

Best uniform approximation by analytic functions

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Introduction

Let $F \in L^p(-\pi, \pi)$, $1 \leq p \leq \infty$, and consider the extremal problem $\inf \|\bar{F} + f\|_p$, f in H^p . For notations and basic results on H^p -spaces, see [6]. This problem was extensively treated by Rogosinski and H. S. Shapiro in [10] and also by Havinson in a series of papers. Havinson studied the corresponding problem in general domains. We refer to the survey [13] by Toumarkine and Havinson, which also contains a quite complete bibliography.

The case $p = \infty$ is of a special interest. It can also be formulated as a problem on so called Hankel matrices and is in this way of importance in probability. We wish to mention in particular the papers by Nehari [9], Hartman [4], Helson and Szegö [5] and Adamyan, Arov and Krein [1]. We shall here consider *continuous* F and look for results on the best approximation f , whereas in the above mentioned papers (except [5]) the results are expressed in terms of the matrix. It is easy to see that in this case a unique best approximation f exists in H^∞ . One might ask to what extent do F and f have the same regularity. The investigation of these problems is the main object of the present paper. We shall see that the answer is about the same as for conjugate functions. We shall also restrict ourselves to the case of the unit disc U . However, the function-theoretic proofs in sections 2, 3 and 4 are all of a local character, and so all the results can easily be carried over to any region which has in each case a sufficiently regular boundary.

In section 1 we have stated the dual problem and collected some well-known material. Theorem 1 is originally due to Bonsall [3] and Shapiro [11]. Section 2 is devoted to the study of the extremal functions of the dual problem in H^1 . In the case $F \in L^\infty$, de Leeuw and Rudin [8] have examined the question of uniqueness for the corresponding extremal function in H^1 . We give a complete solution of this problem, provided F is in C . In section 3 we treat our main problem and in section 4 we give an example which shows that the conditions in Theorem 3 a can not be weakened as long as the regularity of F is expressed only in terms of

its modulus of continuity. In particular we construct an $F \in C$ whose best approximation f is not in $A = H^\infty \cap C$. This was first done by Adamyan, Arov and Krein [1], Remark 3.2 p. 13.

1. The dual problem

We start with the following simple but useful corollary of the Hahn-Banach theorem, a technique which has now become standard.

LEMMA 1. *Let B be a Banach space, M a linear subspace and denote by M^0 the annihilator subspace of B^* corresponding to M . Then for any $L \in B^*$ we have*

$$\inf_{l \in M^0} \|L - l\| = \sup_{x \in M, \|x\| \leq 1} |L(x)|,$$

and there is always an $l \in M^0$ for which the inf is attained.

For a proof see e.g. [12], theorem 4.3-F, p. 188. If we apply this lemma to $B = L^1$, $M = H_0^1$ (the subspace of H^1 of functions vanishing at the origin), we get

$$m = \inf_{f \in H^\infty} \|\bar{F} + f\|_\infty = \sup_{h \in H_0^1, \|h\|_1 \leq 1} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{F(e^{i\theta})} h(e^{i\theta}) d\theta \right|,$$

since in this case $B^* = L^\infty$, $M^0 = H^\infty$. Thus, there is always an $f \in H^\infty$ for which $\|\bar{F} + f\|_\infty = m$. Since we avoid the trivial case $m = 0$, we may suppose $m = 1$ and the sup can of course be taken of the real part of the integral as well.

The next question is whether a maximizing $h \in H_0^1$ exists. With the aid of Fejér's theorem and the theory of normal families, it is easy to prove the following, cf [3] Theorem 4 or [8] Theorem 10 a.

THEOREM 1. *If $\bar{F} \in C + H^\infty$, in particular if $F \in C$, then at least one h in the unit ball of H_0^1 exists maximizing*

$$\operatorname{Re} \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{F(e^{i\theta})} h(e^{i\theta}) d\theta.$$

COROLLARY. *Under the above conditions on F the minimizing $f \in H^\infty$ is unique.*

This follows from Theorem 14 in [10] and also simply from the following fact. For any pair of extremals f and h , we have

$$1 = \operatorname{Re} \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{(F(e^{i\theta}) + f(e^{i\theta}))h(e^{i\theta})} d\theta \leq 1, \text{ and so}$$

$$\overline{(F(e^{i\theta}) + f(e^{i\theta}))h(e^{i\theta})} = |h(e^{i\theta})| \text{ a.e.} \tag{*}$$

From this variational equation and the fact that $F \in C$, we shall obtain all our information.

