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THE PERFORMANCE OF A SEQUENTIAL PROCEDURE FOR THE ESTIMATION OF THE MEAN

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Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with mean μ and unknown variance σ^2. We want to estimate μ by $\bar{X}_n$ with a loss function of $\sigma^{2\delta-2}(\bar{X}_n - \mu)^2 + \lambda n$, where $\delta > 0$ and $\lambda \to 0+$. For $n_\lambda = o(\lambda^{-1/2})$ and $n_\lambda(\log \lambda)^{-1} \to -\infty$ as $\lambda \to 0+$, set $T = \inf\{n \geq n_\lambda : n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n) + b_n \leq \lambda^{1/\delta} a_n\}$. If $a_n n^{-2/\delta} \to 1$ and $0 < b_n \to 0$ as $n \to \infty$, we prove that T is asymptotically risk efficient, that is, as $\lambda \to 0+$, $E[(2\lambda^{1/2}\sigma^\delta)^{-1}(\sigma^{2\delta-2}(\bar{X}_T - \mu)^2 + \lambda T)] \to 1$. When the X_n's are normal, the asymptotic risk efficiency of T was established by Starr. By introducing the delay factor n_λ, we are able to drop the condition of X_n's being normal.

1. Introduction. Let $X_1, X_2, \cdots$ be independent observations from some population with mean μ and variance σ^2. With a sample $(X_1, \cdots, X_n)$ of size n, we want to estimate the unknown mean μ by $\bar{X}_n$ with a loss structure

$$L_n = \sigma^{2\delta-2}(\bar{X}_n - \mu)^2 + \lambda n,$$

where $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$, $\delta > 0$ and $\lambda > 0$. The risk

$$R_n = EL_n = \frac{\sigma^{2\delta}}{n} + \lambda n$$

is minimized by taking a sample size n_0, where

$$[\lambda^{-1/2}\sigma^\delta] \leq n_0 \leq [\lambda^{-1/2}\sigma^\delta] + 1, \; R_{n_0} = 2\lambda^{1/2}\sigma^\delta + 0(\lambda) \qquad \text{as } \lambda \to 0+.$$

But if σ is unknown, there is no fixed sample size procedure that will attain the minimum risk. Robbins ([10], 1959, for $\delta = 1$) proposed to replace σ^2 by its estimator

$$V_n = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2,$$

and to determine the sample size by

$$T' = \inf\{n \geq m : n \geq \lambda^{-1/2} V_n^{\delta/2}\}$$
$$= \inf\{n \geq m : V_n \leq \lambda^{1/\delta} n^{2/\delta}\},$$

where $m \geq 2$ and to estimate μ by $\bar{X}_{T'}$. Let $R_{T'} = EL_{T'}$. The performance of T' is usually measured by

(i) the risk efficiency: $R_{n_0}/R_{T'}$, and
(ii) the regret: $R_{T'} - R_{n_0}$.

Let the observations come from a normal population and $\delta = 1$. Using the loss structure

$$L_n^* = \lambda^{-1}\big((\bar{X}_n - \mu)^2 + \lambda n\big).$$

Robbins [10] obtained some numerical and Monte Carlo results which suggested the boundedness of the regret, and Starr [11] established that, with $R_{n_0}^* = EL_{n_0}^*$ and $R_{T'}^* = EL_{T'}^*$, $R_{n_0}/R_{T'} = R_{n_0}^*/R_{T'}^* \to 1$ as $\lambda \to 0$ (asymptotically risk efficient), if and only if $m \geq 3$. As $\lambda \to 0$, Starr

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and Woodroofe [12] found that the regret is bounded if and only if $m \geq 3$. Woodroofe [13] proved that if $m \geq 4$, then

$$ET' = \lambda^{-1/2}\sigma + \tfrac{1}{2}\sigma^{-2}\nu - \tfrac{3}{4} + o(1),$$

$$R^*_{T'} = 2\lambda^{-1/2}\sigma + \tfrac{1}{2} + o(1),$$

where ν is a constant which can be computed. All these strongly indicate the good performance of the sequential procedure for the normal case. But is this procedure good in general? Let $P(X_1 = 1) = \mu = 1 - P(X_1 = 0)$, $0 < \mu < 1$, and $\delta = 1$. Then for $m \geq 2$

$$R_{T'} = E\big((\bar{X}_{T'} - \mu)^2 + \lambda T'\big)$$

$$\geq \int_{\{X_1=1,\cdots,X_m=1\}} (\bar{X}_m - \mu)^2 \, dP$$

$$= (1-\mu)^2\mu^m > 0,$$

and $R_{n_0} \cong 2(\lambda\mu(1-\mu))^{1/2} \to 0$ as $\lambda \to 0$. Hence $\lim_{\lambda\to 0} R_{n_0}/R_{T'} = 0$ and T' is not asymptotically risk efficient.

