Extremal Analytic Functions in the Unit Circle

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1. Introduction

Let u be subharmonic in $\{|z| < 1\}$ and put $A(r) = \inf_{|z|=r} u(z)$, $B(r) = \max_{|z|=r} u(z)$ when $0 < r < 1$. Let λ be a fixed number in (0, 1) and suppose that

$$A(r) \leq \cos \pi\lambda B(r), \quad 0 < r < 1 \tag{1.1}$$

$$u \leq c < +\infty . \tag{1.2}$$

Here c is a positive constant. Under these assumptions Hellsten, Kjellberg and Norstad [4] proved

THEOREM A. *There exists a subharmonic function*

$$U(z) = \frac{2c}{\pi} \tan\left(\frac{\pi\lambda}{2}\right) \operatorname{Re} \int_0^z \frac{t^{\lambda-1} - t^{1-\lambda}}{1 - t^2}\, dt, \quad |\arg z| \leq \pi ,$$

in $\{|z| < 1\}$ *for which* (1.1) *holds with equality and such that for* $0 < r < 1$

$$B(r) \leq U(r) \leq \frac{2c}{\pi\lambda} \tan\left(\frac{\pi\lambda}{2}\right) r^{\lambda} .$$

We note that the author [5] and Essén [1] have proved related theorems. Here we consider a similar problem for analytic functions. More specifically, let σ be a step function on [0, 1], i.e.; a piecewise constant function with a finite number of jumps, which satisfies

(*) σ is upper semicontinuous,
(**) $-1 < \sigma(r) < 1$ when $0 \leq r \leq 1$.

Let f be analytic in $\{|z|<1\}$ and put $m(r)=\min_{|z|=r}|f(z)|$, $M(r)=\max_{|z|=r}|f(z)|$ when $0<r<1$. Let a and r_0 be fixed numbers satisfying $0<a<1$ and $0<r_0\leq 1$. Finally, we suppose that

$$m(r)M(r)^{\sigma(r)}\leq a^{1+\sigma(r)},\quad 0<r<r_0\ , \tag{1.3}$$

$$|f|\leq 1\ . \tag{1.4}$$

Then we shall prove

THEOREM 1. *There exists an analytic function* $F=F(\cdot,a,\sigma,r_0)$ *in* $\{|z|<1\}$ *satisfying* (1.3) *and* (1.4) *with the following properties:*

(i) *If* f *satisfies* (1.3) *and* (1.4), *then* $M(r,f)\leq F(r)$, $0<r<1$,
(ii) F *is the unique analytic function in* $\{|z|<1\}$ *for which* (1.3), (1.4) *and* (i) *are true,*
(iii) *all the zeros of* F *are negative and simple,*
(iv) *between two zeros of* F *in* $[-r_0,0]$ *there exists at least one point* $-r$ *where* $|F(-r)|F(r)^{\sigma(r)}=a^{1+\sigma(r)}$,
(v) F *has at most one zero* $-t$ *for which* $r_0\leq t<1$,
(vi) *if* F *has zeros* $-s$, $-t$, *such that* $0<s<r_0<t<1$, *then* $|F(-r)|F(r)^{\sigma(r)}=a^{1+\sigma(r)}$ *for some* $r\in(s,r_0]$,
(vii) $F(0)=a$,
(viii) F *is a finite or infinite Blaschke product depending on whether* $r_0<1$ *or* $r_0=1$.

Now let u be subharmonc in $\{|z|<1\}$ and satisfy (1.1) and (1.2). In addition assume that

$$u=\log|g|,\ \text{ where } g \text{ is analytic in } \{|z|<1\}\ . \tag{1.5}$$

Put $a=e^{-c}$, $f=ag$, and $\sigma=-\cos\pi\lambda$. Applying Theorem 1 with $F=F(\cdot,a,\sigma,1)$ we obtain $M(r,f)\leq F(r)$ when $0<r<1$ and thereupon that

$$B(r)\leq c+\log F(r),\quad 0<r<1\ . \tag{1.6}$$

Since all steps are reversible, it follows that $U=\log|F|+c$ is the unique member of the class of subharmonic functions u satisfying (1.1), (1.2) and (1.5) for which (1.6) is true.

To obtain a more general theorem let $\lambda(r)$, $0\leq r\leq 1$, be an upper semicontinuous step function on [0, 1] and suppose that $0<\lambda(r)<1$. Let u be subharmonic in $\{|z|<1\}$ and satisfy (1.2), (1.5) and the condition

$$A(r)\leq\cos\pi\lambda(r)B(r)\ , \tag{1.7}$$

when $0<r<1$. Again, we apply Theorem 1 with $a=e^{-c}$, $f=ag$ and $\sigma(r)=-\cos\pi\lambda(r)$ for $0<r<1$. Using this theorem we see that (1.6) is true when

$F = F(\cdot, a, \sigma, 1)$. It now follows that in the class of subharmonic functions u satisfying (1.2), (1.5) and (1.7), there exists a member with largest maximum modulus.

Here we remark that the corresponding problem for the class of subharmonic functions satisfying only (1.2) and (1.7), has not been solved. That is, we do not know whether there exists a member of this class with largest maximum modulus.

We note that Heins (see [2, § 7] and [3, Theorem 3.2]) proved Theorem 1 when $\sigma \equiv 0$. He also gave two methods for determining F when $\sigma = 0$ and $r_0 = 1$. In § 9 we shall discuss the problem of determining F when σ is constant on $[0, r_0]$.

Now let n be a positive integer and suppose that Γ_n is the set of all analytic functions b_n in $\{|z| < 1\}$ which can be written in the form

$$b_n(z) = e^{i\gamma} \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \; (\gamma \text{ real}; \; |\alpha_k| < 1, \; 1 \le k \le n)\,. \tag{1.8}$$

Then we prove

THEOREM 2. *Let σ and r_0 be fixed numbers satisfying $-1 < \sigma < 1$ and $0 < r_0 \le 1$. Let*

$$\mu(n) = \inf_{b_n \in \Gamma_n} \sup_{r \in (0, r_0]} m(r, b_n) M(r, b_n)^\sigma$$

and put $a^{1+\sigma} = \mu(n)$. Then $F = F(\cdot, a, \sigma, r_0)$ is a member of Γ_n. Moreover, if $f \in \Gamma_n$ and $\sup_{0<r<r_0} m(r, f) M(r, f)^\sigma = a^{1+\sigma}$, then for some θ, $0 \le \theta < 2\pi$, we have $f = e^{i\theta} F$.

Here we remark that Theorem 2 is similar to Theorem 7.1 of Heins [3] when $\sigma = 0$. Finally, the author would like to thank Professor Heins for suggesting these problems to him and Professor Matts Essén for several helpful suggestions in presenting this paper.

2. Preliminary reductions

Let $f \not\equiv 0$ be as in (1.3) and (1.4). Then it is well known (see Nevanlinna [6]) that f can be written in the form,

$$f(z) = \exp\left(-\int_0^{2\pi} \frac{1}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\alpha(\theta) + i\gamma\right) B(z); \quad |z| < 1\,. \tag{2.1}$$

Here α is a nondecreasing function on $[0, 2\pi]$, $0 \le \gamma < 2\pi$, and either B is a Blaschke product or $B = 1$.

Let

$$f^*(z) = \exp\left[\frac{z-1}{z+1}\frac{1}{2\pi}\int_0^{2\pi} d\alpha(\theta)\right] B^*(z), \quad |z| < 1 . \tag{2.2}$$

If $B = 1$, then $B^* = 1$. Otherwise if (a_i) denotes the zeros of B, then B^* is the associated Blaschke product with zeros at $\{-|a_i|\}$.

We have the following inequalities between f and f^*:

$$|f^*(-r)| \leq |f^*(z)| \leq f^*(r), \quad |z| = r \quad (0 < r < 1), \tag{2.3}$$

$$|f^*(-r)| \leq m(r,f) \leq M(r,f) \leq f^*(r), \quad 0 < r < 1 , \tag{2.4}$$

$$|f^*(-r)|f^*(r)^{\sigma(r)} \leq m(r,f)M(r,f)^{\sigma(r)}, \quad 0 < r < 1 , \tag{2.5}$$

$$|f^*(-r)|f^*(r)^{\sigma(r)} \leq |f(-z)||f(z)|^{\sigma(r)}, \quad |z| = r \quad (0 < r < 1) . \tag{2.6}$$

Here σ is as in Theorem 1. (2.3)—(2.6) are easily verified (see for example Hellsten, Kjellberg and Norstad [4]).

