for a measure μ with finite support. In this case we may also write $Q_q(\mathbf{a})$, for a certain set $\mathbf{a} = \{a_k\}_{k=1}^n$. By (4.4.1) we may assume that $-(\pi/q) < \arg(a_k) \leq (\pi/q)$, and it is also clear that we may assume that $\arg(a_k) \leq \arg(a_{k+1})$, all k. We now take $z_0 = \sum_{k=1}^n a_k$, and we see that z_0 is a boundary point of $Q_q(\mathbf{a})$. In fact z_0 maximizes the real part in $Q_q(\mathbf{a})$, and maximizes the imaginary part among all such points. We now define consecutively

$$z_k = z_{k-1} + a_k(\varrho - 1), \quad \varrho = e^{2\pi i/q} .$$

All the points $\{z_k\}_{k=0}^n$ are then extreme points of $Q_q(\mathbf{a})$, and we have further $z_n = \varrho \cdot z_0$. Defining now $z_{k+n} = \varrho \cdot z_k$, we get all the extreme points of $Q_q(\mathbf{a})$. See Figures 1 and 2.

Fig. 1. $Q_2(\mathbf{a})$
$\mathbf{a} = (a_1, a_2, a_3)$

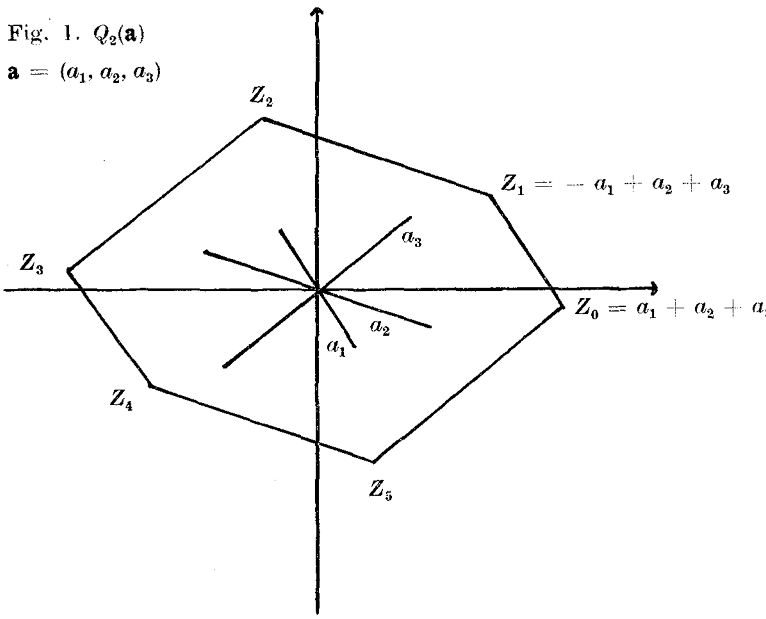
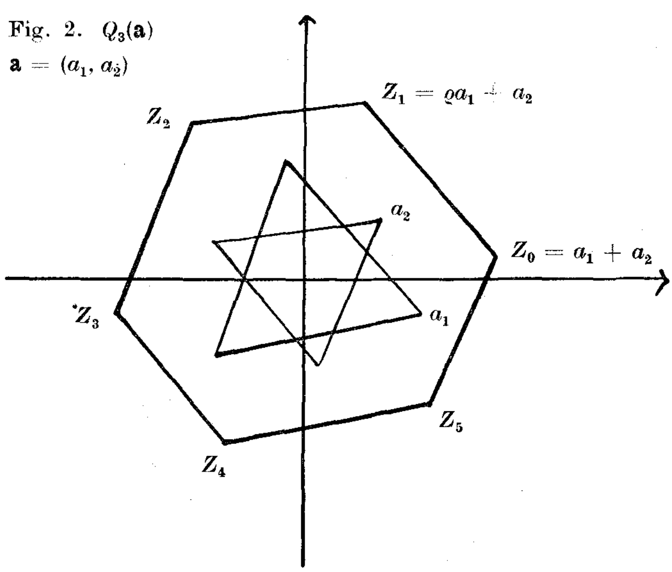


Fig. 2. $Q_3(\mathbf{a})$
$\mathbf{a} = (a_1, a_2)$



Conversely, we see that if Q is a convex polygon, invariant under multiplication by q^{th} roots of unity, then Q is a $Q_q(\mathbf{a})$, for some set of complex numbers. To construct one such set, we assume that Q has say $q \cdot n$ vertices. We enumerate then the vertices consecutively, starting from an arbitrary vertex by $z_0, z_1, \ldots,$ $z_n, \ldots z_{qn}$. We have then $z_n = \varrho z_0$, and we define $a_k = (z_k - z_{k-1})/(\varrho - 1)$, $1 \leq k \leq n$.

We shall now prove that if the given measure μ is continuous, or more precisely if μ can be decomposed into sufficiently small mutually singular parts, then the conclusion of the theorem holds. Towards this we first define a real number a, by the relation

$$1 + \frac{3q}{4\pi}(\tan a - a) = (1 - \alpha)^{-1} \tag{4.4.3}$$

or equivalently,

$$\tan a - a = \frac{1}{3} \cdot \frac{4\pi}{1 - \alpha} \cdot \frac{\alpha}{q}.$$

Using now the elementary estimate $\pi^2 < 10$, and the assumption $\alpha < \pi^2/8q^2 \leq \pi^2/32$, we have

$$a^3 \leq 3(\tan a - a) < \frac{4\pi}{\left(1 - \frac{5}{16}\right)} \cdot \frac{\alpha}{q}.$$

This implies then

$$a < \left(\frac{64\,\pi\alpha}{11q}\right)^{1/3} < \left(\frac{8\pi^3}{11q^3}\right)^{1/3} < \frac{\pi}{q}. \tag{4.4.4}$$

On the other hand, we have also $\alpha \leq 1/15$ and therefore $a < \pi/4$.