To remedy the situation, Chow and Robbins [6] proposed the stopping rule (with $\delta = \beta = 1$)

$$T = \inf\{n \geq 2 : V_n + n^{-\beta} \leq \lambda^{1/\delta} n^{2/\delta}\}, \qquad \beta > 0,$$

by introducing the term $n^{-\beta}$. How good is the performance of the sequential procedure T? In this note, we shall prove the following theorem which shows that, no matter what the population is, T is asymptotically risk efficient as $\lambda \to 0$.

THEOREM. *Let $X, X_1, X_2, \cdots$ be independent identically distributed random variables with $EX = \mu$ and* Var $X = \sigma^2 \in (0, \infty)$. *For $\delta > 0$, let a_n and b_n be sequences of constants such that $a_n n^{-2/\delta} \to 1$ and $0 < b_n = o(1)$ as $n \to \infty$. For $\lambda > 0$ and $n_\lambda \geq 1$, define*

$$T \equiv T_\lambda = \inf\{n \geq n_\lambda : n^{-1} \textstyle\sum_{i=1}^n (X_i - \bar{X}_n)^2 + b_n \leq \lambda^{1/\delta} a_n\}.$$

Then,

(i) *If $n_\lambda = o(\lambda^{-1/2})$ as $\lambda \to 0$, then* $\lim_{\lambda\to 0}\lambda^{1/2}T = \sigma^\delta$ a.s., *and* $\lim_{\lambda\to 0} E(\lambda^{1/2}T) = \sigma^\delta$.
(ii) *If $E|X|^{2p} < \infty$ for some $p > 1$ and $-K \log \lambda \leq n_\lambda = o(\lambda^{-1/2})$ for some $K > K_{\alpha,p}$, then as $\lambda \to 0$,*

$$\frac{R_T}{R_{n_0}} = E\left(\frac{\sigma^{2\delta-2}(\bar{X}_T - \mu)^2 + \lambda T}{2\lambda^{1/2}\sigma^\delta}\right) \to 1.$$

It should be pointed out that a key tool in proving this theorem under the mere assumption of a little bit more than the second moment on X is Lemma 5, whose proof will be given in detail in the next section.

2. Proof of the theorem.

LEMMA 1. *Let $\{T \equiv T_\lambda, 1 \geq \lambda > 0\}$ be a family of random variables such that $P(T \geq 1) = 1$. If for some $0 < \gamma < 1, p > 0, \beta > 0$*

$$(1) \qquad P(\lambda^\beta T < \gamma) = o(\lambda^{\beta p}) \quad \text{as} \quad \lambda \to 0,$$

then $\{(\lambda^\beta T)^{-p}, 1 \geq \lambda > 0\}$ is uniformly integrable.

PROOF. Since $(\lambda^\beta T)^{-p} I_{[\lambda^\beta T \geq \gamma]} \leq \gamma^{-p}$ and by (1), $E(\lambda^\beta T)^{-p} I_{[\lambda^\beta T < \gamma]} \leq \lambda^{-\beta p} P(\lambda^\beta T < \gamma) = o(1)$, as $\lambda \to 0$, $\{(\lambda^\beta T)^{-p}, 1 \geq \lambda > 0\}$ is uniformly integrable.

LEMMA 2. *Let $X, X_1, X_2, \cdots$ be independent identically distributed random variables with*

$EX = \mu \in (0, \infty)$ and $E(X^+)^p < \infty$ for some $p \geq 1$. Assume $\{a_n\}$ and $\{b_n\}$ are constants such that as $n \to \infty$, $b_n = o(1)$ and $n^{-\alpha}a_n \to 1$ for some $\alpha > 0$. Assume n_λ to be a positive integer-valued function of λ, α and p, with $n_\lambda = 0(\lambda^{-1/\alpha})$ as $\lambda \to 0$. For $\lambda > 0$, put

$$T \equiv T_\lambda = \inf\{n \geq n_\lambda : n^{-1}S_n + b_n \leq \lambda a_n\},$$

where $S_n = \sum_{j=1}^n X_j$. Then $\{(\lambda^{1/\alpha}T)^p,\ 1 \geq \lambda > 0\}$ is uniformly integrable.