2. Properties of the extremals in the dual problem

Everything in this section will depend on the fact that if $F \in C$ and if $F(e^{i\tau}) = 0$, then by (*), fh has its values in a small sector for θ close to τ and we can make use of the following theorem.

LEMMA 2. *If v is real valued and harmonic in the simply connected domain D and $|v(z)| < \mu$, $z \in D$, then the analytic function $g = e^{u+iv}$ is in $H^p(D)$ for $p < \pi/2\mu$.*

A proof which easily carries over to any simply connected domain D may be found in [7], Theorem 1.9, p. 70.

Before announcing the theorems we want to make a definition.

Definition. Let f be analytic in a region D having a smooth boundary ∂D , D not necessarily simply connected and assume e.g. $f \in H^1(D)$. We say that f is outer in D if for $z \in D$

$$\log |f(z)| = \frac{1}{2\pi} \int_{\partial D} \frac{\partial G(z, \zeta)}{\partial n} \log |f(\zeta)| ds,$$

G being the Green's function of D .

THEOREM 2. *If $F \in C$, then we have for the corresponding $h \in H^1_0$*

- a. $h \in H^p$ for every $p < \infty$.
- b. h is outer in some annulus $R_0 = \{z | r_0 < |z| < 1\}$.

Remark. For later use we observe that the assumption on h can be localized to $h \in H^1(R)$ for some annulus $R = \{z | r < |z| < 1\}$. The conclusions are then valid in some R_0 with $r_0 > r$.

The following corollaries are consequences of the proof of Theorem 2 b rather than of the result.

COROLLARY 1. *If $f_\tau = f + \overline{F(e^{i\tau})}$, then f_τ is outer in subregions D_τ of $D^0_\tau = \{z = re^{i\theta} | r_0 < r < 1, |\theta - \tau| < \delta\}$ with smooth boundaries. We can choose δ and r_0 independently of τ .*

COROLLARY 2. If $h^{(1)}$ and $h^{(2)}$ are extremals corresponding to the same $F \in C$, then $h^{(1)}/h^{(2)}$ is a rational function.

COROLLARY 3. If $F \in C$, a necessary and sufficient condition for $h \in H^1$ satisfying (*) to be unique is that $h^{-1} \in H^1$, and this implies that $h^{-1} \in H^p$ for every $p < \infty$.

Remark 1. If we restrict h to be in H^1_0 , Corollary 3 should be formulated with h replaced by h_0 , where $h_0(z) = h(z)/z$.

Remark 2. In the case $F \in L^\infty$ de Leeuw and Rudin have proved, [8] Theorem 8, that a sufficient condition for $h \in H^1$ to be unique is that $h^{-1} \in H^\infty$. Necessary is that h is outer and that for $h_\alpha(z) = h(z)/(z - e^{i\alpha})^2$, h_α fails to be in H^1 for every real α . We shall give a proof which shows that $h^{-1} \in H^1$ is sufficient also in the general case $F \in L^\infty$.

For the proof of Theorem 2, set $F_\tau = F - F(e^{i\tau})$ and $f_\tau = f + \overline{F(e^{i\tau})}$ so that $F_\tau(e^{i\tau}) = 0$ and (*) reads

$$\overline{F_\tau(e^{i\theta})} + f_\tau(e^{i\theta})h(e^{i\theta}) = |h(e^{i\theta})| \text{ a.e.}$$

Fix an arbitrary $p < \infty$, choose $\varepsilon > 0$ so small that $\arctg \varepsilon/(1 - \varepsilon) < \pi/2p$ and let δ be such that $|F_\tau(e^{i\theta})| < \varepsilon$ for $\theta \in \gamma_\tau = \{\theta \mid |\theta - \tau| < \delta\}$. Observe that this implies $|f_\tau(e^{i\theta})| > 1 - \varepsilon$ a.e. on γ_τ . Put $f_\tau h = u + iv$. For $\theta \in \gamma_\tau$ we then have $u(e^{i\theta}) > (1 - \varepsilon)|h(e^{i\theta})|$ and $|v(e^{i\theta})| < \varepsilon|h(e^{i\theta})|$ a.e. Thus, if $(1 + \beta)\varepsilon = 1$, $1 + u(e^{i\theta}) \pm \beta v(e^{i\theta}) > 1$ a.e. on γ_τ , and since $f_\tau h \in H^1$, this implies that $1 + u(z) \pm \beta v(z) > 1$ in $D_\tau^0 = \{z \mid |\theta - \tau| < \delta_1 < \delta, r_\tau < |z| < 1\}$.