We next denote by $A_q(a)$ the set defined as the convex hull of the union of the unit disc and the points $\varrho^k(1/\cos a)$, $1 \leq k \leq q$. We further define a function $F(x)$ (or more properly $F_{q,a}(x)$) by

Fig. 3. $A_2(a)$

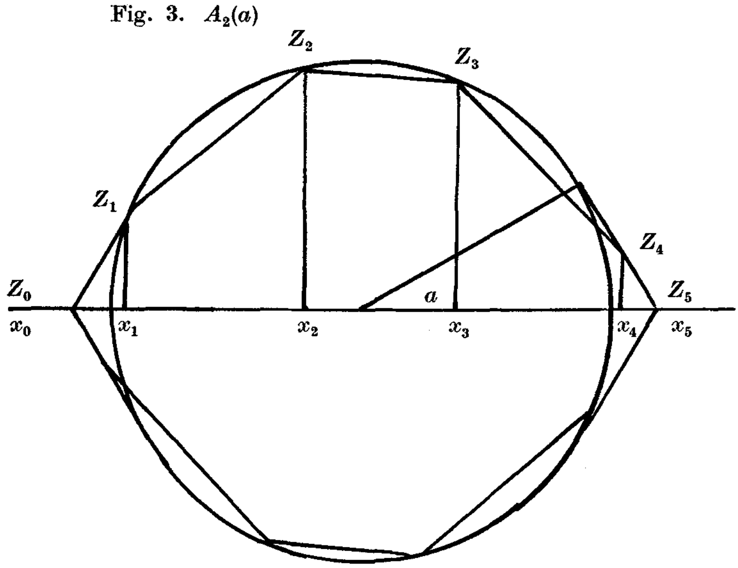
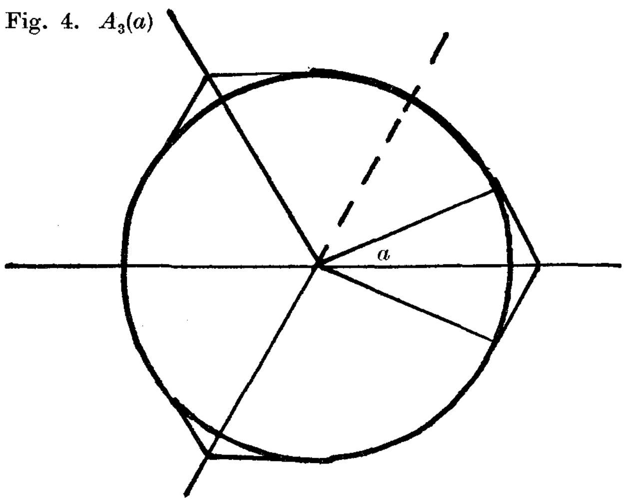


Fig. 4. $A_3(a)$



$$F(x) = \sup y \quad \text{over all} \quad (x, y) \in A_q(a), \tag{4.4.5}$$

for

$$\frac{\cos(2\pi/q)}{\cos a} \leq x \leq \frac{1}{\cos a}.$$

(Figures 3 and 4.)

We have, since $a < \pi/q$, $p(A_q(a)) = 2\pi + 2q(\tan a - a)$, so if A' is a polygon with sufficiently small sides and with vertices on the boundary of $A_q(a)$, then $p(A') \geq 2\pi + \frac{3}{4} 2q(\tan a - a)$. We shall presently construct a ν whose $Q_q(\nu)$ is such an inscribed polygon. We shall then have

$$\|\nu\| = p(Q_q(\nu))/p(P_q) = p(Q_q(\nu))/2\pi\gamma_q \geq \gamma_q^{-1}\left(1 + \frac{3q}{4\pi}(\tan a - a)\right), \tag{4.4.6}$$

and by (4.4.3) and the assumptions on $\mathcal{F}$, we can then find $g \in \mathcal{F}$ with $|I_\nu(g)| > 1$.

To make the construction, we first break μ into small pieces. We partition K into a finite number of disjoint Borel sets $\{E_i\}_{i=1}^N$, and we assume for simplicity that $|\mu|(E_i) = r_i > 0$, all i. We denote $\mu_i = r_i^{-1} \cdot \mu|_{E_i}$, so that we have $\mu = \sum r_i\mu_i$. Now, for all i we have $Q_q(\mu_i) = P_q$, and this implies that if $\mu(\mathbf{a}) = \sum a_i\mu_i$, then $Q_q(\mu(\mathbf{a})) = Q_q(\mathbf{a})$. The measure ν will be such a linear combination, and for $q = 2$ the construction is illustrated geometrically in Figure 3. We shall carry out the construction first in the somewhat simpler case $q = 2$, and then for an arbitrary q.

