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WEAK CONVERGENCE OF SOME QUANTILE PROCESSES ARISING IN PROGRESSIVELY CENSORED TESTS[1]

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For progressive censoring schemes pertaining to a general class of (parametric as well as nonparametric) testing situations, one encounters a (partial) sequence of linear combinations of functions of order statistics where the coefficients are themselves stochastic variables. Weak convergence of such a quantile process to an appropriate Gaussian function is studied here, and the same is incorporated in the formulation of suitable (time-) sequential tests based on these quantile processes.

1. Introduction. Let $X_1, \cdots, X_N$, the *survival times* of $N(>1)$ items under *life testing*, be independent random variables (rv) with continuous distribution functions (df) $F_1, \cdots, F_N$, respectively, all defined on the real line $(-\infty, \infty)$. In a life testing problem, the smallest observation comes first, the second smallest next, and so on, until the largest one emerges last. Thus, the observable random variables can be represented as

$$\{(Z_{N1}, Q_{N1}), \cdots, (Z_{NN}, Q_{NN})\} \tag{1.1}$$

where Z_{Nj} is the jth smallest observation among $X_1, \cdots, X_N$ $(1 \leqslant j \leqslant N)$ and

$$Z_{Nj} = X_{Q_{Nj}} \quad \text{for} \quad j = 1, \cdots, N; \tag{1.2}$$

by virtue of the assumed continuity of the F_i, ties among the X_i (and hence, the Z_{Ni}) may be neglected, in probability, so that the Q_{Nj} are uniquely (in probability) defined by (1.2) and $(Q_{N1}, \cdots, Q_{NN})$ represents a permutation of $(1, \cdots, N)$. Since a complete collection of (1.1) demands the span of the experimentation until Z_{NN} is observed, while practical considerations often set time and cost limitations on the duration of experimentation, the experiment may be terminated at the rth failure Z_{Nr}, where

$$r = [Np] + 1 \quad \text{for some} \quad 0 < p < 1 \tag{1.3}$$

($[s]$ being the largest integer contained in s). Thus, here, the observable random variables are

$$\mathbf{Z}_N^{(r)} = (Z_{N1}, \cdots, Z_{Nr}) \quad \text{and} \quad \mathbf{Q}_N^{(r)} = (Q_{N1}, \cdots, Q_{Nr}). \tag{1.4}$$

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(We also know the complementary set $\mathbf{Q}_N^{(N)} - \mathbf{Q}_N^{(r)}$, but without any idea of the order in which the elements appear.) For testing suitable statistical hypotheses concerning the df's $F_1, \cdots, F_N$, a terminal test based on $(\mathbf{Z}_N^{(r)}, \mathbf{Q}_N^{(r)})$ is termed a *censored test*; we denote the corresponding test statistic by T_{Nr}.

In a progressive censoring scheme (PCS), the experiment is monitored from the very beginning with the objective of an early termination (prior to Z_{Nr}) whenever statistically feasible, i.e., one observes T_{Nk} at each failure time Z_{Nk} $(1 \leqslant k \leqslant r)$, and, if for some k $(\leqslant r)$, T_{Nk} provocates a clear statistical decision in favor of one of the hypotheses, experimentation is terminated at that time-point; if no such $k(< r)$ exists, the experimentation is stopped at the rth failure Z_{Nr} along with an appropriate statistical decision. Thus, by constitution, a PCS test is based on the entire partial sequence

$$\{Z_N^{(k)}, Q_N^{(k)}; 1 \leqslant k \leqslant r\}, \tag{1.5}$$

and is time-sequential in nature. Since the updated sequence $\{T_{Nk}, 1 \leqslant k \leqslant r\}$ involves dependent random elements and the PCS involves repeated testing on these dependent statistics, statistical analysis of such a problem often becomes complicated. In this context suitable invariance principles for $\{T_{Nk}, 1 \leqslant k \leqslant r\}$ provide us with convenient tools for formulating a PCS test and studying its (asymptotic) properties.

In the context of *nonparametric life testing*, Chatterjee and Sen (1973) have studied PCS tests based on a general class of linear rank statistics; the theory rests on an invariance principle for PCS linear rank statistics. For the case of $F_1 = \cdots = F_N = F$ involving an unknown parameter θ (form of F assumed to be specified), Sen (1976) has constructed PCS tests for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq$ (or $>$ or $<$) θ_0 based on PCSLR (likelihood ratio-) statistics; here also, the theory is based on an invariance principle for the PCSLR. The object of the present investigation is to focus on a general class of location, scale and regression models where the PCSLR statistics yield suitable *quantile processes* (QP) and to develop suitable invariance principles for such PCQP's. These models are introduced in Section 2 and the corresponding PCSLR statistics are derived and incorporated in the construction of appropriate PCQP's. By nature, such a PCQP involves a partial sequence of linear combinations of functions of order statistics with stochastic coefficients depending on the various censoring stages. Further, the distribution of these stochastic coefficients is generated by the classical permutation distribution arising in the theory of rank tests and this enables one to study the limiting behavior of PCQP's under certain regularity conditions which are less stringent than the ones pertaining to the asymptotic normality of linear functions of order statistics (as has been studied by Chernoff, Gastwirth and Johns (1967), Stigler (1969) and Shorack (1972), among others). Invariance principles for the PCQP are studied in Section 3. The last section deals with some applications of the main theory to some time-sequential tests.

2. A class of PCQP's. Let Θ be an open interval containing 0 and let $\{f(x;\theta),\ \theta \in \Theta\}$ be a family of absolutely continuous probability density functions (pdf), and for every $f: -\infty < x < \infty$, let us denote by

(2.1)
$$g(x) = -(\partial/\partial\theta)\log f(x;\theta)|_0 \quad \text{and}$$
$$\bar{G}(x) = [1 - F(x;0)]^{-1}\textstyle\int_x^\infty g(z)\,dF(z;0)$$

where $F(x;\theta) = \int_{-\infty}^x f(z;\theta)\,dz$. We conceive of the model where the df F_i admits of the pdf f_i and

(2.2)
$$f_i(x) = f(x;\Delta(c_i - \bar{c}_N)),\ -\infty < x < \infty, \quad 1 \leqslant i \leqslant N,$$

where $c_1, \cdots, c_N$ are given constants (not all equal), $\bar{c}_N = N^{-1}\Sigma_{i=1}^N c_i$ and Δ is an unknown parameter. We intend to test

(2.3)
$$H_0: \Delta = 0 \quad \text{vs.} \quad H_1: \Delta \neq (\text{or} > \text{or} <)0.$$

Let us also denote by

(2.4)
$$C_N^2 = \Sigma_{i=1}^N (c_i - \bar{c}_N)^2 \quad \text{and} \quad c_{Ni}^* = C_N^{-1}(c_i - \bar{c}_N), \quad 1 \leqslant i \leqslant N,$$

so that $\Sigma_{i=1}^N c_{Ni}^* = 0$ and $\Sigma_{i=1}^N (c_{Ni}^*)^2 = 1$. Then, the likelihood function for $(\mathbf{Z}_N^{(k)}, \mathbf{Q}_N^{(k)})$ is given by

(2.5)
$$L_{N,k}(\mathbf{Z}_N^{(k)}, \mathbf{Q}_N^{(k)}) = \Pi_{i=1}^k f_{Q_{Ni}}(Z_{Ni})\Pi_{i=k+1}^N[1 - F_{Q_{Ni}}(Z_{Nk})]$$
$$= \Pi_{i=1}^k f(Z_{Ni};\Delta(c_{Q_{Ni}} - \bar{c}_N))\Pi_{i=k+1}^N[1 - F(Z_{Nk};\Delta(c_{Q_{Ni}} - \bar{c}_N))].$$

Defining C_N and the c_{Ni}^* as in (2.4), we have from (2.5)

$$T_{Nk} = C_N^{-1}\{(-\partial/\partial\Delta)\log L_{N,k}(\mathbf{Z}_N^{(k)}, \mathbf{Q}_N^{(k)})|_{\Delta=0}\}$$
$$= \Sigma_{i=1}^k c_{NQ_{Ni}}^* g(Z_{Ni}) + C_N^{-1}\Sigma_{i=k+1}^N[1 - F(Z_{Nk};0)]^{-1}$$

