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ESTIMATION OF QUANTILES IN CERTAIN NONPARAMETRIC MODELS

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The deficiency of sample quantiles with respect to quasiquantiles is investigated under the assumption that the true density function has bounded derivatives. Then the sample quantile is still an efficient estimator of the true quantile but the relative deficiency of sample quantiles with respect to suitably defined quasiquantiles quickly tends to infinity for increasing sample sizes. If the second derivative of the true density function is bounded, then adaptive estimators will be found which are of a better performance than quasiquantiles. Corresponding results are derived for two-sided confidence intervals which are based on quasiquantiles and adaptive estimators.

1. Introduction. The estimation of quantiles is usually considered either in parametric models or in models with hardly any restriction concerning the distributions. In the first situation an efficient estimator for the unknown quantile can be derived from the efficient estimator of the unknown parameter; in the second case it is intuitively clear (and can be proved) that the "natural" estimator, namely the sample quantile, cannot be beaten. The only exception dealt with in the literature is the case of symmetric distributions. Then the sample median—as an estimator of the center of the distribution—has different nonparametric competitors.

For symmetric distributions a comparison of the sample median and the sample mean was given by Laplace (1818). Let M denote the median and p the density function of a symmetric distribution. Laplace proved that the asymptotic relative efficiency of the sample median with respect to the sample mean (based on their asymptotic variances) is given by $e = 4p^2(M)\int x^2 p(x)\,dx$. Given a parametric family of p-measures with location parameter (w.l.g. being equal to the median) the sample median is usually inefficient; i.e. $e^* < 1$ where e^* denotes the asymptotic relative efficiency with respect to the best obtainable estimator. An interesting exception is the sample median in the case of the double-exponential distribution with unknown location parameter.

If $e^* < 1$ then the sample median has still favourable properties in the following situations: (a) The sample median gives a rough estimate of the true median without laborious computations. Such an estimate might be satisfactory in preliminary investigations. (b) If the costs of the experiment are low, then it might be better—from an economical point of view—to raise the number of observations by the amount of $(1 - e^*)n$ instead of using an efficient estimator for the sample size n.

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A compromise between the two requirements to find an estimator which, on the one hand, has efficiency close to one and, on the other hand, can be easily handled, is achieved by using systematic statistics. Mosteller (1946) investigated the asymptotic efficiency of a linear combination of k order statistics (with k not depending on the sample size). For normal distributions with variance 1 and unknown location parameter, the asymptotic efficiency of optimal quasimedians is equal to .810 compared to .637 for the sample median (see Mosteller (1946), page 387, Table III). Quasimedians M_n are defined by $M_n = (Z_{[n/2]-m+1:n} + Z_{[n/2]+m+1:n})/2$ for $m \in \{1, \cdots, [n/2]\}$ with $Z_{i:n}$ denoting the ith order statistic for the sample size n. The choice of the optimal m—with $m = m(n) \sim n\lambda$ for some $\lambda \in (0, \frac{1}{2})$—depends on the assumption that the true distribution is normal. The proof of this result was enabled by computing the asymptotic joint distribution of the order statistics $Z_{[n\lambda_i]:n}$, $0 < \lambda_1 < \cdots < \lambda_k < 1$ (this distribution was earlier found by Smirnov ((1944), (29) page 184). For estimating the median M as a location parameter (the distributions are given by their density functions $p(x - M)$, p known) an asymptotically efficient estimator is given by

$$\hat{M}_n - \left(n\int (p^{(1)}(x))^2/p(x)\, dx\right)^{-1}\Sigma_{i=1}^n p^{(1)}(Z_{i:n} - \hat{M}_n)/p(Z_{i:n} - \hat{M}_n)$$

with $\hat{M}_n$ denoting the sample median (see LeCam (1956) page 139).

If p is symmetric, we do not even need to know the form of p to get an efficient estimator. Estimating $p(x - M)$ by a suitable density estimator $\hat{p}_n(x)$, we get the asymptotically efficient adaptive estimator

$$\hat{M}_n - \left(n\int (\hat{p}_n^{(1)}(x))^2/\hat{p}_n(x)\, dx\right)^{-1}\Sigma_{i=1}^n \hat{p}_n^{(1)}(Z_{i:n})/\hat{p}_n(Z_{i:n})$$

(see Stone (1975), (1.11) page 270). This method is not applicable without the symmetry condition.

It is proved in Pfanzagl (1975) that the sample median is an efficient estimator in the following nonparametric cases: let $\mathcal{P}$ denote the family of all p-measures P with positive differentiable density function $p(\cdot, P)$. For $Q \in \mathcal{P}$ and $\varepsilon > 0$ define

$$\mathcal{P}(Q, \varepsilon) := \left\{P \in \mathcal{P} : \left|\frac{p(\cdot, P)}{p(\cdot, Q)} - 1\right| < \varepsilon\right\}.$$

Among all estimators T_n which are translation-equivariant and asymptotically median unbiased uniformly over $\mathcal{P}(Q, \varepsilon)$ the sample median is optimal in the following sense: let $P_{1,n}$ and $P_{2,n}$ denote the distribution of $\hat{M}_n$ and T_n, respectively. Then

$$P_{2,n}(M - t', M + t'') \leqslant P_{1,n}(M - t', M + t'') + o(1)$$

uniformly for all $t', t'' > 0$. A corresponding result holds true for quantiles, in general.

We remark that the result of Pfanzagl (1975) still holds true if p-measures are considered, only, for which (a) all derivatives of the distribution function exist, or (b) finitely many derivatives exist and are uniformly bounded. The present paper deals with the second case showing that estimators and two-sided confidence intervals can be found, which are considerably better than sample medians. Our starting point will be quasimedians. Quasimedians M_n, with m fixed, were proposed by Hodges and Lehmann (1967) for estimating the center of a symmetric distribution (instead of symmetry we could as well assume that the first derivative of the density function at the median is equal to zero). Under the assumption that the distribution function has a continuous third derivative, they proved by means of heuristic arguments that the ratio of the variances of M_n and $Z_{[n/2]+1:n}$, n odd, is equal to $1 - 2m/n + o(n^{-1})$.

We shall drop the symmetry condition and let $m(n)$ tend to infinity with $m(n) = o(n)$ facing the problem that the quasimedians are no longer unbiased.

The paper is organized as follows: the notations are collected in Section 2. In Section 3 we discuss the results of the Sections 4 and 5 (concerning estimators and confidence intervals for quantiles). Section 6 contains some auxiliary results. In Section 7 we mention some unsolved questions and make some concluding remarks.

2. Notations and the models. Let $\mathbb{R}$ (respectively, $\mathbb{N}$) denote the set of all real numbers (positive integers). Let P^m denote the independent product of m identical p-measures P on the Borel-algebra of $\mathbb{R}^m$. For any map $g : \mathbb{R}^m \to \mathbb{R}^k$ and $B \subset \mathbb{R}^k$ let $\{g \in B\} := \{(x_1, \cdots, x_m) \in \mathbb{R}^m : g(x_1, \cdots, x_m) \in B\}$. A function $T_n : \mathbb{R}^n \to \mathbb{R}$ is called an estimator if it is Borel-measurable.

The normal distribution with mean μ and variance σ^2 is denoted by $N_{(\mu, \sigma^2)}$. The distribution function and the density function of $N_{(0,1)}$ are denoted by Φ and φ, respectively. The ith order statistic $Z_{i:n} : \mathbb{R}^n \to \mathbb{R}$ for the sample size n is defined by $Z_{i:n}(x_1, \cdots, x_n) = z_{i:n}$ where $z_{1:n} \leqslant \cdots \leqslant z_{n:n}$ are the components of $(x_1, \cdots, x_n) \in \mathbb{R}^n$ arranged in the increasing order.

The usual definition of the sample median is $\hat{M}_n = Z_{[n/2]+1:n}$ if n is odd and $\hat{M}_n = (Z_{[n/2]:n} + Z_{[n/2]+1:n})/2$ if n is even. We remark that the distribution of $\hat{M}_n$ differs from that of $Z_{[n/2]:n}$ in terms of order $0(n^{-\frac{1}{2}})$, uniformly over all intervals, under the condition (2.3) stated below. Thus, the definition of the sample q-quantile, as $Z_{[nq]:n}$ for $q \in (0, 1)$, is in the case of $q = \frac{1}{2}$ asymptotically conforming with the notion of sample medians.

