Local maxima of Gaussian fields*

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0. Summary

The structure of a stationary Gaussian process near a local maximum with a prescribed height u has been explored in several papers by the present author, see [5]—[7], which include results for moderate u as well as for $u \to \pm\infty$. In this paper we generalize these results to a homogeneous Gaussian field $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbf{R}^n\}$, with mean zero and the covariance function $r(\mathbf{t})$. The local structure of a Gaussian field near a high maximum has also been studied by Nosko, [8], [9], who obtains results of a slightly different type.

In Section 1 it is shown that if ξ has a local maximum with height u at $\mathbf{0}$ then $\xi(\mathbf{t})$ can be expressed as

$$\xi_u(\mathbf{t}) = uA(\mathbf{t}) - \boldsymbol{\zeta}_u' \mathbf{b}(\mathbf{t}) + \Delta(\mathbf{t}),$$

where $A(\mathbf{t})$ and $\mathbf{b}(\mathbf{t})$ are certain functions, $\boldsymbol{\zeta}_u$ is a random vector, and $\Delta(\mathbf{t})$ is a non-homogeneous Gaussian field. Actually $\xi_u(\mathbf{t})$ is the old process $\xi(\mathbf{t})$ conditioned in the horizontal window sense to have a local maximum with height u for $\mathbf{t} = \mathbf{0}$; see [4] for terminology.

In Section 2 we examine the process $\xi_u(\mathbf{t})$ as $u \to -\infty$, and show that, after suitable normalizations, it tends to a fourth degree polynomial in $t_1, \ldots, t_n$ with random coefficients. This result is quite analogous with the one-dimensional case.

In Section 3 we study the locations of the local minima of $\xi_u(\mathbf{t})$ as $u \to \infty$. In the non-isotropic case $r(\mathbf{t})$ may have a local minimum at some point $\mathbf{t}^0$. Then it is shown in 3.2 that $\xi_u(\mathbf{t})$ will have a local minimum at some point $\boldsymbol{\tau}^u$ near $\mathbf{t}^0$, and that $\boldsymbol{\tau}^u - \mathbf{t}^0$ after a normalization is asymptotically n-variate normal as $u \to \infty$. This is in accordance with the one-dimensional case.

*) This research was supported in part by the Office of Naval Research under Contract N00014-67-A-0002.

If the process is isotropic, then $r(\mathbf{t})$ depends only on $|\mathbf{t}|$, and it can have no strict local minima. We can then obtain some results without direct counterpart in the one-dimensional case. Consider $\xi_u(\mathbf{t})$ as $\mathbf{t}$ runs along a direction $\boldsymbol{\theta}$ from the origin, and let $\tau_u(\boldsymbol{\theta})$ be the value of $|\mathbf{t}|$ at the first local minimum of $\xi_u(\mathbf{t})$ along that direction. Then it is shown in 3.3 that if $r(\mathbf{t})$ has (non-strict) local minima for $|\mathbf{t}| = t_0$ then the random field $\{\tau_u(\boldsymbol{\theta}) - t_0, \boldsymbol{\theta} \in \mathbf{R}^n, |\boldsymbol{\theta}| = 1\}$, defined on the unit sphere, after a normalization tends to an isotropic Gaussian field on the unit sphere.

Section 3 also deals with the values of $\xi_u(\mathbf{t})$ at the minimum points.

1. General results

1.1. *Some definitions*

Let $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbf{R}^n\}$ be a separable, homogeneous Gaussian field with n-dimensional parameter, with mean zero and with the covariance function r $(r(\mathbf{0}) = 1)$. Define, for $\mathbf{k} = (k_1, \ldots, k_n)$, $K = \sum k_i$, the spectral moments

$$\lambda_{\mathbf{k}} = (-1)^{K/2} \frac{\partial^K r(\mathbf{t})}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}}\bigg|_{\mathbf{t}=\mathbf{0}} \qquad (K \text{ even}).$$

If r has continuous partial derivatives of every order $K \leq 4$, and if furthermore, for every $\mathbf{k}$ with $K = 4$

$$\left| \frac{\partial^4 r(\mathbf{t})}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}} - \lambda_{\mathbf{k}} \right| = O(|\mathbf{t}|) \quad \text{as} \quad \mathbf{t} \to \mathbf{0}\,, \tag{1.1}$$

then $\xi(\mathbf{t})$ has, with probability one, sample functions with continuous first and second order partial derivatives. Thus we can define $\xi_i'(\mathbf{t}) = \partial\xi(\mathbf{t})/\partial t_i$, $\xi_{ij}''(\mathbf{t}) = \partial^2\xi(\mathbf{t})/\partial t_i \partial t_j$, and $\xi_{ij}''(\mathbf{t}) = \xi_{ji}''(\mathbf{t})$ for all $\mathbf{t}$ holds almost surely. This can be shown from the one-dimensional analogues, (see the book by Cramér and Leadbetter [3], Ch. 4), combined with a result by Winkler on continuity of Gaussian fields [10].

Now arrange the column vectors

$$\begin{aligned} \xi'(\mathbf{0}) &= (\xi_i'(\mathbf{0}), 1 \leq i \leq n)'\,, \\ \xi_d''(\mathbf{0}) &= (\xi_{ii}''(\mathbf{0}), 1 \leq i \leq n)'\,, \\ \xi_0''(\mathbf{0}) &= (\xi_{ij}''(\mathbf{0}), 1 \leq i < j \leq n)'\,, \end{aligned}$$

and

$$\xi''(\mathbf{0}) = \begin{pmatrix} \xi_d''(\mathbf{0}) \\ \xi_0''(\mathbf{0}) \end{pmatrix}.$$

As in the one-dimensional case the covariances among $\xi(\mathbf{0})$, $\xi'(\mathbf{0})$, and $\xi''(\mathbf{0})$ are given by the spectral moments $\lambda_{\mathbf{k}}$. If we write λ_{ii} or λ_{ij} ($k_i = 2$ and $k_i = k_j = 1$ respectively) instead of $\lambda_{\mathbf{k}}$ when $K = 2$ and correspondingly λ_{ijkl} if $K = 4$ then

$$\begin{aligned}
&V(\xi(\mathbf{0})) = 1\,,\\
&\mathrm{Cov}\,(\xi_i'(\mathbf{0}), \xi_j'(\mathbf{0})) = -\,\mathrm{Cov}\,(\xi(\mathbf{0}), \xi_{ij}''(\mathbf{0})) = \lambda_{ij}\,,\\
&\mathrm{Cov}\,(\xi_{ij}''(\mathbf{0}), \xi_{kl}''(\mathbf{0})) = \lambda_{ijkl}\,,\\
&\mathrm{Cov}\,(\xi(\mathbf{0}), \xi_i'(\mathbf{0})) = \mathrm{Cov}\,(\xi_i'(\mathbf{0}), \xi_{kl}''(\mathbf{0})) = 0\,.
\end{aligned}$$

The covariance matrix of the $(1 + n + \frac{1}{2}n(n+1))$-variate normal variable $(\xi(\mathbf{0}), \xi'(\mathbf{0}), \xi''(\mathbf{0}))'$ is then partitioned in a natural way:

$$\begin{pmatrix} 1 & \mathbf{0} & S_{02} \\ \mathbf{0} & S_{11} & \mathbf{0} \\ S_{20} & \mathbf{0} & S_{22} \end{pmatrix} \tag{1.2}$$

say, where S_{11} and S_{22} are the internal covariance matrices for $\xi'(\mathbf{0})$ and $\xi''(\mathbf{0})$ respectively, while S_{20} (and $S_{02} = S_{20}'$) gives the covariances between $\xi(\mathbf{0})$ and $\xi''(\mathbf{0})$. Here we make the additional assumption that $\xi(\mathbf{0})$, $\xi'(\mathbf{0})$, $\xi''(\mathbf{0})$ have a non-singular distribution.

In the special case when ξ is isotropic, the covariance function $r(\mathbf{t})$ depends only on the distance $|\mathbf{t}| = (\sum t_i^2)^{1/2}$. The Taylor expansion of $r(\mathbf{t})$ will then contain only terms of the type $t_i^2, t_i^4, t_i^2 t_j^2$ and higher. This implies that most of the spectral moments vanish, and the S-matrices are considerably simplified:

$$\lambda_{ij} = \begin{cases} \lambda_2^* & \text{if } i = j \\ 0 & \text{if } i \neq j\,, \end{cases}$$

$$\lambda_{ijkl} = \begin{cases} \lambda_{40}^* & \text{if } i = j = k = l \\ \lambda_{22}^* & \text{if } i = k,\ j = l,\ i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

This means especially that the mixed second order derivatives $\xi_0''(\mathbf{0})$ are independent of both $\xi(\mathbf{0})$ and of the unmixed derivatives $\xi_d''(\mathbf{0})$. If we define

$$\begin{aligned}
&\varrho = \lambda_{22}^*/\lambda_{40}^* && (0 < \varrho < 1),\\
&\mathbf{1} = (1, \ldots, 1)' && (n \times 1)\,,\\
&D = \begin{pmatrix} 1 & \varrho & \cdots & \varrho \\ \varrho & 1 & \cdots & \varrho \\ \vdots & \vdots & \ddots & \vdots \\ \varrho & \varrho & \cdots & 1 \end{pmatrix} && (n \times n)\,,\\
&I_m = \text{unit matrix of order } m\,,
\end{aligned}$$

then

$$S_{02} = (-\lambda_2^* \mathbf{1}', \mathbf{0}), \quad S_{22} = \begin{pmatrix} \lambda_{40}^* D_\varrho & \mathbf{0} \\ \mathbf{0} & \lambda_{22}^* I_{n(n-1)/2} \end{pmatrix}. \tag{1.3}$$

The inverse of D_ϱ is easily computed:

$$(D_\varrho^{-1})_{ij} = \begin{cases} \dfrac{1}{1-\varrho} \cdot \dfrac{1+(n-2)\varrho}{1+(n-1)\varrho} & \text{if } i = j\,, \\ -\dfrac{\varrho}{1-\varrho} \cdot \dfrac{1}{1+(n-1)\varrho} & \text{if } i = j\,. \end{cases} \tag{1.4}$$

1.2. *The structure near maximum*

We want to consider a »conditional process»

$$\xi_u(\mathbf{t}) = \xi(\mathbf{t}) | \xi \text{ has a local maximum with height } u \text{ at } \mathbf{0}\,.$$

Since the probability of the conditioning event has probability zero we have to approximate it by the »horizontal window» event

$A(h, h')$: ξ has a local maximum with height in $(u, u+h)$ at some point $\mathbf{s}$ in the sphere $|\mathbf{s}| \le h'$.

Let $\tau = (\mathbf{t}^1, \ldots, \mathbf{t}^m)$ be m different points in $\mathbf{R}^n$, and let $\mathbf{x} = (x_1, \ldots, x_m)'$ be a matrix of real numbers. Then we can compute the conditional probability that $\xi(\mathbf{t}^i) \le x_i$, $i = 1, \ldots, m \mid A(h, h')$, after which we can let h, $h' \to 0$. Under the assumption (1.1) the stream of local maxima with heights in $(u, u+h)$ is regular (cf. Belyaev, [2], Theorem 4), and

$$P(\xi(\mathbf{t}^i) \le x_i, i = 1, \ldots, m \mid A(h, h')) \to E(N_1(\mathbf{x}, u, h))/E(N_1(u, h)) \text{ as } h' \to 0, \tag{1.5}$$

where $N_T(u, h) =$ the number of local maxima $\mathbf{s}$ with $|\mathbf{s}| \le T$ with heights in $(u, u+h)$, while $N_T(\mathbf{x}, u, h) =$ the number of those maxima $\mathbf{s}$ for which $\xi(\mathbf{s} + \mathbf{t}^i) \le x_i$, $i = 1, \ldots, m$.

If furthermore the process is ergodic, which is the case if $r(\mathbf{t}) \to 0$ as $|\mathbf{t}| \to \infty$, then the right hand side in (1.5) is the (a.s.) limit of $N_T(\mathbf{x}, u, h)/N_T(u, h)$ as $T \to \infty$, (cf. Nosko [8]). This means that it has a natural frequency interpretation as the distribution of the ξ-values at points $\mathbf{s}^\nu + \mathbf{t}^i$ where the points $\mathbf{s}^\nu$ are the locations of local maxima. Thus we have motivated the use of (1.5), or equivalently its limit as $h \to 0$, as a conditional distribution.