(2.6)
$$\times\{(-\partial/\partial\Delta)\textstyle\int_{Z_{Nk}}^\infty f(x;\Delta(c_{Q_{Ni}} - \bar{c}_N))\,dx|_{\Delta=0}\}$$
$$= \Sigma_{i=1}^k c_{NQ_{Ni}}^* g(Z_{Ni}) + \Sigma_{i=k+1}^N[1 - F(Z_{Nk};0)]^{-1}\{\textstyle\int_{Z_{Nk}}^\infty g(x)f(x;0)\,dx\}c_{NQ_{Ni}}^*$$
$$= \Sigma_{i=1}^k c_{NQ_{Ni}}^* g(Z_{Ni}) + \bar{G}(Z_{Nk})\Sigma_{i=k+1}^N c_{NQ_{Ni}}^*$$
$$= \Sigma_{i=1}^k c_{NQ_{Ni}}^*[g(Z_{Ni}) - \bar{G}(Z_{Nk})] \quad (\text{as} \quad \Sigma_{i=1}^N c_{Ni}^* = 0).$$

Note that the differentiation under the integral sign in (2.6) is permissible under the assumption that there exists an open interval Θ (containing 0 as an inner point) such that $f(x;\theta)$ is a continuously differentiable function of θ and for every $\theta \in \Theta$, $|(\partial/\partial\theta)f(x;\theta)| \leqslant U(x)$ where $\int_{-\infty}^\infty U(x)\,dx < \infty$. For the time-being, we make this assumption. Also, conventionally, we let $T_{N0} = 0$ with probability 1. Note that

(2.7)
$$T_N = T_{NN} = \Sigma_{i=1}^N c_{NQ_{Ni}}^* g(Z_{Ni}) = \Sigma_{i=1}^N c_{Ni}^* g(X_i).$$

Thus, the LMP (locally most powerful) test statistic based on $(\mathbf{Z}_N^{(k)}, \mathbf{Q}_N^{(k)})$ is T_{Nk}, and in the setup of progressive censoring, the sequence $\{T_{Nk};\ 0 \leqslant k \leqslant r\}$ relates to a sequence of linear combinations of functions of order statistics with the coefficients $\{c^*_{NQ_{Ni}}\}$ all stochastic in nature. By reference to Hájek and Sidák (1967, pages 70–71), we may remark that the model (2.2) includes, as special cases, the classical two-sample location and scale models as well as the so-called regression model in location and scale.

We are primarily concerned here with weak convergence of suitable stochastic processes constructed from the partial sequence $\{T_{Nk};\ 0 \leqslant k \leqslant r\}$ where r satisfies (1.3). In statistical applications we often face some related PCQP's which we pose below.

Note that under the usual Cramér-regularity conditions, $\int_{-\infty}^{\infty} g(x)dF(x;\,0) = 0$, so that by (2.1)

$$\bar{G}(x) = -\{1 - F(x;\,0)\}^{-1}\int_{-\infty}^{x} g(z)dF(z;\,0), \quad -\infty < x < \infty. \tag{2.8}$$

Let now $u(t)$ be equal to 1 or 0 according as t is $\geqslant$ or < 0, and let

$$S_N(x) = N^{-1}\Sigma_{i=1}^{N} u(x - X_i), \qquad -\infty < x < \infty \tag{2.9}$$

be the empirical df. We define

$$\begin{aligned} \bar{G}_N(x) &= -\{1 - S_N(x)\}^{-1}\int_{-\infty}^{x} g(x)dS_N(x), && x < Z_{NN}, \\ &= g(Z_{NN}), && x \geqslant Z_{NN}. \end{aligned} \tag{2.10}$$

Then, in (2.6), we replace $\bar{G}(Z_{nk})$ by $\bar{G}_N(Z_{Nk})$, and obtain a related sequence

$$\begin{aligned} T^*_{Nk} &= 0 && k = 0, \\ &= \Sigma_{i=1}^{k} c^*_{NQ_{Ni}}\left[g(Z_{Ni}) - \bar{G}_N(Z_{Nk})\right], && 1 \leqslant k \leqslant N-1, \\ &= T_{NN-1}, && k = N. \end{aligned} \tag{2.11}$$

Note that one can rewrite $T^*_{Nk}(1 \leqslant k \leqslant N-1)$ as

$$\begin{aligned} T^*_{Nk} &= \Sigma_{i=1}^{k} c^*_{NQ_{Ni}}\left[g(Z_{Ni}) + \frac{1}{N-k}\Sigma_{i=1}^{k} g(Z_{Ni})\right] \\ &= \Sigma_{i=1}^{k} g(Z_{Ni})\left[c^*_{NQ_{Ni}} + \frac{1}{N-k}\Sigma_{\alpha=1}^{k} c^*_{NQ_{N\alpha}}\right], \end{aligned} \tag{2.12}$$

so that it is a linear combination of a function of order statistics with stochastic coefficients depending on the censoring stage.

We conclude this section with the asymptotic stochastic equivalence of the two sequences $\{T_{Nk};\ 0 \leqslant k \leqslant r\}$ and $\{T^*_{Nk};\ 0 \leqslant k \leqslant r\}$. We specifically provide the proof for the null hypothesis situation and we shall see in Section 3 that the conclusion remains true for contiguous alternatives. We denote by $F_0(x) = F(x;\,0)$ and consider first the following.

LEMMA 2.1. *Let $\{d_{Ni},\ 1 \leqslant i \leqslant N;\ N \leqslant 1\}$ be a triangular array of real numbers satisfying*

$$\Sigma_{i=1}^{N} d_{Ni} = 0 \quad \text{and} \quad \Sigma_{i=1}^{N} d_{Ni}^2 = 1. \tag{2.13}$$

Also, let $q = q(t) : 0 < t < 1\}$ *be a continuous, nonnegative, U-shaped and square integrable function inside* $I = [0, 1]$. *Finally, let* $\mathbf{Q} = (Q_1, \cdots, Q_N)$ *take on each permutation of* $(1, \cdots, N)$ *with equal probability* $(N!)^{-1}$. *Then*

$$P\{\max_{1 \leqslant k \leqslant N-1} q(k/N)|\Sigma_{i=1}^k d_{NQ_i}| \geqslant 1\} \leqslant \int_0^1 q^2(t)\, dt. \tag{2.14}$$

PROOF. Let

$$U_{Nk} = (N-k)^{-1}\Sigma_{i=1}^k d_{NQ_i} \quad \text{for} \quad k = 1, \cdots, N-1. \tag{2.15}$$

Then, it follows from the results of Sen (1970) and Serfling (1974) that if $\mathcal{P}_N$ be the uniform probability measure over the set of $N!$ permutations of $(1, \cdots, N)$, then under $\mathcal{P}_N$, $\{U_{Nk}\}$ is a martingale. Let

$$h_{Nk} = (N-k)q(k/N), \qquad 1 \leqslant k \leqslant N-1. \tag{2.16}$$

Then, by the U-shapedness of q, there exists an $\alpha : 0 < \alpha < 1$, such that $(N - k)q(k/n)$ is $\searrow$ in k for $1 \leqslant k \leqslant N\alpha$. Hence, by the Chow (1960) extension of the Hájek–Rényi inequality,

$$\begin{aligned}
(2.17) \quad & P\{\max_{1 \leqslant k \leqslant N\alpha} q(k/N)|\Sigma_{i=1}^k d_{NQi}| \geqslant 1\} \\
&= P\{\max_{1 \leqslant k \leqslant N\alpha} h_{Nk}|U_{Nk}| \geqslant 1\} \\
&\leqslant \{h_{N1}^2 E(U_{N1}^2) + \Sigma_{k=2}^{[N\alpha]} h_{Nk}^2[E(U_{Nk}^2) - E(U_{Nk-1}^2)]\} \\
&= \{N^{-1}q(1/N) + \Sigma_{k=2}^{[N\alpha]} q^2(k/N)[(N-k)/(N-1)(N-k+1)]\} \\
&\leqslant N^{-1}\Sigma_{k=1}^{[N\alpha]} q^2(k/N) \quad (\text{as} \quad (N-k)/(N-1)(N-k+1) \leqslant N^{-1}, \quad k \geqslant 1), \\
&\leqslant \int_0^\alpha q^2(t)\, dt, \quad \text{as} \quad q \quad \text{is} \quad U\text{-shaped}.
\end{aligned}$$

Since $\Sigma_{i=1}^k d_{NQ_i} = -\Sigma_{i=k+1}^N d_{NQ_i}$, $1 \leqslant k \leqslant N-1$, the case of $N\alpha < k \leqslant N-1$ can be reduced by reflection and an inequality for this complementary part be obtained in the same manner. $\square$

In particular, if we let $q(t) = K^{-1}$, $0 \leqslant t \leqslant 1$ where $0 < K < \infty$ and choose K large, we obtain from (2.14) that

$$\max_{1 \leqslant k \leqslant N}|\Sigma_{i=1}^k d_{NQ_i}| = 0_p(1) \qquad \text{uniformly in} \quad N. \tag{2.18}$$

Note that if the X_i are i.i.d. with df F_0, then by the Glivenko–Cantelli lemma, as $N \to \infty$,

$$\max_{1 \leqslant k \leqslant N}|F_0(Z_{Nk}) - k/N| \to 0 \qquad \text{almost surely (a.s.).} \tag{2.19}$$

