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A ROBUSTNESS PROPERTY OF HOTELLING'S T^2-TEST

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This paper shows that Hotelling's T^2-test for testing $\mu = 0$ in the one-sample problem is robust against departures from normality in the following sense. It is still UMPI in a broad class of distributions, and the null distribution under any member of the class is the same as that under normality.

1. Introduction. As is well known, when $(y_1, \cdots, y_n)$ is a random sample from a p-dimensional normal distribution $N_p(\mu, \Sigma)$, for testing $\mu = 0$ Hotelling's T^2-test is UMPI (uniformly most powerful invariant), where

$$T^2 = n\bar{y}'S^{-1}\bar{y} \tag{1.1}$$

$$\bar{y} = \textstyle\sum_{i=1}^n y_i/n \qquad \text{and} \qquad S = \sum_{i=1}^n (y_i - \bar{y})(y_i - \bar{y})'. \tag{1.2}$$

A stronger result due to Simaika (1941) is that it is UMP in the class of tests whose power functions depend on (μ, Σ) only through $\mu'\Sigma^{-1}\mu$ (see Anderson (1959) pages 115–118). Further, the admissibility of the T^2-test has been proved by Stein (1959) and Kiefer and Schwartz (1965). Giri and Kiefer (1964) showed its local and asymptotic minimaxity and Salaevski (1968) verified its minimaxity, which had been shown for a special case by Giri, Kiefer and Stein (1963).

In this paper, as one more optimality of the T^2-test, a robustness property is studied in the following set-up. Let $Y = (y_1, \cdots, y_n)'$ be an $n \times p$ random matrix with a pdf h, let $\mathscr{C}_{np}$ be the class of pdf's on R^{np} with respect to Lebesgue measure dY, and let Q be the set of nonincreasing convex functions from $[0, \infty)$ into $[0, \infty)$. We assume $n \geq p + 1$. For $\mu \in R^p$ and $\Sigma \in \mathscr{S}_p$, define a class of pdf's on R^{np} as follows:

$$\mathscr{C}_{np}(\mu, \Sigma) = \{f \in \mathscr{C}_{np} \mid f(Y \mid \mu, \Sigma) = |\Sigma|^{-(n/2)} q(\textstyle\sum_{i=1}^n (y_i - \mu)'\Sigma^{-1}(y_i - \mu)),\ q \in Q\}, \tag{1.3}$$

where $\mathscr{S}_p$ denotes the set of $p \times p$ positive definite matrices. In this model, we consider the following testing problem

$$H_0{:}h \in \mathscr{C}_{np}(0, \Sigma),\ \Sigma \in \mathscr{S}_p \quad \text{vs.} \quad H_1{:}h \in \mathscr{C}_{np}(\mu, \Sigma),\ \mu \neq 0,\ \Sigma \in \mathscr{S}_p \tag{1.4}$$

and show that Hotelling's T^2-test is UMPI. Clearly if $(y_1, \cdots, y_n)$ is a random sample from $N_p(\mu, \Sigma)$ or $Y \sim N_{np}(e_n\mu', I_n \otimes \Sigma)$, where $e_n = (1, \cdots, 1)' \in R^n$, the pdf h of Y belongs to $\mathscr{C}_{np}(\mu, \Sigma)$. Further if $f(Y \mid \mu, \Sigma)$ belongs to $\mathscr{C}_{np}(\mu, \Sigma)$, then

$$g(Y \mid \mu, \Sigma) = \int_0^\infty f(Y \mid \mu, a\Sigma)\, dG(a)$$

also belongs to $\mathscr{C}_{np}(\mu, \Sigma)$ where G is a distribution function on $(0, \infty)$, and so $\mathscr{C}_{np}(\mu, \Sigma)$ contains the (np-dimensional) multivariate t-distribution, the multivariate Cauchy distribution, the contaminated normal distribution, etc. On the other hand, Dawid (1977) has shown that for any $\Sigma \in \mathscr{S}_p$ and any $h \in \mathscr{C}_{np}(0, \Sigma)$, the null distribution of T^2 is exactly the same as that when $Y \sim N(0, I_n \otimes \Sigma)$, that is, the distribution of $(n - p)T^2/p$ under H_0 is $F(p, n - p)$, the F-distribution with d.f.'s (degrees of freedom) p and $n - p$. In fact, the null distributions of the tests appearing in the MANOVA problem as well as the T^2-test do not depend on normality but on the invariance of the distribution of Y under the transformation $Y \to \rho Y$ for $\rho \in \mathscr{O}(n)$,

