PERMUTATION TESTS FOR DIRECTIONAL DATA

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The two-sample problem for directional data with dimension greater than one is considered. A large family of permutation tests is proposed and studied. The statistics upon which the tests are based are related to those introduced by Giné in the context of tests for uniformity, and are defined in terms of Sobolev norms. Examples treated include the unit spheres S^p and hemispheres H^p for directions in $(p + 1)$-dimensional Euclidean space, and the torus $T^2 = S^1 \times S^1$ for pairs of directions in two dimensions. Computable forms of the statistics with specified consistency properties are obtained for each of these examples. Sampling from the permutation distribution is proposed as a means of implementing the tests in practice. Several tests for uniformity on the torus T^2 are also obtained.

1. Introduction. Several recent papers in the area of directional data have addressed the problem of testing for uniformity on the circle, sphere or, more generally, a compact Riemannian manifold. Giné (1975) introduced a class of invariant tests for uniformity based on Sobolev norms and showed that his class of tests contains tests previously proposed by Rayleigh, Watson, Ajne, Beran and Bingham for testing uniformity on the circle, sphere and hemisphere. He also produced several new tests consistent against all alternatives. Prentice (1978) applied Giné's results to obtain tests for uniformity on the sphere S^p and hemisphere H^p in $(p + 1)$-dimensional Euclidean space.

Another type of problem confronting the statistician is the two-sample problem of comparing two distributions on the circle, sphere or some other specified Riemannian manifold. In the case of directions in two dimensions (observations on the unit circle S^1 in $\mathbb{R}^2$) several nonparametric two-sample tests which are invariant already exist: e.g., Watson's form of the two-sample Cramér-von Mises statistic (cf. Watson (1962), Durbin ((1973), page 47), Mardia ((1972), page 201)), the two-sample Kuiper statistic (Kuiper (1960), Durbin ((1973), page 46), Mardia ((1972), page 201)), or the tests due to Beran (1969) and Schach (1969). This is a one dimensional situation. For higher dimensional situations, such as directions in three dimensions (observations on the unit sphere S^2 in $\mathbb{R}^3$), nonparametric two-sample tests are generally unavailable. The small number of available two-sample tests are for parametric models or for specific classes of distributions: see, for example, Watson and Williams (1956), Mardia ((1972), page 263 ff.) and Wellner (1978).

The purpose of the present paper is to introduce a class of permutation tests for two-sample problems involving directional data in higher dimensional situations. To allow for flexibility in applications, we follow Giné (1975) and formulate our tests for an arbitrary compact (connected) Riemannian manifold $\mathbb{X}$; important cases of interest are the unit spheres S^p and hemispheres H^p in $(p + 1)$-dimensions with $p \geqslant 2$ and the torus $T^2 = S^1 \times S^1$. The test statistics defined in Section 2 are based on the Sobolev norms introduced by Giné in the context of tests for uniformity. These statistics are related to Giné's statistics for testing uniformity in much the same way that Watson's two-sample statistic on the circle is related to the invariant form of the Cramér-von Mises statistic (also due to Watson) for testing uniformity on the circle (cf. Durbin (1973), page 36 and 47). There are clear connections between the statistics proposed here and in Giné (1975), and the "components" approach to the ordinary Cramér-von Mises statistic due to Durbin and Knott (1972).

Our permutation tests are also related to the test introduced by Bickel (1969) in the context of the two-sample problem for observations in $\mathbb{R}^k$, $k \geqslant 2$. Bickel used the Kolmogorov or supremum distance between distribution functions together with the permutation principle to obtain a two-sample test consistent against all alternatives. Here we deal with a family of statistics determined by a sequence of weights $\{a_k\}$. As will be shown in Section 3 and the Appendix, our permutation tests are consistent against all alternatives if all the a_k's are nonzero and consistent against specified alternatives if only some a_k's are nonzero.

It will usually not be possible to carry out the permutation tests proposed in Section 2 exactly unless the sample sizes are quite small; this is a drawback of any permutation test. In practice, however, the tests can be carried out by sampling from the permutation distribution. The idea of implementing a permutation test by sampling from the permutation distribution is apparently due to Dwass (1957), and has been suggested before in the context of directional data by Watson and Beran (1967). For examples of the use of such procedures (in problems not involving directional data) see Gabriel and Feder (1969) and Green (1977).

The paper is organized as follows: Section 2 contains the tests and the necessary permutation theory. Consistency and other asymptotic properties of the tests are stated in Section 3, and examples are considered in Section 4. Invariant two-sample tests consistent against all alternatives are obtained in computable form for the sphere S^p and hemisphere H^p in $(p + 1)$-dimensions (Examples 1c and 1d), and for the torus $T^2 = S^1 \times S^1$ (Example 2c). Several invariant tests consistent against specified alternatives on each of S^p, H^p, and T^2 are also given; see Examples 1a, 1b, 2a, and 2b. Finally, the Appendix contains proofs of the consistency and asymptotic properties stated in Section 3. Our consistency results are based on Theorems A and B of the Appendix; these theorems are permutational versions of Theorem 3.4 of Giné (1975).

For precise definitions and detailed technical information concerning Riemannian manifolds and their isometry groups, the Laplacian, and Sobolev norms, we refer the reader to Sections 2 and 5 of Giné (1975) and the references given there.

2. The permutation tests. Let X be a compact (connected) Riemannian manifold with Riemannian metric d and isometry group G. Let ρ and σ be two Borel probability measures on X. Denote the "uniform measure" on X by μ. Suppose we observe $X_1, \cdots, X_m$ i.i.d. ρ-distributed random variables with values in X, and $Y_1, \cdots, Y_n$ i.i.d. σ-distributed random variables with values in X. Denote the set of all Borel probability measures on X by $\mathcal{P}(X)$; let $\Theta = \{(\rho, \sigma) : \rho, \sigma \in \mathcal{P}(X)\}$, the collection of all pairs of Borel measures on X; and set $\Theta_0 = \{\theta \in \Theta : \theta = (\rho, \rho)$ for some $\rho \in \mathcal{P}(X)\}$. Our goal is to find level $\alpha > 0$ invariant tests of

$$(2.0) \qquad\qquad H : \theta \in \Theta_0 \text{ versus } K : \theta \notin \Theta_0$$

(i.e., $H : \rho = \sigma$ unspecified, versus $K : \rho \neq \sigma$) which are consistent against all (or specified) alternatives in $\Theta_1 \equiv \Theta \setminus \Theta_0$. Stated in the language of test functions, we want to find a sequence of measurable critical or test functions $\phi_{m,n} : (X)^m \times (X)^n \to [0, 1]$ which, for a given $0 < \alpha \leq 1$, satisfy

$$(2.1) \qquad\qquad E_\theta\{\phi_{m,n}(X, Y)\} \leq \alpha \qquad \text{for all} \quad \theta \in \Theta_0,$$

$$(2.2) \qquad\qquad \phi_{m,n}(X, Y) = \phi_{m,n}(gX, gY) \qquad \text{for all} \quad g \in G,$$

and

$$(2.3) \qquad\qquad \lim_{m,n \to \infty} E_\theta\{\phi_{m,n}(X, Y)\} = 1 \qquad \text{for all} \quad \theta \notin \Theta_0.$$