Given $q \in (0, 1)$ and $m \in \{1, \cdots, \min\{[nq] - 1, n - [nq]\}\}$ the quasiquantile Q_n is defined by

$$Q_n := (Z_{[nq]-m:n} + Z_{[nq]+m:n})/2. \tag{2.1}$$

Another class of estimators is given by the adaptive quasiquantiles $\hat{Q}_n$ which are defined by

$$\hat{Q}_n := (-2Z_{[nq]-2m:n} + 8Z_{[nq]-m:n} + 13Z_{[nq]:n} + 8Z_{[nq]+m:n} - 2Z_{[nq]+2m:n})/25 \tag{2.2}$$

for $q \in (0, 1)$ and $m \in \{1, \cdots, \min\{([nq] - 1)/2, (n - [nq])/2\}\}$.

Given a p-measure P with distribution function F a solution of the equation $F(y) = q$, $q \in (0, 1)$, is called a q-quantile of P. Hereafter we shall always assume that P has a density function p such that for some q-quantile $q(P)$ of P the following condition is fulfilled:

$$\text{(2.3)}\qquad \begin{aligned} &p(q(P)) > 0 \\ &\text{and} \\ &|p(x) - p(y)| \leqslant C|x - y| \quad \text{for all } x, y \in [q(P) - \varepsilon, q(P) + \varepsilon] \\ &\qquad\qquad \text{for some constants } C, \varepsilon > 0. \end{aligned}$$

(2.3) obviously implies that $q(P)$ is the unique q-quantile of P. Let, furthermore, $\sigma(P) = (q(1 - q))^{\frac{1}{2}}/p(q(P))$.

Q_n and $\hat{Q}_n$ will be appropriate estimators of the q-quantile of p-measures which fulfill the smoothness condition (2.4). For every $k \in \mathbb{N}$, $q \in (0, 1)$ and for positive real numbers $A, C, \varepsilon, u, v_i$, $i = 0, \cdots, k$, we define a family of p-measures $\mathcal{P}(q, k, A, C, u, \varepsilon, v_i, i = 0, \cdots, k)$ which contains exactly those p-measures P which have a density function p that fulfills condition (2.3) (for C and ε) and

$$\text{(2.4)}\qquad p(x) = \sum_{i=0}^{k-1} \frac{a_i}{i!}(x - q(P))^i + \frac{a_k + g_k(x)}{k!}(x - q(P))^k$$

with

$$\begin{aligned} |g_k| &\leqslant A a_0^{k+1}, \\ u \leqslant a_0 &\leqslant v_0, \\ |a_i| &\leqslant v_i \qquad\qquad \text{for} \quad i = 1, \cdots, k. \end{aligned}$$

Such models could, e.g., be accepted if we started from a theoretical model which is defined by p-measures P which have a smooth density function. If higher derivatives of the density function exist then $a_i = p^{(i)}(q(P))$ in (2.4). To protect ourselves against small deviations of the actual situation from the idealized one —in other words, to base our considerations on a model which is possibly more related to the actual situation—we include in the model all density functions which deviate from the idealized density function in the kth derivative. Then our results are little affected by the additional treatment of letting the values a_i vary within certain ranges.

As far as our results are not depending (asymptotically) on the special values of $C, u, \varepsilon, v_i, i = 0, \cdots, k$, we shall write $\mathcal{P}(q, k, A)$ instead of $\mathcal{P}(q, k, A, C, u, \varepsilon, v_i, i = 0, \cdots, k)$ for notational simplicity. Whenever necessary we include again some of the variables which define $\mathcal{P}(q, k, A)$.

Other reasonable models can be obtained if conditions are used where $(a_k + g_k(x))(x - q(P))^k$ in (2.4) is replaced by $(a_{k+\alpha} + g_{k+\alpha}(x))(x - q(P))^{k+\alpha}$ for some $\alpha \in (0, 1)$. In order not to overload this paper with details we abstain from this possible generalization of (2.4). Conditions of this type (with $\alpha \neq 0$ and $a_k = 0$) were used by Weiss and Wolfowitz (1967), page 328, and Woodroofe (1970), page 1666, in connection with density estimation.

3. Discussion of the results.

A. *Quasiquantiles in the case of* $k = 1$. Estimating an unknown parameter we are usually not interested in the fact whether the estimate falls left or right of the parameter. Thus, we base our measure of accuracy of estimators on the value of the distributions on symmetric intervals about the true parameter.

For the double-exponential distribution E with the Lebesgue-density $p(x) = e^{-|x|}/2$ it is easy to see that $E \in \mathcal{P}(\frac{1}{2}, 1, 2)$. As an immediate consequence of Theorem 2.7 in Reiss (1976) and Theorem 4.1 we obtain for every $t > 0$

$$\text{(3.1)}\quad \begin{aligned} E^n\left\{-\frac{t}{n^{\frac{1}{2}}} \leqslant T_n \leqslant \frac{t}{n^{\frac{1}{2}}}\right\} &\leqslant E^n\left\{-\frac{t}{n^{\frac{1}{2}}} \leqslant Z_{[n/2]:n} \leqslant \frac{t}{n^{\frac{1}{2}}}\right\} + 0(n^{-\frac{1}{2}}) \\ &= 2\Phi(t) - 1 + 0(n^{-\frac{1}{2}}) \end{aligned}$$

for every sequence of estimators $T_n : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, which are equivariant under translations, i.e.

$$\text{(3.2)}\qquad T_n(x_1 + u, \cdots, x_n + u) = T_n(x_1, \cdots, x_n) + u$$

for every $(x_1, \cdots, x_n, u) \in \mathbb{R}^{n+1}$. It is obvious that sample quantiles and the estimators which are defined in (2.1) and (2.2) fulfill (3.2).

Next, the performance of quasiquantiles Q_n with $m(n) \to \infty$ for $n \to \infty$ is investigated. Lemma 6.13 implies that the distribution of Q_n can be approximated by a normal distribution $N_{(\mu_n, \nu_n^2)}$ where μ_n and ν_n are explicitly given in (4.4) and (4.5). Thus, by Lemma 6.12

$$\text{(3.3)}\qquad P^n\left\{|Q_n - q(P)| \leqslant \frac{t\sigma(P)}{n^{\frac{1}{2}}}\right\} = 2\Phi(t(1 + s_n)) - 1 + o(s_n)$$

where

$$s_n = s_n(m(n), q, P) = \frac{1}{2}\left(1 - \frac{n}{\sigma(P)^2}(\nu_n^2 + (\mu_n - q(P))^2)\right).$$

To find the estimator Q_n for which the distribution is maximally concentrated on intervals $[q(P) - t\sigma(P)/n^{\frac{1}{2}}, q(P) + t\sigma(P)/n^{\frac{1}{2}}]$ we have to minimize the "mean square error" $\nu_n^2 + (\mu_n - q(P))^2$. Notice that, in general, $\nu_n^2 + (\mu_n - q(P))^2$ is not an approximation of the mean square error $\int(Q_n - q(P))^2\, dP^n$.

To get an optimum sequence $m^*(n)$, $n \in \mathbb{N}$, which is independent of the particular p-measure $P \in \mathcal{P}(q, 1, A)$ we use the inequality

$$\text{(3.4)}\quad \begin{aligned} s_n \geqslant{}& \frac{m(n)}{n}\left(\frac{1}{4q(1-q)} - A\right) - \frac{m(n)^4}{n^3}\frac{1}{8q(1-q)}\left(\frac{v_1}{u^2} + A\right)^2 \\ &+ o\left(\frac{m(n)}{n} + \frac{m(n)^4}{n^3}\right) \end{aligned}$$

which holds uniformly over $\mathcal{P}(q, 1, A, C, \varepsilon, v_0, v_1)$. We remark that the estimate in (3.4) is sharp in the sense that the largest lower bound is given for which only the term of order $o\left(\frac{m(n)}{n} + \frac{m(n)^4}{n^3}\right)$ depends on C.

Hereafter, let $A < 1/4q(1 - q)$ whenever the case of $k = 1$ is considered. It can easily be proved that sequences $m^*(n)$, $n \in \mathbb{N}$, with

$$(3.5) \qquad m^*(n) \sim n^{\frac{2}{3}}(1 - 4Aq(1 - q))^{\frac{1}{3}}\left(\frac{v_1}{u^2} + A\right)^{-\frac{2}{3}}$$

are optimal in the sense of asymptotically maximizing the right-hand side of (3.4). Using $m^*(n)$, the covering probabilities of quasiquantiles exceed the respective covering probabilities of sample quantiles by terms of order $n^{-\frac{1}{3}}$. Remember that in the case of the double-exponential distribution—where $A = 2$—it is only possible to find translation-equivariant estimators which improve the covering probabilities of sample medians by terms of order $n^{-\frac{1}{2}}$.