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where $\mathcal{O}(n)$ denotes the set of $n \times n$ orthogonal matrices (see Dempster (1969) and Dawid (1977)). In this sense, the T^2-test is robust against departures from normality. The result obtained here is regarded as a multivariate extension of the result in Kariya and Eaton (1977) by invariance. To obtain an expression of the distribution of a maximal invariant, Wijsman's (1967) theorem is used.

Finally we remark that in terms of a canonical form, the problem is stated as follows. Let X be an $n \times p$ random matrix whose pdf, say h, belongs to the class

$$(1.5) \qquad \mathscr{D}_n(\mu, \Sigma) = \{f \in \mathscr{C}_{np} \mid f(X \mid \mu, \Sigma) = |\Sigma|^{-n/2} q(\operatorname{tr} \Sigma^{-1}(X - \epsilon_1 \mu')'(X - \epsilon_1 \mu')),\ q \in Q\}$$

where $\epsilon_1 = (1, 0, \cdots, 0)' \in R^n$. Then the problem is to show that for testing

$$(1.6) \qquad H_0: h \in \mathscr{D}_{np}(0, \Sigma),\ \Sigma \in \mathscr{S}_p \quad \text{vs.} \quad H_1: h \in \mathscr{D}_{np}(\mu, \Sigma),\ \mu \neq 0,\ \Sigma \in \mathscr{S}_r$$

the T^2-test defined by

$$(1.7) \qquad T^2 = X_1(X_2'X_2)^{-1}X_1' > c$$

is UMPI, where

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

with $X_1: 1 \times p$ and $X_2: (n-1) \times p$. This canonical form is simply obtained by transforming Y into $X = \rho_0 Y$ where $\rho_0 \in \mathcal{O}(n)$ with $e_n'/(e_n'e_n)^{1/2}$ as its first row. It is noted that the space of X can be restricted to the set of $n \times p$ matrices of rank p since the complement of this set in R^{np} has Lebesgue measure 0.

2. Null distributions. In this section, we consider the null distributions of the T^2-test. Let $\mathscr{L}(Z)$ denote the distribution law of a random matrix Z. An $n \times p$ random matrix is said to be left $\mathcal{O}(n)$-invariant if $\mathscr{L}(\rho Z) = \mathscr{L}(Z)$ for all $\rho \in \mathcal{O}(n)$. Let $\mathscr{Z} = \{Z: n \times p \mid \operatorname{rank}(Z) = p\} \subset R^{np}$ where $n \geq p + 1$ is assumed. The next theorem is a result due to Dawid (1977) (see also Dempster (1969)).

THEOREM 2.1. *Let Z be an $n \times p$ left $\mathcal{O}(n)$-invariant random matrix over $\mathscr{Z}$, and let*

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

where $Z_1: k \times p$ and $Z_2: (n-k) \times p$ with $n - k \geq p \geq k$. Then $\mathscr{L}(Z_1(Z_2'Z_2)^{-1}Z_1')$ is the same as that under $\mathscr{L}(Z) = N(0, I_n \otimes \Sigma)$. Especially when $k = 1$, $\mathscr{L}((n-p)Z_1(Z_2'Z_2)^{-1}Z_1'/p) = F(p, n-p)$.

This theorem shows that the null distributions appearing in the MANOVA problem as well as the T^2-test do not depend on normality but on the left invariance of Z (see, e.g., Anderson (1958) for the MANOVA tests). The readers are also referred to Eaton (1979) for invariance of random matrices.