Note that our testing problem is invariant and hence (2.2) is a natural requirement. (For g in the isometry group G of X define $\bar{g} : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\bar{g}\nu = \nu \cdot g^{-1}$ for $\nu \in \mathcal{P}(X)$ (i.e., $(\bar{g}\nu)(A) = \nu(g^{-1}(A))$ for $A \in \Sigma$ where Σ denotes the Borel sigma field of X); and define $g^* : \Theta \to \Theta$ by $g^*\theta = (\bar{g}\rho, \bar{g}\sigma)$ where $\theta = (\rho, \sigma)$. Then $G^* \equiv \{g^* : g \in G\}$ is a group, and $g^*\Theta_0 = \Theta_0$ and $g^*\Theta_1 = \Theta_1$ for all $g^* \in G^*$.)

The statistics upon which our tests will be based are defined as follows: Set $N \equiv m + n$ and $\lambda_N = mN^{-1}$. Let $\rho_m = m^{-1}\Sigma_{i=1}^m \delta_{X_i}$ and $\sigma_n = n^{-1}\Sigma_{j=1}^n \delta_{Y_j}$ denote the empirical measures associated with the X and Y samples respectively. (Here δ_x denotes the probability measure with mass 1 at x.) Now define statistics $T_{m,n}^{(s)} = T_{m,n}^{(s)}(\{a_k\})$ by

$$(2.4) \qquad T_{m,n}^{(s)}(\{a_k\}) \equiv \frac{mn}{N} \Sigma_{k=1}^\infty a_k^2 \Sigma_{f_i \in E_k} \left\{ \int_X f_i d(\rho_m - \sigma_n) \right\}^2.$$

Here, as in Giné (1975), the functions $\{f_i\}$ are an orthonormal basis of $L_2(X, \mu)$ consisting of eigenfunctions of the Laplace-Beltrami operator (Laplacian) Δ of X, E_k denotes the kth eigenspace of Δ with eigenvalue e_k, $s > (\dim X)/2$ is a real number, and $\{a_k\}$ is a sequence of real numbers or weights (which may be chosen by the statistician) subject only to the restriction that $\sup_k |a_k e_k^{s/2}| < \infty$.

Note that $T_{m,n}^{(s)}$ is simply a weighted sum of squared Fourier coefficients of $(\rho_m - \sigma_n)$ with respect to the orthonormal system $\{f_i\}$ (equivalently, a weighted sum of squared differences of the Fourier coefficients $\int_{\mathbb{X}} f_i d\rho_m$ and $\int_{\mathbb{X}} f_i d\sigma_n$) with weights which depend only on the eigenspaces. This last requirement yields the desired invariance of the statistics, while the restriction $\sup_k |a_k e_k^{s/2}| < \infty$ with $s > (\dim X)/2$ guarantees convergence of the series.

The results of Giné (1975) imply that

$$T_{m,n}^{(s)}(\{a_k\}) = \frac{mn}{N} \|(\Sigma_{k=1}^{\infty} a_k e_k^{s/2} \pi_k)(\rho_m - \sigma_n)\|_{-s}^2$$

where $\| \cdot \|_{-s}$ is a Sobolev norm of negative index $-s < -(\dim \mathbb{X})/2$ and π_k is the orthogonal projection of $L_2(\mathbb{X}, \mu)$ onto the kth eigenspace E_k of the Laplacian Δ. Our proofs (given in the Appendix) will make use of this fact, but for the most part we will work with the statistics $T_{m,n}^{(s)}$ in the form (2.4) or in the alternative forms (4.1) and (4.2) given at the beginning of Section 4.

The tests we propose and study are simply two-sample permutation tests based on the statistics $T_{m,n}^{(s)}(\{a_k\})$: i.e., condition on the "pooled sample", compute $T_{m,n}^{(s)}$ for all $\binom{N}{m}$ relabelings of the N elements of the pooled sample as X's and Y's, and reject the null hypothesis $\rho = \sigma$ if the "observed" $T_{m,n}^{(s)}$ is "too big" relative to the resulting (conditional) distribution. As mentioned in Section 1, this type of procedure may be implemented in practice by sampling from the permutation distribution.

The following development makes this more precise: Let $\tau_N \equiv \lambda_N \rho_m + (1 - \lambda_N)\sigma_n$ denote the empirical measure of the "pooled sample". Then let $P_\theta(\cdot | \tau_N)$ and $E_\theta(\cdot | \tau_N)$ denote the regular conditional probabilities and conditional expectations given τ_N; i.e., given the values of the first m X's and first n Y's without regard to their sample origin. If $(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ have been observed, τ_N is a sufficient statistic for $\theta \in \Theta_0$: for Borel sets A in $(\mathbb{X})^m \times (\mathbb{X})^n$, there is a version of

$$P_\theta\{(X_1, \cdots, X_m, Y_1, \cdots, Y_n) \in A | \tau_N\}$$

which is independent of $\theta \in \Theta_0$. In fact, conditional on τ_N, for $\theta \in \Theta_0$ all $N!$ permutations of $(X_1, \cdots, X_m, Y_1, \ldots Y_n)$ are equally likely. In view of this sufficiency of τ_N for $\theta \in \Theta_0$, we shall drop the subscript θ from $P_\theta(\cdot | \tau_N)$ or $E_\theta(\cdot | \tau_N)$ when these are computed for $\theta \in \Theta_0$.

Now the test functions $\phi_{m,n}$ may be defined by

$$
\begin{aligned}
\phi_{m,n}(\mathbf{X}, \mathbf{Y}) &= 1 & &\text{if } \left(\frac{mn}{N}\right)^{-1} T_{m,n}^{(s)} > c_N \\
&= \gamma_N & &\text{if } \left(\frac{mn}{N}\right)^{-1} T_{m,n}^{(s)} = c_N \\
&= 0 & &\text{if } \left(\frac{mn}{N}\right)^{-1} T_{m,n}^{(s)} < c_N
\end{aligned}
$$

(2.5)

where $T_{m,n}^{(s)} = T_{m,n}^{(s)}(\{a_k\})$ is defined by (2.4) and $c_N = c_N(\tau_N(\omega))$ and $\gamma_N = \gamma_N(\tau_N(\omega))$ are the uniquely determined smallest numbers satisfying

$$(2.6) \qquad\qquad E\{\phi_{m,n}(\mathbf{X}, \mathbf{Y})|\tau_N\} = \alpha.$$

The test functions $\phi_{m,n}$ defined by (2.5) and (2.6) clearly satisfy (2.1). Because the Sobolev norms $\|\cdot\|_{-s}$ are invariant under isometries (cf. Giné (1975) pages 1246–1248), $T_{m,n}^{(s)}$ is invariant, and hence (2.2) holds. The invariance of the statistics $T_{m,n}^{(s)}$ (and hence the tests) will become obvious in the examples considered in Section 4. In Section 3 we will give conditions on the weight sequence $\{a_k\}$ which will ensure that (2.3) holds.