The case $q = 2$. We assume $\|\mu\| = (\cos a)^{-1}$, and we have then positive numbers $\{r_i\}_{i=1}^N$, such that $\sum r_i = (\cos a)^{-1}$. We denote now

$$x_k = -(\cos a)^{-1} + 2\sum_{i=1}^{k} r_i, \quad 0 \leq k \leq N \quad (\text{i.e. } x_0 = -(\cos a)^{-1}),$$

and we denote, in terms of the $F(x)$ defined in (4.4.4) $z_k = x_k + iF(x_k)$, $0 \leq k \leq N$. We can now finally define

$$a_k = \frac{z_k - z_{k-1}}{2}, \quad 1 \leq k \leq N.$$

The set $\mathbf{a} = \{a_k\}_{k=1}^N$ is constructed in such a way that $Q_2(\mathbf{a})$ is a polygon inscribed in $A_2(a)$, and we have $\mathrm{Re}\,(a_k) = r_k$. We define now $\nu = \sum a_k \mu_k$, and we have then $\mu = \mathrm{Re}\,(\nu)$. We assume now that all r_k are so small that $\|\nu\| \geq (\pi/2)(1-\alpha)^{-1}$, so that there exists $g \in \mathcal{G}$ with $|I_\nu(g)| > 1$. Assuming $\mathrm{Re}\,(I_\nu(g)) \geq 0$, we have $\mathrm{Re}\,(I_\nu(g)) > \cos a$, and since g is real and since $\mu = \mathrm{Re}\,(\nu)$ we have

$$I_\mu(g) > \cos a = \cos^2 a \cdot \|\mu\| \geq (1 - 5\cdot\alpha^{2/3})\|\mu\|$$

Arbitrary q. We assume again $\|\mu\| = (\cos a)^{-1}$, and we have then also in this case positive numbers $\{r_i\}_{i=1}^N$, such that $\sum r_i = (\cos a)^{-1}$. In this case we shall however first pass over to the numbers $\{s_i\}_{i=1}^N$ defined by

$$s_i = 2\,(1 - \cos a \cos(\pi/q)) r_i.$$

We denote now

$$x_k = +(\cos a)^{-1} - \sum_{i=1}^{k} s_i, \quad 0 \leq k \leq N.$$

We define now L by $x_L \geq \cos(\pi/q) > x_{L+1}$, and

$$z_k = x_k + iF(x_k), \quad 0 \leq k \leq L.$$

We denote further

$$y_k = (\cos a)^{-1} - \sum_{i=k+1}^{N} s_i, \quad L+1 \leq k \leq N,$$

and
$$w_k = y_k - iF(y_k), \quad L+1 \leq k \leq N.$$

Defining now z_k for $L+1 \leq k \leq N$ by $z_k = \varrho \cdot w_k$, $\varrho = e^{2\pi i/q}$, we have z_k defined for all k, $0 \leq k \leq N$. We can now define

$$a_k = \frac{z_k - z_{k-1}}{\varrho - 1}, \quad 1 \leq k \leq N.$$

This finishes the construction of the numbers a_k. We set $\nu = \sum a_k \mu_k$, so that $Q_q(\nu)$ is a polygon inscribed in $A_q(a)$, and assuming all r_k sufficiently small we have then $\gamma_q(1-\alpha)\|\nu\| \geq 1$. By assumption, there then exists $g \in \mathcal{G}$ such that $|I_\nu(g)| > 1$. Assuming $-(\pi/q) < \arg(I_\nu(g)) \leq (\pi/q)$, we have, since

$I_\nu(g) \in Q_q(\nu) \subset A_q(a)$, $\operatorname{Re}(I_\nu(g)) > \cos a$. We shall estimate $\operatorname{Re}(I_\mu(g))$ and towards this we first define $\int g\, d\mu_k = t_k = 1 - u_k$, and since $t_k \in P_q$, we have $-(\pi/2 - \pi/q) \leq \arg(u_k) \leq (\pi/2 - \pi/q)$. We can now write

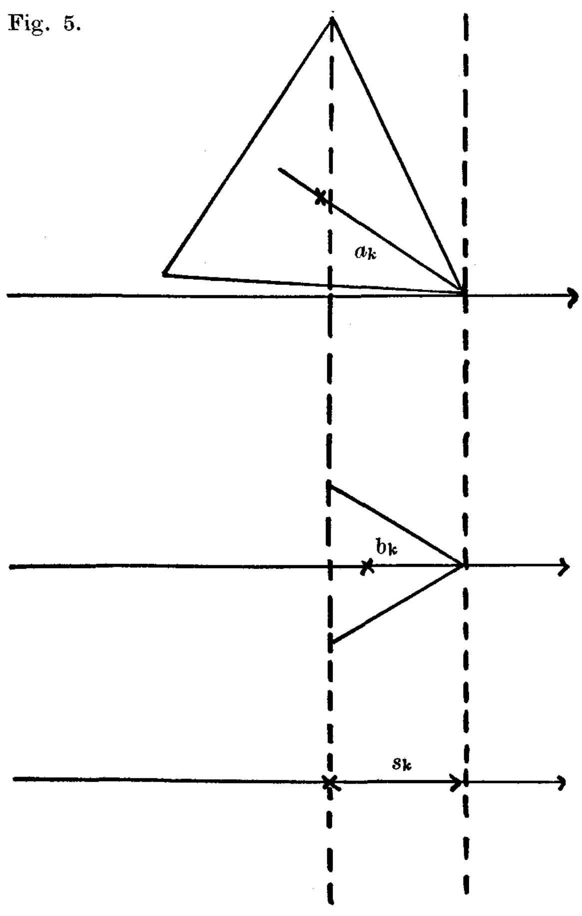
$$\operatorname{Re}(I_\nu(g)) = \operatorname{Re}\left(\sum a_k(1-u_k)\right) = \frac{1}{\cos a} - \operatorname{Re}\left(\sum a_k u_k\right)$$

and
$$\operatorname{Re}(I_\mu(g)) = \operatorname{Re}\left(\sum r_k(1-u_k)\right) = \frac{1}{\cos a} - \operatorname{Re}\left(\sum r_k u_k\right). \tag{4.4.7}$$