It remains to express the expectations in (1.5) in closed form. This can be done by generalizing the formulas for the expected number of local maxima given by Belyaev [2]. First we need some definitions. Besides the set τ of m points in $\mathbf{R}^n$ and the matrix $\mathbf{x}$ we define $\mathbf{y} = (y_1, \ldots, y_m)'$, $\mathbf{v} = (v_1, \ldots, v_n)'$, $\mathbf{z}_d = (z_{ii}, 1 \le i \le n)'$, $\mathbf{z}_0 = (z_{ij}, 1 \le i < j \le n)'$, and $\mathbf{z} = (\mathbf{z}_d', \mathbf{z}_0')'$. Also let $\mathbf{Z} = (z_{ij})$

be the symmetric matrix formed by the elements in $\mathbf{z}_d$ and $\mathbf{z}_0$. In the sequel we will often exploit the convention that if a capital letter denotes a symmetric matrix, then the corresponding small letter denotes the column matrix formed of its $n(n+1)/2$ different elements.

Define the following probability densities and conditional probability densities

$$\begin{aligned}
&p_\tau(u, \mathbf{v}, \mathbf{z}, \mathbf{x}) && \text{for } \xi(0), \xi'(0), \xi''(0), \xi(\mathbf{t}^1), \ldots, \xi(\mathbf{t}^m), \\
&p_\tau(\mathbf{x} \mid u, \mathbf{v}, \mathbf{z}) && \text{for } \xi(\mathbf{t}^1), \ldots, \xi(\mathbf{t}^m) \mid \xi(0) = u, \ \xi'(0) = \mathbf{v}, \ \xi''(0) = \mathbf{z}, \\
&p(u, \mathbf{v}, \mathbf{z}) && \text{for } \xi(0), \ \xi'(0), \ \xi''(0), \\
&p(\mathbf{z} \mid u, \mathbf{v}) && \text{for } \xi''(0) \mid \xi(0) = u, \ \xi'(0) = \mathbf{v}.
\end{aligned}$$

Then

$$\frac{E(N_1(\mathbf{x}, u, h))}{E(N_1(u, h))} = \frac{\int_{y_i \le x_i} (m) \int_{\xi=u}^{u+h} \int_{\mathbf{Z} \prec 0} |\det \mathbf{Z}| p_\tau(\xi, \mathbf{0}, \mathbf{z}, \mathbf{y}) d\mathbf{z} d\xi d\mathbf{y}}{\int_{\xi=u}^{u+h} \int_{\mathbf{Z} \prec 0} |\det \mathbf{Z}| p(\xi, \mathbf{0}, \mathbf{z}) d\mathbf{z} d\xi},$$

where $\mathbf{Z} \prec 0$ means that the matrix $\mathbf{Z}$ is negative definite.

By letting $h \to 0$ we finally get the following fundamental theorem.

THEOREM 1.1. *The conditional distribution of* $\xi(\mathbf{t}^1), \ldots, \xi(\mathbf{t}^m)$ *given that* ξ *has a local maximum with height* u *at* $\mathbf{0}$ *has the density*

$$\frac{\int_{\mathbf{Z} \prec 0} |\det \mathbf{Z}| p_\tau(u, \mathbf{0}, \mathbf{z}, \mathbf{x}) d\mathbf{z}}{\int_{\mathbf{Z} \prec 0} |\det \mathbf{Z}| p(u, \mathbf{0}, \mathbf{z}) d\mathbf{z}} \tag{1.6}$$

In the rest of this section we will simplify (1.6) to a form which gives considerable insight in the structure of ξ near a maximum. To begin with write

$$q_u(\mathbf{z}) = \begin{cases} \dfrac{\det \mathbf{Z}\, p(-\mathbf{z} \mid u, \mathbf{0})}{\int_{\mathbf{Z} \succ 0} \det \mathbf{Z}\, p(-\mathbf{z} \mid u, \mathbf{0}) d\mathbf{z}} & \text{if } \mathbf{Z} > 0 \\ 0 & \text{otherwise}. \end{cases} \tag{1.7}$$

Then (1.6) can be replaced with

$$\int q_u(\mathbf{z}) p_\tau(\mathbf{x} \mid u, \mathbf{0}, -\mathbf{z}) d\mathbf{z}. \tag{1.8}$$

Up to now we have made limited use of the normality of ξ. Let us now use it in full to derive the conditional densities in (1.7) and (1.8). Let

$$\begin{aligned} S_1(\mathbf{t}) &= (-r_i'(\mathbf{t}),\ 1 \le i \le n)' , \\ S_2(\mathbf{t}) &= (r_{ii}''(\mathbf{t}),\ 1 \le i \le n;\ r_{ij}''(\mathbf{t}),\ 1 \le i < j \le n)' . \end{aligned} \tag{1.9}$$

Then

$$\left|\begin{matrix} 1 & \mathbf{0} & S_{02} & r(\mathbf{t}^1) & \dots & r(\mathbf{t}^m) \\ \mathbf{0} & S_{11} & \mathbf{0} & S_1(\mathbf{t}^1) & \dots & S_1(\mathbf{t}^m) \\ S_{20} & \mathbf{0} & S_{22} & S_2(\mathbf{t}^1) & \dots & S_2(\mathbf{t}^m) \\ r(\mathbf{t}^1) & S_1'(\mathbf{t}^1) & S_2'(\mathbf{t}^1) & 1 & \dots & r(\mathbf{t}^m - \mathbf{t}^1) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ r(\mathbf{t}^m) & S_1'(\mathbf{t}^m) & S_2'(\mathbf{t}^m) & r(\mathbf{t}^1 - \mathbf{t}^m) & \dots & 1 \end{matrix}\right|$$

is the covariance matrix in the density $p_\tau(u, \mathbf{y}, \mathbf{z}, \mathbf{x})$. The mean and covariances in the conditional density in (1.8) can then be expressed in terms of simple functions of r: define $A(\mathbf{t})$, $C(\mathbf{s}, \mathbf{t})$, and the matrix function $\mathbf{b}(\mathbf{t}) = (b_{ii}(\mathbf{t}), 1 \le i \le n; b_{ij}(\mathbf{t}), 1 \le i < j \le n)'$ by

$$\begin{aligned} (A(\mathbf{t}), \mathbf{b}(\mathbf{t})') &= (r(\mathbf{t}), S_2'(\mathbf{t})) \begin{pmatrix} 1 & S_{02} \\ S_{20} & S_{22} \end{pmatrix}^{-1} , \\ C(\mathbf{s}, \mathbf{t}) &= r(\mathbf{s} - \mathbf{t}) - (r(\mathbf{s}), S_2'(\mathbf{s})) \begin{pmatrix} 1 & S_{02} \\ S_{20} & S_{22} \end{pmatrix}^{-1} \begin{pmatrix} r(\mathbf{t}) \\ S_2(\mathbf{t}) \end{pmatrix} - S_1'(\mathbf{s}) S_{11}^{-1} S_1(\mathbf{t}) . \end{aligned} \tag{1.10}$$

Then $p_\tau(\cdot \mid u, \mathbf{0}, -\mathbf{z})$ is normal with means $uA(\mathbf{t}^i) - \mathbf{z}'\mathbf{b}(\mathbf{t}^i)$ and covariances $C(\mathbf{t}^i, \mathbf{t}^j)$. The density q_u comes out similarly: $p(\cdot \mid u, \mathbf{0})$ is normal with mean

$$(S_{20}, \mathbf{0}) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & S_{11} \end{pmatrix}^{-1} \begin{pmatrix} u \\ \mathbf{0} \end{pmatrix} = uS_{20}$$

and covariance matrix

$$S_{22} - (S_{20}, \mathbf{0}) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & S_{11} \end{pmatrix}^{-1} \begin{pmatrix} S_{02} \\ \mathbf{0} \end{pmatrix} = S_{22} - S_{20} S_{02} = S_{2\cdot 0} , \quad \text{say.} \tag{1.11}$$

Thus, for $\mathbf{Z} \succ 0$ we have

$$q_u(\mathbf{z}) = k_u^{-1} \det \mathbf{Z} \exp\{-\tfrac{1}{2}(\mathbf{z} + uS_{20})' S_{2\cdot 0}^{-1} (\mathbf{z} + uS_{20})\} \tag{1.12}$$

where k_u is the normalizing constant.

Note that the assumptions made about the non-singularity of $\xi(\mathbf{t}), \boldsymbol{\xi}'(\mathbf{t}), \boldsymbol{\xi}''(\mathbf{t})$ guarantee the non-singularity of every interesting matrix.

We finally can conclude that (1.8) is the density for a process $\xi_u(\mathbf{t})$ defined in the following theorem.

THEOREM 1.2. *Given a local maximum with height u at $\mathbf{0}$ the conditional process $\xi(\mathbf{t})$ has the same finite-dimensional distributions as the process $\{\xi_u(\mathbf{t}), \mathbf{t} \in \mathbf{R}^n\}$ defined as follows:*

$$\xi_u(\mathbf{t}) = uA(\mathbf{t}) - \boldsymbol{\zeta}_u'\mathbf{b}(\mathbf{t}) + \Delta(\mathbf{t})$$

where Δ is a non-homogeneous, zero-mean, Gaussian field with the covariance function C, and $\boldsymbol{\zeta}_u$ is an $n(n+1)/2$-variate random variable, independent of Δ and with the density q_u.

2. Asymptotic properties as $u \to -\infty$

For large negative u the maximum of ξ_u at $\mathbf{0}$ will exert a strong influence over $\xi_u(\mathbf{t})$ for small $\mathbf{t}$. Actually, the normalized process

$$\xi_u^*(\mathbf{t}) = |u|^3(\xi_u(\mathbf{t}/|u|) - u) = u^4(1 - A(\mathbf{t}/|u|)) + |u|^3\Delta(\mathbf{t}/|u|) - |u|\boldsymbol{\zeta}_u'u^2\mathbf{b}(\mathbf{t}/|u|) \quad (2.1)$$

has asymptotically the same distributions as a certain fourth degree polynomial

$$\Gamma(\mathbf{t}) = -\frac{1}{4!}\left(\sum_{\nu=1}^{n} t_\nu \frac{\partial}{\partial t_\nu}\right)^4 A(\mathbf{0}) + \frac{1}{3!}\left(\sum_{\nu=1}^{n} t_\nu \frac{\partial}{\partial t_\nu}\right)^3 \Delta(\mathbf{0}) - \frac{1}{2}\,\mathbf{t}'\mathbf{Z}\mathbf{t} \quad (2.2)$$

(where $(\sum t_\nu\partial/\partial t_\nu)^3 f(\mathbf{0}) = \sum_{\nu,\mu,\gamma} t_\nu t_\mu t_\gamma \partial^3 f(\mathbf{t})/\partial t_\nu \partial t_\mu \partial t_\gamma|_{\mathbf{t}=\mathbf{0}}$, etc.).

In $\Gamma(\mathbf{t})$ the first sum contains non-stochastic, fourth order terms in $t_1, \ldots, t_n$, while $1/3!(\sum t_\nu\partial/\partial t_\nu)^3\Delta(\mathbf{0})$ contains third order terms with stochastic coefficients, which essentially are the third order derivatives of Δ at $\mathbf{0}$. It is a normal »deterministic» random field with mean zero and with the singular covariance function $1/3!3!(\sum s_\nu\partial/\partial s_\nu)^3(\sum t_\nu\partial/\partial t_\nu)^3 C(\mathbf{0}, \mathbf{0})$. Finally the third term in $\Gamma(\mathbf{t})$ is the quadratic one where the symmetric random matrix $\mathbf{Z} = (\zeta_{ij}, 1 \le i, j \le n)$ is positive definite, and the corresponding column matrix $\boldsymbol{\zeta} = (\zeta_{ii}, 1 \le i \le n; \zeta_{ij}, 1 \le i < j \le n)'$ has the density

$$\tilde{q}(\mathbf{z}) = \tilde{k}_{-\infty}^{-1} \det \mathbf{Z} \exp S_{02}S_{2\cdot 0}^{-1}\mathbf{z} \quad (\mathbf{Z} \succ 0)\,, \quad (2.3)$$

with $\tilde{k}_{-\infty}$ as a normalizing constant.