3. Consistency; asymptotic power. The "natural parameter" corresponding to the statistic $T_{m,n}^{(s)}(\{a_k\})$ is the number $t^{(s)} \equiv t^{(s)}(\{a_k\})(\theta)$ defined for $\theta = (\rho, \sigma) \in \Theta$ by

$$t^{(s)}(\{a_k\})(\theta) \equiv \Sigma_{k=1}^{\infty} a_k^2 \Sigma_{f_i \in E_k} \{\int_{\mathbf{X}} f_i d(\rho - \sigma)\}^2.$$

Note that $t^{(s)} = 0$ for $\theta \in \Theta_0$, but it may be zero or positive for $\theta \notin \Theta_0$ depending on the measures ρ and σ and the weights $\{a_k\}$.

THEOREM 1. (*Consistency*). *Suppose that* $\theta \notin \Theta_0$. *Then*

$$(3.0) \qquad\qquad \lim_{m,n\to\infty} E_\theta\{\phi_{m,n}(\mathbf{X}, \mathbf{Y})\} = 1$$

if and only if $t^{(s)}(\{a_k\})(\theta) > 0$. *If the weights* $\{a_k\}$ *satisfy* $a_k \neq 0$ *for all* k, *then* $t^{(s)}(\{a_k\})(\theta) > 0$, *and* (3.0) *holds for all* $\theta \notin \Theta_0$.

The proof of Theorem 1 depends on a permutational version of Theorem 3.4 of Giné (1975). That result and proofs of the three theorems of this section are given in the Appendix.

The following two theorems give asymptotic distributions of the test statistics $T_{m,n}^{(s)}$ under local and fixed alternatives respectively. In principle they may be used to obtain large sample approximations to the power of our tests for specified alternatives. But the asymptotic distribution under the null hypothesis (local alternatives) is that of a weighted sum of dependent (noncentral) chi-square random variables, and hence asymptotic power properties of the tests are rather intractable in practice. Whereas the tests of Beran (1968) and Giné (1975) for testing uniformity are locally most powerful against certain alternatives, our two-sample tests do not seem to have a comparable property. It would be of interest to find invariant two-sample tests with specified optimality properties. In this connection, see Wellner (1978) for power computations in the case of a somewhat more specific two-sample testing problem on the sphere.

In the following we shall denote weak (star) convergence, or convergence in distribution, of a sequence of measures by $'w^* - \lim'$.

THEOREM 2. (*Local alternatives*). *Suppose that ρ is a fixed measure in $\mathscr{P}(\mathbb{X})$, and that $\{\sigma_N\}$ is a sequence of measures satisfying $w^* - \lim \sigma_N = \rho$ and $\lim_{m, n \to \infty} (mn/N)^{\frac{1}{2}} \int_{\mathbb{X}} f_i d(\rho - \sigma_N) = d_i$ for all $f_i \in E_k$, $k = 1, 2, \cdots$ with $\sum_{k=1}^{\infty} a_k^2 \sum_{\{i : f_i \in E_k\}} d_i^2 < \infty$. Then*

$$T_{m,n}^{(s)}(\{a_k\}) \to_d \sum_{k=1}^{\infty} a_k^2 \sum_{f_i \in E_k} \{X^{(\rho)}(f_i) + d_i\}^2$$

where $X^{(\rho)}(f)$ is a mean 0 Gaussian process, indexed by $f \in L_2(\mathbb{X}, \rho)$, with $E\{X^{(\rho)}(f)X^{(\rho)}(g)\} = \int_{\mathbb{X}} (f - \int_{\mathbb{X}} f d\rho)(g - \int_{\mathbb{X}} g d\rho) d\rho$ for $f, g \in L_2(\mathbb{X}, \rho)$; in particular, $X^{(\rho)}(f_i) \sim N(0, V_\rho(f_i))$ where $V_\rho(f_i) \equiv \int_{\mathbb{X}} (f_i - \int_{\mathbb{X}} f_i d\rho)^2 d\rho$.

For fixed $\theta = (\rho, \sigma) \notin \Theta_0$ define

$$g(x, y) \equiv \sum_{k=1}^{\infty} a_k \sum_{f_i \in E_k} f_i(x) f_i(y),$$

(3.1) $$u(x) \equiv 2 \sum_{k=1}^{\infty} a_k^2 \sum_{f_i \in E_k} \{\int_{\mathbb{X}} f_i d(\rho - \sigma)\} f_i(x),$$

$$r(x) \equiv \int_{\mathbb{X}} g(x, y) d\rho(y), \quad \text{and} \quad s(x) \equiv \int_{\mathbb{X}} g(x, y) d\sigma(y)$$

for $x, y \in \mathbb{X}$.

THEOREM 3. (*Fixed alternatives*). *If $\theta \notin \Theta_0$, $t^{(s)} \equiv t^{(s)}(\{a_k\})(\theta) > 0$, and $\lambda_N \to \lambda$, $0 < \lambda < 1$ as $m, n \to \infty$, then*

$$(mn/N)^{\frac{1}{2}} \{(mn/N)^{-1} T_{m,n}^{(s)} - t^{(s)}\} \to_d N(0, V^2)$$

where $V^2 = (1 - \lambda)\mathrm{Var}_\rho(u) + \lambda \, \mathrm{Var}_\sigma(u)$ and

$$\mathrm{Var}_\rho(u) = 4\{\int_{\mathbb{X}} (\int_{\mathbb{X}} (r(x) - s(x)) g(x, y) d\mu(x))^2 d\rho(y) - (\int_{\mathbb{X}} r(r - s) d\mu)^2\},$$

$$\mathrm{Var}_\sigma(u) = 4\{\int_{\mathbb{X}} (\int_{\mathbb{X}} (r(x) - s(x)) g(x, y) d\mu(x))^2 d\sigma(y) - (\int_{\mathbb{X}} s(s - r) d\mu)^2\}.$$

4. Examples. Now the goal is to obtain computable forms of the statistics $T_{m,n}^{(s)}$ for specific weights a_k when the Riemannian manifold $\mathbb{X}$ is the sphere S^p or hemisphere H^p in $(p + 1)$-dimensional Euclidean space or the torus $T^2 = S^1 \times S^1$. It follows easily from Giné (1975) that the statistics $T_{m,n}^{(s)}$ may be reexpressed in two useful alternative forms: first,