Our task is thus to estimate $\operatorname{Re}\left(\sum r_k u_k\right)$ in terms of $\operatorname{Re}\left(\sum a_k u_k\right)$. Examining carefully the construction of the numbers a_k, we see that $-(\pi/q) < \arg(a_k) < \pi/q$, so that $\operatorname{Re}(a_k u_k) > 0$, unless $u_k = 0$. We define now for every k a positive number b_k, by the condition that b_k is the largest real number, such that

$$\operatorname{Re}(b_k u) \leq \operatorname{Re}(a_k u), \text{ for all } u \text{ with } -(\pi/2 - \pi/q) \leq \arg(u_k) \leq (\pi/2 - \pi/q), \tag{4.4.8}$$

Fig. 5.



thus

$$b_k = \frac{\sin(\pi/q - \arg|a_k|)}{\sin \pi/q}$$

Notice also that for $k \leq L$, we have $\arg(a_k) < 0$.

(See Fig. 5.)

Now it follows from this definition of the numbers b_k that $s_k \leq (1 - \cos 2\pi/q) b_k$, with equality unless $k = L + 1$.

We have then

$$r_k \leq \frac{1 - \cos(2\pi/q)}{2(1 - \cos a \cos(\pi/q))} b_k \leq 2b_k ,$$

and therefore $\mathrm{Re}\left(\sum r_k u_k\right) \leq 2 \,\mathrm{Re}\left(\sum b_k u_k\right) \leq 2 \,\mathrm{Re}\left(\sum a_k u_k\right) \leq 2 \sin a \tan a$.

We have therefore finally

$$\mathrm{Re}\, I_\mu(g) > (1 - 2\sin^2 a)\|\mu\| \geq (1 - 14(\alpha/q)^{2/3})\|\mu\| .$$

5. *A geometric lemma.*

So far we have proved that the conclusions of the theorem hold for every measure that can be decomposed into sufficiently small mutually singular parts. Our next task is to decide how large »sufficiently small» can be. This is a purely geometric problem, and one solution is given by the following lemma. The lemma is stated so as to fit with the construction of the measure ν in the case $q = 2$, but it is easy to see that the estimates obtained will be valid for arbitrary q.

LEMMA 4.2. *Let a, $0 < a < \pi/4$, be a real number and let $F(x)$ or more properly $F_{2,a}(x)$ be the function defined in (4.4.4). Let $\{x_k\}_{k=1}^M$ be real numbers, such that*

$$x_0 = \frac{-1}{\cos a}, \quad x_M = \frac{1}{\cos a} \tag{4.5.1}$$

and

$$0 < x_k - x_{k-1} \leq \frac{1}{\cos a} - \cos a = \sin a \tan a . \tag{4.5.2}$$

Let further $z_k = x_k + iF(x_k)$, $0 \leq k \leq M$, and let

$$a_k = \frac{z_k - z_{k-1}}{2}, \quad 1 \leq k \leq M .$$

Then $Q_2(\mathbf{a})$ is a polygon with vertices on the boundary of $A_2(a)$, and

$$p(Q_2(\mathbf{a})) \geq 4(\pi/2 + \tfrac{3}{4}(\tan a - a)) . \tag{4.5.3}$$

Proof. We shall denote the set $A_2(a)$ simply by A, and the set $Q_2(\mathbf{a})$ by A'. We shall estimate $p(A')$, so we write

$$p(A') = p(A) - (p(A) - p(A')) = (2\pi + 4(\tan a - a)) - 4D .$$

We shall first make the additional assumption that

$$x_1 = -\cos a, \quad x_{M-1} = \cos a \tag{4.5.4}$$

and we shall under this additional assumption prove that $D \leq (\tan a - a)/8$.

To estimate D we parametrize the upper boundary of A, by using arc-length (to the right) from $z = i$ as parameter. We have then

$$x_k = \sin s_k, \quad z_k = \sin s_k + i \cos s_k, \quad 1 \leq k \leq M-1 .$$

We denote further

$$m_k = \frac{s_{k-1} + s_k}{2}, \quad d_k = s_k - s_{k-1}, \quad 2 \leq k \leq M-1 .$$

We have then

$$2D = \sum_{k=2}^{M-1} \left(d_k - 2 \sin \frac{d_k}{2}\right) \tag{4.5.5}$$

By condition (4.5.2), we have now

$$x_k - x_{k-1} = \sin s_k - \sin s_{k-1} = 2 \sin \frac{d_k}{2} \cos m_k \leq \sin a \tan a = B ,$$

and therefore $2 \sin d_k/2 \leq B/\cos m_k$. We have also the trivial estimate $d_k \leq \tan a \leq 2a$, and therefore

$$d_k = \frac{d_k}{2 \sin d_k/2} \cdot 2 \sin \frac{d_k}{2} \leq \frac{2a}{2 \sin a} \frac{B}{\cos m_k} = \frac{a \tan a}{\cos m_k} . \tag{4.5.6}$$