LEMMA 2.2. *If the* X_i *are* i.i.d. *with* df F_0, *then under* (1.3) *and* $\int|g|dF_0 < \infty$,

$$\max_{1 \leqslant k \leqslant r}|\bar{G}(Z_{Nk}) + (N-k)^{-1}\Sigma_{i=1}^k g(Z_{Ni})| \to 0 \qquad \text{a.s. as} \quad N \to \infty. \tag{2.20}$$

PROOF. First note that under the hypothesis of Lemma 2.2,

$$\sup_{-\infty < z < \infty}|\int_{-\infty}^z g(x)dS_N(x) - \int_{-\infty}^z g(x)dF_0(x)| \to 0 \qquad \text{a.s. as} \quad N \to \infty; \tag{2.21}$$

the proof is straightforward (see, for example, Basu and Borwankar (1971)), and hence, is omitted. Secondly, under (1.3), $r/N \to p;\ 0 < p < 1$,

(2.22)

$$\max_{1 \leqslant k \leqslant r} |\{1 - F_0(Z_{Nk})\}N/(N-k) - 1| \to 0 \quad \text{a.s. as} \quad N \to \infty; \text{ by (2.19).}$$

The rest of the proof follows from (2.8) and (2.20)–(2.22). □

Note that for i.i.d. $X_1, \cdots, X_N$ (with df F_0), $\mathbf{Q}_N^{(N)} = (Q_{N1}, \cdots, Q_{NN})$ takes on each permutation of $(1, \cdots, N)$ with equal probability $1/N!$ Thus, by (2.6), (2.12), (2.16) and (2.20), we obtain that under H_0 and (1.3)

(2.23)
$$\max_{1 \leqslant k \leqslant r} |T_{Nk} - T_{Nk}^*|$$
$$\leqslant \left\{\max_{1 \leqslant k \leqslant r} |\Sigma_{i=1}^k c_{NQ_{Ni}}^*|\right\} \left\{\max_{1 \leqslant k \leqslant r} |\bar{G}(Z_{Nk}) + \frac{1}{N-k} \Sigma_{\alpha=1}^k g(Z_{N\alpha})|\right\}$$
$$\to 0, \qquad \text{in probability.}$$

With these results at our disposal, we are tempted to consider a more general class of PCQP's and then to study invariance principles for this class, leading to similar results for $\{T_{Nk}\}$ as special cases.

3. An invariance principle PCQP. Instead of considering PCQP's derivable from some PCSLR statistics, we study here a broader class of PCQP's.

Let $J = \{J(x), -\infty < x < \infty\}$ be absolutely continuous (on finite intervals) and be a difference of two nondecreasing and square integrable (with respect to F_0) functions, so that

(3.1)
$$\delta^2 = \int_{-\infty}^{\infty} J^2(x) dF_0(x) (< \infty).$$

Further, let $\{d_{N1}, \cdots, d_{NN}; N \geqslant 1\}$ be a triangular array of real numbers satisfying the conditions:

(3.2)

$$\Sigma_{i=1}^N d_{Ni} = 0, \qquad \Sigma_{i=1}^N d_{Ni}^2 = 1 \qquad \text{and} \quad \max_{1 \leqslant i \leqslant N} |d_{Ni}| \to 0 \qquad \text{as} \quad N \to \infty.$$

Finally, let $\{\mathbf{Z}_N^{(k)}, \mathbf{Q}_N^{(k)}, 1 \leqslant k \leqslant r\}$ and r be defined as in (1.3) and (1.4), and let

(3.3)
$$\begin{aligned} \mathfrak{L}_{Nk} &= 0, \qquad k = 0 \\ &= \Sigma_{i=1}^k J(Z_{Ni}) \left[d_{NQ_{Ni}} + \frac{1}{N-k} \Sigma_{\alpha=1}^k d_{NQ_{N\alpha}} \right], \qquad 1 \leqslant k \leqslant N-1 \\ &= \mathfrak{L}_{NN-1}, \qquad k = N. \end{aligned}$$

Our primary concern is to develop an invariance principle for $\{\mathfrak{L}_{Nk}; 0 \leqslant k \leqslant r\}$, and we consider first the case of the null hypothesis (H_0) where the X_i are i.i.d. rv with an absolutely continuous df F_0. We denote the expectation and variance under H_0 by E_0 and V_0 respectively. Let

(3.4)
$$\delta_{Nk}^2 = E_0(\mathfrak{L}_{Nk}^2), \qquad 0 \leqslant k \leqslant N,$$

and for every $N(> r \geqslant 1)$, we consider a stochastic process $W_N = \{W_N(t), t \in I\}(I = [0, 1])$ by introducing a sequence of nondecreasing, right-continuous and integer-valued functions $\{k_N(t), t \in I\}$, where

$$k_N(t) = \max\{k : \delta_{Nk}^2 \leqslant t\delta_{Nr}^2\}, \qquad t \in I, \tag{3.5}$$

and then letting

$$W_N(t) = \delta_{Nr}^{-1}\mathfrak{L}_N k_N(t), \qquad t \in I. \tag{3.6}$$

Note that W_N belongs to the $D[0, 1]$ space endowed with the Skorokhod J_1-topology(for $N = 0, 1$, $W_N(t) = 0$, $\forall t \in I$). Our primary concern is to show that under suitable regularity conditions,

$$W_N \to_{\mathfrak{D}} W, \qquad \text{in the } J_1\text{-topology on } D[0, 1], \tag{3.7}$$

where $W = \{W(t), t \in I\}$ is a standard Brownian motion on I.

Let us define

$$\xi_\alpha = \inf\{x : F_0(x) \geqslant \alpha\} \qquad \text{for} \quad 0 < \alpha < 1 \tag{3.8}$$

and let

$$\nu_\alpha^2 = \int_{-\infty}^{\xi_\alpha} J^2(x)dF_0(x) + (1-\alpha)^{-1}\left(\int_{-\infty}^{\xi_\alpha} J(x)dF_0(x)\right)^2, \quad 0 < \alpha < 1; \tag{3.9}$$

by (3.1), $\nu_\alpha^2 < \infty$ for every $0 < \alpha < 1$. First, we consider the following.

LEMMA 3.1. *Under* (3.1), (3.2) *and* H_0, *as* $N \to \infty$,

$$\left[\frac{k}{N} \to \alpha\right] \Rightarrow E_0(\mathfrak{L}_{Nk}^2) \to \nu_\alpha^2, \quad \forall 0 < \alpha < 1. \tag{3.10}$$

PROOF. Let $\mathfrak{Q}_N$ be the set of all possible $(N!)$ permutations of $(1, \cdots, N)$. Then, under H_0,

$$L_{N,N}(\mathbf{Z}_N^{(N)}, \mathbf{Q}_N^{(N)}) = \Pi_{i=1}^N f_0(Z_{Ni}), \quad \forall \mathbf{Q}_N^{(N)} \in \mathfrak{Q}_N, \tag{3.11}$$

and hence $\mathbf{Z}_N^{(N)}, \mathbf{Q}_N^{(N)}$ are stochastically independent with $\mathbf{Q}_N^{(N)}$ assuming each permutation of $(1, \cdots, N)$ with the same probability $(N!)^{-1}$. This insures that for each $k(= 1, \cdots, N)$, $\mathbf{Q}_N^{(k)}$ is independent of $\mathbf{Z}_N^{(N)}$ (and hence, of $\mathbf{Z}_N^{(k)}$) when H_0 holds. Thus, proceeding as in the proof of Lemma 2.1, we obtain that

$$E_0(\mathfrak{L}_{Nk}|\mathbf{Z}_N^{(k)}) = E_0(\mathfrak{L}_{Nk}) = 0, \qquad 1 \leqslant k \leqslant N; \tag{3.12}$$

$$\begin{aligned} &E_0(\mathfrak{L}_{Nk}^2) \\ &= E_0\{E_0(\mathfrak{L}_{Nk}^2|\mathbf{Z}_N^{(k)})\} = E_0\{V_0(\mathfrak{L}_{Nk}|\mathbf{Z}_N^{(k)})\} \\ &= E_0\left\{V_0\left(\Sigma_{i=1}^k J(Z_{Ni})d_{NQ_{Ni}} + \Sigma_{i=k+1}^N d_{NQ_{Ni}}\left[-\frac{1}{N-k}\Sigma_{\alpha=1}^k J(Z_{N\alpha})\right]\right)\Big|\mathbf{Z}_N^{(k)}\right\} \\ &= \left\{\frac{N}{N-1}\Sigma_{i=1}^N\left[d_{Ni} - \frac{1}{N}\Sigma_{\alpha=1}^N d_{N\alpha}\right]^2\right\} \\ &\quad \times E_0\left\{\frac{1}{N}\Sigma_{i=1}^k J^2(Z_{Ni}) + \frac{1}{N(N-k)}\left[\Sigma_{i=1}^k J(Z_{Ni})\right]^2\right\} \\ &= \frac{N}{N-1}E_0\left\{\int_{-\infty}^{Z_{Nk}} J^2(x)dS_N(x) + \frac{N}{N-k}\left(\int_{-\infty}^{Z_{Nk}} J(x)dS_N(x)\right)^2\right\}, \end{aligned} \tag{3.13}$$