(4.1) $$T_{m,n}^{(s)}(\{a_k\}) = \frac{mn}{N} \int_{\mathbb{X}} [M_m(x) - N_n(x)]^2 d\mu(x)$$

where $M_m(x) \equiv m^{-1} \sum_{j=1}^{m} g(x, X_j)$, $N_n(x) \equiv n^{-1} \sum_{r=1}^{n} g(x, Y_r)$ and $g(x, y)$ is given by (3.1), as in (4.6), (5.4) and (5.4)' of Giné (1975). Second,

(4.2) $$T_{m,n}^{(s)} = \frac{mn}{N}$$
$$\{m^{-2} \sum_{i,j=1}^{m} h(X_i, X_j) - 2m^{-1}n^{-1} \sum_{j=1}^{m} \sum_{r=1}^{n} h(X_j, Y_r) + n^{-2} \sum_{r,s=1}^{n} h(Y_r, Y_s)\}$$

where $h(x, y) \equiv \sum_{k=1}^{\infty} a_k^2 \sum_{f_i \in E_k} f_i(x) f_i(y)$ as in (5.3) and (5.3)' of Giné (1975); once the function $h(x, y)$ is determined, (4.2) gives an effective computing formula for $T_{m,n}^{(s)}$. If $\mathbb{X}$ is a two-point homogeneous manifold then h simplifies as in (5.3)' of Giné, and depends only on $d(x, y)$, the Riemannian distance between x and y. In

this case we denote $h(x, y)$ by $h(\theta)$ with $\theta = d(x, y)$; i.e., for real θ set $h(\theta) = \Sigma_{k=1}^{\infty} a_k^2 (\dim E_k)^{\frac{1}{2}} h_k(\theta)$.

EXAMPLE 1.   Let $\mathbb{X} = S^p = \{x \in \mathbb{R}^{p+1} : x'x = 1\} = \{\theta \in \mathbb{R}^p : \theta_i \in [0, \pi], i = 1, \cdots, p - 1, \theta_p \in [0, 2\pi]\}$ in rectangular or polar coordinates respectively. The uniform measure is $d\mu = [\Gamma(\alpha + 1)/2\pi^{\alpha+1}]\prod_{j=1}^{p}(\sin \theta_j)^{p-j}d\theta$ with $\alpha \equiv \frac{1}{2}(p - 1)$ and we may take $s = \alpha + 1 > p/2 = (\dim S^p)/2$. Here $\dim(E_k) = \begin{pmatrix} p + k - 2 \\ p - 1 \end{pmatrix} + \begin{pmatrix} p + k - 1 \\ p - 1 \end{pmatrix}$, $e_k = k(k + 2\alpha)$, and an orthonormal basis for E_k, the kth eigenspace of the Laplacian, is (Prentice (1978), page 171, Vilenkin (1968), page 468, Yaglom (1961), page 600)

$$\{ f_k^{(\mathbf{m})} : \mathbf{m} = (m_2, \cdots, m_{p-2}, \pm m_{p-1}), k \geqslant m_2 \geqslant \cdots \geqslant m_{p-1} \geqslant 0 \}$$

where $f_k^{(\mathbf{m})}$ is a product of harmonics on spheres of smaller dimension. Since S^p is two-point homogeneous, the simplification entailed by (5.3)′ of Giné applies, and

$$h(\theta) = \Sigma_{k=1}^{\infty} a_k^2 \left(1 + \frac{k}{\alpha}\right) C_k^{\alpha}(\cos \theta)$$

where

$$C_k^{\alpha}(z) = \frac{\Gamma\left(\alpha + \frac{1}{2}\right)\Gamma(k + 2\alpha)(1 - z^2)^{\frac{1}{2} - \alpha}}{\Gamma\left(k + \alpha + \frac{1}{2}\right)\Gamma(2\alpha)k!\,(-2)^k} \frac{d^k}{dz^k}\left[(1 - z^2)^{k + \alpha - \frac{1}{2}}\right]$$

is the Gegenbauer polynomial (zonal ultraspherical harmonic) of index α and order k (cf. Prentice (1978), page 170, Whittaker and Watson (1929), page 329 and Szegö (1939), page 80).

EXAMPLE 1a.   (Two-sample analogue of Rayleigh's statistic for S^p).   If $\{a_k\} = \{1, 0, \cdots\}$, then $h(\theta) = (1 + (1/\alpha))C_1^{\alpha}(z) = (p + 1)z$, $z = \cos \theta$, and hence

$$T_{m,n}^{(\alpha+1)}(\{1, 0, \cdots\}) = (p + 1)\frac{mn}{N}|m^{-1}\mathbf{R}^X - n^{-1}\mathbf{R}^Y|^2$$

where $\mathbf{R}^X \equiv \Sigma_{i=1}^{m}\mathbf{x}_i$ and $\mathbf{R}^Y \equiv \Sigma_{j=1}^{n}\mathbf{y}_j$ are the resultant vectors of the two samples. By Theorem 1 the permutation test based on $T_{m,n}^{(\alpha+1)}(\{1, 0, \cdots\})$ is consistent against all alternatives $(\rho, \sigma) \notin \Theta_0$ with different mean vectors; i.e., if $\int_{S^p}\mathbf{x}d\rho(\mathbf{x}) \neq \int_{S^p}\mathbf{y}d\sigma(\mathbf{y})$, the test will be consistent. See Wellner (1978) for further information about this and related tests for mean vectors.

EXAMPLE 1b.   (Two-sample version of Bingham's test statistic.)   If $\{a_k\} = \{0, 1, 0, \cdots\}$, then $h(\theta) = (1 + (2/\alpha))C_2(\cos \theta) = \frac{1}{2}(p + 3)((p + 1)(\cos \theta)^2 - 1)$, and hence (cf. Prentice (1978), page 172, Giné (1975), page 1263)

$$T_{m,n}^{(\alpha+1)}(\{0, 1, 0, \cdots\}) = \frac{1}{2}(p + 1)(p + 3)\frac{mn}{N} \operatorname{trace}\left[(T^X - T^Y)^2\right]$$

where $T^X \equiv m^{-1}\Sigma_{i=1}^{m}\mathbf{x}_i\mathbf{x}_i'$ and $T^Y \equiv n^{-1}\Sigma_{j=1}^{n}\mathbf{y}_j\mathbf{y}_j'$ are the sample second moment matrices. By Theorem 1 the permutation test based on this statistic is consistent

against all alternatives $(\rho, \sigma) \notin \Theta_0$ with different second-moment matrices; i.e., if $\int_{S^p} \mathbf{x}\mathbf{x}' d\rho(\mathbf{x}) \neq \int_{S^p} \mathbf{y}\mathbf{y}' d\sigma(\mathbf{y})$ the test will be consistent. This statistic is appropriate for distributions on S^p which are symmetric about the origin (the center of the sphere); equivalently the test is appropriate for distributions on H^p, the $(p + 1)$-dimensional hemisphere.