Hence, $\log(1 + f_\tau h)$ is analytic in D_τ and $|\arg(1 + f_\tau(z)h(z))| < \arctg 1/\beta < \pi/2p$, $z \in D_\tau$. By Lemma 2, $f_\tau h \in H^p(D_\tau)$, and since $|f_\tau(e^{i\theta})| > 1 - \varepsilon$ a.e. on γ_τ , $\int_{\gamma_\tau} |h(e^{i\theta})|^p d\theta < \infty$. Now F is uniformly continuous on the unit circle which can thus be covered by a finite number of γ_τ 's. Hence $\int_{-\pi}^\pi |h(e^{i\theta})|^p d\theta < \infty$.

Next, choose $p = 2$ and define φ_τ by $\varphi_\tau(\theta) = -i \arctg v(e^{i\theta})/u(e^{i\theta})$ if $\theta \in \gamma_\tau$ and $h(e^{i\theta}) \neq 0$, and $\varphi_\tau(\theta) = 0$ elsewhere. Form

$$k_\tau(z) = \exp \frac{1}{2\pi} \int_{-\pi}^\pi \frac{e^{i\theta} + z}{e^{i\theta} - z} \varphi_\tau(\theta) d\theta.$$

By Lemma 2, $k_\tau \in H^2$. Thus, for $G_\tau = f_\tau h k_\tau$ we have by Theorem 2 a that $G_\tau \in H^1$, and by the construction of G_τ , that $G_\tau(e^{i\theta}) \geq 0$ a.e. on γ_τ . This implies that G_τ can be analytically continued over γ_τ , and in particular that G_τ is outer in D_τ for any $\delta_1 < \delta$ if only r_τ is sufficiently close to 1. The factors of G_τ are thus all outer in D_τ . This proves Theorem 2 b and also Corollary 1, except that we have to prove that there is an r_0 which will do for every τ . In Theorem 2 b this is evident, since $[-\pi, \pi]$ can be covered by a finite number of intervals of length $2\delta_1 = \delta$. If, however, r_τ is sufficiently close to 1, we also find from the representation formula for

outer functions that $|f_\tau(z)| > \frac{1}{2}(1 - \varepsilon)$ in D_τ . Since $f_\sigma = f_\tau + \overline{F(e^{i\sigma})} - \overline{F(e^{i\tau})}$, f_σ is also outer in D_τ if $|\sigma - \tau| < \delta - \delta_1$. The same r_τ will therefore do for $|\sigma - \tau| < \min(\delta_1, \delta - \delta_1)$, and by the same argument as above the proof of Corollary 1 is finished.

To prove Corollary 2, suppose that $h^{(1)}$ and $h^{(2)} \in H^1$ both satisfy (*). They must then have the same argument on γ_τ . Thus $\varphi_\tau^{(1)} = \varphi_\tau^{(2)}$, $k_\tau^{(1)} = k_\tau^{(2)}$ and so the function $R = h^{(1)}/h^{(2)} = G_\tau^{(1)}/G_\tau^{(2)}$ is analytic in a neighbourhood of γ_τ except for possible poles on γ_τ and R takes positive values there. It now follows that R is a rational function.

If $F \in C$, we find almost immediately by the preceding arguments that a sufficient condition for the uniqueness of h is that $h^{-1} \in H^1$. Let us, however, for the moment suppose only that (*) is satisfied with F in L^∞ . If $h^{-1} \in H^1$, then $h = k^2$, k outer, $k^{-1} \in H^2$. Let $h_1 = g_1 k_1^2$ be any extremal function corresponding to the same F , where g_1 is an inner function and k_1 is outer with $k_1 \in H^2$. By (*), $g_1 k_1 k_1^{-1} = \overline{k_1 k_1^{-1}}$ a.e. Hence $g_1 k_1 k_1^{-1} = \text{const.}$, whence $g_1 = e^{i\alpha}$, $k_1 = e^{-i\alpha/2} k$ and $h_1 = h$.