From (2.3) and (2.5) we see that f^* also satisfies (1.3) and (1.4). In view of (2.4) it follows that it suffices to prove Theorem 1 for f^*.

3. Proof of Theorem 1

The proof of Theorem 1 is long. In this section we construct F for $r_0 < 1$. In § 4 we show F is a finite Blaschke product with negative zeros. In § 5 we prove Lemma 2. Using this lemma we show in § 6 that F has properties (iii)—(viii) when $r_0 < 1$. In § 7 we prove our main lemma. We then deduce properties (i) and (ii) of F for $r_0 < 1$ from this lemma. In § 8 we prove Theorem 1 for $r_0 = 1$.

First assume that $0 < r_0 < 1$. In this case we let E denote the class of analytic functions in $\{|z| < 1\}$ satisfying (1.3) and (1.4). Then $f = a$ is a member of E, and hence E has a nonzero member. Using this fact and a normal family argument it follows for fixed ϱ, $0 < \varrho < 1$, that there exists $F \in E$ for which

$$0 < F(\varrho) = \sup_{f \in E} M(\varrho, f) . \tag{3.1}$$

Moreover, if F^* is the function associated with F as in (2.2), then $F^* = F$ since otherwise $F^* \in E$ and $F(\varrho) < F^*(\varrho)$. This inequality follows from the fact that strict inequality holds in (2.4) unless $f = e^{i\theta} f^*$ for some real θ. Hence from (2.3) we have $m(r, F) = |F(-r)|$ and $M(r, F) = F(r)$ when $0 < r < 1$.

We assert that

$$F(0) > 0 . \tag{3.2}$$

To verify this assertion we shall want the following lemma.

LEMMA 1. *Let* $0 \leq u < 1$ *and put*

$$\theta(z, u) = \frac{z + u}{1 + uz}, \quad |z| < 1 .$$

Then if $0 \leq \alpha_1 < \alpha < 1$ *we have*

$$|\theta(-r, \alpha)|\theta(r, \alpha)^{\sigma(r)} < |\theta(-r, \alpha_1)|\theta(r, \alpha_1)^{\sigma(r)}$$

when $\alpha \leq r < 1$ *while if* $0 \leq r \leq \alpha_1$, *the reverse inequality holds.*

Proof: The lemma is a direct consequence of the formula

$$\frac{\partial}{\partial u} \log \{|\theta(-r, u)|\theta(r, u)^{\sigma(r)}\} =$$

$$= \frac{(1 - r^2)[r(1 - \sigma(r))u^2 + (1 + \sigma(r))(1 + r^2)u + (1 - \sigma(r))r]}{(u^2 - r^2)(1 - u^2r^2)} .$$

We omit the details.

The following remark will be used both in the proof of (3.2) and in the proof of Lemma 3.

$$r \to \frac{r + \varrho}{1 + r\varrho} \text{ is an increasing function on } [0, 1] . \tag{3.2a}$$

We now prove (3.2). If (3.2) is false, then from (2.2) we see that $F(0) = 0$. We write, $F(z) = zh(z)$, where h is analytic in $\{|z| < 1\}$ and $|h| \leq 1$. Let $0 < \alpha < 1$ and put $H = \theta(\cdot, \alpha)h$. Here θ is as in Lemma 1. Using the lemma with $\alpha_1 = 0$, we find that

$$|H(-r)|H(r)^{\sigma(r)} \leq |F(-r)|F(r)^{\sigma(r)}$$

when $\alpha \leq r < 1$. It follows for α near 0 that $H \in E$. Also by (3.2a) we see that $F(\varrho) < H(\varrho)$. We have reached a contradiction. Hence (3.2) is true.

We also assert that

$$\sup_{0 \leq r \leq r_0} \frac{|F(-r)|F(r)^{\sigma(r)}}{a^{1+\sigma(r)}} = 1 . \tag{3.3}$$

Indeed, suppose that

$$\sup_{0 \leq r \leq r_0} \frac{|F(-r)|F(r)^{\sigma(r)}}{a^{1+\sigma(r)}} = c_1 < 1 .$$

In this case let $n \geq 2$ be a positive integer and put $f_n = F \circ \theta(\cdot, 1/n)$. Here θ is as in Lemma 1. The sequence $(f_n)_2^\infty$ converges uniformly to F on $[-r_0, r_0]$ and since $F(0) > 0$, $f_n(r)/F(r) \to 1$ uniformly for $0 \leq r \leq r_0$. Let $\varepsilon > 0$ be a small positive number. Choose n_0 large enough such that for $n > n_0$ and $0 \leq r \leq r_0$,

$$1 - \varepsilon \le \frac{f_n(r)}{F(r)} \le 1 + \varepsilon ,$$

$$|F(-r)| - \varepsilon \le |f_n(-r)| \le |F(-r)| + \varepsilon .$$

Then if ε is small enough we have

$$\begin{aligned} |f_n(-r)| f_n(r)^{\sigma(r)} &\le (|F(-r)| + \varepsilon) \left[\frac{f_n(r)}{F(r)}\right]^{\sigma(r)} F(r)^{\sigma(r)} \\ &\le (|F(-r)| + \varepsilon)(1-\varepsilon)^{-1} F(r)^{\sigma(r)} \\ &\le c_1(1-\varepsilon)^{-1} a^{1+\sigma(r)} + \varepsilon(1-\varepsilon)^{-1} F(0)^{-1} < a^{1+\sigma(r)} \end{aligned}$$

when $n > n_0$ and $0 \le r \le r_0$. Hence for $n > n_0$, $f_n \in E$. Since $F(\varrho) < f_n(\varrho)$, we have reached a contradiction. We conclude from this contradiction that (3.3) is true.

From (3.3) we see there exists a sequence $(r_n)_1^\infty$, $0 \le r_n \le r_0$, such that $r_n \to s$ $(0 \le s \le r_0)$ and

$$\frac{|F(-r_n)| F(r_n)^{\sigma(r_n)}}{a^{1+\sigma(r_n)}} \to 1 .$$

Since $|F(-r_n)| \le F(r_n)$ and σ is a step function it follows that $F(s) \ge a$. Using this inequality and the fact that σ is upper semicontinuous, we obtain

$$\frac{|F(-s)| F(s)^{\sigma(s)}}{a^{1+\sigma(s)}} \ge \lim_{r_n \to s} \frac{|F(-r_n)| F(r_n)^{\sigma(r_n)}}{a^{1+\sigma(r_n)}} = 1 .$$

Since $0 \le s \le r_0$ and F satisfies (1.3) we conclude that

$$|F(-s)| F(s)^{\sigma(s)} = a^{1+\sigma(s)} . \tag{3.4}$$

We have shown that F satisfies (3.4) for at least one point in $[0, r_0]$. Next we show there are at most a finite number of distinct points in $[0, r_0]$ for which (3.4) is true. Indeed if (3.4) were true for an infinite number of points in $[0, r_0]$, then by the Identity Theorem for analytic functions we would have for some $b \in (-1, 1)$,

$$F(-z)F(z)^b = \pm a^{1+b}, \quad z \in \{|z| < 1\} - (-1, 0] .$$

Here F^b denotes the analytic b-th power of F in $\{|z| < 1\} - (-1, 0]$ for which $F(\frac{1}{2})^b > 0$. Since by (2.2), $\lim_{z \to e^{i\theta}} |F(-z)||F(z)|^b = 1$, $0 < |\theta| < \pi$, we would then have a contradiction. We let $\{p_i\}_1^n$ denote the finite set of points in $[-r_0, r_0]$ for which (3.4) is true when either $s = p_i$ or $s = -p_i$.

4. A property of F

Next we shall show that F is a finite Blaschke product. Since $F = F^*$ and $F(0) \neq 0$, it then follows that F has negative real zeros.

We consider two possibilities. First if $\varrho \notin \{p_i\}_1^n$, put $\mu = F(\varrho)$. Using the same argument as Heins [2, p. 353—354], we find that either there exists a unique analytic function ϕ in $\{|z| < 1\}$ such that

(i) $|\phi| < 1$
(ii) $\phi(p_i) = F(p_i),\ 1 \leq i \leq n,\ \phi(\varrho) = F(\varrho)$,

or there exists μ^*, $\mu < \mu^* < 1$, and an analytic function Φ_0 in $\{|z| < 1\}$ such that

(i′) $|\Phi_0| < 1$
(ii′) $\Phi_0(p_i) = F(p_i),\ 1 \leq i \leq n,\ \Phi_0(\varrho) = \mu^*$.