PROOF. We can assume $\mu = 1$. Define $\tau = \inf\{n \geq 1 : S_n < 2n\}$ and let $\tau_1, \tau_2, \cdots$ be the copies of τ, and $t_n = \tau_1 + \cdots + \tau_n$. Then $S_{t_n} < 2t_n$ and since $E(X^+)^p < \infty$, $E\tau^p < \infty$. (See [8]). By Doob's martingale theorem $\{(n^{-1}t_n)^p,\ n \geq 1\}$ is uniformly integrable. Choose a positive integer n_1 such that $a_n > \frac{1}{2} n^\alpha$ if $n \geq n_1$, and choose M such that $M \geq 2(2 + c)$ and $M \geq \lambda n_\lambda^\alpha$, where $c \geq \sup_{n\geq 1} |b_n|$. Put

$$q = [(\lambda^{-1}M)^{1/\alpha}] + 1. \tag{2}$$

For $\lambda > 0$, and $q \geq n_1$, $t_q \geq q \geq n_\lambda$ and

$$t_q^{-1}S_{t_q} + b_{t_q} \leq 2 + c \leq \frac{2(2+c)}{\lambda}\frac{\lambda}{2} \leq \frac{M}{\lambda}\frac{\lambda}{2} \leq q^\alpha \frac{\lambda}{2} \leq t_q^\alpha \frac{\lambda}{2} \leq \lambda a_{t_q}.$$

Hence $T \leq t_q$ a.s. By (2) the uniform integrability of $\{(n^{-1}t_n)^p,\ n \geq 1\}$ implies that of $\{(\lambda^{1/\alpha}t_q)^p,\ 1 \geq \lambda > 0\}$, and it follows that $\{(\lambda^{1/\alpha}T)^p,\ 1 \geq \lambda > 0\}$ is uniformly integrable.

LEMMA 3. *Let $X, X_1, X_2, \cdots$ be independent identically distributed random variables which are bounded from below with $EX = \mu \in (0, \infty)$. Let $Y, Y_1, Y_2, \cdots$ be independent identically distributed bounded random variables with $EY = 0$. Assume $\{a_n\}$ and $\{b_n\}$ are constants such that as $n \to \infty$, $b_n = o(1)$ and $n^{-\alpha}a_n \to 1$ for some $\alpha > 0$. For $\lambda > 0$, let*

$$T \equiv T_\lambda = \inf\{n \geq n_\lambda :\ n^{-1}\textstyle\sum_{j=1}^n X_j - \bar{Y}_n^2 + b_n \leq \lambda a_n\},$$

where $n_\lambda \geq -K \log \lambda$ for some $K > 0$. Then for $p > 0$, there exists $K_{\alpha,p} > 0$ such that if $K > K_{\alpha,p}$, $\{(\lambda^{1/\alpha}T)^{-p},\ 1 \geq \lambda > 0\}$ is uniformly integrable.

The proof involves bounding the probability of $(\lambda^{1/2}T < \gamma)$ for $\gamma \in (0, 1)$ in the right order of λ. This can be done by a truncation argument and then applying the Kolmogorov's exponential bound. The details are omitted.

LEMMA 4. *Let $Z, Z_1, Z_2, \cdots$ be independent identically distributed random variables with variance* Var $Z = \sigma^2$, *which is positive and finite. Assume $\{a_n\}$ and $\{b_n\}$ are constants such that as $n \to \infty$, $b_n = o(1)$ and $n^{-\alpha}a_n \to 1$ for some $\alpha > 0$. For $\lambda > 0$, let*

$$T \equiv T_\lambda = \inf\{n \geq n_\lambda : n^{-1}\textstyle\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 + b_n \leq \lambda a_n\},$$

where $n_\lambda \geq -K \log \lambda$, for some $K > 0$. Then for $p > 0$, there exists $K_{\alpha,p} > 0$ such that if $K > K_{\alpha,p}$, $\{(\lambda^{1/\alpha}T)^{-p},\ 1 \geq \lambda > 0\}$ is uniformly integrable.