We use the concept of deficiency (as introduced by Hodges and Lehmann (1970)) to describe the performance of quasiquantiles in a different way. Let $T_n : \mathbb{R}^n \to \mathbb{R}$ be an estimator and $r(n) \equiv r(n, T_n, P)$ be a positive integer which minimizes

$$\left| P^{r(n)}\left\{ |Z_{[r(n)q]:r(n)} - q(P)| \leqslant \frac{t\sigma(P)}{n^{\frac{1}{2}}} \right\} - P^n\left\{ |T_n - q(P)| \leqslant \frac{t\sigma(P)}{n^{\frac{1}{2}}} \right\} \right|.$$

The integer $d(n, T_n, P) := r(n, T_n, P) - n$ is called the relative deficiency of the sample quantile with respect to T_n. Thus, $d(n, T_n, P)$ is the number of observations which are additionally required such that the distribution of the sample quantile is as concentrated about $q(P)$ as that of T_n based on n observations. To get a concept of deficiency which only depends on a given family $\mathcal{P}$ of p-measures we define

$$(3.6) \qquad d(n, T_n) := \inf\{d(n, T_n, P) : p \in \mathcal{P}\}.$$

Then, T_n has an at least equally good performance as the sample quantile for the sample size $n + d(n, T_n)$ for every $P \in \mathcal{P}$. Put $d(n) = d(n, Q_n)$. By means of Theorem 4.6 and Lemma 6.13 it can be proved in an elementary way that the sequences $m^*(n)$, $n \in \mathbb{N}$, as characterized by (3.5), are the only sequences $m(n)$, $n \in \mathbb{N}$, such that

$$(3.7) \qquad \liminf_{n \in \mathbb{N}} d(n)/n^{\frac{2}{3}} \geqslant (1 - 4Aq(1 - q))^{\frac{1}{3}}/2\left(\frac{v_1}{u} + A\right)^{\frac{2}{3}}$$

for $\mathcal{P} := \mathcal{P}(q, 1, A, C, u, \varepsilon, v_0, v_1)$ for every $C > 0$. Thus, the relative deficiency of the sample quantile with respect to the quasiquantiles Q_n, defined with $m^*(n)$, is of order $n^{\frac{2}{3}}$.

B. *Adaptive quasiquantiles in the case of* $k = 2$ *and* $k = 3$. If $P \in \mathcal{P}(q, 2, B)$ for some $B > 0$, then (3.4) will hold for $A = 0$. Even in this case we cannot find quasiquantiles which are considerably better than those for $k = 1$ due to the fact that the "expectation" of the quasiquantiles Q_n is equal to

$$q(P) + \frac{a_1}{2a_0^3}\left(\frac{m(n)}{n}\right)^2 + o\left(\left(\frac{m(n)}{n}\right)^3\right).$$

By means of Q_n and an appropriate estimator for a_1/a_0^3 we shall define another estimator for the q-quantile. Notice that $-a_1/a_0^3 = (F^{-1})^{(2)}(q(P))$ under appropriate differentiability conditions on the distribution function F of P. By means of estimators of

$$1/p\left(F^{-1}\left(q + \frac{m'(n)}{n}\right)\right) \quad \text{and} \quad 1/p\left(F^{-1}\left(q - \frac{m'(n)}{n}\right)\right)$$

we find an appropriate estimator $\hat{a}_{2,n}$ of $-a_1/a_0^2$, namely,

$$\begin{aligned}\hat{a}_{2,n} &:= \left(\frac{n(Z_{[nq]+m'(n):n} - Z_{[nq]:n})}{m'(n)} - \frac{n(Z_{[nq]:n} - Z_{[nq]-m'(n):n})}{m'(n)}\right) \Big/ \left(\frac{m'(n)}{n}\right)\\ &= \left(\frac{n}{m'(n)}\right)^2 (Z_{[nq]-m'(n):n} - 2Z_{[nq]:n} + Z_{[nq]+m'(n):n}).\end{aligned}$$

The choice of the sequence $m'(n)$, $n \in \mathbb{N}$, determines the asymptotic properties of $Q_n - \dfrac{\hat{a}_{2,n}}{2}\left(\dfrac{m(n)}{n}\right)^2$ which is identical to the sample quantile for $m(n) = m'(n)$. Hereafter, we take in rather an arbitrary way $m'(n) = 2m(n)$ obtaining the estimator

$$\begin{aligned}\hat{Q}_n^\gamma = &-\frac{\gamma}{4} Z_{[nq]-2m(n):n} + \gamma Z_{[nq]-m(n):n}\\ &+ \left(1 - \frac{3}{2}\gamma\right) Z_{[nq]:n} + \gamma Z_{[nq]+m(n):n} - \frac{\gamma}{4} Z_{[nq]+2m(n):n}\end{aligned} \tag{3.8}$$

with $\gamma = \frac{1}{2}$.

The covering probabilities of the estimators Q_n^γ are given in Theorem 4.12. For the families $\mathcal{P}(q, 2, A)$ the optimum choice of γ and $m(n)$—in the sense of maximizing the covering probabilities—is $\gamma = \frac{8}{25}$ and

$$m^*(n) \sim n^{\frac{4}{5}} A^{-\frac{2}{5}} \left(\tfrac{9}{20}\right)^{\frac{1}{5}} \tag{3.9}$$

(see Theorem 4.12 and Remark 4.16). Since the optimum γ does not depend on q and A we can restrict our attention to the estimators $\hat{Q}_n = \hat{Q}_n^{\frac{8}{25}}$ as defined in (2.2). It can easily be proved that $\gamma = \frac{8}{25}$ and $m^*(n)$, as characterized by (3.9), have also the property of asymptotically maximizing the relative deficiency $d(n, \hat{Q}_n^\gamma)$ (of the sample quantile with respect to $\hat{Q}_n^\gamma$) given $\mathcal{P} = \mathcal{P}(q, 2, A)$. For $\hat{Q}_n$, defined with $m^*(n)$, we have

$$d(n, \hat{Q}_n) \sim n^{\frac{4}{5}} \tfrac{4}{25} \left(\tfrac{9}{20}\right)^{\frac{1}{5}} (q(1-q))^{-1} A^{-\frac{2}{5}}. \tag{3.10}$$

For $k = 3$ estimators $\hat{Q}_n^\gamma$ with $m(n)$ of order $n^{\frac{6}{7}}$ can be found such that $d(n, \hat{Q}_n^\gamma)$ —defined with respect to $\mathcal{P} = \mathcal{P}(q, 3, A)$—is of order $n^{\frac{6}{7}}$. For details we refer to Theorem 4.18 where the covering probabilities of $\hat{Q}_n^\gamma$ are given if $P \in \mathcal{P}(q, 3, A)$. Estimators $\hat{Q}_n^\gamma$ should not be used in the case of $k = 3$. The performance of $\hat{Q}_n^\gamma$ depends on $P \in \mathcal{P}(q, 3, A)$ through the numbers a_4' which are given in (6.2). Thus, the optimum choice of $m(n)$ also depends on a_4'. The situation is similar to that of $k = 1$: we expect that there exists an appropriate estimator $\hat{a}_{4,n}$ of a_4' (in analogy to the estimator $\hat{a}_{2,n}$ of $a_2' = -a_1/a_0^3$) such that an estimator $\hat{Q}_n^\gamma + \frac{3}{12}\gamma\hat{a}_{4,n}(m(n)/n)^4$ has (a) the favourable properties of an optimum estimator $\hat{Q}_n^\gamma$ and (b) depends on the family $\mathcal{P}(q, 3, A, \cdots)$ only through the number A.