EXAMPLE 1c. (Two sample analogue of an Ajne-Beran type statistic for S^p.) If $a_{2k} = 0$, $a_{2k-1} = [(-1)^k \Gamma(k + \alpha + 1)/(2k + 1)(k + \alpha + \frac{1}{2})k! \Gamma(\alpha)]$, then $h(\theta) = \frac{1}{4}(1 - (2/\pi)\theta)$ (cf. Prentice (1978), page 172) and hence

$$T_{m,n}^{(\alpha+1)}(\{a_k\}) = \frac{1}{2\pi} \frac{mn}{N}$$

$$\left\{ 2m^{-1}n^{-1}\Sigma_{j=1}^m\Sigma_{r=1}^n \widehat{X_j Y_r} - m^{-2}\Sigma_{i=1}^m\Sigma_{j=1}^m \widehat{X_i X_j} - n^{-2}\Sigma_{r=1}^n\Sigma_{s=1}^n \widehat{Y_r Y_s} \right\}$$

where $\widehat{XY} = \arccos(\mathbf{x}'\mathbf{y})$, the angle between X and Y. Alternatively, the statistic may be written as

$$T_{m,n}^{(\alpha+1)}(\{a_k\}) = \frac{mn}{N} \int_{S^p} \left[M_m(\mathbf{x}) - N_n(\mathbf{x}) \right]^2 d\mu(\mathbf{x})$$

where $M_m(\mathbf{x}) \equiv m^{-1}\Sigma_{i=1}^m 1_{[0,1]}(\mathbf{x}'\mathbf{x}_i)$ and $N_n(x) \equiv n^{-1}\Sigma_{r=1}^n 1_{[0,1]}(\mathbf{x}'\mathbf{y}_r)$ denote the proportions of X's and Y's respectively in the hemisphere centered at x.

EXAMPLE 1d. (Two sample analogue of a Giné type statistic for S^p and H^p.) If $a_{2k-1} = 0$, $a_{2k}^2 = \{p(2q - 1)/8\pi(2q + p)\}\{\Gamma(\alpha + \frac{1}{2})\Gamma(q - \frac{1}{2})/\Gamma(q + \alpha + \frac{1}{2})\}^2$, then $h(\theta) = \frac{1}{2} - K_p \sin\theta$ with $K_p = (p/4)\{\Gamma(\alpha + \frac{1}{2})/\Gamma(\alpha + 1)\}^2$ (cf. Prentice (1978), page 172), and hence

$$T_{m,n}^{(\alpha+1)} = K_p \frac{mn}{N}$$

$$\left\{ 2m^{-1}n^{-1}\Sigma_{j=1}^m\Sigma_{r=1}^n \sin(\widehat{X_j Y_r}) - m^{-2}\Sigma_{i,j=1}^m \sin(\widehat{X_i X_j}) - n^{-2}\Sigma_{r,s=1}^n \sin(\widehat{Y_r Y_s}) \right\}.$$

The permutation test based on this statistic is consistent against all alternatives $(\rho, \sigma) \notin \Theta_0$ on S^p with at least one differing even spectral moment; equivalently the test is consistent against all alternatives on H^p. By Theorem 1, any linear combination of the statistics of Examples 1c and 1d is consistent against all alternatives $(\rho, \sigma) \notin \Theta_0$ on S^p.

EXAMPLE 2. Let $\mathbb{X} = T^2 = S^1 \times S^1$. Then $d\mu = (4\pi^2)^{-1}d\theta d\phi$, the Riemannian metric d is just the ordinary Euclidean metric in $\mathbb{R}^2/(2\pi\mathbb{Z})^2$, and the Laplacian is $-\Delta = (\partial^2/\partial\theta^2) + \partial^2/\partial\phi^2$. Here it is convenient to index the eigenspaces of Δ by the eigenvalues k: if k is an integer admitting a representation of the form $a^2 + b^2 = k, a, b \in \mathbb{Z}$, then the functions $\{f_{a,b}(\theta, \phi) \equiv \exp(i(a\theta + b\phi)): a^2 + b^2 = k\}$ are orthonormal (with respect to μ) eigenfunctions of Δ with eigenvalue k. Let $\mathfrak{R}$ denote the set of positive integers which can be represented as the sum of two squares. Then the dimension of the eigenspace E_k with eigenvalue $k, k \in \mathfrak{R}$, is

just the number of representations of k as the sum of two squares: if k factors as

$$k = 2^u \prod_{\substack{t_j \\ p_j = 1 (\text{mod } 4)}}^{t_j} \prod_{\substack{s_j \\ q_j = 3 (\text{mod } 4)}}^{s_j} \equiv 2^u k_1 k_3,$$

the (total) number of such representations of k is known to be zero if k_3 is not a square, and four times the number of divisors of k_1 if k_3 is a square (cf. Hardy and Wright (1960), page 241).

The torus T^2 is not two-point homogeneous, however, and in this respect it differs from the circle, sphere and projective plane (cf. Giné (1975), page 1256). It is easily checked that the isotropy group of a point (the subgroup of isometries leaving the point fixed), for example $\mathbf{0} = (0, 0)$, consists of only eight elements: reflections in each coordinate, reversal of coordinates, the identity, and compositions of these (which yield four more distinct elements). Hence, if K_0 denotes the isotropy group of $\mathbf{0}$, the orbit under K_0 of a point x on a sphere centered at $\mathbf{0}$ consists of only eight points. Thus the isotropy subgroup K_0 does not act transitively on spheres centered at $\mathbf{0}$ and T^2 is not two-point homogeneous.

A consequence is that the addition formula ((5.1) of Giné (1975)) for eigenfunctions on the torus does not enjoy the simplification entailed by Giné's (5.2) for two-point homogeneous manifolds. Nonetheless, the addition formula does hold in the form (5.1):

$$\Sigma_{\{a, b \in \mathbb{Z}: a^2 + b^2 = k\}} f_{a, b}(x) \overline{f_{a, b}(y)} = (\dim E_k)^{\frac{1}{2}} f_0^{(k)}(g_{x, 0}(y))$$

where $g_{x, 0}$ denotes the element of the isometry group G of T^2 defined by $g_{x, 0}(y) = x - y$, $x, y \in T^2$. Hence the (zonal with respect to $\mathbf{0}$) functions appearing in the addition formula are given by

$$f_0^{(k)}(x) = (\dim E_k)^{-\frac{1}{2}} \Sigma_{\{a, b \in \mathbb{Z}: a^2 + b^2 = k\}} f_{a, b}(x)$$

for $x \in T^2$ and $k \in \mathfrak{R}$. For example, for $k = 1$, $\dim(E_1) = 4$, and, for $x = (\theta, \phi) \in T^2$, $f_0^{(1)}(x) = \cos \theta + \cos \phi$; for $k = 2$, $\dim(E_2) = 4$ and $f_0^{(2)}(x) = 2 \cos \theta \cos \phi$.