By a series expansion and (4.5.6), we have now

$$d_k - 2 \sin \frac{d_k}{2} \leq \frac{d_k^3}{24} \leq \frac{1}{24} \cdot d_k \cdot \left(\frac{a \tan a}{\cos m_k}\right)^2 = \frac{a^2 \tan^2 a}{24} \frac{d_k}{\cos^2 m_k} . \tag{4.5.7}$$

The function $\varphi(t) = 1/\cos^2 t$ is convex, and by Jensen's inequality, we have therefore

$$\frac{d_k}{\cos^2 m_k} \leq \int_{m_k - d_k/2}^{m_k + d_k/2} \frac{1}{\cos^2 t} \, dt . \tag{4.5.8}$$

Combining now (4.5.5), (4.5.7) and (4.5.8), we have

$$2D = \sum_{k=2}^{M-1}\left(d_k - 2\sin\frac{d_k}{2}\right) \le \frac{a^2\tan^2 a}{24}\int_{a-\pi/2}^{\pi/2-a}\frac{dt}{\cos^2 t}$$

$$= \frac{a^2\tan^2 a}{24}\,2\tan(\pi/2 - a) = \frac{a^2\tan^2 a}{12}\cot a = \frac{a^2\tan a}{12} \tag{4.5.9}$$

$$\le \frac{\tan a - a}{4}\,, \quad \text{and so} \quad D \le \frac{\tan a - a}{8}\,.$$

We now prove (4.5.3) without additional assumptions on the x_i's. Towards this we first denote by A'', the polygon that besides the vertices of A' has additional vertices at $\pm e^{\pm ia}$, and we observe that by the previous case we have $p(A) - p(A'') \le (\tan a - a)/2$. To complete the proof we must show that $4D_2 = p(A'') - p(A') \le (\tan a - a)/2$.

To fix notations we assume $x_{M-2} < \cos a \le x_{M-1}$, and we shall denote

$$z_{M-2} = \cos(a+s) + i\sin(a+s)$$
$$Z_{M-1} = \cos a + t\sin a + i(\sin a - t\cos a)\,.$$

We further write $l = 2\sin(s/2)$.

We have then, by the cosine theorem

$$D = |Z_{M-1} - e^{ia}| + |e^{ia} - Z_{M-2}| - |Z_{M-1} - Z_{M-2}|$$
$$= (l+t) - \sqrt{l^2 + t^2 + 2\left(\cos\frac{s}{2}\right)lt} =$$
$$= (l+t)\left(1 - \sqrt{1 - \frac{2lt}{(l+t)^2}\left(1 - \cos\frac{s}{2}\right)}\right)$$
$$\le (l+t)\left(1 - \left(1 - \frac{lt}{(l+t)^2}\cdot\frac{s^2}{8}\right)\right) \le \frac{1}{8}\cdot\frac{s^3 t}{s+t}\,.$$

We further have $s + t \le \tan a$, and therefore

$$D_2 \le \frac{(\tan a)^3}{8}\cdot\max_{0<x<1}[x^3(1-x)] = \frac{(\tan a)^3\cdot 27}{8\cdot 256}\,.$$

Since we have assumed $a \le \pi/4$, we have now

$$\frac{(\tan a)^3}{3(\tan a - a)} \le \frac{(\tan \pi/4)^3}{3(\tan\pi/4 - \pi/4)} = \frac{1}{3(1-\pi/4)} = \frac{4}{3(4-\pi)}$$

and therefore

$$D_2 \le \frac{81}{1024}\cdot\frac{4}{3(4-\pi)}\cdot(\tan a - a) \le \frac{1}{8}(\tan a - a)\,.$$

and this proves the lemma.

6. *Conclusions of Lemma* 4.2.

The polygon in Lemma 4.2 is the $Q_2(\nu)$ constructed earlier. Combining the lemma with the previous construction we see therefore that: If $\mu = \sum r_k\mu_k$, μ_k mutually singular real measures $\|\mu_k\| = 1$, and if $0 < r_k \leq \frac{1}{2}\sin^2 a \cdot \|\mu\|$, all k, where a is defined by (4.4.3), then the conclusion of the theorem holds for μ.

Let now E be a finite subset of K with card $(E) = N \leq 2/\sin^2 a$. We define a real number $a' \geq a$, by $\sin^2 a' = 2/N$, and we define $\mu \in M(E) \subset M(K)$ by

$$\mu = \sum_{x_k \in E} \left(\frac{1}{N\cos a'}\right) \varepsilon_k \delta(x_k)\ ,$$

where $\delta(x_k)$ is the point mass at x_k, and $\varepsilon_k^2 = 1$. We now construct as above a measure ν with vertices of $Q_2(\nu)$ on $A_2(a')$. We have then

$$\|\nu\| \geq (\pi/2 + \tfrac{3}{4}(\tan a' - a')) \geq (\pi/2 + \tfrac{3}{4}(\tan a - a)) = (\gamma_2(1-\alpha))^{-1}\ ,$$

and there exists then $g \in \mathcal{F}$, such that $|I_\nu(g)| > 1$. Assuming as usual $\mathrm{Re}\,(I_\nu(g)) \geq 0$ we have $I_\mu(g) = \mathrm{Re}\,(I_\nu(g)) > \cos a'$. If now $g(x_k) = \varepsilon_k$, all k, then $I_\mu(g) = 1/\cos a'$, otherwise, we have in fact $I_\mu(g) \leq 1/\cos a' - 2 \cdot \sin a' \tan a'/2 = \cos a'$, and this would contradict the choice of g. This proves thus, that $\mathcal{F}|_E = S_2(E)$.