by (2.9) and (3.2), where the penultimate step follows by using the facts that under H_0, $\mathbf{Q}_N^{(N)}$ (and hence, $\mathbf{Q}_N^{(k)}$) and $\mathbf{Z}_N^{(k)}$ are independent (so that given $\mathbf{Z}_N^{(k)}$, the $J(Z_{Ni})$, $i \leqslant k$ are also given, while $\mathbf{Q}_N^{(N)}$ assumes all possible permutations of $(1, \cdots, N)$ with the common probability $(N!)^{-1}$), under this permutational law, for arbitrary $a_{N1}, \cdots, a_{NN}$, the variance of $\Sigma_{i=1}^N a_{Ni} d_{NQ_{Ni}}$ is equal to $(N-1)^{-1}\{\Sigma_{i=1}^N [a_{Ni} - \bar{a}_N]^2 \cdot \Sigma_{i=1}^N [d_{Ni} - \bar{d}_N]^2\}$ (where $\bar{a}_N = N^{-1}\Sigma_{i=1}^N a_{Ni}$ and $\bar{d}_N = N^{-1}\Sigma_{i=1}^N d_{Ni}$) and, by our choice, $a_{Ni} = J(Z_{Ni})$ for $i \leqslant k$ while $a_{Nk+1} = \cdots = a_{NN} = -(N-k)^{-1}\Sigma_{i=1}^k J(Z_{Ni})$.

Now, $k/N \to \alpha : 0 < \alpha < 1 => N/(N-k) \to (1-\alpha)^{-1} < \infty$, and, by (2.19)–(2.21),

$$(3.14) \qquad \int_{-\infty}^{Z_{Nk}} J^r(x)\,dS_N(x) \to \int_{-\infty}^{\xi_\alpha} J^r(x)\,dF_0(x) \quad \text{a.s. as } N \to \infty \ (r = 1, 2).$$

Finally, for $r = 1, 2$,

$$(3.15) \qquad \left[\int_{-\infty}^{Z_{Nk}} J^r(x)\,dS_N(x)\right]^{2/r} \leqslant \left[\int_{-\infty}^{\infty} |J^r(x)|\,dS_N(x)\right]^{2/r} \leqslant \int_{-\infty}^{\infty} J^2(x)\,dS_N(x) = N^{-1}\Sigma_{i=1}^N J^2(X_i)$$

where under (3.1), $N^{-1}\Sigma_{i=1}^N J^2(X_i)$ (being a reverse-martingale) is uniformly (in N) integrable. Hence, (3.10) follows from (3.13)–(3.15) and the dominated covergence theorem (cf. Loeve (1963, page 124)). □

Let $\mathfrak{B}_{Nk} = \mathfrak{B}(\mathbf{Z}_N^{(k)}, \mathbf{Q}_N^{(k)})$ be the σ-field generated by $(\mathbf{Z}_N^{(k)}, \mathbf{Q}_N^{(k)})$ and $\mathfrak{B}_{Nk}^* = \mathfrak{B}(\mathbf{Z}_N^{(N)}, \mathbf{Q}_N^{(k)})$ be the σ-field generated by $(\mathbf{Z}_N^{(N)}, \mathbf{Q}_N^{(k)})$, for $k = 1, 2, \cdots, N$.

LEMMA 3.2. *Under H_0, $\{\mathfrak{L}_{Nk}, \mathfrak{B}_{Nk}^*; 0 \leqslant k \leqslant N\}$ (and hence, $\{\mathfrak{L}_{Nk}, \mathfrak{B}_{Nk}; 0 \leqslant k \leqslant N\}$) are martingales for every $N(\geqslant 1)$.*

PROOF. Note that $\mathfrak{L}_{NN} = \mathfrak{L}_{NN-1}$, while for $k \leqslant N-2$, by (3.2),

$$(3.16) \qquad \begin{aligned} &\mathfrak{L}_{Nk+1} - \mathfrak{L}_{Nk} \\ &= \Sigma_{i=1}^k J(Z_{Ni})\left[\frac{1}{N-k-1}\Sigma_{\alpha=1}^{k+1} d_{NQ_{N\alpha}} - \frac{1}{N-k}\Sigma_{\alpha=1}^k d_{NQ_{N\alpha}}\right] \\ &\quad + J(Z_{Nk+1})\left[d_{NQ_{Nk+1}} + (N-k-1)^{-1}\Sigma_{j=1}^{k+1} d_{NQ_{Nj}}\right] \\ &= \frac{(N-k)}{(N-k-1)}\left[d_{NQ_{Nk+1}} + (N-k)^{-1}\Sigma_{j=1}^k d_{NQ_{Nj}}\right] \\ &\quad \times \left[J(Z_{Nk+1}) + (N-k)^{-1}\Sigma_{j=1}^k J(Z_{Nj})\right]. \end{aligned}$$

Since, under H_0, $\mathbf{Q}_N^{(k+1)}$ is independent of $\mathbf{Z}_N^{(k+1)}$ and

$$(3.17) \qquad \begin{aligned} E_0\left[d_{NQ_{Nk+1}} \mid \mathbf{B}_{Nk}^*\right] &= E_0\left[d_{NQ_{Nk+1}} \mid \mathbf{Q}_N^{(k)}\right] = (N-k)^{-1}\Sigma_{j=k+1}^N d_{NQ_{Nj}} \\ &= -(N-k)^{-1}\Sigma_{j=1}^k d_{NQ_{Nj}} \qquad [\text{by } (3.2)], \end{aligned}$$

it follows from (3.16) and (3.17) that

$$(3.18) \qquad E_0\left[\mathfrak{L}_{Nk+1} - \mathfrak{L}_{Nk} \mid \mathfrak{B}_{Nk}^*\right] = 0, \qquad \forall 0 \leqslant k \leqslant N-2.$$

□

LEMMA 3.3. *Under* (3.1), (3.2) *and* H_0, $k/N \to \alpha : 0 < \alpha < 1$ *insures that*

$$\Sigma_{s=0}^{k} E_0\{(\mathfrak{L}_{Ns+1} - \mathfrak{L}_{Ns})^2 | \mathfrak{B}_{Ns}\} \to_p \nu_\alpha^2 \quad \text{as} \quad N \to \infty. \tag{3.19}$$

PROOF. Note that by (3.16) and the stochastic independence of $\mathbf{Q}_N^{(N)}$, $\mathbf{Z}_N^{(N)}$, we have for $0 \leq s \leq N-2$,

$$\begin{aligned} E_0\{(\mathfrak{L}_{Ns+1} - \mathfrak{L}_{Ns})^2 | \mathfrak{B}_{Ns}\} &= \{(N-s)/(N-s-1)^2\}\left\{\Sigma_{i=s+1}^{N}\left[d_{NQ_{Ni}} - \frac{1}{N-s}\Sigma_{l=s+1}^{N} d_{NQ_{Nl}}\right]^2\right\} \\ &\cdot E_0\left\{\left[J(Z_{Ns+1}) + \frac{1}{N-s}(\Sigma_{i=1}^{s} J(Z_{Ni}))\right]^2 | \mathbf{Z}_N^{(s)}\right\}. \end{aligned} \tag{3.20}$$

Now, by (3.1), for every $\eta > 0$, there exists an $\varepsilon : 0 < \varepsilon < 1$ such that

$$\int_{-\infty}^{\xi_\varepsilon} |J(x)|^r dF_0(x) < \eta \quad \text{for} \quad r = 1, 2. \tag{3.21}$$

For $s \leq N\varepsilon$, we note that $\Sigma_{i=s+1}^{N}[d_{NQ_{Ni}}^2 - (1/N-s)\Sigma_{l=s+1}^{N} d_{NQ_{N}}]^2 \leq \Sigma_{i=s+1}^{N} d_{NQ_{Ni}}^2 \leq \Sigma_{i=1}^{N} d_{Ni}^2 = 1$, so that

$$\begin{aligned} &\Sigma_{s=0}^{[N\varepsilon]} E_0\{(\mathfrak{L}_{Ns+1} - \mathfrak{L}_{Ns})^2 | \mathfrak{B}_{Ns}\} \\ &\leq \frac{1}{N}\left[\Sigma_{s=0}^{[N\varepsilon]} E_0\left\{\left[J(Z_{Ns+1}) + \frac{1}{N-s}\Sigma_{i=1}^{s} J(Z_{Ni})\right]^2 | \mathbf{Z}_N^{(s)}\right\}\right]\left[(1-\varepsilon)^{-1} + 0(N^{-1})\right] \end{aligned} \tag{3.22}$$

and proceeding as in (3.14)–(3.15), it can be shown on using (3.21) that the right hand side of (3.22) can be made arbitrarily small, in probability (when $N \to \infty$).