3. UMP property of T^2-test. Without loss of generality, we treat the problem in terms of the canonical form stated in Section 1. Obviously the problem remains invariant under the group $\mathscr{G} = \mathcal{O}(n-1) \times \mathscr{G}\ell(p)$ acting on

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

by

$$(3.1) \qquad g \circ X = (X_1 A', \rho X_2 A') \qquad \text{for} \quad g = (\rho, A) \in \mathscr{G},$$

where $X_1: 1 \times p$ and $X_2: (n-1) \times p$ and $\mathscr{G}\ell(p)$ denotes the set of $p \times p$ nonsingular matrices. As is well known, a maximal invariant under $\mathscr{G}$ is $T^2 = X_1(X_2'X_2)^{-1}X_1'$ and a maximal invariant parameter is $\delta = \mu'\Sigma^{-1}\mu$. To obtain a formal expression of the distribution of T^2, a theorem due to Wijsman (1967) is applied.

LEMMA 3.1. *Let $h \in \mathscr{D}_{np}(\mu, \Sigma)$ be the* pdf *of X, let $W = w(X)$ be a maximal invariant under the transformation* (3.1) *and let P_δ^W be the distribution induced by W under $\delta = \mu'\Sigma^{-1}\mu$. Then the* pdf *of W with respect to P_0^W evaluated at $W = w(X)$ is given by*

$$\frac{dp_\delta^W}{dP_0^W}(w(X)) = h_W(w(X) \mid \delta) = \frac{\int_{\mathscr{G}} h(g \circ X \mid \mu, \Sigma) \mid A'A \mid^{n/2} d\nu(g)}{\int_{\mathscr{G}} h(g \circ X \mid 0, \Sigma) \mid A'A \mid^{n/2} d\nu(g)} \tag{3.2}$$

where ν is a left invariant measure on $\mathscr{G}$. Naturally $h_W(w(X) \mid 0) \equiv 1$.

Theorem 4 in Wijsman (1967) states the conditions for which (3.2) holds. But checking the conditions is included in the proof of Lemma 5.1 in Kariya (1978) and so it is omitted here.

Write the pdf h of X as

$$h(X \mid \mu, \Sigma) = \mid \Sigma \mid^{-n/2} q(\text{tr}\, \Sigma^{-1}(X - \epsilon_1\mu')'(X - \epsilon_1\mu')), \tag{3.3}$$

where q is convex and nonincreasing, and take $\nu = \nu_1 \times \nu_2$ where ν_1 is the invariant probability measure on $\mathscr{O}(n-1)$ and $d\nu_2(A) = dA/\mid A'A \mid^{p/2}$.

LEMMA 3.2. *The* pdf h_W *in* (3.2) *is evaluated as*

$$h_W(w(X) \mid \delta) = \frac{\int_{\mathscr{G}\ell(p)} q(\text{tr}\, A'A + \delta - 2a_{11}\delta^{1/2}v) \mid A'A \mid^{(n-p)/2} dA}{\int_{\mathscr{G}\ell(p)} q(\text{tr}\, A'A) \mid A'A \mid^{(n-p)/2} dA} \tag{3.4}$$

where $v = T/(1 + T^2)^{1/2}$ with $T^2 = X_1(X_2'X_2)^{-1}X_1'$, a_{ij} denotes the (i, j) element of A and a_i denotes the ith *column of A.*

PROOF. First we note that the integrals in (3.4) are finite and positive. To evaluate (3.2), let N_δ denote the numerator of (3.2). From (3.1) and (3.3) N_δ is written as

$$N_\delta = \int_{\mathscr{O}(n-1) \times \mathscr{G}\ell(p)} \mid \Sigma \mid^{-n/2} q(\text{tr}\, \Sigma^{-1}(X_1A' - \mu')'(X_1A' - \mu') + \text{tr}\, \Sigma^{-1}AX_2'X_2A') \mid A'A \mid^{n/2} d\nu_1(\rho)\, d\nu_2(A). \tag{3.5}$$