If we have $q > 2$, then the estimates used in the proof of Lemma 4.2 are still valid, and combining the lemma with our previous construction, we have: If $\mu = \sum r_k\mu_k$, $\mu_k = f_k|\mu_k|$, f_k a Borel function, $f_k^q = 1$, $\|\mu_k\| = 1$, μ_k mutually singular, and if $0 < r_k \leq \dfrac{\sin^2 a}{2(1-\cos a \cos \pi/q)} \cdot \|\mu\|$, then the conclusion of the theorem holds for μ. We have also, by the above arguments, that if E is finite subset of K, card $(E) \leq 2(1 - \cos a \cos(\pi/q))/\sin^2 a$, then $\mathcal{F}|_E = S_q(E)$.

We have thus proved, that if $\mu = f|\mu|$, f a Borel function with $f^q = 1$, and if either the support of μ is a small finite set, or if μ can be decomposed into sufficiently small parts, then the conclusion of the theorem holds for μ. In the general case, we shall first replace μ in a suitable way by a measure μ' for which the theorem holds. We take then a $g \in \mathcal{F}$, satisfying the conclusions with respect to μ', and we shall see that such a g satisfies the conclusions also for the original measure μ. To construct the measure μ', we shall use the following lemma.

LEMMA 4.3. *Let b be a real number, $0 < b < 1$, and let $\{c_k\}_{k=1}^N$ be real numbers, such that $c_k \geq c_{k+1} > 0$, and such that $\sum c_k \leq 1$. Let $M \leq N$ be the natural number defined by the conditions*

$$\sum_{k=1}^{M} c_k \geq Mb, \qquad \sum_{k=1}^{M+1} c_k < (M+1)b \tag{4.6.1}$$

(If $\sum_{k=1}^{N} c_k \geq Nb$, then we put $M = N$)

Finally, let $t = (1 - Mb) \cdot (1 - \sum_{k=1}^{M} c_k)^{-1}$.
Then $tc_{M+1} < b$.

Proof. By the definitions we have $t \geq 1$. We denote $R = 1 - \sum_{k=1}^{N} c_k$, and we see that t is defined so that

$$Mb + t\Big(R + \sum_{k=M+1}^{N} c_k\Big) = Mb + t\Big(1 - \sum_{k=1}^{M} c_k\Big) = 1 . \tag{4.6.2}$$

If now, $M = N$, then there is nothing to prove, and if $(M + 1)b > 1$, then $1 - Mb < b$, so that $t(c_{M+1} + \sum_{k=M+2}^{N} c_k + R) = 1 - Mb < b$, and in particular $tc_{M+1} < b$. If $(M + 1)b = 1$, then by the choice of M, we have either $R > 0$ or $N > M + 1$, and in both cases do we have $tc_{M+1} < b$.

We assume now $(M + 1)b < 1$, and by definition of M we have then

$$R + \sum_{k=M+2}^{N} c_k = 1 - \sum_{k=1}^{M+1} c_k < 1 - (M + 1)b ,$$

and therefore

$$\begin{aligned} tc_{M+1} &= 1 - Mb - t(R + \sum_{k=M+2}^{N} c_k) < 1 - Mb - t(1 - (M + 1)b) \\ &\leq 1 - Mb - (1 - (M + 1)b) = b . \end{aligned}$$

This proves the lemma.

End of proof of Theorem 4.1.

Let $\mu \in M(K)$, $\mu = f|\mu|$, f a Borel function, $f^q = 1$, $\|\mu\| = 1$. We write $\mu = \mu_d + \mu_c$, where μ_c is continuous, and

$$\mu_d = \sum_{k=1}^{N} c_k \bar{\varrho}_k \delta(x_k), \quad \varrho_k^q = 1, \quad c_k \geq c_{k+1} > 0 .$$

We assume that μ_d has finite support, but since there is otherwise nothing new to prove, we shall also assume that either $\mu_c \neq 0$, or

$$N > \frac{2(1 - \cos a \cos \pi/q)}{\sin^2 a} .$$

We now use Lemma 4.3 on the numbers c_k and with

$$b = \frac{\sin^2 a}{2(1 - \cos a \cos \pi/q)} . \tag{4.6.3}$$

The lemma gives us numbers M and t, such that writing

$$\mu' = \mu_1 + t\mu_2 = b \cdot \sum_{k=1}^{M} \bar{\varrho}_k \delta(x_k) + t\Big(\sum_{k=M+1}^{N} c_k \bar{\varrho}_k \delta(x_k) + \mu_c \Big) ,$$

(b defined by (4.6.3)), we have $\|\mu'\| = 1$, and μ' can be decomposed into small parts.