Note that the conditional pdf of Z_{Ns+1} given $\mathbf{Z}_N^{(s)}$ is

$$(N-s) f_0(Z)[1 - F_0(Z)]^{N-s-1}/[1 - F_0(Z_{Ns})]^{N-s}, \quad Z_{Ns} \leq Z < \infty. \tag{3.23}$$

It is easy to show that $E[J(Z_{Ns+1}) | \mathbf{Z}_N^{(s)}]$ exists for all $0 \leq s \leq N-1$ (under (3.1)) and further by the absolute continuity of $J(x)$ (on finite intervals) and the a.s. convergence of $|Z_{Ns} - \xi_{s/N}|$ to 0 for every $N\varepsilon \leq s \leq N\alpha$, $\alpha < 1$, it follows as in Theorem 3.1 of Sen (1.61) that

$$\max_{s : \varepsilon \leq s/N \leq \alpha} |E[J(Z_{Ns+1}) | Z_{Ns}] - J(Z_{Ns})| \to 0 \quad \text{a.s. as} \quad N \to \infty. \tag{3.24}$$

Let us then denote by

$$U_{Ns} = (N-s)^{-1}\Sigma_{s+1}^{N} d_{NQ_{Ni}} = -(N-s)^{-1}\Sigma_{i=1}^{s} d_{NQ_{Ni}}, \quad 1 \leq s \leq N-1, \tag{3.25}$$

$$\tilde{U}_{Ns} = (N-s)^{-1}\Sigma_{s+1}^{N} d_{NQ_{Ni}}^2, \quad s = 1, \cdots, N-1. \tag{3.26}$$

It follows from (2.19) that $\{U_{Ns}, \mathfrak{B}(\mathbf{Q}_N^{(s)}); 1 \leq s \leq N-1\}$ is a martingale when H_0 holds and as in (2.14) and (2.26),

$$\max_{s \leq N\alpha} |U_{Ns}| = 0_p(N^{-1}) \quad \text{for every} \quad 0 < \alpha < 1. \tag{3.27}$$

Also, note that

$$(3.28)\quad E_0[\tilde{U}_{Ns+1}|\mathfrak{B}(\mathbf{Q}_N^{(s)})]$$
$$= (N-s-1)^{-1}\{(N-s)\tilde{U}_{Ns} - E_0[d^2_{NQ_{Ns+1}}|\mathfrak{B}(\mathbf{Q}_N^{(s)})]\}$$
$$= (N-s-1)^{-1}\left\{(N-s)\tilde{U}_{Ns} - \frac{1}{N-s}\Sigma_{\alpha=s+1}^{N} d^2_{NQ_{N\alpha}}\right\} = \tilde{U}_{Ns},$$
$$1 \leqslant s \leqslant N-2.$$

Using the martingale property in (3.28) and the Kolmogorov inequality, we obtain that under H_0, for every $\varepsilon > 0$

$$(3.29)\quad p\{\max_{1\leqslant s\leqslant k}|N\tilde{U}_{Ns} - 1| > \varepsilon\} \leqslant \varepsilon^{-2}E_0\{N\tilde{U}_{Nk} - 1\}^2,$$

where

$$(3.30)\quad E_0[N\tilde{U}_{Nk} - 1]^2 = E_0\left\{\frac{N}{N-k}\Sigma_{s=k+1}^{N}\left[d^2_{NQ_{Ns}} - \frac{1}{N}\right]\right\}^2$$
$$= \frac{N^2}{(N-k)^2}\frac{k(N-k)}{N(N-1)}\Sigma_{i=1}^{N}\left(d^2_{Ni} - \frac{1}{N}\right)^2$$
$$= [Nk/(N-1)(N-k)]\left\{\Sigma_{i=1}^{N} d^4_{Ni} - \frac{1}{N}\right\}$$
$$\leqslant [Nk/(N-1)(N-k)]\left\{(\max_{1\leqslant i\leqslant n} d^2_{Ni})\Sigma_{i=1}^{N} d^2_{Ni} + \frac{1}{N}\right\}$$
$$\to 0$$

by (3.2) and the fact that $k/N \to \alpha : 0 < \alpha < 1$, as $N \to \infty$. Thus from (3.29) and (3.30), we have under H_0,

$$(3.31)\quad \max_{1\leqslant s\leqslant k}|\tilde{U}_{Ns} - \frac{1}{N}| = o_p(1), \quad \text{as} \quad N \to \infty.$$

From (3.20) through (3.31) we obtain that for $k/n \leqslant \alpha : 0 < \alpha < 1$,

$$(3.32)\quad \Sigma_{s=1}^{k} E_0\{(\mathfrak{L}_{Ns+1} - \mathfrak{L}_{Ns})^2|\mathfrak{B}_{Ns}\}$$
$$= N^{-1}\left[\Sigma_{s=1}^{k}\left\{J(Z_{Ns}) + \frac{1}{N-s}\Sigma_{i=1}^{s} J(Z_{Ni})\right\}^2\right][1 + o_p(1)]$$
$$= N^{-1}[\Sigma_{s=1}^{k} J^2(Z_{Ns})(N-s+1)/(N-s)$$
$$+ \Sigma_{i=1}^{k}\Sigma_{j=1}^{k} J(Z_{Ni})J(Z_{Nj})((N - i\vee j)^{-1} + \Sigma_{s=i\vee j}^{k}(N-s)^2)][1 + o_p(1)]$$
$$= N^{-1}\left\{\Sigma_{s=1}^{k} J^2(Z_{Ns}) + \frac{1}{N-k}(\Sigma_{i=1}^{k} J(Z_{Ni}))^2\right\}\{1 + 0_p(N^{-1})\}\{1 + o_p(1)\}$$
$$= \left[\textstyle\int_{-\infty}^{Z_{Nk}} J^2(x)dS_N(s) + \frac{N}{N-k}(\int_{-\infty}^{Z_{Nk}} J(x)dS_N(x))^2\right]\{1 + o_p(1)\},$$

while

$$(3.33)\quad E_0(\mathfrak{L}^2_{N1}|\mathfrak{B}_{N0}) = E_0(\mathfrak{L}^2_{N1})$$
$$= N^2(N-1)^{-2}\frac{1}{N}E_0J^2(Z_{N1}) \to 0 \quad \text{as} \quad N \to \infty.$$

Hence, by (2.23) and (3.14), the right hand side of (3.32) converges (in probability) to ν_α^2, as $N \to \infty$, and the proof of the lemma then follows from (3.32) and (3.33). □

REMARK. Note that in (3.33), $N^{-1}E_0J^2(Z_{N1}) \to 0$ follows from the fact that

(3.34) $$N^{-1}E_0J^2(Z_{N1}) \leqslant \max_{1\leqslant i\leqslant N} N^{-1}E_0J^2(Z_{Ni}) \leqslant N^{-1}E_0\{\max_{1\leqslant i\leqslant N} J^2(X_i)\} \to 0, \quad \text{as} \quad N \to \infty,$$

where the last step follows by standard arguments under (3.1).