If conversely h is known to be unique it can have no zeros in U , cf [8], p. 479. Namely, suppose $h(a) = 0$, $a \in U$, and form

$$h^{(1)}(z) = \frac{(z - b)(1 - \bar{b}z)}{(z - a)(1 - \bar{a}z)} \cdot h(z),$$

with $b \neq a$. Then $h^{(1)} \in H^1$ and $h^{(1)}$ also satisfies (*).

Now, assume again $F \in C$ and fix an arbitrary $p < \infty$. Since, by Lemma 2 again, $f_\tau k_\tau \in H^p(D_\tau)$, we will be through if we can prove that $G_\tau^{-1} \in H^\infty(D_\tau)$, and this is fulfilled if G_τ is free from zeros on γ_τ . If $G_\tau(e^{i\alpha}) = 0$, the zero is of even order and so with $G_\tau^{(1)}(z) = -e^{i\alpha z} G_\tau(z)(z - e^{i\alpha})^{-2}$, $G_\tau^{(1)}$ is positive and analytic on γ_τ . Thus $h^{(1)} = G_\tau^{(1)}/f_\tau k_\tau \in H^1(D_\tau)$ and then $h^{(1)} \in H^1$ since also $h^{(1)}(z) = -e^{i\alpha z} h(z)(z - e^{i\alpha})^{-2}$. Moreover $h^{(1)}$ satisfies (*), and we have proved that for a unique extremal h we necessarily have $h^{-1} \in H^p$ for every $p < \infty$, provided that F is continuous.

3. The regularity of the minimizing f in relation to that of the given F

Let C_ω denote the class of functions being Dini-continuous. They form a Banach algebra under the norm $\|F\|_\omega = \max |F(x)| + \int_0^1 \omega_F(t) t^{-1} dt$, ω_F standing for the modulus of continuity of F . A_α is the class of functions which satisfy a Lipschitz condition of order α , and by $F \in C^{n+\alpha}$, $n \in N$, $0 < \alpha < 1$, is meant that $F \in C^n$ with $F^{(n)}$ in A_α .

THEOREM 3. a. $F \in C_\omega \Rightarrow f \in A$ and in general $F^{(n)} \in C_\omega \Rightarrow f^{(n)} \in A$.
 b. If $0 < \alpha < 1$, then $F \in A_\alpha \Rightarrow f \in A_\alpha$ and also $F \in C^{n+\alpha} \Rightarrow f \in C^{n+\alpha}$.

Consequently F in C^∞ implies f in C^∞ , cf. [1], p. 17 and we have moreover c. There is a q independent of n such that

$$\left(\frac{\|f_\theta^{(n)}\|_\infty}{n!}\right)^{1/n} \leq q \cdot \max_{0 \leq k \leq n} \left(\frac{\|F^{(k)}\|_\infty}{k!}\right)^{1/k}.$$

If thus $F \in C(M_n)$ where the sequence $(M_n/n!)^{1/n}$ is non-decreasing, then f , regarded as a function of θ , belongs to $C(M_{n+1})$. A rough estimate by the Cauchy and Bessel inequalities also gives

$$\begin{aligned} \|f_z^{(n)}\|_\infty &\leq \sum_n^\infty n! \binom{k}{n} |c_k| \leq \left(\sum_n^\infty k^{2n+2} |c_k|^2\right)^{1/2} \left(\sum_n^\infty k^{-2}\right)^{1/2} \leq \\ &\leq A \|f_\theta^{(n+1)}\|_2 \leq A \|f_\theta^{(n+1)}\|_\infty \end{aligned}$$

whence f , regarded as a function of z , is in $C(M_{n+2})$. The following theorem has been proved by H. S. Shapiro [11] in the case $M_n = n!$

COROLLARY. *If the sequence $(M_n/n!)^{1/n}$ is non-decreasing and if $\log M_n = O(n^2)$, then F in $C(M_n)$ implies f in $C(M_n)$.*

That $\log M_n = O(n^2)$ is a sufficient condition for $C(M_n)$ and $C(M_{n+1})$ to coincide is proved in [2], Theorem VII.

The proof is based on the result in Corollary 1 in the preceding section. Now if f is outer in D_τ and if $g(z) = f(e^{iz})$, then g is outer in a corresponding region D'_τ in the upper half-plane. Accordingly, it is sufficient to prove the theorem for the upper half-plane under the conditions that $|\overline{F(x)} + f(x)| = 1$ a.e. and that $F(\tau) = 0$ implies f outer in a region D_τ whose boundary contains the interval $(\tau - \delta, \tau + \delta)$, F and f being periodic with period 2π . By doing so we reach a typographic simplification but, above all, we do not have to distinguish between derivatives with respect to arc length and derivatives with respect to z , when studying the behaviour of f on the boundary. Before proving the theorem we shall state a simple lemma on Cauchy integrals.