In the first case it follows as in Heins [2, p. 353] that $F = \phi$ and F is a finite Blaschke product.

In the second case following Heins [2, (2.14)] we introduce the function $\Psi(\cdot, \tau)$ defined for $0 < \tau < 1$ by

$$\Psi(z, \tau) = \tau F(z) + (1 - \tau)\beta\Phi_0(z),\quad |z| < 1,\quad \text{where } \beta = \frac{\mu + \mu^*}{2\mu^*}.$$

We shall show for τ_1 near 1 that

$$|\Psi(-r, \tau)||\Psi(r, \tau)|^{\sigma(r)} \leq a^{1+\sigma(r)} \tag{4.1}$$

whenever $r \in [0, r_0]$ and $\tau \in [\tau_1, 1)$. To do this we let $r_1 \in \{p_i\}_1^n$ and $r_1 \geq 0$. Next for small $\delta > 0$ we let

$$\begin{aligned} I(r_1) &= [r_1 - \delta,\ r_1 + \delta] \quad \text{when } r_1 \neq 0,\ r_0\,, \\ &= [r_1,\ r_1 + \delta] \quad \text{when } r_1 = 0\,, \\ &= [r_1 - \delta,\ r_1] \quad \text{when } r_1 = r_0\,. \end{aligned}$$

Put

$$\frac{\Phi_0(-r)}{F(-r)} = \frac{\Phi_0(-r_1)}{F(-r_1)} + \varepsilon(r) = 1 + \varepsilon(r),\quad r \in I(r_1) \tag{4.2}$$

and

$$\frac{\Phi_0(r)}{F(r)} = \frac{\Phi_0(r_1)}{F(r_1)} + \eta(r) = 1 + \eta(r),\quad r \in I(r_1)\,. \tag{4.3}$$

Then if $r \in I(r_1)$ and $I(r_1) \subset [0, r_0]$, we have since $F \in E$,

$$\begin{aligned} &a^{-(1+\sigma(r))}|\Psi(-r, \tau)||\Psi(r, \tau)|^{\sigma(r)} \leq \\ &\tau^{1+\sigma(r)}\left|1 + \frac{\beta(1-\tau)}{\tau}\frac{\Phi_0(-r)}{F(-r)}\right|\left|1 + \frac{\beta(1-\tau)}{\tau}\frac{\Phi_0(r)}{F(r)}\right|^{\sigma(r)} \qquad (4.4) \\ &= 1 + (\beta - 1)(1 + \sigma(r))(1 - \tau) + \beta[\varepsilon(r) + \sigma(r)\eta(r)](1 - \tau) + O((1 - \tau)^2) \end{aligned}$$

as $\tau \to 1$. The term $O[(1-\tau)^2]$ is independent of r when τ is near 1. Since $\beta < 1$ and $\varepsilon(r)$, $\eta(r)$, are uniformly small when $\delta > 0$ is small, it follows for τ_0 near 1 that (4.1) is true when $r \in I(r_1)$ and $\tau_0 \leq \tau < 1$.

Let $\Lambda = \bigcup I(r_1)$, $r_1 \geq 0$, $r_1 \in \{p_i\}_1^n$. Now

$$\sup_{r \in [0, r_0] - \Lambda} |F(-r)| F(r)^{\sigma(r)} a^{-(1+\sigma(r))} < 1 ,$$

since otherwise there would exist s in the closure of $[0, r_0] - \Lambda$ for which $|F(-s)| F(s)^{\sigma(s)} = a^{1+\sigma(s)}$. Using this fact and the fact that $F(0) > 0$ we find (4.1) is true when $r \in [0, r_0]$ and $\tau_1 \leq \tau < 1$.

Let $\Psi^*(\cdot, \tau)$ be the function associated with $\Psi(\cdot, \tau)$ as in (2.2). Then from (2.3), (2.6) and (4.1) we see that $\Psi^*(\cdot, \tau) \in E$ for $\tau_1 \leq \tau < 1$. Since

$$F(\varrho) = \mu < \tau\mu + (1-\tau) \frac{\mu + \mu^*}{2} = \Psi(\varrho, \tau) \leq \Psi^*(\varrho, \tau)$$

we have reached a contradiction. Hence if $\varrho \notin \{p_i\}_1^n$, then $F = \phi$ and F is a finite Blaschke product.

Now suppose that $\varrho \in \{p_i\}_1^n$. We assume, as we may, that $\varrho = p_n$. Let $F(\varrho) = \mu$. Then as before either there exists a unique analytic function ϕ in $\{|z| < 1\}$ satisfying $|\phi| < 1$ and

$$\phi(p_i) = F(p_i)\ (1 \leq i \leq n-1), \quad \phi(\varrho) = \mu ,$$

or there are an infinite number of such functions. If ϕ is unique, then as previously, $F = \phi$ and F is a finite Blaschke product. Otherwise we define Φ_0, μ^*, and Ψ corresponding to $\{p_i\}_1^{n-1}$ as previously. We also define $\varepsilon(r)$ and $\eta(r)$ corresponding to $r_1 = \varrho$ as in (4.2) and (4.3)

Proceeding as in the first case we obtain for r near ϱ and $0 \leq r \leq r_0$, (see (4.4))

$$\begin{aligned} a^{-(1+\sigma(r))} |\Psi(-r, \tau)| |\Psi(r, \tau)|^{\sigma(r)} \leq 1 + [\beta + \sigma(r)\beta x^{-1} - (1 + \sigma(r))](1-\tau) + \\ + \beta(\varepsilon(r) + \sigma(r)\eta(r))(1-\tau) + O[(1-\tau)^2] \end{aligned} \tag{4.5}$$

where $x = \mu/\mu^*$. Since $\beta = \frac{1}{2}(\mu + \mu^*)/\mu^*$, we have

$$\beta(1 + \sigma(r)x^{-1}) = \tfrac{1}{2}(1 + x)(1 + \sigma(r)/x) = g(x) .$$

Let $\sigma_0 = \max_{0 \leq r \leq 1} |\sigma(r)|$. From the definition of σ we see that $\sigma_0 < 1$. From Heins argument [2, p. 353], it is clear that μ^* may be chosen such that $\sqrt{\sigma_0} < x < 1$. Since $g'(x) = \frac{1}{2}(1 - \sigma(r)x^{-2})$, we have

$$g(1) - g(x) > \tfrac{1}{2}(1 - \sigma_0 x^{-2})(1 - x) > 0 . \tag{4.6}$$

Hence $g(1) - g(x)$ is bounded below by a positive constant which does not depend on r. Since $g(1) = 1 + \sigma(r)$, it follows from (4.6) and (4.5) that for r near ϱ,

τ_1 near 1 and $r \in [0, r_0]$ that (4.1) is true. Using this fact and arguing as in the previous case we obtain a contradiction. Hence in both cases $\phi = F$ and F is a finite Blaschke product.

5. A Lemma

We want to prove that F satisfies (i)—(viii) of Theorem 1. The following lemma will play a fundamental role in this proof. This lemma may be compared with lemma 7.1 in Heins [2].

LEMMA 2. *Let* $0 \leq u \leq v \leq 1$ *and put*

$$\phi(z, u, v) = \frac{z+u}{1+uz} \cdot \frac{z+v}{1+vz}, \quad |z| < 1 .$$

Let $0 \leq \alpha_1 < \alpha \leq \beta < \beta_1 \leq 1$ *and suppose for fixed* ϱ, $0 < \varrho < 1$ *that* $\phi(\varrho, \alpha, \beta) = \phi(\varrho, \alpha_1, \beta_1)$. *Then if either* $0 \leq r \leq \alpha_1$ *or* $\beta_1 \leq r < 1$ *we have*

$$|\phi(-r, \alpha_1, \beta_1)|\phi(r, \alpha_1, \beta_1)^{\sigma(r)} < |\phi(-r, \alpha, \beta)|\phi(r, \alpha, \beta)^{\sigma(r)} ,$$

while if $\alpha \leq r \leq \beta$ *the opposite inequality holds.*

Proof. Consider the set

$$\Gamma = \{(u, v) : \alpha_1 \leq u \leq \alpha, \ \ \beta \leq v \leq \beta_1, \ \text{ and } \ \phi(\varrho, u, v) = \phi(\varrho, \alpha, \beta)\}$$

which defines a function $v \to u(v)$ and a function $u \to v(u)$. Then we shall show for fixed r that

$$\frac{d}{dv} [|\phi(-r, u(v), v)|\phi(r, u(v), v)^{\sigma(r)}] < 0 \tag{5.1}$$

when either $0 \leq r \leq \alpha_1$ or $\beta_1 \leq r < 1$ and

$$\frac{d}{dv} [|\phi(-r, u(v), v)|\phi(r, u(v), v)^{\sigma(r)}] > 0 \tag{5.2}$$

when $\alpha \leq r \leq \beta$. To prove (5.1) and (5.2) we observe that

$$\frac{du}{dv} = -\frac{(\varrho + u)(1 + u\varrho)}{(\varrho + v)(1 + v\varrho)} \tag{5.3}$$

and hence that

$$(1 - s^2)^{-1}(\varrho + v)(1 + v\varrho) \frac{d}{dv} \log |\phi(s, u, v)| = N(s, v) - N(s, u) ,$$

where $N(s, x) = (\varrho + x)(1 + x\varrho)/(s + x)(1 + sx)$. It follows for fixed r where $0 \leq r < 1$ and $r \notin \{u, v\}$ that

$$(1 - r^2)^{-1}(\varrho + v)(1 + v\varrho) \frac{d}{dv} \log\{|\phi(-r, u, v)|\phi(r, u, v)^{\sigma(r)}\} = \psi(r, v) - \psi(r, u), \quad (5.4)$$

where $\psi(r, x) = N(-r, x) + \sigma(r)N(r, x)$.