This gives explicitly the behaviour of $\xi_u(\mathbf{t})$ for $|\mathbf{t}|$ up to order $|u|^{-1}$ from a very low maximum

To show the alleged convergence we need the asymptotic distribution of $|u|\boldsymbol{\zeta}_u$ and the behaviour of A, $\mathbf{b}$, and C for small arguments. We then need to impose the further assumption that r is six times continuously differentiable and that, for $\sum k_i = 6$,

$$\left|\frac{\partial^6 r(\mathbf{t})}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}} - \lambda_{\mathbf{k}}\right| = O(|\mathbf{t}|) \quad \text{as} \quad \mathbf{t} \to \mathbf{0}\,. \quad (2.4)$$

This is only what can be expected from the one-dimensional case in which the sixth derivative of r plays an important role, see [5, Theorem 4, 5].

LEMMA 2.1. *The density* $\tilde{q}_u(\mathbf{z})$ *of* $|u|\zeta_u$ *tends to* $\hat{q}(\mathbf{z})$ *with dominated convergence* as $u \to -\infty$.

LEMMA 2.2. *If* r *fulfills* (2.4) *then, as* $\mathbf{s}, \mathbf{t} \to \mathbf{0}$

$$A(\mathbf{t}) = 1 + \frac{1}{4!}\left(\sum_{\nu=1}^{n} t_\nu \frac{\partial}{\partial t_\nu}\right)^4 A(\mathbf{0}) + O(|\mathbf{t}|^5),$$

$$b_{ij}(\mathbf{t}) = \frac{1}{2}\left(\sum_{\nu=1}^{n} t_\nu \frac{\partial}{\partial t_\nu}\right)^2 b_{ij}(\mathbf{0}) + O(|\mathbf{t}|^3) = \begin{cases} t_i^2/2 + O(|\mathbf{t}|^3) & \text{for } i = j \\ t_i t_j + O(|\mathbf{t}|^3) & \text{for } i \neq j, \end{cases}$$

$$C(\mathbf{s}, \mathbf{t}) = \frac{1}{3!3!}\left(\sum_{\nu=1}^{n} s_\nu \frac{\partial}{\partial s_\nu}\right)^3 \left(\sum_{\nu=1}^{n} t_\nu \frac{\partial}{\partial t_\nu}\right)^3 C(\mathbf{0}, \mathbf{0}) + O(\max(|\mathbf{s}|, |\mathbf{t}|)^7).$$

The proofs of these two lemmas are rather lengthy and are given after the following main result, concerning the convergence of ξ_u^*.

THEOREM 2.1. *If* r *fulfills* (2.4) *then, for any set of times* $(\mathbf{t}^i, \ i = 1, \ldots, m)$, *it holds*

$$(\xi_u^*(\mathbf{t}^i), \ i = 1, \ldots, m) \xrightarrow{\mathscr{L}} (\Gamma(\mathbf{t}^i), \ i = 1, \ldots, m) \ \text{as} \ u \to -\infty.$$

Outline of proof. An impeccable proof could have used the characteristic function convergence as is done in Theorem 4 in [5]. We give here only some pertinent points. Lemma 2.2 gives

$$u^4(1 - A(\mathbf{t}/|u|)) \to -\frac{1}{4!}\left(\sum t_\nu \frac{\partial}{\partial t_\nu}\right)^4 A(\mathbf{0}),$$

$$u^2 b_{ij}(\mathbf{t}/|u|) \to \begin{cases} t_i^2/2 & \text{for } i = j \\ t_i t_j & \text{for } i \neq j, \end{cases}$$

while by Lemma 2.1 $|u|\zeta_u$ has asymptotically the density (2.3). Therefore

$$(|u|\zeta_u' u^2 \mathbf{b}(\mathbf{t}^i/|u|), \ i = 1, \ldots, m) \xrightarrow{\mathscr{L}} (\tfrac{1}{2}\mathbf{t}^{i\prime}\mathbf{Z}\mathbf{t}^i, \ i = 1, \ldots, m).$$

Finally $-u^3\Delta(\mathbf{t}^i/|u|), \ i = 1, \ldots m$ have the covariances

$$u^6 C(\mathbf{t}^i/|u|, \mathbf{t}^j/|u|) \to \frac{1}{3!3!}\left(\sum t_\nu^i \frac{\partial}{\partial t_\nu}\right)^3 \left(\sum t_\nu^j \frac{\partial}{\partial t_\nu}\right)^3 C(\mathbf{0}, \mathbf{0}),$$

which are just the covariances of the Δ-terms in $\Gamma(\mathbf{t}^i), \ i = 1, \ldots, m.$ □

Remark. The function space method used by Lindgren [6] works also in the multidimensional case. Then Theorem 2.1 can be extended to include almost sure convergence and not only convergence in law.

Proof of lemma 2.1. From (1.12) and $S'_{20} = S_{02}$ we get, with $N = n(n+1)/2$,

$$\begin{aligned}\tilde{q}_u(\mathbf{z}) &= |u|^{-N} q_u(|u|^{-1}\mathbf{z}) = |u|^{-N-n} k_u^{-1} \det \mathbf{Z} \cdot \\ &\cdot \exp\{-\tfrac{1}{2}(|u|^{-1}\mathbf{z} + uS_{20})' S_{2\cdot 0}^{-1}(|u|^{-1}\mathbf{z} + uS_{20})\} = \\ &= \tilde{k}_u^{-1} \det \mathbf{Z} \cdot \exp S_{02} S_{2\cdot 0}^{-1}\mathbf{z} \exp(-\tfrac{1}{2}u^{-2}\mathbf{z}' S_{2\cdot 0}^{-1}\mathbf{z}),\end{aligned}$$

where $\tilde{k}_u$ is a new normalizing constant. As $u \to -\infty$ then $\tilde{k}_u \tilde{q}_u(\mathbf{z})$ is dominated by and tends pointwise to

$$\det \mathbf{Z} \cdot \exp S_{02} S_{2\cdot 0}^{-1}\mathbf{z} \quad (\mathbf{Z} \succ 0). \tag{2.5}$$

As is shown below, this function is integrable, and therefore the dominated convergence theorem implies

$$\tilde{k}_u = \tilde{k}_u \int_{\mathbf{Z} \succ 0} \tilde{q}_u(\mathbf{z}) d\mathbf{z} \to \int_{\mathbf{Z} \succ 0} \det \mathbf{Z} \cdot \exp S_{02} S_{2\cdot 0}^{-1}\mathbf{z} d\mathbf{z} = \tilde{k}_{-\infty},$$

which gives the lemma.

It remains to show that (2.5) is integrable: transform to polar coordinates, $r = |\mathbf{z}| = (\sum_{i \leq j} z_{ij}^2)^{1/2}$, $\boldsymbol{\theta} = \boldsymbol{\theta}(\mathbf{z})$, with the inverse $\mathbf{z} = \mathbf{z}(r, \boldsymbol{\theta}) = r \cdot \mathbf{z}(1, \boldsymbol{\theta})$. For the sake of simplicity write $\mathbf{z}(\boldsymbol{\theta}) = \mathbf{z}(1, \boldsymbol{\theta})$ and $\mathbf{Z}(\boldsymbol{\theta})$ for the corresponding symmetric matrix. Let $A = \{\boldsymbol{\theta}; \mathbf{Z}(\boldsymbol{\theta}) \succ 0\}$ and denote by J the functional determinant. Then

$$\int_{\mathbf{Z} \succ 0} \det \mathbf{Z} \cdot \exp S_{02} S_{2\cdot 0}^{-1}\mathbf{z} d\mathbf{z} = \int_{r=0}^{\infty} \int_{\boldsymbol{\theta} \in A} r^n \det \mathbf{Z}(\boldsymbol{\theta}) \cdot \exp\{r \cdot S_{02} S_{2\cdot 0}^{-1}\mathbf{z}(\boldsymbol{\theta})\} |J| dr d\boldsymbol{\theta}. \tag{2.6}$$

The critical term is $S_{02} S_{2\cdot 0}^{-1}\mathbf{z}(\boldsymbol{\theta})$ and we will show that

$$\sup_{\boldsymbol{\theta} \in A} S_{02} S_{2\cdot 0}^{-1}\mathbf{z}(\boldsymbol{\theta}) = c_n < 0. \tag{2.7}$$

Then (2.6) is bounded by

$$\int_{r=0}^{\infty} r^n \exp c_n r \int_{\boldsymbol{\theta} \in A} r^{-1} |J| \det \mathbf{Z}(\boldsymbol{\theta}) d\boldsymbol{\theta} dr.$$

Since $r^{-1}|J|$ is a bounded function over the bounded region A, this shows that (2.5) is integrable.

We still have to prove (2.7). Writing $S = \begin{pmatrix} 1 & S_{02} \\ S_{20} & S_{22} \end{pmatrix}$ we have $\det S = (1 - S_{02} S_{22}^{-1} S_{20}) \det S_{22}$, which implies that $S_{02} \det S = (S_{02} - S_{02} S_{22}^{-1} S_{20} S_{02}) \det S_{22} = S_{02} S_{22}^{-1}(S_{22} - S_{20} S_{02}) \det S_{22} = S_{02} S_{22}^{-1} S_{2\cdot 0} \det S_{22}$ so what we actually need to prove is that

$$\sup_{|\mathbf{z}|=1, \mathbf{Z} \succ 0} S_{02} S_{22}^{-1}\mathbf{z} = d_n < 0. \tag{2.8}$$

But $S_{02}S_{22}^{-1}\mathbf{z} = E(\xi(\mathbf{0}) \mid \xi'(\mathbf{0}) = \mathbf{0},\ \xi''(\mathbf{0}) = \mathbf{z})$ and thus, if $\mathbf{Z} \succ 0$, we are dealing with the expected value of the process at a point where a local minimum with a certain second order derivative occurs. Since the process is zero on the average, and we expect it not to be more at local minima, (2.8) seems reasonable. However, we have not been able to find a proof, unless the process is isotropic, but it is conjectured that (2.8) is true for general processes.

If ξ is isotropic then (1.3) and (1.4) give

$$-\frac{\lambda_2^*}{\lambda_{40}^*} S_{02}S_{22}^{-1}\mathbf{z} = (\mathbf{1}'D_\varrho^{-1}, \mathbf{0})\mathbf{z} = \mathbf{1}'D_\varrho^{-1}\mathbf{z}_d = \sum_j z_{ii}\{\sum_i (D_\varrho^{-1})_{ij}\} =$$
$$= (1 + (n-1)\varrho)^{-1} \sum_i z_{ii} \,. \tag{2.9}$$

If $|\mathbf{z}| = 1$ then $\max |z_{ij}| \geq e_n = (\tfrac{1}{2}n(n+1))^{-1/2}$. Since $\mathbf{Z} \succ 0$ we have $|z_{ij}|^2 \leq z_{ii}z_{jj}$ so that $\max |z_{ii}| \geq e_n$. But $z_{ii} > 0$ implies that $\sum z_{ii} \geq e_n > 0$, and thus (2.9) is positive and bounded away from 0. This shows (2.8) in the isotropic case. □

Proof of lemma 2.2. We concentrate upon the rest term and the terms of order six in $C(\mathbf{s}, \mathbf{t})$. Similar, but simpler, versions of the methods will give the lower order terms (which all vanish) as well as A and $\mathbf{b}$.

Recall the definitions of $S_{11}, S_1(\mathbf{t}), S_2(\mathbf{t})$ from (1.2) and (1.9) and let $S = \begin{pmatrix} 1 & S_{20} \\ S_{02} & S_{22} \end{pmatrix}$, $T = S^{-1}$, $V = S_{11}^{-1}$. The rows and columns in T (except for the first one) will be identified by a double index (ij); e.g. the third row in T will be called row (2,2), since the leading element in the third row in S (the second element in S_{20}) is $-\lambda_{22}$. The first row and column will be numbered 0.