LEMMA 3. *If $|k(\pm t)| \leq \omega(t)$ a.e., where $\int_0^\delta \frac{\omega(t)}{t} dt < \infty$ and if $\alpha(z) = \int_{-\delta}^\delta \frac{k(t)}{t-z} dt$, then for $z \in S_0(\delta) = \{z = x + iy \mid |x| < \delta, |z| < \delta\}$,*

$$|\alpha(z) - \alpha(0)| \leq A \int_0^{|z|} \frac{\omega(t)}{t} dt + A|z| \int_{|z|}^\delta \frac{\omega(t)}{t^2} dt.$$

If instead $|k(\pm t)| \leq |t|^n \omega(t)$, then

$$|\kappa^{(n)}(z) - \kappa^{(n)}(0)| \leq A^n n! \int_0^{|z|} \frac{\omega(t)}{t} dt + A^n n! |z| \int_{|z|}^\delta \frac{\omega(t)}{t} dt \text{ for } z \in S_0(\delta).$$

We only have to estimate the integrals

$$\int_0^{|z|} \left(\frac{1}{|t-z|} + \frac{1}{t} \right) \omega(t) dt \text{ and } \int_{|z|}^\delta \left| \frac{1}{t-z} - \frac{1}{t} \right| \omega(t) dt$$

using that $|t - z| > A|t|$ if $z \in S_0(\delta)$.

If $\omega \neq 0$ is a modulus of continuity, then for every $\varepsilon > 0$ there is an η such that $|\kappa(z) - \kappa(0)| < A\omega(\varepsilon)$ if $z \in S_0(\eta)$, for

$$|z| \int_{|z|}^\varepsilon \frac{\omega(t)}{t^2} dt < \omega(\varepsilon) \text{ while } \int_0^{|z|} \frac{\omega(t)}{t} dt \text{ and } |z| \int_\varepsilon^\delta \frac{\omega(t)}{t^2} dt$$

both tend to zero with $|z|$. If $\omega(t) = t^\alpha$, $0 < \alpha < 1$, we find that $|\kappa(z) - \kappa(0)| < A(\alpha)|z|^\alpha$, $z \in S_0(\delta)$. Also let us observe that the results hold as well for

$$\kappa(z) = \int_{-\delta}^\delta \frac{1 + tz}{t - z} \cdot \frac{k(t)}{1 + t^2} dt$$

since this integral differs from the Cauchy integral only by a constant.

Fix an arbitrary $\tau \in R$ and form F_τ and f_τ as in the proof of Theorem 2. We know that f_τ is bounded away from zero in a region D_τ of length 2δ . Let $L_\tau = \log f_\tau$ and let $z \in D_\tau$, then

$$L_\tau(z) = \frac{1}{i\pi} \int_{-\delta}^\delta \frac{1 + tz}{t - z} \frac{\log |f_\tau(t)|}{1 + t^2} dt + \lambda_\tau(z) = \kappa_\tau(z) + \lambda_\tau(z),$$

where λ_τ can be analytically continued over $(\tau - \delta, \tau + \delta)$. Now $1 - |f_\tau(x)|^2 = |F_\tau(x)|^2 + 2 \operatorname{Re} F_\tau(x)f_\tau(x)$ a.e. and so $|1 - |f_\tau(x)|^2| \leq A\omega_F(|x - \tau|)$ a.e. Thus also $|\log |f_\tau(x)|| \leq A\omega(|x - \tau|)$ a.e. in $(\tau - \delta, \tau + \delta)$. By Lemma 3, it is then possible for any $\varepsilon > 0$ to find an η such that $|\kappa_\tau(z) - \kappa_\tau(\tau)| < \varepsilon$ for z in a sector $S_\tau(\eta)$ with radius η . The arguments after Lemma 3 also show that η can be chosen so as to be independent of τ . Since λ_τ is analytic over $(\tau - \delta, \tau + \delta)$, $|\lambda_\tau(z) - \lambda_\tau(\tau)| \leq A|z - \tau|$, and since the bound of λ_τ in D_τ and the length and height of D_τ are all independent of τ , so is A . Thus $|L_\tau(z) - L_\tau(\tau)| < 2\varepsilon$ for $z \in S_\tau(\eta)$, i.e. the same η will do for every τ . Accordingly