We shall prove that

(a) When $0 \leq r \leq \alpha_1$, $\log |\psi(r, \cdot)|$ is a decreasing function and $\psi(r, \cdot)$ is a positive function on $(\alpha_1, 1)$,

(b) When $\alpha \leq r \leq \beta$, $\psi(r, u) < 0 < \psi(r, v)$,

(c) When $\beta_1 \leq r < 1$, $\log |\psi(r, \cdot)|$ is an increasing function and $\psi(r, \cdot)$ is a negative function on $[0, \beta_1)$.

Since $\alpha_1 < u < \alpha \leq \beta < v < \beta_1$, we have the following consequences of (5.4), (a), (b), and (c), respectively.

(a′) $\psi(r, \cdot)$ is a decreasing function on $(\alpha_1, 1)$ and (5.1) is true for $0 \leq r \leq \alpha_1$,

(b′) (5.2) is true when $\alpha \leq r \leq \beta$,

(c′) $\psi(r, \cdot)$ is a decreasing function on $[0, \beta_1)$ and (5.1) is true when $\beta_1 \leq r < 1$.

To complete the proofs of (5.1) and (5.2), it only remains to prove (a), (b), and (c). To prove (a) we write

$$\psi(r, x) = \psi_1(x)\psi_2(r, x)\psi_3(r, x),$$

where

$$\psi_1(x) = (\varrho + x)(1 + x\varrho),$$

$$\psi_2(r, x) = \{(x^2 - r^2)(1 - x^2r^2)\}^{-1}$$

$$\psi_3(r, x) = r(1 - \sigma(r))x^2 + (1 + \sigma(r))(1 + r^2)x + (1 - \sigma(r))r.$$

We claim that

$$\frac{d}{dx} \log |\psi_1(x)| < \frac{1}{x}, \quad 0 < x < 1,$$

$$\frac{\partial}{\partial x} \log |\psi_2(r, x)| \leq \frac{-2}{x}, \quad 0 \leq r < x < 1,$$

$$\frac{\partial}{\partial x} \log \psi_3(r, x) < \frac{1}{x}, \quad 0 < x < 1.$$

The proofs of these statements are straight forward and are omitted. Using these inequalities we see that (a) is true.

The proof of (b) is immediate from the definition of ψ. The proof of (c) is also simple and we omit the details. This proves (5.1) and (5.2).

Using (5.1) and (5.2) we conclude that

$$|\phi(-r, \alpha_1, \beta_1)|\phi(r, \alpha_1, \beta_1)^{\sigma(r)} = \lim_{v \to \beta_1} |\phi(-r, u, v)|\phi(r, u, v)^{\sigma(r)} <$$
$$< \lim_{v \to \beta} |\phi(-r, u, v)|\phi(r, u, v)^{\sigma(r)} = |\phi(-r, \alpha, \beta)|\phi(r, \alpha, \beta)^{\sigma(r)}$$

when either $0 \leq r \leq \alpha_1$ or $\beta_1 \leq r < 1$, and that the reverse inequality holds when $\alpha \leq r \leq \beta$. This completes the proof of Lemma 2.

6. Descriptive properties of F

To continue the proof of Theorem 1, we recall that we are assuming $r_0 < 1$. Then in § 4 we showed that F is a finite Blaschke product with negative zeros. In this section we prove that F satisfies (iii)—(vii) of Theorem 1. Theorem 1 is thereby established for $r_0 < 1$ save for (i) and (ii) which we treat in § 7. We begin by proving (iii). Let $0 < \alpha < 1$ and suppose that $-\alpha$ is a multiple zero of F.

We write

$$F(z) = \left(\frac{z+\alpha}{1+\alpha z}\right)^2 g(z), \quad |z| < 1 ,$$

where g is analytic in $\{|z| < 1\}$ and $|g| \leq 1$. Let $\phi(\cdot, u, v)$, $0 \leq u \leq v \leq 1$, be as in Lemma 2 and put $\alpha = \beta$. Choose α_1, β_1, such that $0 < \alpha_1 < \alpha = \beta < \beta_1 < 1$ and $\phi(\varrho, \alpha_1, \beta_1) = \phi(\varrho, \alpha, \beta)$. Since $F \in E$, it follows by Lemma 2 that $G = \phi(\cdot, \alpha_1, \beta_1)g$ satisfies

$$|G(-r)|G(r)^{\sigma(r)} < |F(-r)|F(r)^{\sigma(r)} \leq a^{1+\sigma(r)}$$

when $r \in [0, \alpha_1] \cup [\beta_1, 1)$, $0 \leq r \leq r_0$, and $F(-r) \neq 0$. Using this inequality we find that if α_1 and β_1 are near α, then $G \in E$ and

$$|G(-r)|G(r)^{\sigma(r)} < a^{1+\sigma(r)}$$

for $0 \leq r \leq r_0$. However, since $F(\varrho) = G(\varrho)$ and $G \in E$ we must have (see (3.4)), $|G(-s)|G(s)^{\sigma(s)} = a^{1+\sigma(s)}$ for some $s \in [0, r_0]$. Again we have reached a contradiction. Hence (iii) of Theorem 1 is true.

The proof of (iv) is similar to the proof of (iii). Let $-r_0 \leq -\beta < -\alpha < 0$ be two zeros of F and suppose that

$$|F(-r)|F(r)^{\sigma(r)} < a^{1+\sigma(r)}, \quad \alpha \leq r \leq \beta . \tag{6.1}$$

Then $F = \phi(\cdot, \alpha, \beta)f$, where f is analytic in $\{|z| < 1\}$ and $|f| \leq 1$. Choose α_1, β_1, such that $0 < \alpha_1 < \alpha < \beta < \beta_1 < 1$ and $\phi(\varrho, \alpha, \beta) = \phi(\varrho, \alpha_1, \beta_1)$. Let $I = \phi(\cdot, \alpha_1, \beta_1)f$. Using (6.1) and Lemma 2 we find for α_1 near α and β_1 near β that $I \in E$ and

$$|I(-r)|I(r)^{\sigma(r)} < a^{1+\sigma(r)}, \quad 0 \leq r \leq r_0 .$$

However, since $I(\varrho) = F(\varrho)$ we must have by (3.4), $|I(-s)|I(s)^{\sigma(s)} = a^{1+\sigma(s)}$ for some $s \in [0, r_0]$. Again we have reached a contradiction. Hence (iv) is true.

The proofs of (v) and (vi) are exactly the same as the proof of (iv). We omit the details.

To prove (viii), let $t_1 = \min\{r : F(-r) = 0\}$. From (iii) we see that $t_1 > 0$. We assert that

$$|F(-r)|F(r)^{\sigma(r)} \leq F(0)^{1+\sigma(r)} \tag{6.2}$$

when $0 \leq r \leq t_1$. Indeed, for fixed $\sigma(r)$ the function $g(z) = \log |F(-z)||F(z)|^{\sigma(r)}$, $|z| < t_1$, is harmonic in $\{|z| < t_1\}$. Moreover from (2.6) we see that $\min_{|z|=s} g(z) = g(-s)$ for $0 < s < t_1$. Since $\min_{|z|=s} g(z)$ is a nonincreasing function of s, it follows that (6.2) is true.