But since the integrand of (3.5) does not depend on ρ and since $\nu_1(\mathscr{O}(n-1)) = 1$, N_δ is equal to the integration over $\mathscr{G}\ell(p)$ with respect to ν_2. Since ν_2 is left and right invariant, replacing A by $\Sigma^{1/2}A(X_2'X_2)^{-1/2}$ leaves the integral the same;

$$N_\delta = \int_{\mathscr{G}\ell(p)} c_1 q(\text{tr}\,(uA' - \eta')'(uA' - \eta') + \text{tr}\, AA') \mid A'A \mid^{n/2} d\nu_2(A) \tag{3.6}$$

where $(\Sigma^{1/2})^2 = \Sigma$, $[(X_2'X_2)^{-1/2}]^2 = (X_2'X_2)^{-1}$, $c_1 = \mid X_2'X_2 \mid^{-n/2}$, $u = X_1(X_2'X_2)^{-1/2}$ and $\eta = \Sigma^{-1/2}\mu$. Let ρ_1 and ρ_2 be $p \times p$ orthogonal matrices with $\eta'/\| \eta \|$ and $u'/\| u \|$ as their first rows respectively. Since $\text{tr}\,(uA' - \eta')'(uA' - \eta') = uA'Au' - 2uA'\eta + \eta'\eta$, replacing A by $\rho_1'A\rho_2$ yields

$$N_\delta = \int c_1 q((1 + T^2)\| a_1 \|^2 - 2\delta^{1/2}Ta_{11} + \delta + \textstyle\sum_{i=2}^p \| a_i \|^2) \mid A'A \mid^{n/2} d\nu_2(A)$$

where $T^2 = uu' = X_1(X_2'X_2)^{-1}X_1' > 0$ (a.e.) and $\delta = \eta'\eta = \mu'\Sigma^{-1}\mu$. Finally transforming a_1 into $a_1/(1 + T^2)^{1/2}$ and taking the ratio of N_δ and N_0 produces the result (3.4).

Now we shall prove our main theorem.

THEOREM 3.1. *For problem* (1.4), *Hotelling's T^2-test is* UMPI *and the null distribution of* $(n-p)T^2/p$ *is F-distribution with* d.f.'s p *and* $n-p$.

PROOF. By Lemma 3.2, the pdf of a maximal invariant with respect to P_0^W is given by (3.4). Hence for fixed δ_0, a most powerful test for testing $\delta = 0$ versus $\delta_0 > 0$ is given by the c.r. (critical region):

$$h_W(w(X)|\,\delta_0) > c h_W(w(X)|\,0) = c. \tag{3.7}$$

To evaluate this, let $H(v)$ be the numerator of (3.4). Then transforming A into $-A$ yields

$$H(v) = \int q(\operatorname{tr} A'A + \delta + 2a_{11}\delta^{1/2}v)|\,A'A\,|^{(n-p)/2}\,dA \equiv H(-v), \text{ say.}$$

Hence using the convexity of q, for $\frac{1}{2} \le \alpha \le 1$ we obtain

$$H(v) = \alpha H(v) + (1-\alpha)H(-v) \ge H(v(2\alpha - 1)).$$

This implies that $H(v)$ is a nondecreasing function of $v \in [0, 1]$. Therefore (3.7) yields c.r. $v \ge c'$ or $T^2 \ge c''$. Since this c.r. does not depend on δ_0, it is UMPI. The latter part follows from Theorem 2.1. This completes the proof.

When $p = 1$, Theorem 3.1 means that the usual F-test (or equivalently two-sided t-test) is UMPI for the problem (4.1). On the other hand, without invariance Kariya and Eaton (1977) have shown that the usual two-sided t-test is UMP similar (hence unbiased) for (1.4). This is a stronger result since there exists a noninvariant similar test.

Finally we remark that the nonincreasing property of q is not used in the above proof. But since $f(Y\,|\,\mu, \Sigma)$ in (1.3) is a pdf, the convexity of q implies it. Without the convexity, it seems difficult to obtain Theorem 3.1.

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