$$|\exp(L_\tau(z) - L_\tau(\tau)) - 1| < 3\varepsilon, \quad |f_\tau(z) - f_\tau(\tau)| < 3\varepsilon \text{ and } |f(z) - f(\tau)| < 3\varepsilon$$

for z in $S_r(\eta)$. If $|\sigma - \tau| < \eta$ we can then find a $z_0 \in S_r(\eta) \cap S_\sigma(\eta)$ such that $|f(z_0) - f(\tau)| < 3\varepsilon$ and $|f(z_0) - f(\sigma)| < 3\varepsilon$, and we have $|f(\sigma) - f(\tau)| < 6\varepsilon$ for $|\sigma - \tau| < \eta$.

The proof of the first part of Theorem 3 b can be carried out on exactly the same lines. Lemma 3 will now give $|\kappa_r(z) - \kappa_r(\tau)| < A|z - \tau|^\alpha$, $z \in S_r(\delta)$. Thus $|f(z) - f(\tau)| < A|z - \tau|^\alpha$ for z in $S_r(\eta)$ and A and η are independent of τ . As $|z_0 - \tau|^\alpha + |z_0 - \sigma|^\alpha < 2|\sigma - \tau|^\alpha$ for a suitable $z_0 \in S_r(\eta) \cap S_\sigma(\eta)$ we have $|f(\sigma) - f(\tau)| < A|\sigma - \tau|^\alpha$ for $|\sigma - \tau| < \eta$, and this inequality then holds generally with an A independent of σ and τ .

Also the proofs of the second parts of Theorems 3 a and 3 b are almost identical. We prefer to carry out the second of them which is somewhat more involved since we have to check that all the constants occurring are independent of τ .

Now form

$$F_r(x) = F(x) - \sum_0^n \frac{F^{(k)}(\tau)}{k!} (x - \tau)^k \quad \text{and} \quad f_r(z) = f(z) + \sum_0^n \frac{\overline{F^{(k)}(\tau)}}{k!} (z - \tau)^k.$$

As long as we restrict x and τ to some compact set we have $|1 - |f_r(x)|^2| \leq A|x - \tau|^{n+\alpha}$ with A independent of τ . For L_r thus

$$L_r^{(n)}(z) = \frac{n!}{i\pi} \int_{-\delta}^\delta \frac{\log |f_r(t)|}{(t - z)^{n+1}} dt + \lambda_r^{(n)}(z),$$

and $|\log |f_r(t)|| \leq A|t - \tau|^{n+\alpha}$ if $|t - \tau| < \delta$. By Lemma 3 we get

$$|\kappa_r^{(n)}(z) - \kappa_r^{(n)}(\tau)| < A|z - \tau|^\alpha \quad \text{for } z \in S_r(\delta),$$

and by the same argument as above it follows that $|L_r^{(n)}(z) - L_r^{(n)}(\tau)| < A|z - \tau|^\alpha$ for z in $S_r(\eta)$, with A and η independent of τ . Assume that we have proved the theorem up to and including $n - 1$. Then also $|f_r^{(k)}(z) - f_r^{(k)}(\tau)| < A|z - \tau|^\alpha$, $0 \leq k \leq n - 1$, with A independent of τ as long as $|z|$ and τ belong to some compact set. Form $\psi_r = L_r^{(n)} - f_r^{(n)}/f_r$. Since $\psi_r = R(f_r, \dots, f_r^{(n-1)})$, where $R(x_0, \dots, x_{n-1})$ is differentiable with bounded derivatives if $|x_0| \geq m > 0$ and $\sum_1^{n-1} |x_k|^2 \leq M$, it follows that $|\psi_r(z) - \psi_r(\tau)| < A|z - \tau|^\alpha$, with A independent of τ if $z \in S_r(\eta)$ and $\tau \in [-\pi, \pi]$. Thus also

$$\left| \frac{f_r^{(n)}(z)}{f_r(z)} - \frac{f_r^{(n)}(\tau)}{f_r(\tau)} \right| < A|z - \tau|^\alpha, \quad z \in S_r(\eta).$$