EXAMPLE 2a. (Rayleigh type statistics.) If $a_1 = 1$ and $a_k = 0$ for $k \in \mathfrak{R}$ with $k \geqslant 2$, then it follows that $h(x, 0) = \cos \theta + \cos \phi$ for $x = (\theta, \phi) \in T^2$ and hence

$$T_{m, n}^{(\frac{3}{2})}(\{1, 0, \cdots\}) = 2 \frac{mn}{N} |m^{-1} \mathbf{R}^X - n^{-1} \mathbf{R}^Y|^2$$

where $\mathbf{R}^X \equiv \Sigma_{i=1}^m (\cos \theta_i^X, \sin \theta_i^X, \cos \phi_i^X, \sin \phi_i^X)$, $\mathbf{R}^Y \equiv \Sigma_{r=1}^n (\cos \theta_r^Y, \sin \theta_r^Y, \cos \phi_r^Y, \sin \phi_r^Y)$, and $|\cdot|$ denotes the ordinary Euclidean distance in $\mathbb{R}^4$. By Theorem 1 the permutation test based on this statistic is consistent against all alternatives $(\rho, \sigma) \notin \Theta_0$ satisfying $\int_{T^2} f_{a, b}(x) d\rho(x) \neq \int_{T^2} f_{a, b}(x) d\sigma(x)$ for some $(a, b) \in \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$.

(The corresponding one-sample statistic for testing uniformity on T^2 is given by

$$T_n^{(\frac{3}{2})}(\{1, 0, \cdots\}) = 2n^{-1} |\mathbf{R}|^2$$

where $\mathbf{R} \equiv \Sigma_{i=1}^{n}(\cos \theta_i, \sin \theta_i, \cos \phi_i, \sin \phi_i)$, $X_i = (\theta_i, \phi_i)$. If the X_i's are uniformly distributed on T^2, it follows from Theorem 4.1 of Giné (1975) that the asymptotic distribution of $T_n^{(\frac{3}{2})}(\{1, 0, \cdots\})$ is chi-square with four degrees of freedom.)

EXAMPLE 2b. (Bingham type statistics.) If $a_1 = 0$, $a_2 = 1$, and $a_k = 0$ for $k \in \mathcal{R}$ with $k \geqslant 4$, then $h(x, 0) = 4 \cos \theta \cos \phi$ and hence

$$T_{m,n}^{(\frac{3}{2})}(\{0, 1, 0, \cdots\}) = 4 \frac{mn}{N} \text{ trace}\left[(T^X - T^Y)(T^X - T^Y)'\right]$$

where the two by two matrix $T^X \equiv m^{-1}\Sigma_{i=1}^{m}\mathbf{x}_i^{\theta}\mathbf{x}_i^{\phi'}$, $\mathbf{x}_i = (x_i^{\theta'}, x_i^{\phi'}) = (\cos \theta_i^X, \sin \theta_i^X, \cos \phi_i^X, \sin\phi_i^X)$, and similarly for T^Y. By Theorem 1 the permutation test based on $T_{m,n}^{(\frac{3}{2})}(\{0, 1, 0, \cdots\})$ is consistent against all alternatives $(\rho, \sigma) \notin \Theta_0$ satisfying $\int_{T^2} f_{a,b}(x) \, d\rho(x) \neq \int_{T^2} f_{a,b}(x) \, d\sigma(x)$ for some $(a, b) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$.

(The corresponding one-sample statistic for testing uniformity on T^2 is given by

$$T_n^{(\frac{3}{2})}(\{0, 1, 0, \cdots\}) = 4n \text{ trace}(TT')$$

where $T \equiv n^{-1}\Sigma_{i=1}^{n}\mathbf{x}_i^{\theta}\mathbf{x}_i^{\phi'}$; the asymptotic distribution under uniformity is chi-square with four degrees of freedom.)

EXAMPLE 2c. In this final example we proceed somewhat differently than in our preceding examples: let $g : S^1 \to \mathbb{R}^1$ be defined by $g(r) = (\pi^2/6) - \frac{1}{4}r(2\pi - r)$ for $0 \leqslant r \leqslant 2$, and then define $h : T^2 \to \mathbb{R}^1$ by $h(x) = g(\theta) + g(\phi) + g(\theta)g(\phi)$ for $x = (\theta, \phi) \in T^2$. It is easily checked that h is constant on orbits of K_0, the isotropy group of $\mathbf{0}$, and hence is in the space spanned by the functions $\{f_0^{(k)}\}_{k \in \mathcal{R}}$. In fact, a straightforward calculation shows that

$$h(x) = \Sigma_{k \in \mathcal{R}} a_k^2 (\dim E_k)^{\frac{1}{2}} f_0^{(k)}(x)$$

where

$$a_k^2 = (\dim E_k)^{-1}\Sigma\{4^{-1}a^{-2}b^{-2}1_{[a\neq 0, b\neq 0]} + a^{-2}1_{[a\neq 0, b=0]}\}$$
$$\{a, b \in \mathbb{Z} : a^2 + b^2 = k\}$$

is nonzero for all $k \in \mathcal{R}$. Hence

$$T_{m,n}^{(\frac{3}{2})}(\{a_k\}) = \frac{mn}{N}\{m^{-2}\Sigma_{i,j=1}^{m}h(X_i - X_j) - 2m^{-1}n^{-1}$$
$$\Sigma_{j=1}^{m}\Sigma_{r=1}^{n}h(X_j - Y_r) + n^{-2}\Sigma_{r,s=1}^{n}h(Y_r - Y_s)\}$$

where $h(x) = g(\theta) + g(\phi) + g(\theta)g(\phi)$ yields a test which is consistent against all alternatives $(\rho, \sigma) \notin \Theta_0$ on T^2.

(The corresponding one-sample statistic for testing uniformity on T^2 is given by

$$T_n^{(\frac{3}{2})}(\{a_k\}) = n^{-1}\Sigma_{i,j=1}^{n}h(X_i - X_j)$$

where the weights a_k and the function h are as given above. This statistic gives a test for uniformity on T^2 which is consistent against all alternatives $\nu \neq \mu$. By Theorem 4.1 of Giné (1975), under uniformity

$$T_n^{(\frac{3}{2})}(\{a_k\}) \to_d \Sigma_{k \in \Re} a_k^2 H_k$$

as $n \to \infty$ where the H_k's are independent chi-square random variables with $\dim(E_k)$ degrees of freedom. The methods of Hoeffding (1964) may be used to approximate the distribution function of this limiting random variable.)

For other statistical procedures for the torus see Ibero (1975, 1976) and Mardia (1975).

<center>**APPENDIX**</center>

Here we prove the permutational limit theorem upon which our consistency result is based and the theorems of Section 3.

Let $\mathbb{X}$ be a compact Riemannian manifold as in Section 3. Suppose that $\{p_{Ni}\}_{i=1}^N$ are N points in $\mathbb{X}$ and consider the experiment of choosing at random (without replacement) m of these N points. Let τ_N, ρ_m, and σ_n be the measures which assign masses N^{-1}, m^{-1}, and n^{-1} to all the p_{Ni}'s, the m selected points, and the $n = N - m$ remaining points respectively. Denote the probability space for this experiment by $(\tilde{\Omega}, \tilde{\mathcal{C}}, \tilde{P})$, denote points in the sample space $\tilde{\Omega}$ by $\tilde{\omega}$, and write $\rho_m(\tilde{\omega})$ to indicate the dependence of ρ_m on the subset of $\{p_{Ni}\}_{i=1}^N$ selected by the experiment.