C. *Confidence intervals.* In Section 5 asymptotic confidence intervals $[\underline{\kappa}_{n,k}, \bar{\kappa}_{n,k}]$ are constructed by means of the estimators Q_n and $\hat{Q}_n^\gamma$. Thus, the confidence intervals depend on $\gamma \in \mathbb{R}$ in the cases $k = 2, 3$, and on $m(n)$ for $k = 1, 2, 3$. The performance of $[\underline{\kappa}_{n,k}, \bar{\kappa}_{n,k}]$ is measured by the probability that $[\underline{\kappa}_{n,k}, \bar{\kappa}_{n,k}]$ does not contain parameters which are in the complement of symmetric intervals about the true quantile (see Theorem 5.1, (5.3)). Based on this criterion the optimum confidence intervals are determined by those γ and $m(n)$ which have also defined the optimum estimators (see Remark 5.5). In analogy to the relative deficiency of sample quantiles with respect to other estimators we define the relative deficiency $d(n, k, P)$ of the distribution-free confidence intervals $[Z_{[nq]-s(n):n}, Z_{[nq]+s(n):n}]$ with respect to $[\underline{\kappa}_{n,k}, \bar{\kappa}_{n,k}]$ where $s(n)$ is appropriately chosen. We take $d(n, k, P) = r(n) - n$ where $r(n) \equiv r(n, k, P)$ minimizes

$$\Big|P^{r(n)}\Big\{\big[Z_{[r(n)q]-s(r(n)):r(n)}, Z_{[r(n)q]+s(r(n)):r(n)}\big] \subset \big[q(P) - t\sigma(P)n^{-\frac{1}{2}}, q(P) + t\sigma(P)n^{-\frac{1}{2}}\big]\Big\} - P^n\Big\{\big[\underline{\kappa}_{n,k}, \bar{\kappa}_{n,k}\big] \subset \big[q(P) - t\sigma(P)n^{-\frac{1}{2}}, q(P) + t\sigma(P)n^{-\frac{1}{2}}\big]\Big\}\Big|.$$

If $m(n)$ fulfills (3.5) for the case of $k = 1$ or (3.9) for the case of $k = 2$ with $\gamma = \frac{8}{25}$, then the relative "uniform" deficiency (in the sense of (3.6)) of the distribution-free confidence intervals with respect to $[\underline{\kappa}_{n,k}, \bar{\kappa}_{n,k}]$ is again given by (3.7) and (3.10). This can easily be deduced from Theorem 5.1 and (5.4).

4. Estimators. In Sections 4 and 5 we shall always assume that $m(n) \geq n^{\frac{1}{2}+\delta}$ for some $\delta \in (0, \frac{1}{2})$. Furthermore, we shall always write m in place of $m(n)$.

THEOREM 4.1. *Let E be the double-exponential distribution (with the density function $p(x) = e^{-|x|}/2$). For every sequence of translation-equivariant estimators $T_n : \mathbb{R}^n \to \mathbb{R}$ and every $t > 0$*

$$E^n\left\{|T_n| \leq \frac{t}{n^{\frac{1}{2}}}\right\} \leq 2\Phi(t) - 1 + 0(n^{-\frac{1}{2}}).$$

PROOF. According to Corollary 6.11 it suffices to prove that

$$|g_n| \leq a, n \in \mathbb{N}, \quad \text{for some constant} \quad a > 0, \tag{4.2}$$

and

$$\int g_n^2(x)p(x - n^{-\frac{1}{2}}t)\, dx = 4t^2 + 0(n^{-\frac{1}{2}}) \tag{4.3}$$

where

$$g_n(x) = n^{\frac{1}{2}}\left(e^{-|x+n^{-\frac{1}{2}}t|+|x-n^{-\frac{1}{2}}t|} - 1\right).$$

The computations which lead to (4.2) and (4.3) are elementary and therefore omitted. □

Let

$$\mu_n = \left(F^{-1}(\lambda_{1,n}) + F^{-1}(\lambda_{2,n})\right)/2, \tag{4.4}$$

$$\nu_n^2 = \frac{1}{4n}\left(\frac{\lambda_{1,n}(1-\lambda_{1,n})}{p_{1,n}^2} + \frac{2\lambda_{1,n}(1-\lambda_{2,n})}{p_{1,n}p_{2,n}} + \frac{\lambda_{2,n}(1-\lambda_{2,n})}{p_{2,n}^2}\right), \tag{4.5}$$

with

$$\lambda_{1,n} = q - \frac{m}{n}, \quad \lambda_{2,n} = q + \frac{m}{n} \quad \text{and} \quad p_{i,n} = p\left(F^{-1}(\lambda_{i,n})\right)$$

for $i = 1, 2$.

For quasiquantiles Q_n we obtain

THEOREM 4.6. *For every $q \in (0, 1)$*

$$P^n\left\{|Q_n - q(P)| \leqslant \frac{t\sigma(P)}{n^{\frac{1}{2}}}\right\} \geqslant 2\Phi(t(1 + c_{n,1})) - 1 + o\left(\frac{m}{n} + \frac{m^4}{n^3}\right)$$

uniformly for all $t \geqslant 0$ and $P \in \mathcal{P}(q, 1, A)$ where

$$\begin{aligned} c_{n,1} &\equiv c_{n,1}(m, q, P) \\ &= \frac{m}{n}\left(\frac{1}{4q(1-q)} - A\right) - \frac{m^4}{8n^3q(1-q)}\left(\frac{|a_1|}{a_0^2} + A\right)^2. \end{aligned}$$

PROOF. Lemma 6.13, applied for $l = 2$ and $a_{i,n} = \frac{1}{2}p_{i,n}$, $i = 1, 2$, yields that the distribution of Q_n is approximated by a normal distribution $N_{(\mu_n, \nu_n^2)}$ uniformly for all intervals with an error of order $0(n^{-\frac{1}{2}+\varepsilon})$ for every $\varepsilon > 0$. Since the covariances of $N_{r(n)}$ (see Lemma 6.13) are known it is straightforward to see that μ_n and ν_n^2 are given by (4.4) and (4.5).

According to Lemma 6.12, applied for $\mu = q(P)$, it suffices to prove that

$$1 - \frac{n}{\sigma(P)^2}\left(\nu_n^2 + (\mu_n - q(P))^2\right) \geqslant 2c_{n,1} + o\left(\frac{m}{n} + \frac{m^4}{n^3}\right). \tag{4.7}$$

(6.1) implies

$$F^{-1}(\lambda_{i,n}) = q(P) + (-1)^i a_1' \frac{m}{n} + \frac{1}{2}(a_2' + b_{i,n})\left(\frac{m}{n}\right)^2 + 0\left(\left(\frac{m}{n}\right)^3\right), \tag{4.8}$$

with $|b_{i,n}| \leqslant A/a_0$ for $i = 1, 2$.

Thus,

$$(\mu_n - q(P))^2 = \frac{1}{4}(a_2' + (b_{1,n} + b_{2,n})/2)^2\left(\frac{m}{n}\right)^4 + o\left(\left(\frac{m}{n}\right)^4\right). \tag{4.9}$$

Furthermore,

$$p_{i,n} = a_0 - (a_1 + c_{i,n})a_1' \frac{m}{n} + o\left(\left(\frac{m}{n}\right)\right) \tag{4.10}$$

with

$$|c_{i,n}| \leqslant A a_0^2 \quad \text{for} \quad i = 1, 2.$$

Thus,

$$\nu_n^2 = \frac{1}{n}\left[\sigma(P)^2 - \left[\frac{1}{2a_0^2} - \frac{q(1-q)}{a_0^3}(c_{1,n} - c_{2,n})a_1'\right]\frac{m}{n} + o\left(\frac{m}{n}\right)\right]. \tag{4.11}$$

(4.9) and (4.11) immediately imply (4.7). □

A careful examination of (4.8) and (4.10) reveals that the estimate in Theorem 4.6 cannot be improved without depending on the number C (with C as given in (2.3)).

Let $\hat{Q}_n^\gamma$ be the estimator as defined in (3.8).

THEOREM 4.12. *For every* $q \in (0, 1)$

$$\inf_{P \in \mathcal{P}(q,2,A)} P^n\left\{|\hat{Q}_n^\gamma - q(P)| \leqslant \frac{t\sigma(P)}{n^{\frac{1}{2}}}\right\} = 2\Phi(t(1 + c_{n,2}^{(\gamma)})) - 1 + o\left(\frac{m}{n} + \frac{m^6}{n^5}\right)$$

uniformly for all $t \geqslant 0$ *and* $P \in \mathcal{P}(q, 2, A)$ *where*

$$c_{n,2}^{(\gamma)} = \frac{1}{2q(1-q)}\left(\frac{m}{n}\left(\gamma - \frac{5}{4}\gamma^2\right) - \frac{m^6}{n^5}\left(\frac{5}{6}\gamma A\right)^2\right).$$