Further, $\psi_r(\tau)$ and $L_r^{(n)}(\tau)$ are both bounded for $\tau \in [-\pi, \pi]$, and then so is $f_r^{(n)}(\tau)$. Hence $|f_r^{(n)}(z) - f_r^{(n)}(\tau)| < A|z - \tau|^\alpha$ and $|f^{(n)}(z) - f^{(n)}(\tau)| < A|z - \tau|^\alpha$ for $z \in S_r(\eta)$ with A and η still independent of τ . (They of course depend on n .) An application of this result in two sectors S_r and S_σ gives $|f^{(n)}(\sigma) - f^{(n)}(\tau)| < A|\sigma - \tau|^\alpha$ for $|\sigma - \tau| < \eta$.

In the proof of Theorem 3 c we suppose that $\overline{\lim}_{n \rightarrow \infty} (\|F^{(n)}\|_{\omega}/n!)^{1/n} = \infty$. The case when F is analytic is more easily treated with the aid of the principle of reflection as $|\overline{F(x)} + f(x)| = 1$, and we omit it. Let $\delta^{-1} = \max_{0 \leq k \leq n} (\|F^{(k)}\|_{\omega}/k!)^{1/k}$, where n is to be taken so large that δ is sufficiently small. It will mean no restriction if we choose $\tau = 0$. The construction of F_0 and Taylor's theorem yield

$$F_0(x) = \frac{1}{(n-1)!} \int_0^x (F^{(n)}(t) - F^{(n)}(0))(-t)^n dt.$$

With $\omega_n = \omega_{F^{(n)}}$, we have $|F_0(x)| \leq |x|^n \omega_n(|x|)/n!$. This implies that $|F_0(x)| \leq \frac{1}{2}$ if $|x| \leq \delta$. For $\omega_n(|x|) \leq \omega_n(\delta) \leq \frac{1}{2} \int_{\delta}^{\delta^2} \omega_n(t)t^{-1} dt \leq \frac{1}{2} \|F^{(n)}\|_{\omega} \leq \frac{1}{2} n! \delta^{-n}$ if $|x| \leq \delta$. Hence $\frac{1}{2} \leq |f_0(x)| \leq \frac{3}{2}$ and so $|1 - |f_0(x)|^2| \leq A|x|^n \omega_n(|x|)/n!$ and $|\log |f_0(x)|| \leq A|x|^n \omega_n(|x|)/n!$ if $|x| \leq \delta$ with A independent of n . For

$$\kappa_0(z) = \frac{1}{i\pi} \int_{-\delta}^{\delta} \frac{1+tz}{t-z} \frac{\log |f_0(t)|}{1+t^2} dt$$

thus

$$|\kappa_0^{(j)}(0)| \leq \frac{Aj!}{n!} \int_0^{\delta} t^{n-j} \frac{\omega_n(t)}{t} dt \leq Aj! \delta^{-j} \cdot \frac{\delta^n}{n!} \|F^{(n)}\|_{\omega} \leq Aj! \delta^{-j},$$

if $0 \leq j \leq n$. By induction it is now easily proved that $|(\kappa_0^p)^{(j)}(0)| \leq A^p(j+1)^{p-1} j! \delta^{-j}$ for every $p \in N$ and $0 \leq j \leq n$. With $k_0 = e^{\kappa_0}$ this gives

$$|k_0^{(j)}(0)| \leq \frac{j!}{j+1} e^{A(j+1)} \cdot \delta^{-j} \leq j! q^j \delta^{-j}, \quad 0 \leq j \leq n.$$

The maximum principle shows that $|\operatorname{Re} \kappa_0(z)| \leq A$. If for some $a < \delta$ we can prove that $|\operatorname{Re} L_0(z)| \leq A$ for $|z| \leq a$, $\operatorname{Im} z \geq 0$, we find that $\lambda_0 = L_0 - \kappa_0$ is analytic in the disc $|z| \leq a$ with $|\operatorname{Re} \lambda_0(z)| \leq A$ there. Let $l_0 = e^{\lambda_0}$. The Cauchy estimate then gives $|l_0^{(j)}(0)| \leq Aj! a^{-j}$, and for $f_0 = k_0 l_0$ we get $|f_0^{(n)}(0)| \leq n! q^n a^{-n}$. We will find that $a = \delta/3$ will do. For