Then $C(\mathbf{s}, \mathbf{t})$ contains six groups of terms. By (1.10)

$$C(\mathbf{s}, \mathbf{t}) = r(\mathbf{s} - \mathbf{t}) - r(\mathbf{s})T_{00}r(\mathbf{t}) - r(\mathbf{s}) \sum_{\nu' \leq \mu'} T_{0, (\nu'\mu')} \frac{\partial^2 r(\mathbf{t})}{\partial t_{\nu'} \partial t_{\mu'}} -$$
$$- r(\mathbf{t}) \sum_{\nu \leq \mu} T_{(\nu\mu), 0} \frac{\partial^2 r(\mathbf{s})}{\partial s_\nu \partial s_\mu} - \sum_{\nu \leq \mu, \nu' \leq \mu'} \frac{\partial^2 r(\mathbf{s})}{\partial s_\nu \partial s_\mu} T_{(\nu\mu), (\nu'\mu')} \frac{\partial^2 r(\mathbf{t})}{\partial t_{\nu'} \partial t_{\mu'}} -$$
$$- \sum_{\nu, \mu} \frac{\partial r(\mathbf{s})}{\partial s_\nu} V_{\nu\mu} \frac{\partial r(\mathbf{t})}{\partial t_\mu} = I_1 - I_2 - I_3 - I_4 - I_5 - I_6\,, \text{ say.}$$

If $\mathbf{k} = (k_1, \ldots, k_n)$ we also employ the notations

$$\mathbf{k}^i = (k_1, \ldots, k_i + 1, \ldots, k_n)$$
$$\mathbf{k}^{ij} = \begin{cases} (k_1, \ldots, k_i + 1, \ldots, k_j + 1, \ldots, k_n) & \text{if } i < j \\ (k_1, \ldots, k_i + 2, \ldots, k_n) & \text{if } i = j\,, \end{cases}$$

and write $\begin{pmatrix} K \\ \mathbf{k} \end{pmatrix}$ for the multinomial coefficients:

$$(\sum_1^n x_i)^K = \sum_{\Sigma k_i = K} \binom{K}{\mathbf{k}} \prod_1^n x_i^{k_i} .$$

Then, with $K = \sum k_i$, we have as $\mathbf{t} \to \mathbf{0}$

$$r(\mathbf{t}) = 1 + \sum_{K=2,4,6} \frac{(-1)^{K/2}}{K!} \sum_{\mathbf{k}} \binom{K}{\mathbf{k}} \prod_1^n t_i^{k_i} \lambda_{\mathbf{k}} + O(|\mathbf{t}|^7) ,$$

$$\frac{\partial r(\mathbf{t})}{\partial t_\nu} = \sum_{K=1,3,5} \frac{(-1)^{(K+1)/2}}{K!} \sum_{\mathbf{k}} \binom{K}{\mathbf{k}} \prod_1^n t_i^{k_i} \lambda_{\mathbf{k}^\nu} + O(|\mathbf{t}|^6) ,$$

$$\frac{\partial^2 r(\mathbf{t})}{\partial t_\nu \partial t_\mu} = -\lambda_{\nu\mu} - \sum_{K=2,4} \frac{(-1)^{K/2}}{K!} \sum_{\mathbf{k}} \binom{K}{\mathbf{k}} \prod_1^n t_i^{k_i} \lambda_{\mathbf{k}^{\nu\mu}} + H_{\nu\mu}(\mathbf{t}) ,$$

where the unspecified rest terms $H_{\nu\mu}$ are $O(|\mathbf{t}|^5)$. Introducing these expansions into $C(\mathbf{s}, \mathbf{t})$ we will see that the unspecified rest terms sum up to a total rest term of order $O(\max(|\mathbf{s}|, |\mathbf{t}|)^7)$. Obviously this is true for I_1, I_2, and I_6. In I_3, I_4, I_5 however, there will seemingly appear terms of order $O(\max(|\mathbf{s}|, |\mathbf{t}|)^5)$. Fortunately, these all cancel. The questionable terms are

$$\begin{aligned} &1 \cdot \sum_{\nu' \le \mu'} T_{0,(\nu'\mu')} H_{\nu'\mu'}(\mathbf{t}) + 1 \cdot \sum_{\nu \le \mu} T_{(\nu\mu),0} H_{\nu\mu}(\mathbf{s}) + \\ &+ \sum_{\nu \le \mu, \nu' \le \mu'} T_{(\nu\mu),(\nu'\mu')}(-\lambda_{\nu\mu} H_{\nu'\mu'}(\mathbf{t}) - \lambda_{\nu'\mu'} H_{\nu\mu}(\mathbf{s})) = \\ &= \sum_{\nu \le \mu} H_{\nu\mu}(\mathbf{s}) \{ T_{(\nu\mu),0} - \sum_{\nu' \le \mu'} T_{(\nu\mu),(\nu'\mu')} \lambda_{\nu'\mu'} \} + \\ &+ \sum_{\nu' \le \mu'} H_{\nu'\mu'}(\mathbf{t}) \{ T_{0,(\nu'\mu')} - \sum_{\nu \le \mu} T_{(\nu\mu),(\nu'\mu')} \lambda_{\nu\mu} \} . \end{aligned}$$

The bracketed sums are zero, since they are off-diagonal elements in TS and ST, and thus the fourth order rest terms cancel.

Turning to the sixth order terms in $C(\mathbf{s}, \mathbf{t})$, take $\bar{\mathbf{k}}$ and $\bar{\bar{\mathbf{k}}}$ with $\sum (\bar{k}_i + \bar{\bar{k}}_i) = 6$, write $\mathbf{k} = \bar{\mathbf{k}} + \bar{\bar{\mathbf{k}}}$, and consider the coefficient $c_{\bar{\mathbf{k}}, \bar{\bar{\mathbf{k}}}}$ for the term

$$c_{\bar{\mathbf{k}}, \bar{\bar{\mathbf{k}}}} \prod_{i=1}^n s_i^{\bar{k}_i} t_i^{\bar{\bar{k}}_i}$$

in the expansion of $C(\mathbf{s}, \mathbf{t})$.

$$\begin{aligned} \sum \bar{k}_i = 6, \sum \bar{\bar{k}}_i = 0: \quad &-\frac{1}{6!}\binom{6}{\bar{\mathbf{k}}} \lambda_{\mathbf{k}} + \frac{1}{6!}\binom{6}{\bar{\mathbf{k}}} \lambda_{\mathbf{k}} T_{00} - \frac{1}{6!}\binom{6}{\bar{\mathbf{k}}} \lambda_{\mathbf{k}} \sum_{\nu' \le \mu'} T_{0,(\nu'\mu')} \lambda_{\nu'\mu'} = \\ &= \frac{1}{6!}\binom{6}{\bar{\mathbf{k}}} \lambda_{\mathbf{k}} \{ -1 + T_{00} - \sum_{\nu' \le \mu'} T_{0,(\nu'\mu')} \lambda_{\nu'\mu'} \} = 0 \end{aligned}$$

since $T_{00} - \sum T_{0,(\nu'\mu')} \lambda_{\nu'\mu'}$ is an on-diagonal element in TS.

$$\sum \bar{k}_i = 5,\ \sum \bar{\bar{k}}_i = 1:\ \frac{1}{6!}\binom{6}{5}\binom{5}{\bar{\mathbf{k}}}\binom{1}{\bar{\bar{\mathbf{k}}}}\lambda_{\mathbf{k}} - \sum_{\nu,\mu}\frac{1}{5!}\binom{5}{\bar{\mathbf{k}}}\lambda_{\bar{\mathbf{k}}^\nu}V_{\nu\mu}\lambda_{\bar{\bar{\mathbf{k}}}^\mu} =$$
$$= \frac{1}{5!}\binom{5}{\bar{\mathbf{k}}}\binom{1}{\bar{\bar{\mathbf{k}}}}\{\lambda_{\mathbf{k}} - \sum_{\nu,\mu}\lambda_{\bar{\mathbf{k}}^\nu}V_{\nu\mu}\lambda_{\bar{\bar{\mathbf{k}}}^\mu}\}.$$

Since $\sum \bar{\bar{k}}_i = 1$ exactly one $\bar{\bar{k}}_i$ is 1, say $\bar{\bar{k}}_{\nu_0} = 1$. Then the spectral moments $\lambda_{\bar{\bar{\mathbf{k}}}^\nu}$ are actually second order moments, and they constitute column ν_0 in S_{11}. Then

$$\sum_\mu V_{\nu\mu}\lambda_{\bar{\bar{\mathbf{k}}}^\mu} = \begin{cases}1 & \text{if } \nu = \nu_0\\ 0 & \text{if } \nu \neq \nu_0\,,\end{cases}$$

so that

$$\sum_{\nu,\mu}\lambda_{\bar{\mathbf{k}}^\nu}V_{\nu\mu}\lambda_{\bar{\bar{\mathbf{k}}}^\mu} = \lambda_{\bar{\mathbf{k}}^{\nu_0}}\,.$$

But $\bar{\mathbf{k}}^{\nu_0} = \bar{\mathbf{k}} + \bar{\bar{\mathbf{k}}} = \mathbf{k}$, and thus also this term vanishes.

$\sum \bar{k}_i = 4$, $\sum \bar{\bar{k}}_i = 2$: After some simplification we get

$$-\frac{1}{4!2!}\binom{4}{\bar{\mathbf{k}}}\binom{2}{\bar{\bar{\mathbf{k}}}}\{\lambda_{\mathbf{k}} - \lambda_{\bar{\mathbf{k}}}T_{00}\lambda_{\bar{\bar{\mathbf{k}}}} + \lambda_{\bar{\mathbf{k}}}\sum_{\nu'\leq\mu'}T_{0,(\nu'\mu')}\lambda_{\bar{\bar{\mathbf{k}}}^{\nu'\mu'}} +$$
$$+ \lambda_{\bar{\bar{\mathbf{k}}}}\sum_{\nu\leq\mu}T_{(\nu\mu),0}\lambda_{\bar{\mathbf{k}}^{\nu\mu}} - \sum_{\nu\leq\mu,\,\nu'\leq\mu'}\lambda_{\bar{\mathbf{k}}^{\nu\mu}}T_{(\nu\mu),(\nu'\mu')}\lambda_{\bar{\bar{\mathbf{k}}}^{\nu'\mu'}}\}\,. \tag{2.10}$$

Now suppose that the non-zero elements in $\bar{\bar{\mathbf{k}}}$ are either $\bar{\bar{k}}_{\nu_0} = \bar{\bar{k}}_{\mu_0} = 1$ $(\nu_0 \neq \mu_0)$ or $\bar{\bar{k}}_{\nu_0} = 2$. Then the fourth order spectral moments $\lambda_{\bar{\bar{\mathbf{k}}}^{\nu'\mu'}}$ in (2.10) make up row number $(\nu_0\mu_0)$ in S_{22}, and a little reflexion will show that

$$\sum_{\nu'\leq\mu'}T_{0,(\nu'\mu')}\lambda_{\bar{\bar{\mathbf{k}}}^{\nu'\mu'}} - T_{00}\lambda_{\bar{\bar{\mathbf{k}}}} = 0\,,$$

$$\sum_{\nu'\leq\mu'}T_{(\nu\mu),(\nu'\mu')}\lambda_{\bar{\bar{\mathbf{k}}}^{\nu'\mu'}} - T_{(\nu\mu),0}\lambda_{\bar{\bar{\mathbf{k}}}} = \begin{cases}1 & \text{if } (\nu\mu) = (\nu_0\mu_0)\\ 0 & \text{otherwise.}\end{cases}$$

Thus the double sum in (2.10) reduces to

$$\lambda_{\bar{\mathbf{k}}^{\nu_0\mu_0}} + \sum_{\nu\leq\mu}\lambda_{\bar{\mathbf{k}}^{\nu\mu}}T_{(\nu\mu),0}\lambda_{\bar{\bar{\mathbf{k}}}}\,,$$

and the bracketed expression is just

$$\lambda_{\mathbf{k}} - \lambda_{\bar{\mathbf{k}}^{\nu_0\mu_0}}\,.$$

Since $\bar{\mathbf{k}}^{\nu_0\mu_0} = \bar{\mathbf{k}} + \bar{\bar{\mathbf{k}}} = \mathbf{k}$ the term vanishes.

The only non-vanishing terms in $C(\mathbf{s}, \mathbf{t})$ are then those with $\bar{k}_i = \bar{\bar{k}}_i = 3$, and similar calculations as above show that they have the coefficients

$$\frac{1}{3!3!}\binom{3}{\bar{\mathbf{k}}}\binom{3}{\bar{\bar{\mathbf{k}}}}\{\lambda_{\mathbf{k}} - \sum_{\nu,\mu}\lambda_{\bar{\mathbf{k}}^\nu}V_{\nu\mu}\lambda_{\bar{\bar{\mathbf{k}}}^\mu}\} \neq 0\,. \square$$

3. Asymptotic properties as $u \to \infty$

3.1. *General results*

In the same way as a very low maximum conferred a nearly deterministic behaviour to the process, a very high maximum will exert a strong influence over the process over a wide range. As in the one-dimensional case (see [7]) we prefer to rewrite the conditional process ξ_u in Theorem 1.2 in a form better adapted for the present purpose:

$$\xi_u(\mathbf{t}) = ur(\mathbf{t}) - \eta_u'\mathbf{b}(\mathbf{t}) + \Delta(\mathbf{t})$$

where $\eta_u = \zeta_u + uS_{20}$. Note that relation (1.10) implies that $r(\mathbf{t}) = \Delta(\mathbf{t}) + S_{20}'\mathbf{b}(\mathbf{t})$ so that the new form of ξ_u is equivalent with the old one.