In the case $q = 2$, we complete the proof as follows. We use the fact that the theorem holds for μ', and we choose g such that

$$I_{\mu'}(g) = I_{\mu_1}(g) + tI_{\mu_2}(g) > 1 - \sin^2 a . \tag{4.6.4}$$

We have now $g^2 = 1$, and this implies that either (i) $I_{\mu_1}(g) = \|\mu_1\|$ or (ii) $I_{\mu_1}(g) \leq \|\mu_1\| - 2b = \|\mu_1\| - \sin^2 a$. Since (ii) contradicts (4.6.4) we have (i), and this implies that $g(x_k) = \varepsilon_k (= \bar{\varrho}_k)$, for $1 \leq k \leq M$. We have then

$$I_\mu(g) = \sum_{k=1}^{M} c_k + I_{\mu_2}(g) \geq \sum_{k=1}^{N} c_k + \left(\|\mu_2\| - \frac{1}{t} \sin^2 a \right) \geq 1 - \sin^2 a , \tag{4.6.5}$$

and this finishes the proof in the case $q = 2$.

If $q > 2$, then things are technically more complicated. In this case we cannot use merely the fact that the theorem holds for μ', but rather the proof. I.e. we must construct a measure ν as in the continuous case, and choose g with $\mathrm{Re}\,(I_\nu(g)) > \cos a$. From this fact we can now conclude that $\mathrm{Re}\,(I_{\mu_1}(g)) = \|\mu_1\|$. Once this is known, the proof can be completed as in the case $q = 2$, and we have then proved Theorem 4.1 in full generality.

7. Theorem 4.1 can in certain instances, be used as a substitute for a condition of type (4.3.3′), but nevertheless we want a complete result on norm relations. To this end we shall need the following proposition.

PROPOSITION 4.4. *Let K be a compact metrizable Hausdorff space, and let $\mathcal{F} \subset P_q(K)$, be a family of functions, such that for every μ in $M(K)$ with $\mu = \bar{f}|\mu|$, f Borel, $f^q = 1$, there exists a $g \in \mathcal{F}$, such that*

$$\mathrm{Re} \int g\bar{f} d|\mu| > (1 - \delta)\|\mu\|_M .$$

Then to every μ in $M(K)$ there corresponds a g in $\mathcal{F}$, such that

$$\left| \int g \, d\mu \right| > (1 - 2\delta)\|\mu\|_q \quad (or \ > (1 - \delta)\|\mu\|_q, \ if \ q = 2)$$

Proof. Let us write $|\mu| = \lambda$ and $\mu = \bar{\varphi}\lambda$, $|\varphi| = 1$ a.e. (λ). There exists a Borel function f such that $f^q = 1$, and

$$|I(f)| = \left| \int f\bar{\varphi}d\lambda \right| = \|\mu\|_q . \tag{4.7.1}$$

We can choose α real such that $e^{i\alpha}I(f) = |I(f)|$ and replacing thus if necessary μ by $e^{i\alpha}\mu$, we assume $I(f) \geq 0$. We next write $f\bar{\varphi} = e^{-i\psi}$, and we have then by (4.7.1), $-\pi/q \leq \psi \leq \pi/q$ a.e. (λ). We have then

$$\mu = \bar{\varphi}\lambda = \bar{f}f\bar{\varphi}\lambda = \bar{f} \cdot e^{-i\psi} \cdot \lambda = \bar{f} \cdot (\cos\psi - i\sin\psi) \cdot \lambda ,$$

and we shall denote the measure $\bar{f} \cdot \cos\psi \cdot \lambda$ by ν. It follows from our assumptions, and from the construction of ν, that

$$\|\mu\|_q = \int f\bar{\varphi}d\lambda = \int f\bar{f}(\cos\psi)d\lambda = \int fd\nu = \|\nu\|_M .$$

By the assumptions of the theorem we can find $g \in \mathcal{F}$, such that

$$\mathrm{Re} \int g\, d\nu = \mathrm{Re} \int g\bar{f}\cos\psi\, d\lambda < (1 - \delta)\|\nu\| .$$

We shall write $g\bar{f} = h$, and we observe that $h \in P'_q(K)$, and we have then $|\mathrm{Im}\, h| \leq \cot(\pi/q) \cdot (1 - \mathrm{Re}\, h)$, if $q < 2$, while $\mathrm{Im}\, h = 0$ if $q = 2$.

We can now estimate

$$\mathrm{Re} \int g\, d\mu = \mathrm{Re} \int g\bar{f}(\cos\psi - i\sin\psi)d\lambda = \mathrm{Re} \int g\bar{f}\cos\psi\, d\lambda + \int \mathrm{Im}\, h \sin\psi\, d\lambda .$$

By the above we have then

$$\mathrm{Re} \int g\, d\mu \geq (1 - \delta)\|\mu\|_q \quad \text{if} \quad q = 2 ,$$

and

$$\mathrm{Re} \int g\, d\mu \geq (1 - \delta)\|\mu\|_q - \int |\mathrm{Im}\, h| \cdot |\sin\psi| d\lambda$$

$$\geq (1 - \delta)\|\mu\|_q - \cot(\pi/q)\tan(\pi/q) \int (1 - \mathrm{Re}\, h)\cos\psi\, d\lambda \geq (1 - 2\delta)\|\mu\|_q, \quad \text{if} \quad q > 2 .$$

This proves the proposition.