$$\sum_1^n \frac{|F^{(k)}(0)|}{k!} a^k \leq \sum_1^n \frac{\|F^{(k)}\|_{\omega}}{k!} a^k \leq \sum_1^n (a\delta^{-1})^k < \frac{1}{2}$$

if $a = \delta/3$. Thus, if n is sufficiently large, Theorem 3 a tells us that $\frac{1}{4} \leq |f_0(z)| \leq \frac{7}{4}$ if $|z| \leq \delta/3$, $\operatorname{Im} z \geq 0$, and so $|\operatorname{Re} L_0(z)| \leq A$ in this region. As $f_0^{(n)}(0) = f^{(n)}(0) + \overline{F^{(n)}(0)}$, we also have $|f^{(n)}(0)| \leq n! q^n a^{-n}$ with

$$a^{-1} = 3 \max_{0 \leq k \leq n} (\|F^{(k)}\|_{\omega}/k!)^{1/k}.$$

This proves Theorem 3 c, since all constants are easily seen to be independent of the choice of τ .

4. The weakest condition on ω_F which guarantees f to be in A

THEOREM 4. *For every continuous, non-decreasing and subadditive function ω with $\omega(0) = 0$ and $\int_0^1 \omega(t)t^{-1}dt$ divergent, there is an $F \in C$ with $\omega_F(t) \leq \omega(t)$ for which the best approximation $f \in H^\infty$ is not in A .*

For the proof we require the following lemma.

LEMMA 4. *Let ω be as in the theorem and put $F(e^{it}) = \omega(t - \tau)$ in $[\tau, \tau + \delta]$, $F(e^{it}) = 0$ in $[\tau - \delta, \tau)$ and continuous elsewhere. If the corresponding $f \in H^\infty$ is in A , then $f(e^{it}) = \pm i$. We assume F to be defined in such a way that $\|\bar{F} + f\|_\infty = 1$.*

We suppose $f \in A$ with $\operatorname{Re} f(e^{it}) \neq 0$. Let $u(\theta) = \log |f(e^{i\theta})|$. Since $F(e^{it}) = 0$ and $|\bar{F}(e^{i\theta}) + f(e^{i\theta})| = 1$, $u(\tau + t) < -A\omega(t)$ or $u(\tau + t) > A\omega(t)$ for $t \in (0, \delta_1)$ and some $A > 0$, depending on whether $\operatorname{Re} f(e^{it})$ is positive or negative. For $t \in (-\delta_1, 0)$, on the other hand, $u(\tau + t)$ equals zero. Thus

$$\lim_{\varepsilon \rightarrow 0^+} \left| \int_{\varepsilon}^1 \frac{u(\tau + t) - u(\tau - t)}{t} dt \right| = \infty,$$

whence, see e.g. [14], Theorem 7.20, p. 103, $\lim_{r \rightarrow 1^-} |\arg f(re^{it})| = \infty$. This contradicts f in A with $|f(e^{it})| = 1$, and the lemma is proved.

Now let $\delta_n = \pi \cdot 2^{-n}$, $n = 0, 1, 2, \dots$, and put $\omega_n(\theta) = \omega(\theta - \delta_n)$ for θ in $\left[\delta_n, \frac{\delta_n + \delta_{n-1}}{2}\right]$, $\omega_n(\theta) = \omega(\delta_{n-1} - \theta)$ for θ in $\left[\frac{\delta_n + \delta_{n-1}}{2}, \delta_{n-1}\right]$ and $\omega_n(\theta) = 0$ elsewhere. Let $c_n = 0, 2^{-\frac{1}{2}}$ or $i \cdot 2^{-\frac{1}{2}}$ depending on whether n is even, $n \equiv 1, \pmod{4}$ or $n \equiv 3, \pmod{4}$ respectively. Put $F(e^{i\theta}) = \sum_1^\infty c_n \omega_n(\theta)$. It is easily seen that $F \in C$ with $\omega_F(t) \leq \omega(t)$ if $t \leq \pi/4$. Assume the corresponding f to be in A . Lemma 4 then tells us that $f(\varepsilon_{4n+1}) = \pm m \cdot i$ and $f(\varepsilon_{4n+3}) = \pm m$, where $\varepsilon_n = e^{i\delta_n}$ and $m = \|\bar{F} + f\|_\infty > 0$, as $F(e^{i\theta}) = 0$ in $(-\pi, 0)$. Since $\lim_{n \rightarrow \infty} \delta_n = 0$, this is, however, inconsistent with f in A .

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