PROOF. Let $\lambda_{1,n} = q - \frac{2m}{n}$, $\lambda_{2,n} = q - \frac{m}{n}$, $\lambda_{3,n} = q$, $\lambda_{4,n} = q + \frac{m}{n}$, $\lambda_{5,n} = q + \frac{2m}{n}$. Lemma 6.13, applied for $l = 5$, implies that the distribution of $\hat{Q}_n^\gamma$ is approximated by the normal distribution $N_{(\mu_n, \nu_n^2)}$ uniformly for all intervals with an error of order $0(n^{-\frac{1}{2}+\varepsilon})$ for every $\varepsilon > 0$ where

$$\mu_n = -\frac{\gamma}{4}(F^{-1}(\lambda_{1,n}) + F^{-1}(\lambda_{5,n})) + \gamma(F^{-1}(\lambda_{2,n}) + F^{-1}(\lambda_{4,n})) + \left(1 - \frac{3}{2}\gamma\right)q(P)$$

and

$$\nu_n^2 = \frac{1}{n}\left(\Sigma_{i=1}^5 \kappa_{i,n}^2 \sigma_{i,i,n} + 2\Sigma_{1 \leqslant i < j \leqslant 5} \kappa_{i,n}\kappa_{j,n}\sigma_{i,j,n}\right)$$

where

$$\begin{aligned}
\kappa_{1,n} &= -\gamma/4p(F^{-1}(\lambda_{1,n})), & \kappa_{2,n} &= \gamma/p(F^{-1}(\lambda_{2,n})),\\
\kappa_{3,n} &= (1 - \gamma 3/2)/a_0, & \kappa_{4,n} &= \gamma/p(F^{-1}(\lambda_{4,n})),\\
\kappa_{5,n} &= -\gamma/4p(F^{-1}(\lambda_{5,n})), & &\\
\sigma_{i,j,n} &= \lambda_{i,n}(1 - \lambda_{j,n}) & &\text{for} \quad 1 \leqslant i \leqslant j \leqslant 5.
\end{aligned}$$

Straightforward computations give

$$(4.13)\qquad \nu_n^2 = \frac{\sigma(P)^2}{n}\left(1 + \frac{m}{nq(1-q)}\left(-\gamma + \tfrac{5}{4}\gamma^2\right)\right) + o\left(\frac{m}{n}\right).$$

Using (6.1) we obtain by some tedious calculations

$$F^{-1}(\lambda_{1,n}) = q(P) - a_1'\frac{2m}{n} + \frac{a_2'}{2}\left(\frac{2m}{n}\right)^2 - \frac{a_3' + b_{1,n}}{6}\left(\frac{2m}{n}\right)^3,$$

$$F^{-1}(\lambda_{2,n}) = q(P) - a_1'\frac{m}{n} + \frac{a_2'}{2}\left(\frac{m}{n}\right)^2 - \frac{a_3' + b_{2,n}}{6}\left(\frac{m}{n}\right)^3,$$

$$F^{-1}(\lambda_{4,n}) = q(P) + a_1'\frac{m}{n} + \frac{a_2'}{2}\left(\frac{m}{n}\right)^2 + \frac{a_3' + b_{4,n}}{6}\left(\frac{m}{n}\right)^3,$$

$$F^{-1}(\lambda_{5,n}) = q(P) + a_1'\frac{2m}{n} + \frac{a_2'}{2}\left(\frac{2m}{n}\right)^2 + \frac{a_3' + b_{5,n}}{6}\left(\frac{2m}{n}\right)^3$$

where

$$(4.14)\qquad |b_{1,n}|, |b_{4,n}| \leqslant \frac{A}{a_0} + o\left(\frac{m}{n}\right),$$

$$|b_{1,n} - b_{2,n}/8|, |b_{5,n} - b_{4,n}/8| \leqslant \tfrac{7}{8}\frac{A}{a_0} + o\left(\frac{m}{n}\right).$$

This implies

$$(4.15)\qquad |\mu_n - q(P)| = \frac{\gamma}{3}\left(\frac{m}{n}\right)^3((b_{1,n} - b_{2,n}/2) - (b_{5,n} - b_{4,n}/2)) + o\left(\left(\frac{m}{n}\right)^3\right)$$

$$\leqslant \frac{5}{6}\frac{\gamma A}{a_0}\left(\frac{m}{n}\right)^3 + o\left(\left(\frac{m}{n}\right)^3\right).$$

Since the bounds given in (4.14) are attained for p-measures in $\mathcal{P}(q, 2, A)$ it follows that the bound given in (4.15) is attained on $\mathcal{P}(q, 2, A)$. (4.13), (4.15) and Lemma 6.12 imply the assertion.

REMARK 4.16. If $m = m(n)$ fulfills (3.9) then

$$(4.17)\qquad c_{n,2}^{(\frac{8}{25})} \sim n^{-\frac{1}{5}}\left(\tfrac{9}{20}\right)^{\frac{1}{5}}\tfrac{2}{25}A^{-\frac{2}{5}}(q(1-q))^{-1}.$$

Elementary computations show that this is the optimum choice of values γ and m.

By means of (4.13) and a rough estimate of $|\mu_n - q(P)|$ with $P \in \mathcal{P}(q, 3, P)$ we obtain

THEOREM 4.18. *For every $q \in (0, 1)$*

$$(4.19)\qquad P^n\left\{|\hat{Q}_n^\gamma - q(P)| \leqslant \frac{t\sigma(P)}{n^{\frac{1}{2}}}\right\} \geqslant 2\Phi(t(1 + c_{n,3}^{(\gamma)})) - 1 + o\left(\frac{m}{n} + \frac{m^8}{n^7}\right)$$

uniformly for all $t \geqslant 0$ and $P \in \mathcal{P}(q, 3, A)$ where

$$c_{n,3}^{(\gamma)} = \frac{1}{2q(1-q)}\left(\frac{m}{n}\left(\gamma - \tfrac{5}{4}\gamma^2\right) - \frac{m^8}{n^7}\frac{\gamma^2}{16}\left(B + \tfrac{5}{3}A\right)^2\right),$$

B is the supremum of $a_0|a_4'|$ over $\mathcal{P}(q, 3, A)$ and a_4' is defined in (6.2).

5. Confidence intervals. Let $c_{n,k} = c_{n,k}^{(\gamma)}$, $k = 1, 2$, be defined as in the Theorems 4.12 and 4.18. Let $c_{n,1}$ be defined as in Theorem 4.6 with v_1/u^2 in place of a_1/a_0^2. For $k = 1, 2, 3$ let

$$\alpha_{n,k} = t_\alpha(1 - c_{n,k})(q(1-q))^{\frac{1}{2}} n^{\frac{1}{2}}/2m$$

where

$$t_\alpha = \Phi^{-1}(1 - \alpha/2) \qquad \text{for } \alpha \in (0, 1).$$

For $k = 1, 2, 3$ we define the confidence intervals $[\underline{\kappa}_{n,k}, \bar{\kappa}_{n,k}]$ by

$$\underline{\kappa}_{n,k} = T_{n,k} - \alpha_{n,k}(Z_{[nq]+m:n} - Z_{[nq]-m:n})$$

and

$$\bar{\kappa}_{n,k} = T_{n,k} + \alpha_{n,k}(Z_{[nq]+m:n} - Z_{[nq]-m:n})$$

with

$$T_{n,1} = Q_n \quad \text{and} \quad T_{n,k} = \hat{Q}_n^\gamma \quad \text{for} \quad k = 2, 3.$$

THEOREM 5.1. *For every $k = 1, 2, 3$ and every $q \in (0, 1)$*

$$P^n\{\underline{\kappa}_{n,k} \leqslant q(P) \leqslant \bar{\kappa}_{n,k}\} \geqslant 1 - \alpha + o\left(\frac{m}{n} + \frac{m^{2(k+1)}}{n^{2k+1}}\right), \tag{5.2}$$

and for every $t \geqslant t_\alpha$

$$P^n\left\{[\underline{\kappa}_{n,k}, \bar{\kappa}_{n,k}] \subset \left[q(P) - \frac{t\sigma(P)}{n^{\frac{1}{2}}}, q(P) + \frac{t\sigma(P)}{n^{\frac{1}{2}}}\right]\right\} \tag{5.3}$$
$$\geqslant 2\Phi(t(1 + c_{n,k}) - t_\alpha) - 1 + o\left(\frac{m}{n} + \frac{m^{2(k+1)}}{n^{2k+1}}\right)$$

uniformly for every $P \in \mathcal{P}(q, k, A)$.

PROOF. We shall only prove the assertion for the case of $k = 1$. Then the proofs for the cases $k = 2, 3$ will be apparent. The index "1" in $\underline{\kappa}_{n,1}$ etc. is suppressed. Let $\alpha_n' = n^{-\frac{1}{2}} t_\alpha(1 - c_n)\sigma(P)$. Lemma 6.13 implies that $\underline{\kappa}_n + \alpha_n'$ and $\bar{\kappa}_n - \alpha_n'$ are distributed as Q_n uniformly over all intervals within an error of order $o(\delta_n)$ where $\delta_n = \dfrac{m}{n} + \dfrac{m^4}{n^3}$. This implies

$$\begin{aligned} P^n\{\underline{\kappa}_n &\leqslant q(P) \leqslant \bar{\kappa}_n\} \\ &= P^n\{\underline{\kappa}_n + \alpha_n' \leqslant q(P) + \alpha_n'\} - P^n\{\bar{\kappa}_n - \alpha_n' \leqslant q(P) - \alpha_n'\} \\ &= N_{(\mu_n, \nu_n^2)}(-\infty, q(P) + \alpha_n') - N_{(\mu_n, \nu_n^2)}(-\infty, q(P) - \alpha_n') + o(\delta_n) \\ &= N_{(\mu_n, \nu_n^2)}(q(P) - \alpha_n', q(P) + \alpha_n', q(P) + \alpha_n') + o(\delta_n) \end{aligned}$$

where μ_n and ν_n^2 are given in (4.4) and (4.5). Lemma 6.12 together with (4.7) immediately implies (5.2) for $k = 1$.