Next we note that $\sup_{0 \leq r \leq t_1} \{|F(-r)|F(r)^{\sigma(r)} \cdot a^{-(1+\sigma(r))}\} = 1$. Indeed, otherwise the function $G(z) = F(z)\theta(z, t)/\theta(z, t_1)$ is in E for t near t_1, $t > t_1$, and $G(\varrho) > F(\varrho)$, as follows easily from Lemma 1. This fact and (6.2) imply $F(0) = a$. Hence (vii) is true.

7. The final proof for $r_0 < 1$

To prove (i) and (ii) of Theorem 1 we shall want the following lemma.

LEMMA 3. *Let n and m be two positive integers. Let $0 < t_1 < \ldots < t_n < 1$ and $0 \leq s_1 \leq \ldots \leq s_m < 1$. Put $f(z) = \prod_1^n \left(\frac{z + t_i}{1 + t_i z}\right)$, and $g(z) = \prod_1^m \left(\frac{z + s_i}{1 + s_i z}\right)$. Let $t_0 = 0$. For given $\varrho \in (0, 1)$ suppose that $f(\varrho) \leq g(\varrho)$. Then either $f = g$ or there exists a positive integer j, $1 \leq j \leq n$, such that*

$$|f(-r)|f(r)^{\sigma(r)} < |g(-r)|g(r)^{\sigma(r)}, \quad r \in [t_{j-1}, t_j) .$$

Proof. We first assume that $n = 1$.

If $m = 1$, then from (3.2a) and the fact that $f(\varrho) \leq g(\varrho)$, we deduce $t_1 \leq s_1$. If $t_1 = s_1$, then clearly $f = g$. If $t_1 < s_1$, we apply Lemma 1 with $\alpha_1 = t_1$, and $\alpha = s_1$. Using this lemma, we obtain

$$|f(-r)|f(r)^{\sigma(r)} < |g(-r)|g(r)^{\sigma(r)} \tag{7.1}$$

when $r \in [0, t_1]$. Hence Lemma 3 is true for $n = 1$ and $m = 1$.

Let k be a positive integer and suppose that Lemma 3 is true when ever $n = 1$ and $1 \leq m \leq k$. Then if $m = k + 1$ we see from (3.2a) that $t_1 < s_1 \leq s_2 < 1$. Let $\phi(\cdot, u, v)$, $0 \leq u \leq v \leq 1$, be as in Lemma 2. We also consider the function Γ_0 defined by

$$\Gamma_0 = \{(u, v) : t_1 \leq u \leq v \leq 1 \text{ and } \varphi(\varrho, u, v) = \varphi(\varrho, s_1, s_2)\} .$$

Since $dv/du < 0$ (see (5.3)), there exist x_0, y_0, such that $(x_0, y_0) \in \Gamma_0$ and

$$x_0 \leq u \leq v \leq y_0 \text{ when } (u, v) \in \Gamma_0 . \tag{7.2}$$

We claim that $y_0 = 1$. Clearly either $x_0 = t_1$ or $y_0 = 1$, since otherwise (7.2) would be contradicted. If $x_0 = t_1$ then since $f(\varrho) \leq g(\varrho)$ we must have $y_0 = 1$. Hence $y_0 = 1$.

We consider the function

$$h(z) = \left(\frac{z + x_0}{1 + x_0 z}\right) \frac{g(z)}{\phi(z, s_1, s_2)} , \quad |z| < 1 .$$

Using Lemma 2 with $\alpha_1 = x_0$, $\alpha = s_1$, $\beta = s_2$, and $\beta_1 = 1$, we obtain

$$|h(-r)|h(r)^{\sigma(r)} < |g(-r)|g(r)^{\sigma(r)}, \quad 0 \leq r \leq t_1 . \tag{7.3}$$

Moreover h has k zeros and $h(\varrho) = g(\varrho)$. Hence either $h \equiv f$ or

$$|f(-r)|f(r)^{\sigma(r)} < |h(-r)|h(r)^{\sigma(r)}, \quad 0 \leq r < t_1 .$$

In either case we conclude from (7.3) that Lemma 3 is true when $n = 1$ and $m = k + 1$. Hence by induction Lemma 3 is true for $n = 1$.

Let q be a positive integer and suppose that

$$\text{Lemma 3 is true for } n \leq q . \tag{+}$$

Then if $n = q + 1$, we first assume that $m = 1$. In this case if $t_1 \leq s_1$, then from Lemma 1 with $\alpha_1 = t_1$ and $\alpha = s_1$, we see that (7.1) is valid for $0 \leq r < t_1$. If $t_1 > s_1$, then from Lemma 1 with $\alpha = t_1$ and $\alpha_1 = s_1$, we see that (7.1) is valid for $t_1 \leq r \leq t_2$. Hence Lemma 3 is true for $m = 1$ and $n = q + 1$.

Let k be a positive integer and suppose in addition to (+) that

$$\text{Lemma 3 is true for } n = q + 1 \text{ and } m \leq k . \tag{++}$$

Then if $n = q + 1$ and $m = k + 1$, we assume, as we may, that f and g have no common zeros. Indeed, if s, $0 < s < 1$, is such that $f(-s) = g(-s) = 0$, then $f_1(z) = \dfrac{1 + sz}{z + s} f(z)$ and $g_1(z) = \dfrac{1 + sz}{z + s} g(z)$, $|z| < 1$, have q and k zeros respectively in $\{|z| < 1\}$. Using this fact and (+) we obtain that Lemma 3 is true for f_1, g_1, and thereupon for f, g.

We proceed under the above assumptions. We consider the situation,

(a) For some positive integer i $(1 \leq i \leq n - 1)$ there exists a positive integer p $(1 \leq p \leq m - 1)$ such that s_p, s_{p+1} are in (t_i, t_{i+1}).

If situation (a) occurs, we define the function Γ_1 by

$$\Gamma_1 = \{(u, v) : t_i \leq u \leq v \leq t_{i+1} \text{ and } \phi(\varrho, u, v) = \phi(\varrho, s_p, s_{p+1})\} .$$

Then since $dv/du < 0$ we see there exist x_1, y_1, such that $(x_1, y_1) \in \Gamma_1$ and

$$x_1 \leq u \leq v \leq y_1 \text{ when } (u, v) \in \Gamma_1 . \tag{7.4}$$

Clearly either $x_1 = t_i$ or $y_1 = t_{i+1}$, since otherwise (7.4) would be contradicted. If $x_1 = t_i$, and $y_1 < t_{i+1}$, we put

$$f_2(z) = \left(\frac{1 + t_i z}{z + t_i}\right) f(z), \quad |z| < 1 ,$$

$$g_2(z) = \frac{(z + y_1)g(z)}{(1 + y_1 z)\phi(z, s_p, s_{p+1})}, \quad |z| < 1 .$$

Then $f_2(\varrho) \leq g_2(\varrho)$ and f_2, g_2, have q and k zeros respectively in $\{|z| < 1\}$. Using these facts and (+) we find that either $f_2 = g_2$ in which case $n = m = 2$ or for some positive integer j $(1 \leq j \leq n, \ j \neq i + 1)$

$$|f_2(-r)| f_2(r)^{\sigma(r)} < |g_2(-r)| g_2(r)^{\sigma(r)}, \quad t_{j-1} \leq r < t_j . \tag{7.5}$$

If $f_2 = g_2$, then $f = \phi(\cdot, t_1, t_2)$ and $g = \phi(\cdot, s_1, s_2)$. Applying Lemma 2 with $\alpha_1 = t_1$, $\alpha = s_1$, $\beta = s_2$, and $\beta_1 = t_2$, we obtain that (7.1) is valid for $0 \leq r \leq t_1$. If (7.5) is true, we again apply Lemma 2 with $\alpha_1 = t_i$, $\alpha = s_p$, $\beta = s_{p+1}$ and $\beta_1 = y_1$. Then by this lemma and (7.5) we have for $t_{j-1} \leq r < t_j$,

$$|f(-r)| f(r)^{\sigma(r)} = \left|\frac{r - t_i}{1 - r t_i}\right| \left(\frac{r + t_i}{1 + r t_i}\right)^{\sigma(r)} |f_2(-r)| f_2(r)^{\sigma(r)} <$$

$$< \frac{|\phi(-r, t_i, y_1)| \phi(r, t_i, y_1)^{\sigma(r)}}{|\phi(-r, s_p, s_{p+1})| \phi(r, s_p, s_{p+1})^{\sigma(r)}} \cdot |g(-r)| g(r)^{\sigma(r)} < |g(-r)| g(r)^{\sigma(r)}$$

Hence in either case Lemma 3 is valid for $x_1 = t_i$.