Let $I(A)$ be the indicator function of the set A. Then, we have the following

LEMMA 3.4. *For every* $\varepsilon' > 0$, $k/N \to \alpha : 0 < \alpha < 1$, *as* $N \to \infty$

(3.35) $$\Sigma_{s=0}^{k} E_0\{(\mathcal{L}_{Ns+1} - \mathcal{L}_{Ns})^2 I(|\mathcal{L}_{Ns+1} - \mathcal{L}_{Ns}| > \varepsilon')|\mathfrak{B}_{Ns}\} \to_p 0.$$

PROOF. We break up the sum into two subsets $\{s \leqslant N\varepsilon\}$ and $\{N\varepsilon < s \leqslant k\}$. Then, by arguments similar to that in (3.22),

(3.36) $$\Sigma_{s=0}^{[N\varepsilon]} E_0\{(\mathcal{L}_{Ns+1} - \mathcal{L}_{Ns})^2 I(|\mathcal{L}_{Ns+1} - \mathcal{L}_{Ns}| > \varepsilon|\mathfrak{B}_{Ns})\} \leqslant (\Sigma_{s=0}^{[N\varepsilon]} E_0\{(\mathcal{L}_{Ns+1} - \mathcal{L}_{Ns})^2|\mathfrak{B}_{Ns}\} \to_p 0.$$

Since $J(x)$ is the difference of two nondecreasing, absolutely continuous and square integrable functions, it can be shown easily that for every $0 < \varepsilon < \alpha < 1$, there exists a $C = C(\varepsilon, \alpha)(< \infty)$, such that

(3.37) $$\max_{s\,:\,\varepsilon\leqslant s/N\leqslant\alpha} |J(Z_{Ns+1}) + \frac{1}{N-s}\Sigma_{i=1}^{s} J(Z_{Ni})| \leqslant C \quad \text{a.s. as} \quad N \to \infty.$$

On the other hand, by (3.2), for every $\mathbf{Q}_N^{(N)} \in \mathcal{Q}_N$,

(3.38)
$$\max_{1\leqslant i\leqslant N}\left\{|d_{NQ_{Ni}} + \frac{1}{N-i}\Sigma_{s=1}^{i} d_{NQ_{Ns}}|\right\} = \max_{1\leqslant i\leqslant N}\left\{|d_{NQ_{Ni}} - \frac{1}{N-i}\Sigma_{s=i+1}^{N} d_{NQ_{Ns}}|\right\} \leqslant 2\{\max_{1\leqslant i\leqslant N}|d_{Ni}|\} \to 0 \quad \text{as} \quad N \to \infty.$$

Hence, for every C and $\varepsilon' > 0$, there exists an integer $N_0(= N_0(\varepsilon', C))$, such that

(3.39) $$P\left\{\max_{s\leqslant N\alpha}|d_{NQ_{Ns}} + \frac{1}{N-s}\Sigma_{i=1}^{s} d_{NQ_{Ni}}| < \varepsilon'/C\right\} = 1, \qquad \forall N \geqslant N_0.$$

From (3.16), (3.37) and (3.39), it follows that for $N \geqslant N_0$,

(3.40) $$\Sigma_{s=[N\varepsilon]+1}^{k} E_0\{(\mathcal{L}_{Ns+1} - \mathcal{L}_{Ns})^2 I(|\mathcal{L}_{Ns+1} - \mathcal{L}_{Ns}| > \varepsilon')|\mathfrak{B}_{Ns}\} \to_p 0,$$

and (3.35) follows from (3.36) and (3.40). □

We are now in a position to formulate and prove our main theorem of this section.

THEOREM 1. *Under* (1.3), (3.1) *and* (3.2), *when the* X_i *are* i.i.d. rv *with an absolutely continuous* df F_0, (3.7) *holds.*

PROOF. By virtue of the martingale property of $\{\mathcal{L}_{Nk}\}$, when H_0 holds, we are in a position to use Theorem 2 of Scott (1973), and, to prove the theorem, all we need is to show that

$$(3.41) \qquad \delta_{Nr}^{-2}\Sigma_{i=1}^{k_N(t)} V_0[\mathcal{L}_{Ni}|\mathfrak{B}_{Ni-1}] \to_p t, \qquad \text{as} \quad N \to \infty \ (0 < t < 1),$$

$$(3.42) \quad \delta_{Nr}^{-2}\Sigma_{i=1}^{r} E_0\{[\mathcal{L}_{Ni} - \mathcal{L}_{Ni-1}]^2 I(|\mathcal{L}_{Ni} - \mathcal{L}_{Ni-1}| > \varepsilon)|\mathfrak{B}_{Ni-1}\} \to_p 0 \quad (\forall \varepsilon > 0),$$

where $k_N(t)$ and r are defined by (3.5) and (1.3) respectively. Now (3.42) follows directly from Lemmas 3.1 and 3.4 (where we note that (1.3) insures that $0 < p = \alpha < 1$). By the martingale property in Lemma 3.2, δ_{Nk}^2 is $\nearrow$ in $k \leqslant N$ (for every N), while by (3.5) and Lemma 3.1,

$$(3.43) \qquad \delta_{Nk_N(t)}^2 / \delta_{Nr}^2 \to t \qquad \text{as} \quad N \to \infty.$$

Hence, (3.42) follows by using Lemma 3.3 for $k = k_N(t)$ along with (3.43). □

REMARKS. The condition that J is the difference of two nondecreasing functions, though quite general, can be dispensed at the cost of strengthening (3.1) to

$$(3.44) \qquad \int_{-\infty}^{\infty} |J(x)|^m dF_0(x) < \infty \qquad \text{for some} \quad m > 2.$$

In that case, in Lemma 3.4, a Liapounoff-type condition can be obtained (which implies (3.42)) and the rest of the proof remains the same. Secondly, by (1.3), we have limited ourselves to $0 < p < 1$. Though p may be arbitrarily close to 1, there are a few technical barriers for allowing p to be equal to one. Note that ν_p^2 may tend to ∞ as $p \to 1$ (viz., $J(x) = 1, \forall x \Rightarrow \nu_p^2 = p/(1-p)$). If, however, we impose the additional condition [as in before (2.8)] that

$$(3.45) \qquad \int_{-\infty}^{\infty} J(x) dF_0(x) = 0,$$

we obtain from (3.9) and (3.45) that for every $0 < p < 1$,

$$(3.46) \qquad \nu_p^2 = \int_{-\infty}^{\xi_p} J^2(x) dF_0(x) + (1-p)^{-1}\left(\int_{\xi_p}^{\infty} J^2(x) dF_0(x)\right)^2$$
$$\leqslant \int_{-\infty}^{\xi_p} J^2(x) dF_0(x) + \int_{\xi_p}^{\infty} J^2(x) dF_0(x) = \delta^2 \qquad \text{and} \quad \lim_{p \uparrow 1} \nu_p^2 = \delta^2.$$

Even so, $\int_{-\infty}^{\infty} J(x) dS_N(x)$ is not necessarily equal to 0; in fact, it is $0_p(N^{-\frac{1}{2}})$. Further, $\{\max_{1 \leqslant i \leqslant N} J^2(Z_{Ni})\} = E_0\{\max_{1 \leqslant i \leqslant N} J^2(X_i)\} = o(N)$, under (3.9), while under (3.9) and (3.45), $E_0(\int_{-\infty}^{\infty} J(x) dS_N(x))^2 = N^{-1}\delta^2$. Hence, it follows from (3.13) that for every (fixed) $s (\geqslant 1)$, as $N \to \infty$,

$$(3.47) \qquad E_0(\mathcal{L}_{NN-s}^2) \to \delta^2 + s^{-1}\delta^2 = (1 + s^{-1})\delta^2 > \delta^2 = \nu_1^2.$$

This apparent anomaly can be straightened out with the help of (2.8) and (2.10). Note that if g satisfies (3.1), then for every (fixed) $s (\geqslant 1)$, as $N \to \infty$,

$$(3.48) \qquad \bar{G}(Z_{NN-s}) = o_p([N/s]^{\frac{1}{2}}) \qquad \text{while} \quad |\int_{-\infty}^{Z_{NN-s}} g(x) dS_N(x)| = 0_p(N^{-\frac{1}{2}}).$$

Thus, whereas (2.8) relates to a term (for $x = Z_{NN-s}$)$o_p([N/s])$, (2.10) leads us to a term $0_p((N/s)N^{-\frac{1}{2}}) = 0_p(N^{\frac{1}{2}}/s)$, and hence, $\bar{G}(Z_{NN-s})$ and $\bar{G}_N(Z_{NN-s})$ are of different (stochastic) orders of magnitude, and the stochastically larger order of magnitude for $\bar{G}_N(Z_{NN-s})$ pushes up the variance of $\mathcal{L}_{NN-s}$; in fact, here (2.20) and hence, (2.23) may not hold. But, if $s = s(N)$ be such that $s(N) \to \infty$ but $s(N)/N \to 0$ as $N \to \infty$, then it can be shown that

$$(3.49) \qquad E_0(\mathcal{L}^2_{NN-s(N)}) = \delta^2(1 + 1/s(N) + o(1)) \to \delta \qquad \text{as} \quad N \to \infty.$$