Before we can present the analysis of $\xi_u(\mathbf{t})$ for large u we need some definitions. Let D denote the differential operator

$$D = \left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right)'$$

so that if f is differentiable the function Df is defined by

$$Df(\mathbf{t}) = \left(\frac{\partial f(\mathbf{t})}{\partial t_1}, \dots, \frac{\partial f(\mathbf{t})}{\partial t_n}\right)'.$$

Thus, if $D\xi_u(\mathbf{t}) = \mathbf{0}$ then $\mathbf{t}$ obviously is a stationary point of ξ_u. We also let $\|\mathbf{x}\|$ denote the maximal element of a matrix $\mathbf{x}$ so that

$$\|Df(\mathbf{t})\| = \max_i \left|\frac{\partial f(\mathbf{t})}{\partial t_i}\right| .$$

Define the random functions

$$\begin{aligned} \psi_u(\mathbf{t}) &= \eta_u'\mathbf{b}(\mathbf{t}) - \Delta(\mathbf{t}) \\ \psi(\mathbf{t}) &= \eta'\mathbf{b}(\mathbf{t}) - \Delta(\mathbf{t}) , \end{aligned} \tag{3.1}$$

and the n-variate random functions

$$\begin{aligned} \chi_u(\mathbf{t}) &= D\psi_u(\mathbf{t}) = D(\eta_u'\mathbf{b}(\mathbf{t}) - \Delta(\mathbf{t})) \\ \chi(\mathbf{t}) &= D\psi(\mathbf{t}) = D(\eta'\mathbf{b}(\mathbf{t}) - \Delta(\mathbf{t})) , \end{aligned} \tag{3.2}$$

where η_u is the random vector just defined, while η is a new $n(n+1)/2$-variate random variable, which is independent of the process Δ and has a normal distribution $N(\mathbf{0}, S_{2\cdot 0})$, i.e. has mean zero and the covariance matrix $S_{2\cdot 0}$, defined by (1.11).

Our interest thus concentrates upon the zeros of $uDr(\mathbf{t}) - \chi_u(\mathbf{t})$ and the value of $ur(\mathbf{t}) - \psi_u(\mathbf{t})$ at such a zero.

LEMMA 3.1. $\eta_u \xrightarrow{\mathscr{L}} \eta$ *as* $u \to \infty$.

Proof. η_u has the density $q_u^*(\mathbf{y}) = q_u(\mathbf{y} - uS_{20})$. If $\mathbf{z} = \mathbf{y} - uS_{20}$ then the corresponding symmetric matrix $\mathbf{Z}$ is equal to $\mathbf{Y} + uS_{11}$, so that by (1.12)

$$u^{-n}k_u q_u^*(\mathbf{y}) = u^{-n} \det(\mathbf{Y} + uS_{11}) \exp(-\tfrac{1}{2}\mathbf{y}' S_{2\cdot 0}^{-1}\mathbf{y}) \tag{3.3}$$

for $\mathbf{Y} + uS_{11} \succ 0$. This function tends pointwise, and with dominated convergence to $\det S_{11} \exp(-\frac{1}{2}\mathbf{y}' S_{2\cdot 0}^{-1}\mathbf{y})$, and it follows that

$$u^{-n}k_u \to \det S_{11} \int \exp(-\tfrac{1}{2}\mathbf{y}' S_{2\cdot 0}^{-1}\mathbf{y}) d\mathbf{y} = \det S_{11} \cdot k_\infty ,$$

say. We get

$$q_u^*(\mathbf{y}) \to k_\infty^{-1} \exp(-\tfrac{1}{2}\mathbf{y}' S_{2\cdot 0}^{-1}\mathbf{y}) ,$$

with dominated convergence, which is the content of the lemma. □

The lemma implies that the stochastic term $\psi_u(\mathbf{t})$ is of moderate order for all u. The behaviour of $\xi_u(\mathbf{t})$ is therefore well determined by the behaviour of $r(\mathbf{t})$ as is reflected in the following Lemma 3.2.

Let I be any bounded measurable subset of $\mathbf{R}^n$ and define

$$I_\varepsilon = I \cap \{\mathbf{t} \in \mathbf{R}^n, \ |\mathbf{t}| \geq \varepsilon\} .$$

LEMMA 3.2. *If, for all* $\varepsilon > 0$

$$\inf_{\mathbf{t} \in I_\varepsilon} \|Dr(\mathbf{t})\| > 0$$

then, as $u \to \infty$,

$$P(D\xi_u(\mathbf{t}) = \mathbf{0} \ \textit{for some} \ \mathbf{t} \in I, \ \mathbf{t} \neq \mathbf{0}) \to 0 .$$

The lemma implies especially that the probability of at least one local minimum in I tends to zero.

Proof. We prove the lemma in two steps.

a) $P(D\xi_u(\mathbf{t}) = \mathbf{0}$ for some $\mathbf{t} \neq \mathbf{0}, \ |\mathbf{t}| \leq \varepsilon) \to 0$: since

$$\inf_{0 \neq |\mathbf{t}| \leq \varepsilon} |\mathbf{t}|^{-1} \|Dr(\mathbf{t})\| = M_\varepsilon' > 0$$

if ε is small enough, we have

$$\inf_{0 \neq |\mathbf{t}| \leq \varepsilon} |\mathbf{t}|^{-1} \|D\xi_u(\mathbf{t})\| \geq uM_\varepsilon' - \sup_{0 \neq |\mathbf{t}| \leq \varepsilon} |\mathbf{t}|^{-1} \|\chi_u(\mathbf{t})\| .$$

Therefore, the probability in question is less or equal

$$P(\inf_{0 \neq |\mathbf{t}| \leq \varepsilon} |\mathbf{t}|^{-1} \|D\xi_u(\mathbf{t})\| = 0) \leq P(\sup_{0 \neq |\mathbf{t}| \leq \varepsilon} |\mathbf{t}|^{-1} \|\chi_u(\mathbf{t})\| \geq uM_\varepsilon') . \tag{3.4}$$

It is easily shown that $\partial b_{ij}(\mathbf{t})/\partial t_\nu = O(|\mathbf{t}|)$ for small $|\mathbf{t}|$, (cf. Lemma 2.2), so that $\|D\eta_u'\mathbf{b}(\mathbf{t})\| \le K|\mathbf{t}|\|\eta_u\|$ for some $K > 0$. Furthermore $\partial\Delta(\mathbf{t})/\partial t_i$, $i = 1, \ldots, n$ are continuously differentiable (a.s) and $\partial\Delta(\mathbf{0})/\partial t_i = 0$. This implies that

$$\sup_{0 \ne |\mathbf{t}| \le \varepsilon} |\mathbf{t}|^{-1}\|D\Delta(\mathbf{t})\|$$

is a finite (a.s) random variable. Thus the supremum in the right hand probability in (3.4) is finite (a.s) which gives that the probability itself tends to zero as u goes to infinity.

b) Now take an ε such that the proof of part a) goes through, and let

$$\inf_{\mathbf{t} \in I_\varepsilon} \|Dr(\mathbf{t})\| = M_\varepsilon'' > 0 .$$

As before

$$P(D\xi_u(\mathbf{t}) = \mathbf{0} \text{ for some } \mathbf{t} \in I_\varepsilon) \le P(\inf_{\mathbf{t} \in I_\varepsilon} \|D\xi_u(\mathbf{t})\| = 0) \le P(\sup_{\mathbf{t} \in I_\varepsilon} \|\chi_u(\mathbf{t})\| \ge uM_\varepsilon'') .$$

Since I_ε is bounded and the process $\chi_u(\mathbf{t})$ is continuous (a.s.) the right hand probability above tends to zero as u tends to infinity. This implies that there are no stationary points in I_ε. □

3.2. *The non-isotropic pitfall case*

We have seen that the equation $D\xi_u(\mathbf{t}) = \mathbf{0}$ has possible solutions only near stationary points of r. Even then, the behaviour of the solutions depends greatly on the character of the stationary point. The »pitfall» case is most simply defined as follows:

P: r has a strict local minimum at $\mathbf{t}^0 = (t_1^0, \ldots, t_n^0)'$; the matrix $R_\mathbf{t} = (\partial_1^2 r(\mathbf{t})/\partial t_i \partial t_j)$ is positive definite for $\mathbf{t}$ near $\mathbf{t}^0$; as $\mathbf{t} \to \mathbf{t}^0$

$$r(\mathbf{t}) = r(\mathbf{t}^0) + \tfrac{1}{2}\left(\sum_1^n (t_\nu - t_\nu^0)\frac{\partial}{\partial t_\nu}\right)^2 r(\mathbf{t}^0) + o(|\mathbf{t} - \mathbf{t}^0|^2) ,$$

$$\frac{\partial r(\mathbf{t})}{\partial t_i} = \left(\sum_1^n (t_\nu - t_\nu^0)\frac{\partial}{\partial t_\nu}\right)\frac{\partial r(\mathbf{t}^0)}{\partial t_i} + o(|\mathbf{t} - \mathbf{t}^0|) ,$$

$$\frac{\partial^2 r(\mathbf{t})}{\partial t_i \partial t_j} = \frac{\partial^2 r(\mathbf{t}^0)}{\partial t_i \partial t_j} + o(1) .$$

The first two expansions can be written

$$r(\mathbf{t}) = r(\mathbf{t}^0) + \tfrac{1}{2}(\mathbf{t} - \mathbf{t}^0)' R_{\mathbf{t}^0}(\mathbf{t} - \mathbf{t}^0) + o(|\mathbf{t} - \mathbf{t}^0|^2) ,$$

$$Dr(\mathbf{t}) = R_{\mathbf{t}^0}(\mathbf{t} - \mathbf{t}^0) + o(|\mathbf{t} - \mathbf{t}^0|) .$$

The »degenerate» case when R_{t^0} is not definite or possibly vanishes will be dealt with later on.

Write $S(\mathbf{t}, \varepsilon) = \{\mathbf{s} \in \mathbf{R}^n; |\mathbf{s} - \mathbf{t}| \le \varepsilon\}$ for the sphere around $\mathbf{t}$ with radius ε, and let $S(\varepsilon) = S(\mathbf{t}^0, \varepsilon)$. Then, the condition P implies that there is an $\varepsilon^0 > 0$ such that r has no stationary points in $S(\varepsilon^0)$ except $\mathbf{t}^0$. Theorem 3.1 below states that for any ε, $0 < \varepsilon < \varepsilon^0$, with a probability tending to one, ξ_u has exactly one local minimum in $S(\varepsilon)$. If so, let $\boldsymbol{\tau}^u$ be the location of that minimum; otherwise let $\boldsymbol{\tau}^u = \mathbf{t}^0$. Then $\xi_u(\boldsymbol{\tau}^u)$ essentially denotes the value of the process at the minimum. Theorem 3.1 also expresses the asymptotic properties of $\boldsymbol{\tau}^u$ and $\xi_u(\boldsymbol{\tau}^u)$ in terms of the random variables $\psi(= \psi(\mathbf{t}^0))$ and $\chi(= \chi(\mathbf{t}^0))$ defined in Section 3.1.