The proof for $y_1 = t_{i+1}$ is similar. We omit the details. We conclude that if situation (a) occurs, then Lemma 3 is valid for $n = q + 1$ and $m = k + 1$.

We now suppose that situation (a) does not occur. We first assume that $m = k + 1 \leq n = q + 1$. In this case we claim there exists a positive integer i $(1 \leq i \leq n)$ such that

$$s_j \notin [t_{i-1}, t_i] \ \text{ when } \ 1 \leq j \leq m . \tag{7.6}$$

If $m < n$, then clearly (7.6) is true. If $m = n$, then (7.6) is true, since otherwise it follows that $s_j < t_j$, $1 \leq j \leq n$, and thereupon from (3.2a) that $g(\varrho) < f(\varrho)$.

Let p be the minimum of the set of positive integers i for which (7.6) is true. We shall prove that (7.1) is valid for $t_{p-1} \leq r \leq t_p$. If $p = 1$, then $t_j < s_j$ for $1 \leq j \leq m$. Using this fact and applying Lemma 1 with $\alpha_1 = t_j$ and $\alpha = s_j$, $1 \leq j \leq m$, we obtain for $0 \leq r \leq t_1$

$$\begin{aligned} |f(-r)| f(r)^{\sigma(r)} &\leq \prod_1^m |\theta(-r, t_j)| \theta(r, t_j)^{\sigma(r)} < \\ &< \prod_1^m |\theta(-r, s_j)| \theta(r, s_j)^{\sigma(r)} = |g(-r)| g(r)^{\sigma(r)} . \end{aligned} \tag{7.7}$$

If $p = m + 1$, then $s_j < t_j$ for $1 \leq j \leq m$. Using this fact and applying Lemma 1 with $\alpha_1 = s_j$ and $\alpha = t_j$, $1 \leq j \leq m$, we find that (7.7) holds for $t_m \leq r \leq t_{m+1}$.

If $1 < p \leq m$, then from the definition of p we see that $s_j < t_j$ when $1 \leq j \leq p - 1$ and $t_j < s_j$ when $p \leq j \leq m$. We apply Lemma 1 with $\alpha_1 = s_j$, $\alpha = t_j$, when $1 \leq j \leq p - 1$, and with $\alpha_1 = t_j$, $\alpha = s_j$, when $p \leq j \leq m$. We obtain that (7.7) is true for $t_{p-1} \leq r \leq t_p$. Hence, if situation (a) does not occur and if $m = k + 1 \leq n = q + 1$, then Lemma 3 is true.

Finally suppose that situation (a) does not occur and $n = q + 1 < m = k + 1$. In this case we assert that $t_n < s_{m-1} \leq s_m < 1$. Indeed, otherwise it follows that $s_j < t_j$, $1 \leq j \leq n$, and thereupon that $g(\varrho) < f(\varrho)$.

We let Γ_2 be the function defined by

$$\Gamma_2 = \{(u, v) : t_n \leq u \leq v \leq 1 \text{ and } \phi(\varrho, s_{m-1}, s_m) = \phi(\varrho, u, v)\}.$$

Since $dv/du < 0$ we see there exist x_2, y_2, such that $(x_2, y_2) \in \Gamma_2$ and

$$x_2 \leq u \leq v \leq y_2 \text{ when } (u, v) \in \Gamma_2. \tag{7.8}$$

Clearly either $x_2 = t_n$ or $y_2 = 1$, since otherwise (7.8) would be contradicted. If $y_2 = 1$, we put

$$g_3(z) = \frac{z + x_2}{1 + x_2 z} \frac{g(z)}{\varphi(z, s_{m-1}, s_m)}, \quad |z| < 1.$$

Then $g_3(\varrho) = g(\varrho)$ and g_3 has k zeros in $\{|z| < 1\}$. Using (++) it follows that either $f \equiv g_3$ or for some positive integer j $(1 \leq j \leq n)$

$$|f(-r)|f(r)^{\sigma(r)} < |g_3(-r)|g_3(r)^{\sigma(r)}, \quad t_{j-1} \leq r < t_j.$$

Applying Lemma 2 with $\alpha_1 = x_2$, $\alpha = s_{m-1}$, $\beta = s_m$ and $\beta_1 = 1$, we find in either case that Lemma 3 is true for $y_2 = 1$.

If $x_2 = t_n$ and $y_2 < 1$, we let

$$f_4(z) = \left(\frac{1 + t_n z}{z + t_n}\right) f(z), \quad |z| < 1,$$

$$g_4(z) = \left(\frac{z + y_2}{1 + y_2 z}\right) \frac{g(z)}{\phi(z, s_{m-1}, s_m)}, \quad |z| < 1.$$

Then f_4, g_4, have q and k zeros respectively in $\{|z| < 1\}$ and $f_4(\varrho) \leq g_4(\varrho)$. Using these facts and (+) we see that either $f_4 = g_4$ or for some positive integer j, $1 \leq j \leq n - 1$,

$$|f_4(-r)|^{\sigma(r)} f_4(r)^{\sigma(r)} < |g_4(-r)|^{\sigma(r)} g_4(r)^{\sigma(r)}, \quad t_{j-1} \leq r < t_j.$$

Applying Lemma 2 with $\alpha_1 = t_n$, $\alpha = s_{m-1}$, $\beta = s_m$, and $\beta_1 = y_2$, we find in either case that Lemma 3 is valid for $x_2 = t_n$. Hence if situation (a) does not occur, then

Lemma 3 is true for $n = q + 1$ and $m = k + 1$. Since we have already considered the possibility that situation (a) does occur, we conclude that Lemma 3 holds for $n = q + 1$ and $m = k + 1$.

Now by induction on m we see that Lemma 3 is valid for $n = q + 1$ and m a positive integer. Then by induction on n it follows that Lemma 3 is true whenever n and m are positive integers. This completes the proof of Lemma 3.

To continue the proof of Theorem 1, let E be as in § 3 and suppose that $F \in E$ satisfies (3.1). Then in § 6 we proved that F has properties (iii)—(viii) of Theorem 1 for $r_0 < 1$. We now use Lemma 3 to show that F satisfies (i) and (ii).

Let τ be a fixed number and $0 < \tau < 1$. Let G be an analytic function in $\{|z| < 1\}$ for which $G \in E$ and

$$G(\tau) = \sup_{f \in E} M(\tau, f) .$$

We see that G satisfies (iii)—(viii) of Theorem 1 for $r_0 < 1$. Applying Lemma 3 with $f = F$, $g = G$, and $\varrho = \tau$, we obtain that either $F = G$ or for some positive integer j,

$$|F(-r)||F(r)^{\sigma(r)} < |G(-r)||G(r)^{\sigma(r)}, \quad t_{j-1} \le r < t_j . \tag{7.9}$$

The latter situation cannot occur. Indeed, if (7.9) were true, then from properties (iv)—(vii) of F and the fact that $G \in E$, we would have for some $r \in [0, r_0] \cap [t_{j-1}, t_j)$,

$$a^{1+\sigma(r)} = |F(-r)||F(r)^{\sigma(r)} < |G(-r)||G(r)^{\sigma(r)} \le a^{1+\sigma(r)} .$$

Hence, $F \equiv G$. From this equality we conclude that F satisfies (i).

Property (ii) of F (unicity) is now an immediate consequence of (i) and the Identity Theorem for analytic functions.

This completes the proof of Theorem 1 for $r_0 < 1$.

8. Proof of Theorem 1 for $r_0 = 1$

We wish to prove Theorem 1 when $r_0 = 1$. To do so it will be necessary to indicate the dependence of F on r_0 when $0 < r_0 < 1$. Therefore we shall often write $F(\cdot, r_0)$ for F.

Let $0 < r_0 < r_0^* < 1$. Then since

$$|F(-r, r_0^*)||F(r, r_0^*)^{\sigma(r)} \le a^{1+\sigma(r)}$$

when $0 \le r \le r_0$, we see from Theorem 1 that $F(r, r_0^*) \le F(r, r_0)$ for $0 < r < 1$. Hence, $F(r, 1) = \lim_{r_0 \to 1} F(r, r_0)$ exists for each $r \in (0, 1)$. Using this fact and a normal family argument, we see that $F(z, 1) = \lim_{r_0 \to 1} F(z, r_0)$ exists and is analytic for $|z| < 1$. Moreover, $F(\cdot, 1)$ satisfies (1.3) and (1.4) when $r_0 = 1$. Also, $F(\cdot, 1)$

satisfies (i) of Theorem 1 when $r_0 = 1$. Indeed, if f satisfies (1.3) and (1.4) for $r_0 = 1$, then $M(r,f) \leq F(r, r_0)$ when $0 < r < 1$ and $0 < r_0 < 1$. Hence $M(r,f) \leq F(r, 1)$. Property (ii) of $F(\cdot, 1)$ is now an immediate consequence of (i) and the Identity Theorem for analytic functions.

