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AN INADMISSIBLE BEST INVARIANT ESTIMATOR: THE I.I.D. CASE

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An example is presented of an inadmissible best invariant point estimator of a location parameter. The extra moment condition of Brown and Fox (1974b) is violated. The example involves two independent, identically distributed observations, an improvement over Perng's (1970) example.

1. Introduction. Let $Z_1, \cdots, Z_n$ be i.i.d. random variables with density $f(z - \theta)$. We call this the *i.i.d. model*. Set $X = \bar{Z}$ (or Z_n, etc.) and $Y = (Z_1 - \bar{Z}, \cdots, Z_{n-1} - \bar{Z})$ (or $(Z_1 - Z_n, \cdots, Z_{n-1} - Z_n)$, etc.), a maximal invariant. Motivated by this substitution, consider the *abstract model* in which Y takes values in an arbitrary space, distributed according to the probability measure ν, and X is a random variable with conditional density $g(x - \theta, y)$ given $Y = y$.

Assume a location invariant statistical problem. It is a quite general phenomenon that the best invariant procedure is admissible if X has finite moments of order up to one larger than sufficient for finite risk. The precise conditions in general and references to earlier special cases will be found in Brown and Fox (1974b). If X and θ are two-dimensional vectors, the moment condition becomes, roughly, finiteness of moments of order up to two larger than sufficient for finite risk; see Brown and Fox (1974a). In still higher dimensions, for point estimation of θ it is a general phenomenon that the best invariant estimate is inadmissible; see Brown (1966). Portnoy and Stein (1971) have given an example in which this phenomenon holds for testing between two higher dimensional location families.

Counter examples exist when the extra moment conditions are violated. For X and θ one-dimensional such examples were provided by Perng (1970) for point and for confidence interval estimation of θ and by Fox and Perng (1969) for testing between two location families. In the two-dimensional case, similar examples for point estimation of θ and for testing between two location families are found in Brown and Fox (1974a).

All the counter examples fit the abstract model, but not the i.i.d. model. The one-dimensional example in Section 2 fits the i.i.d. model with $n = 2$, has $E|Z_i|^\alpha < \infty$ if, and only if, $\alpha < 3 - \eta$ with $\eta > 0$ arbitrarily small and has inadmissible best invariant estimator under squared error loss.

2. The example. Let Z_1, Z_2 be i.i.d. random variables with density $f(z - \theta)$ where $f(t) = c|t|^{-(4-\eta)}$ if $|t| > 1$ and 0 otherwise. Then, $E|Z_1|^\alpha < \infty$ if, and only if, $\alpha < 3 - \eta$. We wish to estimate θ with loss function $L(\theta, a) = (\theta - a)^2$ when a is the estimate. Let R_0 be the constant risk of $\bar{Z}$, the best invariant estimator.

The alternative estimator is

$$\begin{aligned} \varphi(Z_1, Z_2) &= (1 - \epsilon)\bar{Z} \quad \text{if} \quad Z_1 Z_2 < 0 \\ &= \bar{Z} \quad \text{if} \quad Z_1 Z_2 \geq 0, \end{aligned}$$

where $\epsilon > 0$. This estimator is suggested by the fact that $|\theta|$ small is indicated by $Z_1 Z_2 < 0$. Let $R(\theta)$ be the risk function of φ. We will see that $R(\theta) < R_0$ for all $\theta \geq 0$ when $\eta > 0$ is small and $\epsilon > 0$ is small relative to η. By symmetry, this suffices.

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From the symmetries of f and φ, we obtain

$$\begin{aligned}
R_0 - R(\theta) &= \tfrac{1}{2}\int_0^\infty f(z_1-\theta)\,dz_1 \int_{-\infty}^0 f(z_2-\theta)[(z_1+z_2-2\theta)^2 \\
&\qquad - \{(1-\epsilon)(z_1+z_2)-2\theta\}^2]\,dz_2 \\
&= \tfrac{1}{2}\int_{-\theta}^\infty f(x_1)\,dx_1 \int_{-\infty}^{-\theta} f(x_2)[(x_1+x_2)^2 \\
&\qquad - \{(1-\epsilon)(x_1+x_2)-2\epsilon\theta\}^2]\,dx_2
\end{aligned} \tag{1}$$

Let $\Delta_\eta(\theta) = 2\{R_0 - R(\theta)\}/c^2$.

Case 1. $0 \le \theta \le 1$. We will see that $\Delta_0(\theta) > 0$ when $\epsilon > 0$ is sufficiently small. By continuity of $\Delta_\eta(\theta)$ in η, the desired result follows.

From (1) we obtain

$$\begin{aligned}
\Delta_0(\theta) &= \int_1^\infty x_1^{-4}\,dx_1 \int_{-\infty}^{-1} x_2^{-4}[(x_1+x_2)^2 - \{(1-\epsilon)(x_1+x_2)-2\epsilon\theta\}^2]\,dx_2 \\
&= \int_1^\infty t_1^{-4}\,dt_1 \int_1^\infty t_2^{-4}[(t_1-t_2)^2 - \{(1-\epsilon)(t_1-t_2)-2\epsilon\theta\}^2]\,dt_2 \\
&= \frac{\epsilon}{18}\{3(2-\epsilon) - 8\epsilon\theta^2\}
\end{aligned}$$

which is strictly positive if $\epsilon > 0$ is sufficiently small.

Case 2. $\theta > 1$. In this case (1) yields

$$\begin{aligned}
\Delta_\eta(\theta) &= \left\{\int_{-\theta}^{-1} |x_1|^{-(4-\eta)}\,dx_1 + \int_1^\infty |x_1|^{-(4-\eta)}\,dx_1\right\} \\
&\quad \cdot \int_{-\infty}^{-\theta} |x_2|^{-(4-\eta)}[(x_1+x_2)^2 - \{(1-\epsilon)(x_1+x_2)-2\epsilon\theta\}^2]\,dx_2 \\
&= \int_1^\theta t_1^{-(4-\eta)}\,dt_1 \int_\theta^\infty t_2^{-(4-\eta)}[(t_1+t_2)^2 - \{(1-\epsilon)(t_1+t_2)+2\epsilon\theta\}^2]\,dt_2 \\
&\quad + \int_1^\infty t_1^{-(4-\eta)}\,dt_1 \int_\theta^\infty t_2^{-(4-\eta)}[(t_1-t_2)^2 - \{(1-\epsilon)(t_1-t_2)-2\epsilon\theta\}^2]\,dt_2 \\
&> \frac{2\epsilon}{(1-\eta)(2-\eta)(3-\eta)}\left\{2\eta - \frac{\epsilon(2-\eta+\eta^2)}{3-\eta}\right\}\theta^{-(1-\eta)} + \frac{2\epsilon(2-\epsilon)}{(1-\eta)(3-\eta)}\theta^{-(3-\eta)} \\
&\quad - \frac{2\epsilon}{(1-\eta)(2-\eta)^2(3-\eta)}\left\{2(3-2\eta) - \frac{\epsilon(5-3\eta)}{3-\eta}\right\}\theta^{-(4-2\eta)}
\end{aligned}$$

which is strictly positive for $\eta > 0$ sufficiently small and $\epsilon > 0$ sufficiently small relative to η.

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