Suppose that $w^* - \lim_{N \to \infty} \tau_N = \tau$ where τ is a (Borel) probability measure on $\mathbb{X}$. Since the Sobolev norms $\| \cdot \|_{-s}$ with $s > (\dim \mathbb{X})/2$ metrize the weak-star topology of $\mathscr{P}(\mathbb{X})$ (Giné (1975), Theorem 2.2), $\|\tau_N - \tau\|_{-s} \to 0$ as $N \to \infty$. Define the process $X_N(f)(\tilde{\omega})$ on $(\tilde{\Omega}, \tilde{\mathcal{C}})$, indexed by f in $L_2(\mathbb{X}, \tau)$, by

(A1)
$$X_N(f)(\tilde{\omega}) = \left(\frac{mn}{N}\right)^{\frac{1}{2}} \int_{\mathbb{X}} f d(\rho_m(\tilde{\omega}) - \sigma_n(\tilde{\omega}))$$

$$= \left(\frac{mN}{n}\right)^{\frac{1}{2}} \int_{\mathbb{X}} f d(\rho_m(\tilde{\omega}) - \tau_N).$$

Also, following Giné (1975) and Strassen and Dudley (1969), define the process $X^{(\tau)}(f)$, indexed by f in $L_2(\mathbb{X}, \tau)$, as the mean zero Gaussian process with covariance

(A2) $$E\{X^{(\tau)}(f)X^{(\tau)}(g)\} = \int_{\mathbb{X}}(f - \int_{\mathbb{X}} f d\tau)(g - \int_{\mathbb{X}} g d\tau)\, d\tau$$

for $f, g \in L_2(\mathbb{X}, \tau)$. Let $H_s(\mathbb{X})$ denote the Sobolev space of index s (i.e., the subset of $L_2(\mathbb{X}, \tau)$ having $\|f\|_s < \infty$; see Giné (1975), page 1247), and let B_s denote the closed unit ball of $H_s(\mathbb{X})$. If $\|f\|_\infty \equiv \sup_{x \in \mathbb{X}} |f(x)|$, then $(B_s, \| \cdot \|_\infty)$ is a compact metric space. Giné ((1975), Lemma 3.2) shows that for every Borel probability measure τ on $\mathbb{X}$, the process $X^{(\tau)}$ restricted to B_s is sample continuous and hence defines a probability measure $\mathcal{L}(X^{(\tau)}|B_s) = \mathcal{L}(X^{(\tau)})$ on $C(B_s, \| \cdot \|_\infty) \equiv C(B_s)$.

The following theorem is a permutational version of Giné's Theorem 3.4. It is also related to Bickel's (1969) permutational limit theorem for the empirical distribution function of p-dimensional random vectors.

THEOREM A. *If $w^* - \lim_N \tau_N = \tau$ for some Borel probability measure τ on $\mathbb{X}$, and $s > (\dim \mathbb{X})/2$, then*

$$\mathcal{L}(X_N) \to_{w^*} \mathcal{L}(X^{(\tau)}) \quad in \quad C'(C(B_s))$$

as $N \to \infty$. (Here $\mathcal{L}(X_N) = \mathcal{L}(X_N | B_s)$ is the law of X_N on $C(B_s)$ under $\tilde{P}$, and $\mathcal{L}(X^{(\tau)}) = \mathcal{L}(X^{(\tau)} | B_s)$ is the law of $X^{(\tau)}$ on $C(B_s)$.)

Now let $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ be random variables with values in $\mathbb{X}$ as in Section 2. In a minor change of notation we use ρ_m, σ_n, and τ_N to denote the empirical measures of the X's, Y's, and pooled sample, respectively; all three empirical measures are now random (depend on $\omega \in \Omega$). Define $X_N(f)(\omega)$ as in (A1).

THEOREM B. *If $\theta = (\rho, \rho) \in \Theta_0$ and $s > (\dim X)/2$, then*

$$\mathcal{L}(X_N | \tau_N) \to_{w^*} \mathcal{L}(X^{(\rho)}) \quad in \quad C'(C(B_s))$$

a.s. P_θ.

(Note that we have used "$\mathcal{L}(\cdot \mid \cdot)$" in two ways: in Theorem A, $\mathcal{L}(X_N | B_s)$ denotes the law of the process X_N restricted to B_s; in Theorem B, $\mathcal{L}(X_N | \tau_N)$ denotes the conditional law of the process X_N given τ_N, and then restricted to B_s; this second restriction is suppressed in the above notation.)

PROOF OF THEOREM A. By Prohorov's theorem it suffices to prove (i) that the finite-dimensional distributions of the process X_N converge in distribution to the corresponding finite-dimensional distributions of $X^{(\tau)}$, and (ii) that the laws $\{\mathcal{L}(X_N | B_s)\}_{N=1}^\infty$ on $C(B_s)$ are tight.

(i) *Convergence of the finite-dimensional distributions.* By the Cramér-Wold device, the linearity of X_N, and Minkowski's inequality, it suffices to establish the convergence of the one-dimensional distributions. Let $f \in L_2(\mathbb{X}, \tau)$; some algebra shows that

$$X_N(f)(\tilde{\omega}) =_d \Sigma_{j=1}^n d_{Nj} V_{Nj}$$

where c_{Nj} equals $(n/m)^{\frac{1}{2}}$ for $j = 1, \cdots, m$ and equals $-(m/n)^{\frac{1}{2}}$ for $j = m + 1, \cdots, N$, $\{V_{Nj}\}_{j=1}^N$ is a random permutation of $\{c_{Nj}\}_{j=1}^N$, and $d_{Nj} = N^{-\frac{1}{2}}\{f(p_{Nj}) - \int_X f d\tau_N\}$, $j = 1, \cdots, N$. Then $\Sigma_{j=1}^N c_{Nj} = 0$, $N^{-1}\Sigma_{j=1}^N c_{Nj}^2 = 1$ for all N, $\Sigma_{j=1}^N d_{Nj} = 0$, and

$$\Sigma_{j=1}^N d_{Nj}^2 = \int_X (f - \int_X f d\tau_N)^2 \, d\tau_N \to \int_X (f - \int_X f d\tau)^2 \, d\tau \equiv \sigma_\tau^2(f) \quad as \quad N \to \infty$$

since $w^* - \lim \tau_N = \tau$ and $C(\mathbb{X})$ is dense in $L_2(\mathbb{X}, \tau)$. Also, $N^{-\frac{1}{2}} \max_{1 \leq j \leq N} |c_{Nj}| \to 0$, $\max_{1 \leq j \leq N} |d_{Nj}| \to 0$, and $N^{-1}\Sigma_{\{|\delta_{Nij}| > \varepsilon\}} \delta_{Nij}^2 \to 0$ for all $\varepsilon > 0$ where $\delta_{Nij} \equiv c_{Ni} d_{Nj}$. Hence, by Hájek's (1961) version of the Wald-Wolfowitz-Noether permutational

central limit theorem (or see Lemma 4.1 of Bickel (1969))

$$X_N(f) \to_d N(0, \sigma_\tau^2(f))$$

and this is just the distribution of $X^{(\tau)}(f)$.