PROOF. For $M > 0$, put $Z_n' = (Z_n \wedge M) \vee (-M)$ for each $n \geq 1$, and choose M such that Var $Z_n' > 0$. For $\lambda > 0$, define

$$N \equiv N_\lambda = \inf\{n \geq n_\lambda : n^{-1}\textstyle\sum_{i=1}^n (Z_i' - \bar{Z}_n')^2 + b_n \leq \lambda a_n\}.$$

Put $X_i = (Z_i' - EZ_i')^2$ and $Y_i = Z_i' - EZ_i'$. Then X_i and Y_i are bounded, and by Lemma 3, $\{(\lambda^{1/\alpha}N)^{-p},\ 1 \geq \lambda > 0\}$ is uniformly integrable if $K > K_{\alpha,p}$. But by a result due to Chow and Studden [7],

$$\textstyle\sum_{i=1}^n (Z_i' - \bar{Z}_n')^2 \leq \sum_{i=1}^n (Z_i - \bar{Z}_n)^2.$$

Hence $N \leq T$, and then $\{(\lambda^{1/\alpha}T)^{-p},\ 1 \geq \lambda > 0\}$ is uniformly integrable.

LEMMA 5. *Let $Y_1, Y_2, \cdots$ be independent random variables with $EY_n = 0$ for each $n \geq 1$. Assume that for some $p \geq 2$, $\{|Y_n|^p, n \geq 1\}$ is uniformly integrable. Let $\mathscr{F}_n$ be the σ-algebra generated by $\{Y_1, Y_2, \cdots, Y_n\}$ for each $n \geq 1$, $\mathscr{F}_0 = \{\phi, \Omega\}$, and let $\{M(b), b \in B\}$ be $\mathscr{F}_n$-stopping times with $B \subset (0, \infty)$ such that $\{(b^{-1}M(b))^{p/2}, b \in B\}$ is uniformly integrable. Let $W_n = \sum_{i=1}^n Y_i$. Then $\{|b^{-1/2}W_{M(b)}|^p, b \in B\}$ is uniformly integrable.*

PROOF. For each $b \in B$, let $M' \equiv M'(b) = M(b) \wedge N$, where $N = [Kb]$ for $K \geq 1$. Put $M_n = M' \wedge n$. Since $\{|Y_n|^p, n \geq 1\}$ is uniformly integrable, for any $\delta > 0$, there exists a positive constant K_1 such that

$$\sup_{n\geq 1} E|Y_n I_{[|Y_n|\geq K_1]} - EY_n I_{[|Y_n|\geq K_1]}|^p < \delta.$$

Put $X_n = Y_n I_{[|Y_n|\geq K_1]} - EY_n I_{[|Y_n|\geq K_1]}$ and $Z_n = Y_n - X_n$.
Then

$$\sum_{i=1}^{M_n} X_i = \sum_{i=1}^n X_i I_{[M'\geq i]},$$

and

$$\sum_{i=1}^{M_n} Z_i = \sum_{i=1}^n Z_i I_{[M'\geq i]}$$

are martingales. By a result of Burkholder, Davis and Gundy [2], for some constant A > 0,

$$\begin{aligned} E|\textstyle\sum_{i=1}^{M'} X_i|^p &\leq AE(\textstyle\sum_{i=1}^\infty E(X_i^2 I_{[M'\geq i]}/\mathscr{F}_{i-1}))^{p/2} + AE(\sup_{i\geq 1}|X_i|I_{[M'\geq i]})^p \\ &\leq AE(\textstyle\sum_{i=1}^\infty I_{[M'\geq i]}EX_i^2)^{p/2} + AE(\textstyle\sum_{i=1}^\infty E|X_i|^p I_{[M'\geq i]}) \\ &\leq A\delta E(M')^{p/2} + A\delta E(M') \\ &\leq 2A\delta E(M')^{p/2}. \end{aligned} \tag{3}$$

Similarly

$$E|\textstyle\sum_{i=1}^{M'} Z_i|^{p+1} \leq 2^{p+2}AK_1^{p+1}E(M')^{(p+1)/2}.$$

Therefore

$$\sup_{b\in B} E|b^{-1/2}\textstyle\sum_{i=1}^{M'} X_i|^p \leq 2A\delta \sup_{b\in B} E(b^{-1}M')^{p/2}$$

which can be made arbitrarily small; and

$$\sup_{b\in B} E|b^{-1/2}\textstyle\sum_{i=1}^{M'} Z_i|^{p+1} \leq 2^{p+2}AK_1^{p+1}K^{(p+1)/2} < \infty.$$

Hence

$$\{|b^{-1/2}\textstyle\sum_{i=1}^{M'} Y_i|^p, \quad b \in B\} \tag{4}$$

is uniformly integrable. Now

$$\begin{aligned} W_{M(b)} &= \textstyle\sum_{i=1}^\infty Y_i I_{[M(b)\geq i]} \\ &= \textstyle\sum_{i=1}^N Y_i I_{[M(b)\geq i]} + \textstyle\sum_{i=N+1}^\infty Y_i I_{[M(b)\geq i]}. \end{aligned} \tag{5}$$