To prove (viii) we first note from (2.2) that

$$F(z, 1) = \exp\left[\alpha \frac{(z-1)}{z+1}\right] B^*(z), \quad |z| < 1 . \tag{8.1}$$

Here $0 \leq \alpha < \infty$ and either $B^* = 1$ or B^* is a Blaschke product. If $\alpha > 0$, then from (8.1) we see that $\lim_{r \to 1} |F(-r, 1)| F(r, 1)^{\sigma(r)} = 0$. Using this fact, we find that

$$\sup_{0 < r < 1} \frac{|F(-r, 1)| F(r, 1)^{\sigma(r)}}{a^{1+\sigma(r)}} = 1 ,$$

since otherwise for small $\varepsilon > 0$ the function $f(z) = F((z + \varepsilon)/(1 + \varepsilon z), 1)$, $|z| < 1$, would satisfy (1.3) and (1.4) for $r_0 = 1$, and $F(r, 1) < f(r)$, $0 < r < 1$, (see 3.3)).

It also follows as in § 3 that there are at most a finite number (> 0) of points r in $[0, 1)$ where $|F(-r, 1)| F(r, 1)^{\sigma(r)} = a^{1+\sigma(r)}$. Using the argument of Heins as in § 4 we now obtain that F is a finite Blaschke product. Since $\alpha > 0$, we have reached a contradiction. Hence $\alpha = 0$.

To complete the proof of (vii) for $r_0 = 1$, we observe from (8.1) that $F(\cdot, 1)$ has an infinite number of zeros, since otherwise we would have $\lim_{r \to 1} |F(-r)| F(r)^{\sigma(r)} = 1$, in contradiction to (1.3). Hence, $F(\cdot, 1)$ has property (viii).

The proofs of (iii), (iv), and (vii) are exactly the same as in the case $r_0 < 1$. We omit the details. Here (v) and (vi) are trivially true.

This completes the proof of Theorem 1.

9. Remark

We remark that Heins [3, §§ 4—6] gave two methods for determining F when $r_0 = 1$ and $\sigma(r) = 0$, $0 < r < 1$. Here we consider the problem of determining F when $\sigma(r)$, $0 \leq r \leq 1$, is constant on $[0, 1]$. In this case we put $\sigma(r) = \sigma$, $0 \leq r \leq 1$.

First suppose that $r_0 < 1$. Let $(-t_i)_1^n$, where $0 < t_i < t_{i+1}$, denote the zeros of F and put $t_0 = 0$. We assert that

(*) If $f \in E$ and f satisfies (iii)—(viii) of Theorem 1 for $r_0 < 1$, then $f \equiv F$.

This assertion is verified by using Lemma 3 as in the proof of (7.9). We also assert that

(**) If $1 \leq i \leq n$, then $\log \{ |F(-r)| F(r)^{\sigma} \}$ is a concave function of $\log r$ on (t_{i-1}, t_i).

This assertion is verified by arguing as in the proof of (6.2).

Hence (**) is true.

Using (**) and (iv)—(vii) of Theorem 1 we deduce that

$$\text{If } 1 \le i \le n \text{ and } t_i \in [0, r_0], \text{ then } |F(-r)|F(r)^\sigma = a^{1+\sigma} \qquad (+)$$

for exactly one point in $[t_{i-1}, t_i]$.

$$\text{If } t_n \in (r_0, 1), \text{ then } t_{n-1} \in [0, r_0) \text{ and either } |F(-r)|F(r)^\sigma = a^{1+\sigma} \quad (++)$$

for exactly one point $r \in [t_{n-1}, t_n]$ or $|F(-r_0)|F(r_0)^\sigma = a^{1+\sigma}$.

We observe from (vii) that $r = 0$ is the unique point in $[0, t_1]$ for which (+) is true. From (+) and (++) we see that F is the solution of the equations

(A) $F(0) = a$,

(B) If $n \ge 2$, if $1 \le i \le n-1$, and if $t_{i+1} \in (0, t_0]$, then for some $r \in [t_i, t_{i+1}]$ we have $|F(-r)|F(r)^\sigma = a^{1+\sigma}$ and $d/dr|F(-r)|F(r)^\sigma = 0$.

(C) If $n \ge 2$ and if $t_n \in (r_0, 1)$, then either $|F(-r)|F(r)^\sigma = a^{1+\sigma}$ and $d/dr|F(-r)|F(r)^\sigma = 0$ for some r satisfying $t_{n-1} \le r \le r_0$, or $|F(-r_0)|F(r_0)^\sigma = a^{\sigma+1}$ and for some $r \in (r_0, t_n)$, $|F(-r)|F(r)^\sigma = a^{1+\sigma}$.

From (A), (B), (C), (+) and (++) we see that F is the solution of $2n-1$ equations in $2n-1$ unknowns.

Conversely, suppose f is a finite Blaschke product with simple negative zeros $(-t_i)_1^n$ and that f is a solution of (A), (B), and (C) with $f = F$. Then from (**) with $f = F$ we see that $f \in E$ and f satisfies (iii)—(viii) of Theorem 1. Hence by (*), $f = F$. We conclude that if we can find a solution f of (A), (B), and (C), then we have found F.

In practice an explicit determination of F may be quite difficult even in very simple cases. Consider for example the associated problem of determining for fixed σ, a, and k the largest possible value of r_0 for which $F(\cdot, a, \sigma, r_0)$ has k zeros. If $k = 1$, then from (A) we see that $F(z) = (z+a)/(1+az)$. Moreover, $r_0 = r_0(\sigma, 1)$ is the unique point in (0, 1) for which

$$\left(\frac{r_0 - a}{1 - ar_0}\right)\left(\frac{r_0 + a}{1 + r_0 a}\right)^\sigma = a^{1+\sigma} .$$

If $\sigma = 0$, we see that $r_0(0, 1) = 2a/(1+a^2)$. Also, if σ is replaced by -1 and 1 in the above equation, and if the resulting equations are solved, then we obtain

$$\sqrt{2a^2/(1+a^4)} < r_0(\sigma, 1) < 1 .$$

Of course an explicit determination of $r_0(\sigma, 1)$ for $-1 < \sigma < 1$ is difficult.

If $k = 2$, then

$$F(z) = \left(\frac{z + t_1}{1 + t_1 z}\right)\left(\frac{z + t_2}{1 + t_2 z}\right), \quad 0 < t_1 < t_2 < 1 ,$$

can be found by solving the equations

$$t_1 \cdot t_2 = a \tag{9.1}$$

$$-F(-r)F(r)^\sigma = a^{1+\sigma}, \ \frac{d}{dr} F(-r)F(r)^\sigma = 0 \ , \tag{9.2}$$

for some $r \in (t_1, t_2)$, $r_0(\sigma, 2)$, $0 < r_0 < 1$, is then found by solving the equation, $F(-r_0)F(r_0)^\sigma = a^{1+\sigma}$. For $\sigma = 0$ this solution is given by

$$F(z) = \frac{\left(\dfrac{z + \sqrt{\mu}}{1 + \sqrt{\mu} z}\right)^2 - a}{1 - a\left(\dfrac{z + \sqrt{\mu}}{1 + \sqrt{\mu} z}\right)^2} \ .$$

Here $\mu = 2a/(1 + a^2)$. Moreover, $r_0(0, 2) = 2\sqrt{\mu}/(1 + \mu)$.

Again an explicit determination of F and $r_0(\sigma, 2)$ is difficult when $-1 < \sigma < 1$. However it can be shown that

$$\sqrt{2\sqrt{\mu^*}/(1 + \mu^*)} < r_0(\sigma, 2) < 1 \ .$$

Here $\mu^* = 2a^2/1 + a^4$.

Finally we remark that Heins [3, (4.9)] determined F and r_0 explicitly when $\sigma = 0$ and $k = 2^n$. Here n is a positive integer.