Thus, as regards (2.23), we can proceed as follows. First, doing the same line (of proof) as in Lemma 3.1 of Sen (1976), it can be shown that under H_0,

$$(3.50) \qquad \{T_{Nk}, \mathcal{B}_{Nk};\ 0 \leqslant k \leqslant N\} \qquad \text{is a martingale,}$$

while for $\eta > 0$, arbitrarily small, on letting $r_N = [N(1 - \eta)]$, it can be shown that $E_0(T_N^2) = \nu_1^2$ and $E_0(T^2_{Nr_N}) \to \nu^2_{1-\eta}$, so that by the Kolmogorov-inequality for martingales, for every $\varepsilon > 0$,

$$(3.51) \qquad p\{\max_{r_N \leqslant k \leqslant N}|T_{Nk} - T_{Nr_N}| > \varepsilon\} \leqslant \varepsilon^{-2}(T_N - T_{Nr_N})^2$$
$$= \varepsilon^{-2}\left[\nu_1^2 - \nu^2_{1-\eta} + o(1)\right],$$

which can be made smaller than any given $\delta(> 0)$ by choosing $\eta(> 0)$ sufficiently small (and noting that as $\int g dF_0 = 0$, by (3.46), $\lim_{\eta\to 0}\nu^2_{1-\eta} = \nu_1^2$). On the other hand, for $r \leqslant r_N = [N(1 - \eta)]$, $\eta > 0$, we are in a position to use (2.23), so that the invariance principle for $\{T^*_{Nk};\ 0 \leqslant k \leqslant r_N\}$ leads us to the same for $\{T_{Nk};\ 0 \leqslant k \leqslant r_N\}$, and this along with (3.51) yields the desired result for the entire sequence $\{T_{Nk};\ 0 \leqslant k \leqslant N\}$. In a similar manner, by the martingale property in Lemma 3.2 and (3.49), defining $s(N)$ as in before (3.49), we can replace $\{\mathcal{L}_{Nk};\ 0 \leqslant k \leqslant N - s(N)\}$ by an appropriate $\{\mathcal{L}_{Nk};\ 0 \leqslant k \leqslant N(1 - \eta)\}$ $(\eta > 0)$ and apply our Theorem 1. In view of the fact that $E_0(\mathcal{L}_{NN} - \mathcal{L}_{NN-s(N)})^2 \to \delta^2$, (not to 0), we are, however, unable to replace $N - s(N)$ by N in this case. In actual practice, PCS mostly involves a terminal censoring number (r) corresponding to a value of p quite below 1, and, hence, this technicality is not of much concern to us.

Let us now proceed on to the nonnull case. We shall confine ourselves to local (contiguous) alternatives where parallel results can be derived and these will be incorporated in the next section for the study of asymptotic power of some PCS tests based on such PCQP's. Consider a triangular array $\{X_{Ni}, 1 \leqslant i \leqslant N;\ N \leqslant 1\}$ of (row-wise) independent rv's and assume that X_{Ni} has an absolutely continuous df F_{Ni} with an absolutely continuous pdf f_{Ni} and

$$(3.52) \qquad f_{Ni}(x) = F(x; \Delta c^*_{Ni}), \qquad -\infty < x < \infty, i = 1, \cdots, N,$$

where f, Δ and c^*_{Ni} are all defined as in the beginning of Section 2. Note that, in (3.52), Δ is regarded as fixed while by (2.4), the c^*_{Ni} all go to 0 as $N \to \infty$. We denote such a sequence of alternative hypotheses by $\{H_N\}$, while H_0 relates to $\Delta = 0$. Our concern is to study the weak convergence of $\{W_N\}$, defined by (3.5)–(3.6), when $\{H_N\}$ holds.

We define the d_{Ni} as in (3.2), the c^*_{Ni} as in (2.4), and assume that they satisfy the limits

$$(3.53) \qquad \lim_{N\to\infty}\textstyle\sum_{i=1}^N d_{Ni}c^*_{Ni} = \rho^*(-1 \leqslant \rho^* \leqslant 1), \qquad \max_{1\leqslant i\leqslant N}|c^*_{Ni}| \to 0;$$

in fact, for $d_{Ni} = c^*_{Ni}$, $i = 1, \cdots, N$, $\rho^* = 1$ [by (2.4)]. For $0 < p < 1$, we define

$$\alpha(t, p) = \max\{\alpha : \nu^2_\alpha \leq t\nu^2_p\}, \qquad t \in [0, 1], \tag{3.54}$$

where ν^2_α is defined by (3.9). Note that $\nu^2_{\alpha(t,p)}$ is $\nearrow$ in $t(\in[0, 1])$ and $\nu^2_{\alpha(0,p)} = 0$, $\nu^2_{\alpha(1,p)} = \nu^2_p$, so that $\alpha(0, p) = 0$ and $\alpha(1, p) = p$. Here also, we denote $F(x; 0)$ and $f(x; 0)$ by F_0 and f_0, respectively, and $g(x)$ and $\bar{G}(x)$ as in (2.1). Further, we define

$$J^*(x) = [1 - F_0(x)]^{-1}\textstyle\int_{-\infty}^{x} J(y)dF_0(y), \qquad -\infty < x < \infty; \tag{3.55}$$

$$\zeta^{(p)}_t = \left[\textstyle\int_{-\infty}^{\xi_\alpha} J(x)g(x)dF_0(x) - (1 - \alpha)^{-1}J^*(\xi_\alpha)\bar{G}(\xi_\alpha)\right]_{\alpha = \alpha(t,p)}, \qquad t \in I. \tag{3.56}$$

We also assume that the pdf $f(x; \theta)$ is absolutely continuous in $\theta(\in\Theta)$ for almost all x, $(\partial/\partial\theta)f(x; \theta) = f'_\theta(x; \theta)$ exists and converges to $f'_\theta(x; 0)$ as $\theta \to 0$, and further, defining $g(x)$ as in Section 2 and letting $F_0(x) = F(x; 0)$, we assume that

$$\lim_{\theta\to 0}\textstyle\int_{-\infty}^{\infty}[f'_\theta(x; \theta)]^2[f(x, \theta)]^{-1}\,dx = \int_{\infty}^{\infty} g^2(x)dF_0(x) < \infty. \tag{3.57}$$

Finally, let us denote by

$$\mu = \{\mu(t) = \Delta\rho^*\zeta^{(p)}_t/\nu_p,\ t \in I\} \tag{3.58}$$

and note that by assumptions made on J and g, $\mu \in C[0, 1]$ space. Then we have the following.

THEOREM 2. *Let $\{W_N\}$ and W be defined as in* (3.5)–(3.7). *Then, under* (1.3), (3.2), (3.53), (3.57) *and $\{H_N\}$ in* (3.52), *as $N \to \infty$,*

$$W_N - \mu \to_{\mathfrak{D}} W, \qquad \text{in the } J_1\text{-topology on } D[0, 1]. \tag{3.59}$$

PROOF. Let P_N and P^*_N be respectively the joint df of $(\mathbf{Z}^{(N)}_N, \mathbf{Q}^{(N)}_N)$ when H_0 (i.e., $F_i = F_0$, $\forall i = 1$) and H_N in (3.51) hold. Then, under (3.2), (3.52) and the assumed regularity conditions on f, it can be shown (df. Hájek and Sidák (1967, pages 239–240)) that $\{P^*_N\}$ is *contiguous* to $\{P_N\}$. For $x \in D[0, 1]$ and $\delta \in (0, 1)$, let us define

$$\omega_\delta(x) = \sup\{\min[|x(t) - x(s)|, |x(s) - x(u)|]: 0 \leq u < s < t \leq u + \delta \leq 1\}. \tag{3.60}$$

Since, $W_N(0) = 0$, with probability 1, and, by Theorem 1, under H_0, $\{W_N\}$ is *tight*, it follows that

$$\lim_{\delta\downarrow 0}\limsup_N P\{\omega_\delta(W_N) > \varepsilon | H_0\} = 0, \qquad \forall\varepsilon > 0. \tag{3.61}$$

Also, W_N is a mapping of $(\mathbf{Z}^{(N)}_N, \mathbf{Q}^{(N)}_N)$ into the space $D[0, 1]$. Hence, by the contiguity of $\{P^*_N\}$ to $\{P_N\}$ and (3.61), we conclude that

$$\lim_{\delta\downarrow 0}\limsup_N P\{\omega_\delta(W_N) > \varepsilon | H_N\} = 0, \qquad \forall\varepsilon > 0, \tag{3.62}$$

that is, $\{W_N\}$ remains tight under $\{H_N\}$. Thus, to prove (3.59), we need to establish only the convergence of the finite dimensional distributions of $\{W_N - \mu\}$ to the corresponding ones of W.