Obviously (ψ, χ) is $(n+1)$-variate normal and it has mean zero and the covariance matrix

$$\begin{pmatrix} 1 - r^2(\mathbf{t}^0) & \mathbf{0} \\ \mathbf{0} & S_{11} - R_{\mathbf{t}^0} S_{11}^{-1} R_{\mathbf{t}^0} \end{pmatrix} . \tag{3.5}$$

To show this we compute

$$V(\psi) = V(\eta' \mathbf{b}(\mathbf{t}^0)) + V(\Delta(\mathbf{t}^0)) = \mathbf{b}(\mathbf{t}^0)' S_{2\cdot 0} \mathbf{b}(\mathbf{t}^0) + C(\mathbf{t}^0, \mathbf{t}^0) ,$$

$$\operatorname{Cov}(\psi, \chi_i) = \mathbf{b}(\mathbf{t}^0)' S_{2\cdot 0} \frac{\partial \mathbf{b}(\mathbf{t}^0)'}{\partial t_i} + \left. \frac{\partial C(\mathbf{s}, \mathbf{t})}{\partial t_i} \right|_{\mathbf{s}=\mathbf{t}=\mathbf{t}^0} ,$$

$$\operatorname{Cov}(\chi_i, \chi_j) = \frac{\partial \mathbf{b}(\mathbf{t}^0)'}{\partial t_i} S_{2\cdot 0} \frac{\partial \mathbf{b}(\mathbf{t}^0)}{\partial t_j} + \left. \frac{\partial^2 C(\mathbf{s}, \mathbf{t})}{\partial s_i \partial t_j} \right|_{\mathbf{s}=\mathbf{t}=\mathbf{t}^0} .$$

Now recall definition (1.10) of $\mathbf{b}(\mathbf{t})$ and $C(\mathbf{s}, \mathbf{t})$ and write

$$\begin{pmatrix} T_{00} & T_{02} \\ T_{20} & T_{22} \end{pmatrix} = \begin{pmatrix} 1 & S_{02} \\ S_{20} & S_{22} \end{pmatrix}^{-1} .$$

Using that $\partial r(\mathbf{t}^0)/\partial t_i = 0$ we then get

$$V(\psi) = r(\mathbf{0}) + (r(\mathbf{t}^0), S_2'(\mathbf{t}^0)) \left\{ \begin{pmatrix} T_{02} \\ T_{22} \end{pmatrix} S_{2\cdot 0}(T_{20}, T_{22}) - \begin{pmatrix} T_{00} & T_{02} \\ T_{20} & T_{22} \end{pmatrix} \right\} \begin{pmatrix} r(\mathbf{t}^0) \\ S_2(\mathbf{t}^0) \end{pmatrix} . \tag{3.6}$$

Since $T_{22} = S_{2\cdot 0}^{-1}$, and $T_{00} - T_{02} T_{22}^{-1} T_{20} = S_{00}^{-1} = 1$, the bracketed expression is

$$\begin{pmatrix} T_{02} \\ T_{22} \end{pmatrix} T_{22}^{-1}(T_{20}, T_{22}) - \begin{pmatrix} T_{00} & T_{02} \\ T_{20} & T_{22} \end{pmatrix} = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} ,$$

so that $V(\psi) = 1 - r^2(\mathbf{t}^0)$.

To obtain the remaining covariances we have only to replace the appropriate $(r(\mathbf{t}^0), S_2'(\mathbf{t}^0))$ in (3.6) with the corresponding derivatives. □

THEOREM 3.1. *If r fulfills condition P with $\mathbf{t}^0, R_{\mathbf{t}^0}$, and ε^0, then, for any ε, $0 < \varepsilon < \varepsilon^0$, as $u \to \infty$*

a) *$P(\xi_u(\mathbf{t})$ has exactly one local minimum in $S(\varepsilon)) \to 1$,*

b) $(u(\boldsymbol{\tau}^u - \mathbf{t}^0),\ \xi_u(\boldsymbol{\tau}^u) - ur(\mathbf{t}^0)) \xrightarrow{\mathcal{L}} (R_{\mathbf{t}^0}^{-1}\chi, -\psi)$.

The theorem simply says that $\boldsymbol{\tau}^u \xrightarrow{\mathcal{P}} \mathbf{t}^0$ and that $u(\boldsymbol{\tau}^u - \mathbf{t}^0)$ and $\xi_u(\boldsymbol{\tau}^u) - ur(\mathbf{t}^0)$ are asymptotically normal, independent, and have the covariance matrix and variance $R_{\mathbf{t}^0}^{-1} S_{11} R_{\mathbf{t}^0}^{-1} - S_{11}^{-1}$ and $1 - r^2(\mathbf{t}^0)$ respectively.

Proof. Even if part a) formally follows from part b) we have to give it an independent proof. a) Our concern is the number and locations of the zeros of the mapping $\mathbf{t} \curvearrowright u^{-1}D\xi_u(\mathbf{t}) = Dr(\mathbf{t}) - u^{-1}\chi_u(\mathbf{t})$, where χ_u is defined by (3.2). It is therefore natural to look only at such outcomes for which $\chi_u(\mathbf{t})$ is in a certain sense bounded. Let therefore $\delta > 0$ and M_δ be given constants, and define, for each u, the event $N_\delta (= N_{\delta u})$ so that, for each outcome in N_δ it holds

$$\sup_{\mathbf{t} \in S(\varepsilon^0)} |\chi_u(\mathbf{t})| \leq M_\delta ,$$

$$\sup_{\mathbf{t} \in S(\varepsilon^0)} \max_j \|\partial \chi_u(\mathbf{t}) / \partial t_j\| \leq M_\delta . \tag{3.7}$$

By Lemma 3.1 and the continuity of $\mathbf{b}(\mathbf{t})$ and $\Delta(\mathbf{t})$ and their first and second order partial derivatives we conclude that we can take M_δ so large that, regardless of u, $P(N_\delta) \geq 1 - \delta$. Since δ is arbitrary, the assertion is proved if we can show that $\xi_u(\mathbf{t})$ has exactly one local minimum for all outcomes in N_δ. In the sequel we therefore restrict our attention to such outcomes, even if that is not explicitly mentioned.

We now prove part a) by showing that the range

$$Ra = \{u^{-1}D\xi_u(\mathbf{t});\ \mathbf{t} \in S(\varepsilon)\}$$

contains a sphere $S(\mathbf{0}, d) = \{\mathbf{x} \in \mathbf{R}^n;\ |\mathbf{x}| \leq d\}$, which especially implies that $u^{-1}D\xi_u(\mathbf{t}) = \mathbf{0}$ for at least one $\mathbf{t}$ in $S(\varepsilon)$. It will also follow that there is actually only one such $\mathbf{t}$, and that it represents a local minimum.

We first notice that condition P implies that

$$\inf_{|\mathbf{t} - \mathbf{t}^0| = \varepsilon} |Dr(\mathbf{t})| = d_\varepsilon > 0 .$$

If $u > 2d_\varepsilon^{-1} M_\delta$ then, with inf and sup taken over $|\mathbf{t} - \mathbf{t}^0| = \varepsilon$,

$$\inf |u^{-1}D\xi_u(\mathbf{t})| \geq \inf |Dr(\mathbf{t})| - \sup |u^{-1}\chi_u(\mathbf{t})| > d_\varepsilon - (2d_\varepsilon^{-1}M_\delta)^{-1} M_\delta \geq d_\varepsilon/2 > 0 .$$

We can then take $d = d_\varepsilon/2$ as will now be shown. Write $S^0 = \{\mathbf{x} \in \mathbf{R}^n;\ |\mathbf{x}| < d_\varepsilon/2\}$ for the interior of $S(\mathbf{0}, d_\varepsilon/2)$. Obviously

$$|u^{-1}D\xi_u(\mathbf{0})| \leq |Dr(\mathbf{0})| + |u^{-1}\chi_u(\mathbf{0})| \leq 0 + u^{-1}M_\delta < d_\varepsilon/2 ,$$

so that the set $A = Ra \cap S^0$ is not empty. That in fact $A = S^0$ follows from the »inverse function theorem», see e.g. [1, p. 144]. The matrix $J(\mathbf{t})$ of partial derivatives of the elements of $u^{-1}D\xi_u(\mathbf{t})$ is, for large u, uniformly near $R_\mathbf{t}$ — remember that we are only dealing with outcomes in N_δ — and so the Jacobian $|\det J(\mathbf{t})|$ is non-vanishing over $S(\varepsilon)$. The inverse function theorem then implies that A is open. But since the mapping $u^{-1}D\xi_u(\mathbf{t})$ is continuous the set $B = S^0 \setminus A$ is also open. Then $S^0 = A \cup B$ is the union of two open, disjoint sets, and since S^0 is connected, either A and B is empty. But A is not empty and therefore B is. We conclude that $S^0 \subset Ra$, and that, in fact $S(\mathbf{0}, d_\varepsilon/2) \subseteq Ra$. Thus $\xi_u(\mathbf{t})$ has at least one stationary point in $S(\varepsilon)$.

That any stationary point in fact is a local minimum follows immediately from what was said above about the matrix $J(\mathbf{t})$.

To finish the proof of part a) we still have to show the uniqueness of the minimum. This follows however from the »almost linearity» of the mapping. Since $R_{\mathbf{t}}$ is continuous it follows that $J(\mathbf{t})$ can be made uniformly close, not only to $R_{\mathbf{t}}$, but also to $R_{\mathbf{t}^0}$ by choosing u large. (This might involve choosing a smaller ε than the original one, but by Lemma 3.2, this is not a crucial point.) Since $R_{\mathbf{t}^0}$ is non-singular, this implies that $u^{-1}D\xi_u(\mathbf{t})$ cannot map two different point in $S(\varepsilon)$ on one and the same point in $S(\mathbf{0}, d_\varepsilon/2)$.

It is now a simple task to prove part b). Since part a) implies that $\boldsymbol{\tau}^u \overset{\mathcal{P}}{\rightarrow} \mathbf{t}^0$ we can expand $\xi_u(\boldsymbol{\tau}^u)$ and $u^{-1}D\xi_u(\boldsymbol{\tau}^u)$ in Taylor series for large u:

$$\begin{aligned} \xi_u(\boldsymbol{\tau}^u) - ur(\mathbf{t}^0) &= u(r(\boldsymbol{\tau}^u) - r(\mathbf{t}^0)) - \psi_u(\boldsymbol{\tau}^u) = \\ &= u\{\tfrac{1}{2}(\boldsymbol{\tau}^u - \mathbf{t}^0)' R_{\mathbf{t}^0}(\boldsymbol{\tau}^u - \mathbf{t}^0) + |\boldsymbol{\tau}^u - \mathbf{t}^0|^2 o_p(1)\} - \psi_u(\mathbf{t}^0) + o_p(1)\,, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \mathbf{0} = D\xi_u(\boldsymbol{\tau}^u) = uDr(\boldsymbol{\tau}^u) - \chi_u(\boldsymbol{\tau}^u) &= u\{R_{\mathbf{t}^0}(\boldsymbol{\tau}^u - \mathbf{t}^0) + |\boldsymbol{\tau}^u - \mathbf{t}^0| o_p(1)\} - \\ &- \chi_u(\mathbf{t}^0) + o_p(1)\,, \end{aligned} \tag{3.9}$$

where we have written $o_p(1)$ for any random variable that tends to zero in probability. Then (3.9) implies that $u(\boldsymbol{\tau}^u - \mathbf{t}^0) \sim R_{\mathbf{t}^0}^{-1}\chi_u(\mathbf{t}^0) \overset{\mathcal{L}}{\rightarrow} R_{\mathbf{t}^0}^{-1}\chi$, which is $N(0, R_{\mathbf{t}^0}^{-1}S_{11}R_{\mathbf{t}^0}^{-1} - S_{11}^{-1})$. It also gives that $u|\boldsymbol{\tau}^u - \mathbf{t}^0|^2$ is $o_p(1)$, and therefore (3.8) implies that $\xi_u(\boldsymbol{\tau}^u) - ur(\mathbf{t}^0) \sim -\psi_u(\mathbf{t}^0) \overset{\mathcal{L}}{\rightarrow} -\psi$, which is $N(0, 1 - r^2(\mathbf{t}^0))$.