As in the derivation in (3),

$$\begin{aligned} E|\textstyle\sum_{i=N+1}^\infty Y_i I_{[M(b)\geq i]}|^p &\leq AE(\textstyle\sum_{i=N+1}^\infty I_{[M(b)\geq i]}EY_i^2)^{p/2} + AE(\sup_{i\geq N+1}|Y_i|^p I_{[M(b)\geq i]}) \\ &\leq 2A \sup_{n\geq 1} E|Y_n|^p \int_{M(b)\geq N} (M(b))^{p/2}\, dP \\ &= o(b^{p/2}) \end{aligned}$$

uniformly in b as $K \to \infty$. This, together with (4) and (5), completes the proof that

$$\{|b^{-1/2}W_{M(b)}|^p, b \in B\}$$

is uniformly integrable.

PROOF OF THE THEOREM. Obviously $T < \infty$ a.s., and as $\lambda \to 0$, $T \to \infty$ a.s. By the strong law of large numbers,

$$T^{-1}\sum_{i=1}^{T}(X_i - \bar{X}_T)^2 + b_T \to \sigma^2 \qquad \text{a.s.}$$

Therefore $\lambda^{1/\delta}T^{2/\delta} \to \sigma^2$ a.s., and hence $\lambda^{1/2}T \to \sigma^\delta$ a.s. Define

$$\tau \equiv \tau_\lambda = \inf\{n \geq n_\lambda : n^{-1}\sum_{i=1}^{n}(X_i - \mu)^2 + b_n \leq \lambda^{1/\delta}a_n\}.$$

Let $\lambda' = \lambda^{1/\delta}$. Then $n_\lambda = O(\lambda'^{-\delta/2})$ and by Lemma 2 $\{\lambda^{1/2}\tau, 1 > \lambda > 0\}$ is uniformly integrable. Since $T \leq \tau$, $\{\lambda^{1/2}T, 1 \geq \lambda > 0\}$ is uniformly integrable; consequently $E(\lambda^{1/2}T) \to \sigma^\delta$ as $\lambda \to 0$. This proves (i). For (ii), since $E|X|^{2p} < \infty$ and $n_\lambda = O(\lambda^{-1/2})$, by Lemma 2, $\{(\lambda^{1/2}\tau)^p, 1 \geq \lambda > 0\}$ and hence $\{(\lambda^{1/2}T)^p, 1 \geq \lambda > 0\}$ are uniformly integrable. By Hölder's inequality, Lemmas 4 and 5

$$E\left|(\lambda^{1/4}\sum_{i=1}^{T}(X_i - \mu))^2\left(\frac{1}{\lambda T^2} - \sigma^{-2\delta}\right)\right| \leq E^{1/p}|\lambda^{1/4}\sum_{i=1}^{T}(X_i - \mu)|^{2p}E^{1/q}|(\lambda T^2)^{-1} - \sigma^{-2\delta}|^q$$

$$= O(1)\cdot o(1) = o(1),$$

where $p + q = pq$. Hence by Lemma 5,

$$\lambda^{-1/2}(\bar{X}_T - \mu)^2 = (\lambda^{1/4}\sum_{i=1}^{T}(X_i - \mu))^2\left(\frac{1}{\lambda T^2} - \sigma^{-2\delta}\right) + \sigma^{-2\delta}(\lambda^{1/4}\sum_{i=1}^{T}(X_i - \mu))^2$$

is uniformly integrable; consequently, together with Anscombe's theorem [1],

$$\lambda^{-1/4}\sigma^{\delta/2-1}(\bar{X}_T - \mu) \xrightarrow{\mathscr{L}} N(0, 1), \qquad \text{as} \quad \lambda \to 0,$$

and (i), we have

$$\lim_{\lambda\to 0}\frac{R_T}{R_{n_0}} = \lim_{\lambda\to 0}E\left[\frac{\sigma^{2\delta-2}(\bar{X}_T - \mu)^2 + \lambda T}{2\lambda^{1/2}\sigma^\delta}\right] = 1.$$

REMARKS. It should be noted that if the type of distribution for the population is known, then by direct computation, n_λ can be replaced by a fixed positive integer so that the conditions of Lemma 1 hold. In particular, the case when the X_n's are normally distributed is certainly subsumed under our general results for asymptotic risk efficiency.

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