As an immediate corollary of Theorem 4.1 and Proposition 4.4 we have now

COROLLARY 4.5. *Let $K \subset D_q$ be a Helson-$(\gamma_q(1 - \alpha))$ set, then K is also a $H(q, 1 - \beta)$ set, with $\beta = K < 28(\alpha/q)^{2/3}$.*

§ 5. Tensor algebras in D_q

1. We shall first introduce some notations. Let X be a totally disconnected compact space, and let I_q be the identity map of $C(X)$, considered as an operator from $C(X)$ into $C_q(X)$. By § 4, we have then $\|I_q\| \leq \gamma_q^{-1} = (\pi/q)/\sin(\pi/q)$. Let now $\{X_i\}_{i=1}^n$ be totally disconnected compact spaces, we shall write

$$V_q(X) = C_q(X_1) \hat{\otimes} C_q(X_2) \hat{\otimes} \ldots \hat{\otimes} C_q(X_n) .$$

Let further J_q be the identity map of $V(X)$ considered as a map from $V(X)$ into $V_q(X)$. Since $J_q = (I_q)_1 \otimes (I_q)_2 \otimes \ldots \otimes (I_q)_n$, we have by standard properties of tensor norms $\|J_q\| \leq \gamma_q^{-n}$. Combining this with Theorem 1.2 (resp. Corollary 3.1) we have under the assumptions of Theorem 1.2 (resp. Corollary 3.1) that $\|\mathbf{1} - F\|_{V_q} \leq \gamma_q^{-n}\eta_n(y)$ (resp. $\|g - F\|_{V_q} \leq \gamma_q^{-n}\eta_n(y)$).

We can now state and prove the following theorem.

THEOREM 5.1. *Let δ, $0 < \delta < 1$, be a real number. For every natural number $q \geq 2$, and every natural number $n \geq 2$, there exists a number $\beta_{q,n} > 0$, such that if $\{K_i\}_{i=1}^n$ are disjoint compact subsets of D_q, and if $\bigcup K_i$ is a $H(q, 1-\beta)$ set, $\beta < \beta_{q,n}$, then the map*

$$T : A(\textstyle\sum K_i) \to V_q(\mathbf{K}) \quad (\textit{defined as in } \S\ 1)$$

is a topological isomorphism, and $\|T^{-1}\| \leq \delta^{-1}$.

Proof. Let $\eta_n(y)$ be a function satisfying the conditions of Theorem 1. We define a positive number $y_{q,n}$ by the relation

$$1 - \gamma_q^{-n}\eta_n(y_{q,n}) = (1 + y_{q,n})\delta . \tag{5.1.1}$$

and we define $\beta_{q,n}$ by

$$1 - \beta_{q,n} = \frac{1}{1 + y_{q,n}} . \tag{5.1.2}$$

Let now β, $0 < \beta < \beta_{q,n}$, be a real number, and let $\{K_i\}_{i=1}^n$ be disjoint compact subsets of D_q, such that $\bigcup K_i$ is a $H(q, 1-\beta)$-set. For convenience of notations we shall write $(1-\beta)^{-1} = 1 + y$. In accordance with our previous notations we shall also write $K = K_1 \times K_2 \times \ldots \times K_n$ and $K' = \bigcup K_i$.

To prove the theorem we shall use the same duality argument, that was used in the proof of Theorem 1.1. Let therefore $A \in BM_q(K)$, $\|A\|_q = 1$.

We choose $\varepsilon > 0$, such that $1 - \varepsilon - \gamma_q^{-n}\eta_n(y) > (1 + y)\delta$. Then, by the definition of norm in $BM_q(K)$, there exist functions $g_i \in S_q(K_i)$, such that

$$|A(g)| > 1 - \varepsilon , \tag{5.1.3}$$

where g is defined by (1.1.2). Now the function g' defined by (1.1.3) belongs to $C_q(K')$, and we have $\|g'\|_q = 1$. By assumption we have therefore

$$\|g'\|_{A(K)} < 1 + y\,. \tag{5.1.4}$$

There exists therefore a representation

$$g'(k') = \sum a_\chi \chi(k'), \quad \chi \in \hat{D}_q, \quad \sum |a_\chi| < 1 + y\,. \tag{5.1.5}$$

We shall write $F(k) = \sum a_\chi \chi(k_1)\chi(k_2)\ldots\chi(k_n)$, $k = (k_1, k_2, \ldots, k_n)$, $k_i \in K_i$. By Corollary 3.1 we have then $F \in V$, $\|F\|_V < 1 + y$, $\|g - F\|_V < \eta_n(y)$. Since the functions χ used in the representation of F all belong to S_q, we have in fact $\|F\|_{V_q} < 1 + y$, and we have also $\|g - F\|_{V_q} < \gamma_q^{-n}\eta_n(y)$.

But this implies then

$$\begin{aligned} |A(F)| &= |A(g + F - g)| \geq |A(g)| - |A(F - g)| \\ &\geq 1 - \varepsilon - \|A\|_{BM_q} \cdot \|F - g\|_{V_q} \geq 1 - \varepsilon - \gamma_q^{-n}\eta_n(y)\,. \end{aligned} \tag{5.1.6}$$

By choice of ε, we have therefore $|A(F)| > (1 + y)\delta$.

By the usual map of K into $\sum K_i$, we consider $A(\sum K_i)$ as a subalgebra of $V_q(K)$ $(= V(K))$. Now the function F constructed above belongs to $A(K)$, we have a representation of F, showing that $\|F\|_{A(K)} < 1 + y$, and we have $|A(F)| > (1 + y)\delta$. This implies then $\|A\|_{PM} > \delta$, and this proves the theorem.

Finally, Theorem B follows from Theorem 5.1, combined with Corollary 4.5.

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