Thus far the minimum of r at $\mathbf{t}^0$ has been non-degenerate: the matrix $R_{\mathbf{t}^0}$ has been positive definite. Let us now assume that $R_{\mathbf{t}^0} = 0$ so that the minimum is of higher order than two. Then the mapping Dr is no longer »approximately linear» at $\mathbf{t}^0$, and the arguments used in Theorem 3.1 break down. To remedy this we make a transformation of the region near $\mathbf{t}^0$ as is indicated in the following conditions on r:

P': r has a strict local minimum at $\mathbf{t}^0 = (t_1^0, \ldots, t_n^0)'$; there is a $k > 1$ such that, as $\mathbf{t} \to \mathbf{t}^0$,

$$r(\mathbf{t}) = r(\mathbf{t}^0) + \frac{1}{(2k)!}\left(\sum_1^n (t_\nu - t_\nu^0)\frac{\partial}{\partial t_\nu}\right)^{2k} r(\mathbf{t}^0) + o(|\mathbf{t} - \mathbf{t}^0|^{2k})\,,$$

$$\frac{\partial r(\mathbf{t})}{\partial t_i} = \frac{1}{(2k-1)!}\left(\sum_1^n (t_\nu - t_\nu^0)\frac{\partial}{\partial t_\nu}\right)^{2k-1} \frac{\partial r(\mathbf{t}^0)}{\partial t_i} + o(|\mathbf{t} - \mathbf{t}^0|^{2k-1})\,,$$

$$\frac{\partial^2 r(\mathbf{t})}{\partial t_i \partial t_j} = \frac{1}{(2k-2)!}\left(\sum_1^n (t_\nu - t_\nu^0)\frac{\partial}{\partial t_\nu}\right)^{2k-2} \frac{\partial^2 r(\mathbf{t}^0)}{\partial t_i \partial t_j} + o(|\mathbf{t} - \mathbf{t}^0|^{2k-2})\,;$$

the equation system

$$\frac{1}{(2k-1)!}\left(\sum_1^n x_\nu \frac{\partial}{\partial t_\nu}\right)^{2k-1} \frac{\partial r(\mathbf{t}^0)}{\partial t_i} = h_i\,, \quad i = 1, \ldots, n \tag{3.10}$$

has a unique solution for all $\mathbf{h} = (h_1, \ldots, h_n)'$.

We introduce the notation $\phi(\mathbf{h}) = (\phi_1(\mathbf{h}), \ldots, \phi_n(\mathbf{h}))'$ for the unique solution of the equation system (3.10). If $\alpha = 1/(2k-1)$ then it is easily seen that there are positive constants c_1, c_2 such that

$$c_1|\mathbf{h}|^\alpha \le |\phi(\mathbf{h})| \le c_2|\mathbf{h}|^\alpha\,. \tag{3.11}$$

The meaning of $\phi(\mathbf{h})$ is more clearly understood from the expansion

$$Dr(\mathbf{t}^0 + \phi(\mathbf{h})) = \mathbf{h} + o(|\phi(\mathbf{h})|^{2k-1}) = \mathbf{h} + o(|\mathbf{h}|) \quad \text{as} \quad \mathbf{h} \to \mathbf{0}\,,$$

so that the left hand side is an almost linear function of $\mathbf{h}$.

Now let ψ and χ be defined as before, $= \psi(\mathbf{t}^0), \chi(\mathbf{t}^0)$, and take $\varepsilon^0 > 0$ such that r has no stationary points in $S(\varepsilon^0)$ except $\mathbf{t}^0$. The uniqueness of $\phi(\mathbf{h})$ implies that such an ε^0 exists. Then we have the following theorem, which contains Theorem 3.1 as a special case.

THEOREM 3.2. *If r fulfills condition P' with $\mathbf{t}^0$, k, and ε^0 then, for any ε, $0 < \varepsilon < \varepsilon^0$, it holds as $u \to \infty$*

a) $P(\xi_u(\mathbf{t})$ *has exactly one local minimum in* $S(\varepsilon)) \to 1$,

b) $(u^\alpha(\boldsymbol{\tau}^u - \mathbf{t}^0),\ \xi_u(\boldsymbol{\tau}^u) - ur(\mathbf{t}^0)) \xrightarrow{\mathcal{L}} (\phi(\chi), -\psi)\,.$

Proof. a) In Theorem 3.1 the almost linearity of the function $D_\xi(\mathbf{h}) = u^{-1}D\xi_u(\mathbf{t}^0 + \mathbf{h})$ for $\mathbf{h}$ near $\mathbf{0}$ enabled us to draw simple conclusions about its zeros. Now we rather study the function

$$\begin{aligned} D_\xi^\phi(\mathbf{h}) &= u^{-1}D\xi_u(\mathbf{t}^0 + \phi(\mathbf{h})) = Dr(\mathbf{t}^0 + \phi(\mathbf{h})) - u^{-1}\chi_u(\mathbf{t}^0 + \phi(\mathbf{h})) = \\ &= Dr^\phi(\mathbf{h}) - u^{-1}\chi^\phi(\mathbf{h})\,, \quad \text{say.} \end{aligned}$$

Then there is a one-one correspondence between the zeros of D_ξ and the zeros of D_ξ^ϕ near $\mathbf{t}^0$, and the probability that $D_\xi(\mathbf{h}) = \mathbf{0}$ for exactly one $\mathbf{h} \in S(\mathbf{0}, \varepsilon)$ tends to one for any $\varepsilon > 0$ if and only if the same is true for $D_\xi^\phi(\mathbf{h})$.

We first show that D_ξ^ϕ is essentially linear. Let $J_\Phi(\mathbf{t})$ denote the matrix $(\partial\Phi_i(\mathbf{t})/\partial t_j)$ of partial derivatives of any mapping Φ, $\mathbf{R}^n \curvearrowright \mathbf{R}^n$. Then, condition P' implies that

$$J_{Dr\phi}(\mathbf{h}) = I + o(1) \quad \text{as} \quad \mathbf{h} \to \mathbf{0}\,. \tag{3.12}$$

The relation is easily extended to allow $\mathbf{h} = \mathbf{0}$, and thus $Dr^\phi(\mathbf{h})$ is almost linear for small $\mathbf{h}$. We also have $Dr^\phi(\mathbf{h}) = \mathbf{h} + o(|\mathbf{h}|)$.

Unfortunately the transformation of $\mathbf{h}$ will render the function $u^{-1}\chi^{\phi}(\mathbf{h}) = u^{-1}\chi_u(\mathbf{t}^0 + \phi(\mathbf{h}))$ a highly non-linear behaviour for small $\mathbf{h}$. This obstacle can be avoided by removing a small region near $\mathbf{0}$ in the following way. Take a real number β, $0 < \beta < \alpha/(1-\alpha)$, $(\alpha = 1/(2k-1))$, and consider as before only outcomes in the event N_δ, so that (3.7) is satisfied. Then $D^{\phi}_{\xi}(\mathbf{h})$ will have no zeros in $|\mathbf{h}| \leq u^{-1-\beta}$ as will now be shown. It holds, for some $K_i > 0$, $i = 1, 2, \ldots$

$$\begin{aligned} |D^{\phi}_{\xi}(\mathbf{h}) - D^{\phi}_{\xi}(\mathbf{0})| &\leq |Dr(\mathbf{t}^0 + \phi(\mathbf{h}))| + u^{-1}|\chi_u(\mathbf{t}^0 + \phi(\mathbf{h})) - \chi_u(\mathbf{t}^0)| \leq \\ &\leq K_1|\phi(\mathbf{h})|^{2k-1} + K_2 u^{-1}|\phi(\mathbf{h})| \leq K_3|\mathbf{h}| + K_4 u^{-1}|\mathbf{h}|^{\alpha} \leq \\ &\leq u^{-1}\{K_3 u^{-\beta} + K_4 u^{-\alpha(1+\beta)}\}. \end{aligned} \tag{3.13}$$

Here we used the bounds (3.11) for $|\phi(\mathbf{h})|$. Now restrict the attention to those outcomes $N'_\delta \subseteq N_\delta$ for which it also holds

$$|\chi_u(\mathbf{t}^0)| \geq 2\{K_3 u^{-\beta} + K_4 u^{-\alpha(1+\beta)}\}. \tag{3.14}$$

Since $\beta > 0$, this new restriction becomes negligible as $u \to \infty$, so that $\liminf_{u\to\infty} P(N'_\delta) \geq P(N_\delta) \geq 1 - \delta$. Since $D^{\phi}_{\xi}(\mathbf{0}) = u^{-1}\chi_u(\mathbf{t}^0)$ we can combine (3.13) and (3.14) and conclude that for all outcomes in N'_δ

$$|D^{\phi}_{\xi}(\mathbf{h}) - D^{\phi}_{\xi}(\mathbf{0})| \leq |D^{\phi}_{\xi}(\mathbf{0})|/2 \quad \text{for} \quad |\mathbf{h}| \leq u^{-1-\beta}.$$

Since $D^{\phi}_{\xi}(\mathbf{0}) \neq \mathbf{0}$ this especially means that $D^{\phi}_{\xi}(\mathbf{h}) \neq \mathbf{0}$ for all such $\mathbf{h}$. We can therefore restrict our attention to the region $u^{-1-\beta} \leq |\mathbf{h}| \leq \varepsilon$, in which Dr^{ϕ} trivially is almost linear. We will now show that also $u^{-1}\chi^{\phi}$ is almost linear there. We have

$$\frac{\partial \chi^{\phi}(\mathbf{h})_i}{\partial h_j} = \sum_{\nu} \frac{\partial \chi_u(\mathbf{t}^0 + \phi(\mathbf{h}))_i}{\partial t_\nu} \cdot \frac{\partial \phi_\nu(\mathbf{h})}{\partial h_j}.$$

If $|\mathbf{h}| \leq \varepsilon$ we therefore have, for all outcomes in N_δ,

$$\max_{i,j} \left| \frac{\partial \chi^{\phi}(\mathbf{h})_i}{\partial h_j} \right| \leq n M_\delta \max_{i,j} \left| \frac{\partial \phi_i(\mathbf{h})}{\partial h_j} \right|. \tag{3.15}$$

Since $\phi(\mathbf{h}) = |\mathbf{h}|^{\alpha}\phi(\mathbf{h}/|\mathbf{h}|)$ we get the following expression for its partial derivatives

$$\begin{aligned} \frac{\partial \phi_i(\mathbf{h})}{\partial h_j} &= \frac{\partial |\mathbf{h}|^{\alpha}}{\partial h_j} \cdot \phi_i(\mathbf{h}/|\mathbf{h}|) + |\mathbf{h}|^{\alpha} \cdot \frac{\partial \phi_i(\mathbf{h}/|\mathbf{h}|)}{\partial h_j} = \\ &= \alpha|\mathbf{h}|^{\alpha-1} \cdot \frac{\phi|\mathbf{h}|}{\partial h_j} \cdot \phi_i(\mathbf{h}/|\mathbf{h}|) + |\mathbf{h}|^{\alpha} \sum_{\nu} \frac{\partial \phi_i(\mathbf{h}/|\mathbf{h}|)}{\partial t_\nu} \cdot \frac{\partial h_\nu/|\mathbf{h}|}{\partial h_j}. \end{aligned} \tag{3.16}$$

Here $\partial|\mathbf{h}|/\partial h_j$, $\phi_i(\mathbf{h}/|\mathbf{h}|)$, and $\partial\phi_i(\mathbf{h}/|\mathbf{h}|)/\partial t_\nu$ are uniformly bounded for all $h \neq 0$, while

$$\left| \frac{\partial h_\nu/|\mathbf{h}|}{\partial h_j} \right| = \left| \left\{ \delta_{\nu j}|\mathbf{h}| - h_\nu \frac{\partial |\mathbf{h}|}{\partial h_j} \right\} |\mathbf{h}|^{-2} \right| \leq K|\mathbf{h}|^{-1}.$$

Inserting this into (3.16) gives that for some $K > 0$

$$\left|\frac{\partial \phi_i(\mathbf{h})}{\partial h_j}\right| \leq K|\mathbf{h}|^{\alpha-1} \quad \text{for} \quad \mathbf{h} \neq \mathbf{0}\,.$$

Still with some unspecified $K > 0$ (depending on M_δ) we thus get, for $u^{-1-\beta} \leq |\mathbf{h}| \leq \varepsilon$,

$$u^{-1} \max_{i,j} \left|\frac{\partial \chi^\phi(\mathbf{h})_i}{\partial h_j}\right| \leq u^{-1}K|\mathbf{h}|^{\alpha-1} \leq Ku^{-1+(1-\alpha)(1+\beta)} = Ku^{-(\alpha+\alpha\beta-\beta)}\,. \tag{3.17}$$

Since $\alpha + \alpha\beta - \beta > 0$ this bound tends to zero.

Finally we can combine (3.12), (3.15), and (3.17) and conclude that, for outcomes in N'_δ, the matrix

$$J_{D^\phi_\xi}(\mathbf{h}) = J_{Dr^\phi}(\mathbf{h}) - u^{-1}J_{\chi^\phi}(\mathbf{h})$$

is uniformly near the unity matrix I in the region defined by $u^{-1-\beta} \leq |\mathbf{h}| \leq \varepsilon$, (at least for small ε), and thus that D^ϕ_ξ is almost linear there for all large u.