For this purpose, for any $k : k/N \to \alpha : 0 < \alpha \leqslant p < 1$, we rewrite $\mathfrak{L}_{Nk}$ as

$$(3.63)\qquad \mathfrak{L}_{Nk} = \Sigma_{i=1}^{N} d_{Ni} J(X_i) I(X_i \leqslant Z_{Nk}) + \left\{ \frac{1}{N-k} \Sigma_{i=1}^{k} J(Z_{Ni}) \right\} \left\{ \Sigma_{i=1}^{N} d_{Ni} I(X_i \leqslant Z_{Nk}) \right\},$$

where $I(A)$ stands for the indicator function of the set A. Defining ξ_α and J^* as in (3.8) and (3.55), we introduce

$$(3.64)\qquad \mathfrak{L}_{Nk}^* = \Sigma_{i=1}^{N} d_{Ni} J(X_i) I(X_i \leqslant \xi_\alpha) + J^*(\xi_\alpha) \Sigma_{i=1}^{N} d_{Ni} I(X_I \leqslant \xi_\alpha).$$

If we write $S_N(\xi_\alpha) = N^{-1}k_N$, then by (3.63), (3.64) and the definition of $\mathbf{Q}_N^{(k)}$, we have

$$(3.65)\qquad \mathfrak{L}_{Nk}^* = \mathfrak{L}_{Nk_N} + \left\{ J^*(\xi_\alpha) - \frac{1}{N-k_N} \Sigma_{i=1}^{k_N} J(Z_{Ni}) \right\} \left\{ \Sigma_{i=1}^{k_N} d_{NQ_{Ni}} \right\}.$$

Note that $N^{-1}k_N \to \alpha$, in probability, under H_0 [viz., (2.19)], so that by the same technique as in Lemma 2.2,

$$(3.66)\qquad |J^*(\xi_\alpha) - (N-k_N)^{-1} \Sigma_{i=1}^{k_N} J(Z_{Ni})| \to_p 0, \qquad \text{under} \quad H_0,$$

while by (2.18), $|\Sigma_{i=1}^{k_N} d_{NQ_{Ni}}| = 0_p(1)$, under H_0. Hence, the second term on the right hand side of (3.65) converges in probability to 0 as $N \to \infty$ when H_0 holds. Further, by the martingale property (Lemma 3.2), we have by the Kolmogorov-inequality,

$$(3.67)\qquad P\{\max_{q : |k-q| < \delta N} |\mathfrak{L}_{Nq} - \mathfrak{L}_{Nk}| > \varepsilon | H_0\} \to 0 \qquad \text{as} \quad \delta \downarrow 0,$$

and hence, noting that $|N^{-1}k_N - \alpha| \to_p 0$ and $k/N \to \alpha : 0 < \alpha < 1$, we obtain from the above that

$$(3.68)\qquad \mathfrak{L}_{Nk}^* - \mathfrak{L}_{Nk} \to_p 0 \qquad \text{as} \quad N \to \infty \qquad \text{when } H_0 \text{ holds.}$$

Again, by virtue of the contiguity of $\{P_N^*\}$ to $\{P_N\}$ and (3.68), we conclude that as $N \to \infty$,

$$(3.69)\qquad \mathfrak{L}_{Nk}^* - \mathfrak{L}_{Nk} \to_p 0 \qquad \text{under } \{H_N\} \text{ as well.}$$

Thus, for finitely many k's, say, $k_1 \leqslant \cdots \leqslant k_m$, $m(\geqslant 1)$ given, satisfying

$$(3.70)\qquad N^{-1}k_j \to \alpha(t_j, p), \qquad 0 \leqslant t_1 < \cdots < t_m \leqslant 1,$$

to study the joint distribution of $W_N(t_1), \cdots, W_N(t_m)$, it suffices to consider the joint df of $\mathfrak{L}_{Nk_1}^*, \cdots, \mathfrak{L}_{Nk_m}^*$. Since the $\mathfrak{L}_{Nk}^*$ involve a sum over independent random variables, by the classical (multivariate version of the) central limit theorem, it follows that under (3.1), (3.2), (3.52), (3.53) and (3.57), $[\mathfrak{L}_{Nk_1}^*, \cdots, \mathfrak{L}_{Nk_m}^*]$ converges in law to a multivariate normal distribution with mean vector $[\mu(t_1), \cdots, \mu(t_m)]\nu_p$ and dispersion matrix $\nu_p^2(t_j \wedge t_l)_{j,l=1,\cdots,m}$ which conforms to the desired pattern. □

REMARKS. In (2.23), we proved the stochastic equivalence of $\{T_{Nk}; 0 \leqslant k \leqslant r\}$ and $\{T_{Nk}^*; 0 \leqslant k \leqslant r\}$ when H_0 and (1.3) hold. Here also, we can proceed on the same line as in (3.63)–(3.65) and use the contiguity of $\{P_N^*\}$ to $\{P_N\}$ to show that (2.23) remains true when $\{H_N\}$ (in (3.52), (3.53) and (3.57)) holds along with (1.3).

One could have extended the results of Sen (1976) to the current setup of nonidentically distributed rv's. However, that would have induced more complications in the proof along with some extra (mild) regularity conditions (viz., (2.8) and (3.36) of Sen (1976)). The current approach provides an alternative and simple solution.

Asymptotic normality of linear combinations of functions of order statistics has been studied by Chernoff, Gastwirth and Johns (1967), Bickel (1967), Moore (1968), Stigler (1969), Shorack (1969, 1972) and others. Recently, Sen (1978a) has employed a reverse martingale characterization to strengthen the asymptotic normality to an invariance principle for the tail-sequence of such statistics. A similar invariance principle has also been developed by Sen (1978b) for rank-discounted partial sums (which can be expressed as a linear combination of functions of order statistics with stochastic weights). In all the papers referred to above, the regularity conditions needed to study the asymptotic theory are more restrictive than the ones incorporated in the current study. This is not surprising. The permutational uniform distribution of $\mathbf{Q}_N^{(N)}$ and the stochastic independence of $\mathbf{Z}_N^{(N)}$ and $\mathbf{Q}_N^{(N)}$ (under H_0) account for the main reason for the relaxation of the regularity conditions. For the case of contiguous alternatives in (3.52), this is not quite true, and hence, we need some extra regularity conditions. For general alternatives, contiguity may not hold and, hence, the current method of proving Theorem 2 may not stand valid. A different proof for this type of result, in general, will require more restrictive regularity conditions.

4. Applications to time-sequential tests based on PCQP. A variety of rank based PCS tests is available in the literature. Hájek (1963) has developed the asymptotic theory of Kolmogorov-Smirnov (KS-) type tests for regression alternatives, and his results can be adapted readily in a PCS provided we let $r/N \to 1$. The simple limiting null distributions of these KS-type statistics (viz., (3) and (4) on page 189 of Hájek and Sidák (1967)) are not valid if $r/N \to p$: $0 < p < 1$. However, some recent tabulations of the critical values of the truncated KS-type statistics by Koziol and Byar (1975) and Schey (1977) provides us with these (approximate) critical values. In Chapters V and VI of Hájek and Sidák (1967), some related tests are also considered; in particular, the Rényi-type and Cramér-von Mises-type tests for regression alternatives deserve mention and they can also be adapted in a PCS when we let $r/N \to 1$. Again, for $r/n \to p : 0 < p < 1$, the limiting distributions of these statistics are no longer very simple and extensive simulation studies are being made to provide approximate critical values in such cases. Chatterjee and Sen (1973) have studied the weak convergence of PC linear rank statistics to a Brownian motion and their procedure can be used for any $r/N \to p : 0 < p \leqslant 1$ with simple limiting null distribution theory provided by them. Usually their procedure is better than Hájek's ones. All these procedures share one common feature: namely, they are based solely on the vector $\mathbf{Q}_N^{(r)}$, disregarding any information contained in the vector $\mathbf{Z}_N^{(r)}$ of associated order statistics. Thus, it is quite intuitive to extract this information and in Section 2, we have shown that a PCPLR statistics sequence relates to PCQP's which again can be approximated by more convenient linear combinations of functions of order statistics with stochastic coefficients. Thus, in the same spirit as in Chatterjee and Sen (1973) and Sen (1976), we may be interested in employing the process W_N, defined by (3.5)–(3.6) and use as a test-statistic

$$M_N = M(W_N) \tag{4.1}$$

where $M(x) = M(x(t) : 0 \leqslant t \leqslant 1)$ is a suitable functional. For example, we may take the KS-type statistics as

$$M_N^+ = \sup_{0 \leqslant t \leqslant 1} W_N(t) = \max_{0 \leqslant k \leqslant r} \mathcal{L}_{Nk} / \nu_p, \tag{4.2}$$

$$M_N = \sup_{0 \leqslant t \leqslant 1} |W_N(t)| = \max_{0 \leqslant k \leqslant r} |\mathcal{L}_{Nk}| / \nu_p \tag{4.3}$$

and obtain the limiting distributions of M_N^+ or M_N with the aid of our Theorem 1 and the well-known distributional results on $\sup_{0 \leqslant t \leqslant 1} W(t)$ or $\sup_{0 \leqslant t \leqslant 1} |W(t)|$. Theorem 2 provides us with the asymptotic power of such a test. We may also consider other functional (such as the Rényi-type or Cramér-von Mises type) of W_N and purpose the same as test-statistics. This leads us to the study of the asymptotic behavior of different functionals of PCQP's with different $\{J\}$, and will be studied in a subsequent paper.

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