Furthermore, uniformly over $\mathcal{P}(q, 1, A)$.

$$\begin{aligned}
P^n\{[\underline{\kappa}_n, \bar{\kappa}_n] \subset [q(P) - t\sigma(P)n^{-\frac{1}{2}}, q(P) + t\sigma(P)n^{-\frac{1}{2}}]\} \\
&= P^n\{\bar{\kappa}_n - \alpha_n' \leqslant q(P) + (t - t_\alpha(1 - c_n))\sigma(P)n^{-\frac{1}{2}}\} \\
&\quad - P^n\{\underline{\kappa}_n + \alpha_n' \leqslant q(P) - (t - t_\alpha(1 - c_n))\sigma(P)n^{-\frac{1}{2}}\} \\
&= N_{(\mu_n, \nu_n^2)}\big(q(P) - (t - t_\alpha(1 - c_n))\sigma(P)n^{-\frac{1}{2}}, q(P) \\
&\quad + (t - t_\alpha(1 - c_n))\sigma(P)n^{-\frac{1}{2}}\big) + o(\delta_n) \\
&\geqslant 2\Phi(t(1 + c_n) - t_\alpha) - 1 + o(\delta_n). \qquad \square
\end{aligned}$$

It can easily be proved by means of Lemma 6.13, applied for $l = 1$, that for the distribution-free confidence intervals $[Z_{[nq]-s(n):n}, Z_{[nq]+s(n):n}]$ with confidence coefficient $1 - \alpha + o(n^{-\frac{1}{2}})$ the following holds true:

$$(5.4)\quad \begin{aligned} P^n\{[Z_{[nq]-s(n):n}, Z_{[nq]+s(n):n}] \subset [q(P) - t\sigma(P)n^{-\frac{1}{2}}, q(P) + t\sigma(P)n^{-\frac{1}{2}}]\} \\ = 2\Phi(t - t_\alpha) - 1 + o(n^{-\frac{1}{2}+\varepsilon}) \end{aligned}$$

for every $\varepsilon > 0$ and every $t > t_\alpha$ uniformly for all $P \in \mathcal{P}(q, 1, A)$.

We remark that (5.4) also holds for confidence intervals which are based on the sample quantile as proposed by Siddiqui (1960).

REMARK 5.5. The probabilities in (5.3) as well as the covering probabilities of the estimators Q_n and $\hat{Q}_n^\gamma$ (see Theorems 4.6, 4.12 and 4.18) depend on the family $\mathcal{P}(q, k, A)$ through $c_{n,1}$ and $c_{n,k}^{(\gamma)}$ for $k = 2, 3$. Therefore, the optimum values γ and $m(n)$—in the sense of asymptotically maximizing the right-hand side of (5.3)—are exactly those which were already determined in the case of the optimum estimator sequences.

6. Auxiliary results. Let P fulfill the conditions (2.3) and (2.4) for some $k \in \mathbb{N}$. Denote by F the distribution function of P. Then there exists some $\varepsilon' > 0$ such that the inverse function F^{-1} of F exists on $[q - \varepsilon', q + \varepsilon']$ and

$$(6.1)\qquad F^{-1}(x) = \Sigma_{i=0}^k \frac{a_i'}{i!}(x - q)^i + \frac{a_{k+1}' + \tilde{g}_{k+1}(x)}{(k + 1)!}(x - q)^{k+1}$$

with $|\tilde{g}_{k+1}(x)| = \frac{A}{a_0} + 0(|x - q|)$ uniformly for all $x \in [q - \varepsilon', q + \varepsilon']$ and $P \in \mathcal{P}(q, k, A)$. The numbers a_i' only depend on $a_0, \cdots, a_{i-1}$, $i = 1, \cdots, k$. In particular,

$$(6.2)\qquad \begin{aligned} a_0' &= q(P), \qquad a_1' = a_0^{-1}, \\ a_2' &= -\frac{a_1}{a_0^3}, \qquad a_3' = -\frac{a_2}{a_0^4} + 3\frac{a_1^2}{a_0^4}, \\ a_4' &= -\frac{a_3}{a_0^5} + 10\frac{a_1 a_2}{a_0^6} - 15\frac{a_1}{a_0^7}. \end{aligned}$$

The proof of (6.1) is straightforward and therefore omitted.

LEMMA 6.3. *Let $P_{o,n}$, $P_{1,n}$, $n \in \mathbb{N}$, be p-measures on a measurable space $(X, \mathcal{Q})$. Let*

$$P_{o,n} \ll P_{1,n}, \tag{6.4}$$

$$g_n := n^{\frac{1}{2}}(p_{o,n} - p_{1,n}) / (p_{1,n} + 1_{\{p_{1,n}=0\}}) \tag{6.5}$$

where $p_{i,n}$ are densities of $P_{i,n}$ with respect to some σ-finite measure for $i = 1, 2$, $n \in \mathbb{N}$. If

$$|g_n| \leq a,\ n \in \mathbb{N}, \text{for some constant } a > o \tag{6.6}$$

and

$$\liminf_{n \in \mathbb{N}} \int g_n^2 \, dP_{1,n} > 0 \tag{6.7}$$

then

$$\int \psi_n \, dP_{1,n}^n - \int \psi_n \, dP_{o,n}^n \leq 2\Phi\left(\tfrac{1}{2}\left(\int g_n^2 \, dP_{1,n}\right)^{\frac{1}{2}}\right) - 1 + 0\left(n^{-\frac{1}{2}}\right)$$

for every sequence of $\mathcal{Q}^n$-measurable functions $\psi_n : X^n \to [0, 1]$, $n \in \mathbb{N}$.

PROOF. Let $t_n = g_n - \frac{1}{2}n^{-\frac{1}{2}}g_n^2$, $\mu_{j,n} = \int t_n \, dP_{j,n}$ and $\sigma_{j,n}^2 = \int((t_n - \mu_{j,n})^2) \, dP_{j,n}$ for $j = 1, 2$. Obviously, $P_{o,n}$ has the density $1 + n^{-\frac{1}{2}}g_n$ with respect to $P_{1,n}$ and $\int g_n \, dP_{1,n} = 0$. This implies

$$\mu_{o,n} = \frac{1}{2n^{\frac{1}{2}}} \int g_n^2 \, dP_{1,n} + 0(n^{-1})$$

$$\mu_{1,n} = -\frac{1}{2n^{\frac{1}{2}}} \int g_n^2 \, dP_{1,n},$$

$$\sigma_{j,n}^2 = \int g_n^2 \, dP_{1,n} + 0\left(n^{-\frac{1}{2}}\right) \qquad \text{for } j = 1, 2.$$

For $c > 0$ define

$$C_{n,c} := \left\{(x_1, \cdots, x_n) \in X^n;\ \Pi_{i=1}^n p_{1,n}(x_i) > c\Pi_{i=1}^n p_{o,n}(x_i)\right\}.$$

Applying the Berry-Esseen theorem we obtain for $j = 1, 2$

(6.8) $P_{j,n}^n(C_{n,c})$

$$= P_{j,n}^n\left\{\Sigma_{i=1}^n \log\left(1 + n^{-\frac{1}{2}}g_n(x_i)\right) < -\log c\right\}$$

$$= P_{j,n}^n\left\{\frac{1}{n^{\frac{1}{2}}\sigma_{j,n}}\Sigma_{i=1}^n(t_n(x_i) - \mu_{j,n}) < \left(-\log c - n^{\frac{1}{2}}\mu_{j,n} + 0\left(n^{-\frac{1}{2}}\right)\right)/\sigma_{j,n}\right\}$$

$$= \Phi\left(\left(-\log c - n^{\frac{1}{2}}\mu_{j,n}\right)/\sigma_{j,n}\right) + 0\left(n^{-\frac{1}{2}}\right)$$

uniformly for all $c > 0$.