Suppose now that σ and a are given and $r_0 = 1$. Let $(-t_i)_1^\infty$, $0 < t_i < t_{i+1} < 1$, denote the zeros of $F(\cdot, a, \sigma, 1)$. Then as in the case $r_0 < 1$, we see that $F(0) = a$ and for some $r \in [t_i, t_{i+1}]$,

$$|F(-r)|F(r)^\sigma = a^{1+\sigma} \ , \ \frac{d}{dr} F(-r)F(r)^\sigma = 0 \ . \tag{9.3}$$

Conversely, let f be an infinite Blaschke product with simple negative zeros $(-s_i)_1^\infty$, $0 < s_i < s_{i+1} < 1$. We replace F by f in (9.3). If $f(0) = a$ and (9.3) is true for some $r \in [s_i, s_{i+1}]$, $1 \leq i < \infty$, then from (**) we see that f satisfies (1.3) and (1.4). We assert that $f = F$. Indeed, if $f \neq F$, then for some ϱ, $0 < \varrho < 1$, we would have $f(\varrho) < F(\varrho)$. Let n and k be positive integers and suppose that $n > k$. Put

$$F_n(z) = \prod_1^n \left(\frac{z + t_i}{1 + t_i z}\right) \text{ and } f_k(z) = \prod_1^k \left(\frac{z + s_i}{1 + s_i z}\right) .$$

Since $F_n(z) \to F(z)$ and $f_k(z) \to f(z)$, $|z| < 1$, we see for k sufficiently large that $f_k(\varrho) < F_n(\varrho)$ whenever $n > k$. Applying Lemma 3 we obtain for some positive integer j that

$$|f_k(-r)|f_k(r)^\sigma \leq |F_n(-r)|F_n(r)^\sigma, \ s_{j-1} \leq r < s_j \ .$$

Here if $j = 1$, then $s_0 = 0$.

Letting $n \to \infty$ through a properly chosen sequence we obtain for some positive integer i that

$$|f_k(-r)|f_k(r)^\sigma \le |F(-r)|F(r)^\sigma, \quad s_{i-1} \le r < s_i .$$

However for some $r \in [s_{i-1}, s_i)$ we have

$$a^{1+\sigma} = |f(-r)|f(r)^\sigma < |f_k(-r)|f_k(r)^\sigma \le |F(-r)|F(r)^\sigma \le a^{1+\sigma} .$$

We have reached a contradiction. Hence $F = f$.

We conclude that F is uniquely determined by the equations (9.3) and the condition that $F(0) = a$.

10. Proof of Theorem 2

Let σ, r_0, and Γ_n be as in the statement of Theorem 2. We recall that Γ_n is the set of all analytic functions b_n in $\{|z| < 1\}$ which can be represented in the form (1.8).

Let

$$\mu(n) = \inf_{b_n \in \Gamma_n} \sup_{0 < r \le r_0} m(r, b_n) M(r, b_n)^\sigma \quad \text{and put} \quad a^{1+\sigma} = \mu(n) .$$

We replace n by k in (1.8). Then the class of functions which can be written in the form (1.8) where $0 \le k \le n$ is compact. Hence by a normal family argument there exists a member G of this class with k zeros $(0 \le k \le n)$ for which

$$\sup_{0 < r \le r_0} m(r, G) M(r, G)^\sigma = \mu(n) = a^{1+\sigma} .$$

Now $k = n$, since otherwise $f(z) = z^{n-k} G(z)$ is in Γ_n and

$$\sup_{0 < r \le r_0} m(r, f) M(r, f)^\sigma < \mu(n) .$$

Let G^* be the function associated with G as in (2.2). Then from (2.3) and (2.5) we see that $G^* \in \Gamma_n$ and

$$\sup_{0 < r \le r_0} |G^*(-r)| G^*(r)^\sigma = a^{1+\sigma} = \mu(n) . \tag{10.1}$$

We claim that G^* satisfies (iii)—(viii) of Theorem 1. We first show that G^* satisfies (vii).

If $G^*(0) < a$, let $\alpha = \min\{r \ge 0 : G^*(-r) = 0\}$. Let α_1 be such that $\alpha < \alpha_1 < 1$ and put

$$g(z) = \frac{\theta(z, \alpha_1) G^*(z)}{\theta(z, \alpha)} , \quad |z| < 1 .$$

Here θ is as in Lemma 1. From Lemma 1 we see that

$$\max_{\alpha_1 \le r \le r_0} |g(-r)| g(r)^\sigma < \max_{\alpha_1 \le r \le r_0} |G^*(-r)| G^*(r)^\sigma \le a^{1+\sigma} .$$

Also from (**) of § 9 we deduce $|g(-r)|g(r)^\sigma \leq g(0)^{1+\sigma}$ when $0 \leq r \leq \alpha$. Since $G^*(0) < a$, it follows for α_1 near α that

$$\max_{0 \leq r \leq r_0} |g(-r)|g(r)^\sigma < a^{1+\sigma} .$$

Since $g \in \Gamma_n$ we have reached a contradiction. Hence, $G^*(0) = a$.

Also, G^* has no zeros in $(-1, -r_0]$. Otherwise using Lemma 1, we could obtain a contradiction. Hence, (v) and (vi) are trivially true. Moreover, Lemma 1 and (**) of § 9 imply that

$$|G^*(-r_0)|G^*(r_0)^\sigma = a^{1+\sigma} . \tag{10.2}$$

Finally, we show that G^* has property (iv). Let $-\alpha$, $-\beta$, $0 < \alpha < \beta < 1$, be two zeros of G^*. Choose $\alpha_1, \beta_1, \varrho$, such that $0 < \varrho < 1$ and $0 < \alpha_1 < \alpha \leq \beta < \beta_1 < 1$. Put

$$h(z) = \frac{\phi(z, \alpha_1, \beta_1)}{\phi(z, \alpha, \beta)} G^*(z), \quad |z| < 1 .$$

Here ϕ and α are as in Lemma 2. From Lemma 2 we see that

$$|h(-r)|h(r)^\sigma < |G^*(-r)|G^*(r)^\sigma$$

when $r \in [0, \alpha_1] \cup [\beta_1, 1)$ and $G^*(-r) \neq 0$. Then if

$$\max_{\alpha \leq r \leq \beta} |G^*(-r)|G^*(r)^\sigma < a^{1+\sigma} ,$$

it follows for α_1, β_1, near α, β, that

$$\max_{0 \leq r \leq r_0} |h(-r)|h(r)^\sigma < a^{1+\sigma} .$$

Since $h \in \Gamma_n$, we have a contradiction. Hence, G^* has property (iv) of Theorem 1.

Property (iii) of G^* is an obvious consequence of properties (iv) and (vii). We conclude that G^* satisfies (iii)—(vii) of Theorem 1.

Let $F = F(\cdot, a, \sigma, r_0)$ be as in Theorem 1. Then from the discussion in § 9 we see that $F \equiv G^*$. It remains to show that for some θ, $0 \leq \theta < 2\pi$, we have $G^* = e^{i\theta}G$. To prove this we let $\sigma(r) \to 1$ in (2.5). We obtain that

$$|G^*(-r)|G^*(r) \leq m(r, G)M(r, G), \quad 0 \leq r < 1 . \tag{10.3}$$

Then if $G \neq e^{i\theta}G^*$ for some θ, $0 \leq \theta < 2\pi$, we see from (2.3) that $M(r, G) < G^*(r)$ for $0 < r < 1$. Using this inequality, (10.2), and (10.3), we obtain

$$a^{1+\sigma} = |G^*(-r_0)|G^*(r_0)^\sigma =$$

$$|G^*(-r_0)|G^*(r_0) \cdot G^*(r_0)^{\sigma-1} < m(r_0, G)M(r_0, G) \cdot M(r_0, G)^{\sigma-1} \leq a^{1+\sigma} .$$

We have reached a contradiction. Hence, for some real θ we have $G = e^{i\theta}F$. This concludes the proof of Theorem 2.

References

1. ESSÉN, M., Theorems of (λ, ϱ)-type on the growth of subharmonic functions, to appear.
2. HEINS, M., On a problem of Walsh concerning the Hadamard three circles theorem, *Trans. Amer. Math. Soc.* 55 (1944), 349—372.
3. —»— The problem of Milloux for functions analytic throughout the interior of the unit circle, *Amer. J. Math.* 67 (1945), 212—234.
4. HELLSTEN, U., KJELLBERG, B., & NORSTAD, F., Subharmonic functions in a circle. *Ark. Mat.* 8 (1969) 185—193.
5. LEWIS, J., Some theorems on the $\cos \pi\lambda$ inequality, *Trans. Amer. Math. Soc.* 167 (1972) 171—189.
6. NEVANLINNA, R., *Eindeutige analytische Funktionen*, Berlin, Springer-Verlag, 1953.

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