We can now proceed as in Theorem 3.1: Take $d > 0$ such that $|D^\phi_\xi(\mathbf{h})| \geq d$ for all $|\mathbf{h}| = \varepsilon$; note that the sphere $I_0 = S(D^\phi_\xi(\mathbf{0}),\ |D^\phi_\xi(\mathbf{0})|/2)$ is contained in $S(\mathbf{0}, d)$ for large u; note that $I^0 = \{\mathbf{x} \in \mathbf{R}^n;\ |\mathbf{x}| < d\} \setminus I_0$ is an open connected set that contains $\mathbf{0}$; use the inverse function theorem to prove that the set $A = I^0 \cap \{D^\phi_\xi(\mathbf{h});\ u^{-1-\beta} \leq |\mathbf{h}| \leq \varepsilon\}$ is open, and that $B = I^0 \setminus A$ is open too. Continuity, combined with a little reflexion, will show that A is not empty, and therefore $A = I^0$. Especially $D^\phi_\xi(\mathbf{h})$ is zero for at least one — and in fact for exactly one — $\mathbf{h}$ with $u^{-1-\beta} \leq |\mathbf{h}| \leq \varepsilon$. Since we can exclude the possibility of a zero for $|\mathbf{h}| \leq u^{-1-\beta}$ the uniqueness is clear.

This proves that $D\xi_u(\mathbf{t}^0 + \mathbf{h})(= uD^\phi_\xi(\phi^{-1}(\mathbf{h})))$ has exactly one zero for $\mathbf{h}$ near $\mathbf{0}$ for all outcomes in N'_δ. That the zero corresponds to a local minimum follows as in Theorem 3.1. Since $\liminf P(N'_\delta) \geq 1 - \delta$ and δ is arbitrary, this finishes the proof of part a).

b) This part now presents no further problems. If the unique zero of D^ϕ_ξ is $\boldsymbol{\nabla}$ then $\boldsymbol{\tau}^u - \mathbf{t}^0 = \phi(\boldsymbol{\nabla})$. As in Theorem 3.1 we get $u\boldsymbol{\nabla} \xrightarrow{\mathcal{L}} \chi$, so that

$$\phi(u\boldsymbol{\nabla}) \xrightarrow{\mathcal{L}} \phi(\chi), \quad \text{or} \quad u^\alpha(\boldsymbol{\tau}^u - \mathbf{t}^0) \xrightarrow{\mathcal{L}} \phi(\chi)\,.$$

The proof is complete if we remark that we can incorporate the asymptotic distribution of $\xi_u(\boldsymbol{\tau}^u) - ur(\mathbf{t}^0)$ without further discussion. □

3.3. *The isotropic ditch case*

If ξ is isotropic, and

$$r(\mathbf{t}) = r_*(|\mathbf{t}|), \quad \text{say}$$

then r can have no strict local minimum. This, of course, does not rule out the possibility that ξ_u has strict local minima, but the locations of these are less precisely determined than in the pitfall case. E.g. if r_* has a (strict) local minimum at t_0, then r has (non-strict) minima for all $\mathbf{t}$ with $|\mathbf{t}| = t^0$, and the only thing we can say about ξ_u is that it will have (strict) local minima concentrated near the surface $|\mathbf{t}| = t_0$.

The following account will, at least implicitly, give some idea of the asymptotic spacing of these minima.

Let r_* fulfill the following »ditch» condition.

D: r_* has a strict local minimum at $\mathbf{t}_0$, and there is an integer k such that r_* is $2k$ times continuously differentiable near t_0;

$$\begin{aligned} r_*'(t) &< 0 \quad \text{for} \quad 0 < t < t_0 , \\ r_*^{(j)}(t_0) &= 0 \quad \text{for} \quad j = 1, \ldots, 2k-1, \\ r_*^{(2k)}(t_0) &> 0 . \end{aligned}$$

Because of the circular symmetric character of the problem it is natural to observe ξ_u along radiuses from $\mathbf{0}$. Therefore let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)'$, $|\boldsymbol{\theta}| = 1$, define a direction, and let

$$\xi_u^t(\boldsymbol{\theta}) = \xi_u(t \cdot \boldsymbol{\theta}) \quad \text{for} \quad t \geq 0$$

be the values of ξ_u observed along that direction. To obtain conformity with notations used later on we have written the argument t of the function $\xi^{\cdot}(\boldsymbol{\theta})$ as a superscript. Also define, with the same notations as in (3.1) and (3.2),

$$\begin{aligned} \psi_u^t(\boldsymbol{\theta}) &= \psi_u(t \cdot \boldsymbol{\theta}) , \\ \psi^0(\boldsymbol{\theta}) &= \psi(t_0 \cdot \boldsymbol{\theta}) , \end{aligned}$$

and the derivatives

$$\begin{aligned} \sigma_u^t(\boldsymbol{\theta}) &= \frac{d}{dt}\, \psi_u(t \cdot \boldsymbol{\theta}) = \boldsymbol{\theta}' \cdot \chi_u(t \cdot \boldsymbol{\theta}) = \boldsymbol{\theta}' \cdot D(\eta_u' \mathbf{b}(t \cdot \boldsymbol{\theta}) - \Delta(t \cdot \boldsymbol{\theta})) , \\ \sigma^0(\boldsymbol{\theta}) &= \frac{d}{dt}\, \psi(t_0 \cdot \boldsymbol{\theta}) = \boldsymbol{\theta}' \cdot D(\eta' \mathbf{b}(t_0 \cdot \boldsymbol{\theta}) - \Delta(t_0 \cdot \boldsymbol{\theta})) . \end{aligned} \tag{3.18}$$

Then, for $t \geq 0$

$$\begin{aligned} \xi_u^t(\boldsymbol{\theta}) &= u r_*(t) - \psi_u^t(\boldsymbol{\theta}) , \\ \frac{d}{dt}\, \xi_u^t(\boldsymbol{\theta}) &= u r_*'(t) - \sigma_u^t(\boldsymbol{\theta}) , \end{aligned} \tag{3.19}$$

and, for any set of directions $\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^m$, it holds

$$(\psi_u^{t_0}(\boldsymbol{\theta}^i),\ \sigma_u^{t_0}(\boldsymbol{\theta}^i),\ i = 1, \ldots, m) \xrightarrow{\mathcal{L}} (\psi^0(\boldsymbol{\theta}^i),\ \sigma^0(\boldsymbol{\theta}^i),\ i = 1, \ldots, m) .$$

Now, let the process $\xi_u^t(\boldsymbol{\theta})$ attain its first minimum along the direction $\boldsymbol{\theta}$ at $t = \tau_u(\boldsymbol{\theta})$. By varying $\boldsymbol{\theta}$ we obtain a random field $\{\tau_u(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \mathbf{R}^n,\ |\boldsymbol{\theta}| = 1\}$ defined over the unit sphere, and our main object is now to express the asymptotic distributions of this field in terms of those of the field $\{\sigma^0(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \mathbf{R}^n,\ |\boldsymbol{\theta}| = 1\}$, and similarly for the values of $\xi_u^t(\boldsymbol{\theta})$ at $t = \tau_u(\boldsymbol{\theta})$.

THEOREM 3.3. *If* r *fulfills condition* D *with* t_0, k *then, as* $u \to \infty$
a) *for any* $\varepsilon > 0$

$$P(|\tau_u(\boldsymbol{\theta}) - t_0| \le \varepsilon \text{ for all } \boldsymbol{\theta} \in \mathbf{R}^n,\ |\boldsymbol{\theta}| = 1) \to 1\,,$$

b) $$\{u(\tau_u(\boldsymbol{\theta}) - t_0)^{2k-1},\ \xi_u(\tau_u(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}) - ur_*(t_0),\ \boldsymbol{\theta} \in \mathbf{R}^n,\ |\boldsymbol{\theta}| = 1\} \xrightarrow{\mathcal{L}}$$
$$\left\{\frac{(2k-1)!}{r_*^{(2k)}(t_0)}\,\sigma^0(\boldsymbol{\theta}),\ -\psi^0(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \mathbf{R}^n,\ |\boldsymbol{\theta}| = 1\right\}.$$

Convergence of the random fields means convergence of all finite dimensional distributions.

Proof. a) We need simply to notice that for all small $\varepsilon > 0$

$$P\left(\sup_{\boldsymbol{\theta}} \sup_{0<t<t_0-\varepsilon} t^{-1}\frac{d\xi_u^t(\boldsymbol{\theta})}{dt} < 0,\ \inf_{\boldsymbol{\theta}} \frac{d\xi_u^{t_0+\varepsilon}(\boldsymbol{\theta})}{dt} > 0\right) \to 1, \quad \text{as } u \to \infty$$

which is easily proved if we use (3.19) and apply similar arguments as in the proof of Lemma 3.2.

b) Since, by part a), $\tau_u(\boldsymbol{\theta}) \xrightarrow{\mathcal{P}} t_0$ uniformly in $\boldsymbol{\theta}$ we can expand in Taylor series with a uniformly small rest term:

$$0 = \frac{d}{dt}\,\xi_u^{\tau_u(\boldsymbol{\theta})}(\boldsymbol{\theta}) = u\left\{\frac{r_*^{(2k)}(t_0)}{(2k-1)!}\,(\tau_u(\boldsymbol{\theta}) - t_0)^{2k-1} + o(|\tau_u(\boldsymbol{\theta}) - t_0|^{2k-1})\right\} -$$
$$- \sigma_u^{t_0}(\boldsymbol{\theta}) + o_p(1)\,.$$

This will give the results as far as $\tau_u(\boldsymbol{\theta})$ is concerned, and the rest of the theorem is straightforward as in the one-dimensional case.

We conclude this section with the remark that $\{\psi^0(\boldsymbol{\theta}),\ \sigma^0(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \mathbf{R}^n,\ |\boldsymbol{\theta}| = 1\}$ is a bivariate homogeneous Gaussian field with mean zero and with a covariance structure that depends only on the distance $|\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}| = \sqrt{2}(1 - \boldsymbol{\theta}' \cdot \tilde{\boldsymbol{\theta}})^{1/2}$. In fact, if Θ denotes the angle between $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$, so that $\cos\Theta = \boldsymbol{\theta}' \cdot \tilde{\boldsymbol{\theta}}$, then (with $\lambda_2^* = -r_*''(0)$)

$$\operatorname{Cov}(\psi^0(\boldsymbol{\theta}),\ \psi^0(\tilde{\boldsymbol{\theta}})) = r_*(t_0\sqrt{2}(1 - \cos\Theta)^{1/2}) - r_*^2(t_0)\,,$$
$$\operatorname{Cov}(\psi^0(\boldsymbol{\theta}),\ \sigma^0(\tilde{\boldsymbol{\theta}})) = (1 - \cos\Theta)r_*'(t_0\sqrt{2}(1 - \cos\Theta)^{1/2})\,,$$

$$\operatorname{Cov}(\sigma^0(\theta), \sigma^0(\tilde{\theta})) = -\frac{r'_*(t_0\sqrt{2}(1-\cos\Theta)^{1/2})}{t_0\sqrt{2}(1-\cos\Theta)^{1/2}}\cdot\cos\Theta +$$

$$+\left\{r''_*(t_0\sqrt{2}(1-\cos\Theta)^{1/2}) - \frac{r'_*(t_0\sqrt{2}(1-\cos\Theta)^{1/2})}{t_0\sqrt{2}(1-\cos\Theta)^{1/2}}\right\}\tfrac{1}{2}(1-\cos\Theta) -$$

$$-\lambda_2^{*-1} r''_*(t_0)^2\cos\Theta\,.$$

Especially

$$V(\psi^0(\boldsymbol{\theta})) = 1 - r_*^2(t_0)\,,$$

$$V(\sigma^0(\boldsymbol{\theta})) = \lambda_2^* - r''_*(t_0)^2\lambda_2^*\,,$$

$$\operatorname{Cov}(\psi^0(\boldsymbol{\theta}), \sigma^0(\boldsymbol{\theta})) = 0\,.$$

The proofs of these relations are quite straightforward. One just has to proceed as in the proof of (3.5) and use the simple derivation rules on the function $r(\mathbf{t}) = r_*(|\mathbf{t}|)$.

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Received november 22, 1971

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