(6.8) immediately implies

$$P_{1,n}^n(C_{n,c}) - P_{o,n}^n(C_{n,c}) \leq 2\Phi\left(\tfrac{1}{2}\left(\int g_n^2 \, dP_{1,n}\right)^{\frac{1}{2}}\right) - 1 + 0\left(n^{-\frac{1}{2}}\right) \tag{6.9}$$

uniformly for all $c > 0$ since

$$N_{(0,1)}(\mu - d, \mu + d) \leqslant N_{(0,1)}(-d, d) \qquad \text{for } \mu \in \mathbb{R} \text{ and } d > 0.$$

For $c = 1$ we have equality in (6.9) up to terms of order $0(n^{-\frac{1}{2}})$.

Let $\psi_n : X^n \to [0, 1]$ be a sequence of $\mathscr{Q}^n$-measurable functions. Define $c_n > 0$, $n \in \mathbb{N}$, such that

$$P_{o,n}^n(C_{n,c_n}) - bn^{-\frac{1}{2}} \leqslant \int \psi_n \, dP_{o,n}^n \leqslant P_{o,n}^n(C_{n,c_n}) \tag{6.10}$$

for some $b > 0$. Using a suitable version of the fundamental lemma of Neyman and Pearson (see Pfanzagl (1974), Lemma 6, page 45) we obtain

$$\int \psi_n \, dP_{1,n}^n - P_{1,n}^n(C_{n,c_n}) \leqslant c_n\left(\int \psi_n \, dP_{o,n}^n - P_{o,n}^n(C_{n,c_n})\right) \leqslant 0.$$

(6.9) together with (6.10) implies

$$\begin{aligned} \int \psi_n \, dP_{1,n}^n - \int \psi_n \, dP_{o,n}^n &\leqslant P_{1,n}^n(C_{n,c_n}) - P_{o,n}^n(C_{n,c_n}) + 0(n^{-\frac{1}{2}}) \\ &\leqslant 2\Phi\left(\tfrac{1}{2}\left(\int g_n^2 \, dP_{1,n}\right)^{\frac{1}{2}}\right) - 1 + 0(n^{-\frac{1}{2}}). \end{aligned}$$

COROLLARY 6.11. *Let P be a p-measure on the Borel-algebra of $\mathbb{R}$. Assume that $T_n : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of translation-equivariant estimators. Let $t > 0$. If the assumptions of Lemma 6.3 hold true for $P_{o,n} := {}_{-n^{-\frac{1}{2}}t}P$ and $P_{1,n} := {}_{n^{-\frac{1}{2}}t}P$ (that is P shifted to the left and right, respectively, by the amount of $n^{-\frac{1}{2}}t$) then for every $\kappa \in \mathbb{R}$*

$$P^n\left\{|T_n - \kappa| \leqslant n^{-\frac{1}{2}}t\right\} \leqslant 2\Phi\left(\tfrac{1}{2}\left(\int g_n^2 \, dP_{1,n}\right)^{\frac{1}{2}}\right) - 1 + 0(n^{-\frac{1}{2}}).$$

PROOF. Lemma 6.3 applied for $\psi_n := 1_{\{T_n > \kappa\}}$ implies

$$\begin{aligned} P^n\left\{|T_n - \kappa| \leqslant n^{-\frac{1}{2}}t\right\} &= P^n\left\{T_n + n^{-\frac{1}{2}}t > \kappa\right\} - P^n\left\{T_n - n^{-\frac{1}{2}}t > \kappa\right\} \\ &= P_{1,n}^n\{T_n > \kappa\} - P_{o,n}^n\{T_n > \kappa\} \\ &\leqslant 2\Phi\left(\tfrac{1}{2}\left(\int g_n^2 \, dP_{1,n}\right)^{\frac{1}{2}}\right) - 1 + 0(n^{-\frac{1}{2}}). \end{aligned}$$

LEMMA 6.12. *For all sequences $\mu_n \in \mathbb{R}$, $\nu_n > 0$, $n \in \mathbb{N}$, and constants $\mu \in \mathbb{R}$, $\sigma > 0$*

$$\begin{aligned} N_{(\mu_n, \nu_n^2)}\left(\mu - \frac{t}{n^{\frac{1}{2}}}, \mu + \frac{t}{n^{\frac{1}{2}}}\right) &= 2\phi\left(t\left(1 + \tfrac{1}{2}\left(1 - \frac{n}{\sigma^2}\left(\nu_n^2 + (\mu_n - \mu)^2\right)\right)\right)\right) - 1 \\ &\quad + 0\left(\left(\max\left\{|n^{\frac{1}{2}}\nu_n - \sigma|, n(\mu_n - \mu)^2\right\}\right)^{\frac{3}{2}}\right) \end{aligned}$$

uniformly for all $t \geqslant 0$ and $\sigma \geqslant c > 0$.

PROOF. Use Taylor expansions of Φ about t.$\square$

Given $r_{i,n} \in \mathbb{N}$, $i = 0, \cdots, l+1$, $l \in \mathbb{N}$, with $0 = r_{o,n} < r_{1,n} < \cdots < r_{l+1,n} = n$ let $\underline{r}(n) = (r_{1,n}, \cdots, r_{l+1,n})$. Let $N_{\underline{r}(n)}$ denote the l-variate normal distribution with mean vector zero and covariance matrix $(\sigma_{i,j,n})_{i,j=1,\cdots,l}$ where $\sigma_{i,j,n} = \frac{r_{i,n}}{n}\left(1 - \frac{r_{j,n}}{n}\right)$ for $1 \leqslant i \leqslant j \leqslant l$.

LEMMA 6.13. *Let P be a p-measure which fulfills* (2.3) *for some $q \in (0, 1)$. Let $r_{i,n}$, $i = 1, \cdots, l$, be such that $r_{i,n} \sim nq$, and $r_{i+1,n} - r_{i,n} \sim m$ for $i = 1, \cdots, l-1$ where $n^{1-\delta} \geqslant m \geqslant n^{\delta}$ for some $\delta \in (0, 1)$. Let $a_{i,n} \in \mathbb{R}$, $i = 1, \cdots, l$, $n \in \mathbb{N}$, be such that*

$$\liminf_{n\in\mathbb{N}}\left|\Sigma_{i=1}^{l} a_{i,n}\right| > 0, \tag{6.14}$$

and the sequence $a_{i,n}$, $n \in \mathbb{N}$, is bounded for every $i = 1, \cdots, l$. Then for every $\varepsilon > 0$

$$P^n\left\{n^{\frac{1}{2}}\Sigma_{i=1}^{l} a_{i,n}p\left(F^{-1}\left(\frac{r_{i,n}}{n}\right)\right)\left(Z_{r_{i,n}:n} - F^{-1}\left(\frac{r_{i,n}}{n}\right)\right) < t\right\}$$
$$= N_{\underline{r}(n)}\left\{(x_1, \cdots, x_l) \in \mathbb{R}^l : \Sigma_{i=1}^{l} a_{i,n}x_i < t\right\} + 0\left(n^{-\frac{1}{2}+\varepsilon}\right)$$

uniformly for all $t \in \mathbb{R}$.

PROOF. I. We shall first prove the assertion for the case of the uniform distribution Q on (0, 1). The theorem in Reiss (1975) implies for every $\varepsilon > 0$

$$Q^n\left\{n^{\frac{1}{2}}\Sigma_{i=1}^{l} a_{i,n}\left(Z_{r_{i,n}:n} - \frac{r_{i,n}}{n}\right) < t\right\} \tag{6.15}$$
$$= \left(N_{\underline{r}(n)} + M_{\underline{r}(n)}\right)\left\{(x_1, \cdots, x_1) \in \mathbb{R}^l : \Sigma_{i=1}^{l} a_{i,n}x_i < t\right\} + 0\left(n^{-\frac{1}{2}+\varepsilon}\right)$$

where $M_{\underline{r}(n)}$ is a signed measure with

$$M_{\underline{r}(n)}(B) = 0\left(m^{-\frac{1}{2}+\varepsilon}N_{\underline{r}(n)}(B)\right) \tag{6.16}$$

uniformly over all l-dimensional Borel-sets B.

(6.16) implies that for Borel-measurable functions $S_i : \mathbb{R}^l \to \mathbb{R}$, $i = 1, 2$, and every $\varepsilon > 0$

$$M_{\underline{r}(n)}\{S_1 + S_2 < t\} \tag{6.17}$$
$$= M_{\underline{r}(n)}\{S_1 < t\} + 0\left(m^{-\frac{1}{2}+\varepsilon}N_{\underline{r}(n)}(\{S_1 < t\}\Delta\{S_2 < t\})\right)$$
$$= M_{\underline{r}(n)}\{S_1 < t\} + 0\left(m^{-\frac{1}{2}+\varepsilon}N_{\underline{r}(n)}\{|S_1 - t| < |S_2|\}\right)$$

uniformly for all $t \in \mathbb{R}$.