(ii) Tightness of the laws $\{\mathcal{L}(X_N | B_s)\}_{N=1}^\infty$ on $C(B_s)$. Our proof follows the proof of tightness given by Giné (page 1251) with modifications of the variance calculation necessitated by our present finite sampling situation. We omit the details. □

PROOF OF THEOREM B. If $p_{N1}, \cdots, p_{NN}$ are the values of the first m X's and first n Y's where the X's and Y's have a common probability measure $\rho = \sigma$ on $\mathbb{X}$, then the conditional law of ρ_m given τ_N is exactly the same as that of $\rho_m(\tilde{\omega})$ in the finite sampling context of Theorem A. Hence Theorem B follows from Theorem A and the Sobolev norm version of the Glivenko-Cantelli theorem for empirical measures on $\mathbb{X}$ (cf. Giné (1975), Lemma 4.2): if $\theta = (\rho, \rho) \in \Theta_0$ then

$$P_\theta(\|\tau_N - \rho\|_{-s} \to 0) = 1. \qquad\qquad □$$

PROOF OF THEOREM 1. Define the seminorm h on $C(B_s)$ by

$$h(A) \equiv \sup_{f \in B_s} |A \circ \Sigma_k a_k e_k^{s/2} \pi_k(f)|.$$

Then $h \in C(C(B_s))$ if $\sup_k |a_k e_k^{s/2}| < \infty$ (Giné (1975), page 1252), and $h^2(X_N) = T_{m,n}^{(s)}(\{a_k\})$. Furthermore, for every probability measure ρ on $\mathbb{X}$ and $\alpha > 0$ there are unique finite smallest numbers $c = c(\rho, \alpha)$ and $\gamma = \gamma(\rho, \alpha)$, $0 < \gamma \leqslant 1$, satisfying

(A3) $$P(h^2(X^{(\rho)}) > c) + \gamma P(h^2(X^{(\rho)}) = c) = \alpha.$$

Suppose $\theta = (\rho, \sigma) \notin \Theta_0$ and $t^{(s)}(\{a_k\})(\theta) > 0$. Assume (without loss of generality) that $\lambda_N = mN^{-1} \to \lambda$, $0 \leqslant \lambda \leqslant 1$. Let $\tau \equiv \lambda\rho + (1 - \lambda)\sigma$. By the Sobolev-norm version of the Glivenko-Cantelli theorem, $P_\theta(\|\tau_N - \tau\|_{-s} \to 0) = 1$, and hence Theorem A implies that $(mn/N)c_N(\tau_N) \to c(\tau, \alpha)$ a.s. P_θ with $c = c(\tau, \alpha)$ defined by (A3) with ρ replaced by τ. But

$$\left(\frac{mn}{N}\right)^{-1} T_{m,n}^{(s)}(\{a_k\}) \to t^{(s)}(\{a_k\})(\theta) > 0$$

a.s. P_θ, and hence $\phi_{m,n}(\mathbf{X}, \mathbf{Y}) \to 1$ a.s. P_θ.

Now suppose that $\theta = (\rho, \phi) \notin \Theta_0$, but suppose that $t^{(s)}(\{a_k\})(\theta) = 0$. Then Theorem A again implies that $(mn/N)c_N(\tau_N) \to c(\tau, \alpha)$ a.s. P_θ as above, but now an application of Theorem 3.4 of Giné (1975) shows that

$$T_{m,n}^{(s)}(\{a_k\}) \to_d \Sigma_{k=1}^\infty a_k^2 \Sigma_{f_i \in E_k} \left\{ (1-\lambda)^{\frac{1}{2}} X^{(\rho)}(f_i) - \lambda^{\frac{1}{2}} X^{(\sigma)}(f_i) \right\}^2$$

where $X^{(\rho)}$ and $X^{(\sigma)}$ denote independent Gaussian processes with covariance structure as in (A2). Hence the test fails to be consistent when $t^{(s)}(\{a_k\})(\theta) = 0$.

The second assertion of the theorem follows easily upon noting that $\theta = (\rho, \sigma) \notin \Theta_0$ implies that $\|\rho - \sigma\|_{-s}^2 = \Sigma_{k=1}^\infty e_k^{-s} \Sigma_{f_i \in E_k} \{\int_{\mathbb{X}} f_i \, d(\rho - \sigma)\}^2 > 0$, so $\int_{\mathbb{X}} f_i \, d(\rho - \sigma) \neq 0$ for some f_i, and hence $t^{(s)}(\{a_k\})(\theta) > 0$ (since $a_k \neq 0$ for all k). □

The proof of Theorem 2 is straightforward but lengthy, so we shall omit it.

PROOF OF THEOREM 3.   Let $B_i \equiv \int_{\mathbf{X}} f_i \, d(\rho_m - \sigma_n)$, $b_i \equiv \int_{\mathbf{X}} f_i \, d(\rho - \sigma)$. Then

$$\left(\frac{mn}{N}\right)^{\frac{1}{2}} \left\{ (mn/N)^{-1} T^{(s)}_{m,n} - t^{(s)} \right\}$$

$$= \Sigma^{\infty}_{k=1} a_k^2 \Sigma_{f_i \in E_k} (mn/N)^{\frac{1}{2}} (B_i + b_i)(B_i - b_i)$$

$$= \Sigma^{\infty}_{k=1} a_k^2 \Sigma_{f_i \in E_k} (B_i + b_i)\left((1 - \lambda_N)^{\frac{1}{2}} X^{(\rho)}_m (f_i) - \lambda_N^{\frac{1}{2}} X^{(\sigma)}_n (f_i) \right)$$

$$\to_d \Sigma^{\infty}_{k=1} a_k^2 \Sigma_{f_i \in E_k} 2 b_i \left((1 - \lambda)^{\frac{1}{2}} X^{(\rho)}(f_i) - \lambda^{\frac{1}{2}} X^{(\sigma)}(f_i) \right)$$

$$= (1 - \lambda)^{\frac{1}{2}} X^{(\rho)}(u) - \lambda^{\frac{1}{2}} X^{(\sigma)}(u)$$

where $u(x) \equiv 2\Sigma^{\infty}_{k=1} a_k^2 \Sigma_{f_i \in E_k} b_i f_i(x)$. This limiting random variable is normal with the stated variance by straightforward calculation. ☐ ˋ

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