Applying (6.17) for $S_1(x_1, \cdots, x_l) := (\Sigma_{i=1}^{l} a_{i,n})x_1$ and $S_2(x_1, \cdots, x_l) := \Sigma_{i=1}^{l} a_{i,n}(x_i - x_1)$ we obtain

$$M_{\underline{r}(n)}\left\{(x_1, \cdots, x_l) \in \mathbb{R}^l; \Sigma_{i=1}^{l} a_{i,n}x_i < t\right\} \tag{6.18}$$
$$= 0\left(m^{-\frac{1}{2}+\varepsilon}N_{\underline{r}(n)}\{|S_1 - t| < |S_2|\} + n^{-\frac{1}{2}+\varepsilon}\right)$$

since the theorem in Reiss (1975), applied for $k = 1$ and $k = l$, yields

$$(N_{\underline{r}(n)} + M_{r(n)})\{S_1 < t\} = N_{r(n)}\{S_1 < t\} + 0(n^{-\frac{1}{2}+\varepsilon})$$

for every $\varepsilon > 0$.

Lemma (7.21) in Reiss (1975) implies

$$N_{\underline{r}(n)}\left\{|S_2| > \left(\frac{m}{n}\right)^{\frac{1}{2}+\varepsilon}\right\} = 0(n^{-\frac{1}{2}}) \tag{6.19}$$

for every $\varepsilon > 0$. (6.19) together with (6.14) implies for every $\varepsilon > 0$

$$N_{\underline{r}(n)}\{|S_1 - t| < |S_2|\} = 0\left(2\Phi\left(\left(\frac{m}{n}\right)^{\frac{1}{2}+\varepsilon}\right) - 1\right) = 0\left(\left(\frac{m}{n}\right)^{\frac{1}{2}+\varepsilon}\right).$$

This together with (6.15) and (6.18) implies the assertion for $P = Q$.

II. Using the probability integral transformation for order statistics we obtain

$$\begin{aligned} &P^n\left\{n^{\frac{1}{2}}\Sigma_{i=1}^{l}a_{i,n}p\left(F^{-1}\left(\frac{r_{i,n}}{n}\right)\right)\left(Z_{r_{i,n}:n} - F^{-1}\left(\frac{r_{i,n}}{n}\right)\right) < t\right\} \\ &\quad = Q^n\left\{n^{\frac{1}{2}}\Sigma_{i=1}^{l}a_{i,n}p\left(F^{-1}\left(\frac{r_{i,n}}{n}\right)\right)\left(F^{-1}(Z_{r_{i,n}:n}) - F^{-1}\left(\frac{r_{i,n}}{n}\right)\right) < t\right\}\end{aligned} \tag{6.20}$$

where F^{-1} denotes the generalized inverse of F.

According to Lemma (7.9) in Reiss (1975)

$$Q^n\left\{\left|Z_{r_{i,n}:n} - \frac{r_{i,n}}{n}\right| \leqslant \frac{\log n}{n^{\frac{1}{2}}}\right\} = 1 + 0(n^{-1}).$$

Since F^{-1} is differentiable on $(q - \varepsilon', q + \varepsilon')$ and $(F^{-1})^{(1)}$ fulfills a Lipschitz condition on $(q - \varepsilon', q + \varepsilon')$ for some $\varepsilon' > 0$ we know that

$$\begin{aligned} &Q^n\left\{\left|F^{-1}(Z_{r_{i,n}:n}) - F^{-1}\left(\frac{r_{i,n}}{n}\right)\right.\right. \\ &\quad \left.\left. - \left(Z_{r_{i,n}:n} - \frac{r_{i,n}}{n}\right)\Big/p\left(F^{-1}\left(\frac{r_{i,n}}{n}\right)\right)\right| \leqslant n^{-1+\varepsilon}\right\} = 1 + 0(n^{-1})\end{aligned} \tag{6.21}$$

for every $\varepsilon > 0$.

Part I of the proof, (6.20) and (6.21) immediately imply the assertion for P, in general.

REMARK 6.14. A closer examination of the proof of Lemma 6.13 shows that the assertion holds true uniformly for all p-measures P which fulfill (2.3) and the condition $p(q(P)) \geqslant c > o$ for some fixed constants C, ε, and c, and, furthermore, uniformly over every family L of sequences $(a_{1,n}, \cdots, a_{l,n})$, $n \in \mathbb{N}$, for which $\lim\inf_{n\in\mathbb{N}}|\lim \Sigma_{i=1}^{l}a_{i,n}| > 0$ and $\sup|a_{i,n}|$, $n \in \mathbb{N}$, is bounded for $i = 1, \cdots, l$ where the inf and sup ranges over L.

7. Concluding remarks. The results of the present paper are far away from being the final word on the subject. We mention some open questions and indicate consequences of results in Reiss (1978) concerning one-sided tests for quantiles.

Except for the negative result in (3.1), we have excluded the case of $k = 1$ and $A \geqslant 1/4q(1-q)$. It would be interesting to know whether under these conditions quasiquantiles—defined with m of order $n^{\frac{1}{2}}$—are more concentrated on intervals $[-tn^{-\frac{1}{2}}, tn^{-\frac{1}{2}}]$ than sample quantiles as far as differences of order $n^{-\frac{1}{2}}$ are concerned. For comparison we need asymptotic expansions to two terms of the distribution of sample quantiles and quasiquantiles. Asymptotic expansions of the distribution of sample quantiles can easily be derived from Theorem 2.7 in Reiss (1976) (under appropriate regularity conditions).

In view of the relative deficiencies in the cases $k = 1, 2, 3$ which were obtained in Section 3, we conjecture that sample quantiles have deficiencies of order $n^{2k/(2k+1)}$ for $P \in \mathcal{P}(q, k, A)$, $k \in \mathbb{N}$ (with respect to an optimal estimator sequence). This also leads to the following question: assume that all derivatives of the distribution function exist and all derivatives are uniformly bounded by some fixed number. Are sample quantiles inefficient then?

An answer concerning the order of the deficiency of sample quantiles can be given if we base our measure of performance of estimators on covering probabilities with respect to any interval which contains the unknown quantile. Applying Lemma 6.13 we find for any $t', t'' > 0$ and $k = 2, 3$ sequences $m(n)$, $n \in \mathbb{N}$, of order $n^{(2k-1)/2k}$ such that

$$(7.1)\quad P^n\left\{q(P) - \frac{t'\sigma(P)}{n^{\frac{1}{2}}} \leqslant \hat{Q}_n \leqslant q(P) + \frac{t''\sigma(P)}{n^{\frac{1}{2}}}\right\}$$
$$\geqslant \Phi(t'') - \Phi(t') + \beta n^{-\frac{1}{2k}}(t''\varphi(t'') + t'\varphi(t'))$$

uniformly for every $P \in \mathcal{P}(q, k, A)$ for some $\beta > 0$. Notice that for $t' = t''$ we found estimators such that (7.1) even holds true for $n^{-1/(2k+1)}$ in place of $n^{-1/2k}$. On the other hand, by means of results in Reiss (1978), concerning one-sided tests, one can prove that (7.1) is the best obtainable result in the following sense: let $Q \in \mathcal{P}(q, k, B)$ and $\mathcal{P}(Q, \varepsilon)$ be defined as in Section 1. Let T_n, $n \in \mathbb{N}$, be translation-equivariant estimators which are asymptotically median unbiased of order $o(n^{-\frac{1}{2}k})$ uniformly over $\mathcal{P}(Q, \varepsilon) \cap \mathcal{P}(q, k, A)$ with $B < A$ (e.g. estimators which fulfill (7.1) uniformly for all $t', t'' > 0$ have this property). Then for every $k = 2, 3, 4, \cdots$

$$(7.2)\quad Q^n\left\{q(P) - \frac{t'\sigma(P)}{n^{\frac{1}{2}}} \leqslant T_n \leqslant q(P) + \frac{t''\sigma(P)}{n^{\frac{1}{2}}}\right\}$$
$$\leqslant \Phi(t'') - \Phi(t') + 0\big(n^{-\frac{1}{2}k}(t''\varphi(t'') + t'\varphi(t'))\big)$$

uniformly for all $t', t'' > 0$.

Thus, estimators which are accurate with respect to every interval containing the true quantile are less effective on symmetric intervals (and vice versa).

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