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DENSITY ESTIMATION IN A CONTINUOUS-TIME STATIONARY MARKOV PROCESS

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This paper deals with a general class of recursive estimates of the density function in a continuous-time stationary Markov process. Under the condition G_2 of Rosenblatt sufficient conditions for almost sure convergence of such estimates are given.

1. Introduction. The problem of density estimation in a stationary Markov sequence has been investigated by Rosenblatt (1970) and Roussas (1969). In a stationary and mixing discrete-time process, the problem has been studied by Bosq (1973, 1975).

Recently, in order to solve a class of nonlinear identification problems (of dynamical systems represented by stochastic differential equations) Banon (1976) considered the density estimation in a continuous-time Markov process. In Banon's work, under some specific conditions on the stochastic differential equation, the process which is the solution of this equation is a diffusion process satisfying condition G_2 of Rosenblatt (1970), and recursive estimates of Deheuvels (1973, 1974) are extended to continuous case and used to obtain quadratic mean convergence.

In this note, we consider a general class of recursive estimates of the density function in a continuous-time stationary Markov process satisfying the condition G_2. These estimates have been considered by Deheuvels (1974) in the independent and discrete case, with Yamato's estimates (1971) taken as a particular case. We study the uniform convergence of expectations and the uniform, almost sure convergence of such estimates.

2. Convergence of expectations. Let $\{X_t\}$, $t \in \mathbb{R}^+$, be a stationary stochastic process. Let f be the probability density (on the real line $\mathbb{R}$) of each X_t. In the sequel, we shall consider recursive estimates of f of the following general form:

$$x \in \mathbb{R}, \qquad t > 0, \qquad f_t(x) = \left[\int_0^t h(s)H(h(s))\,ds\right]^{-1}\int_0^t H(h(s))K\{(X_s - x)/h(s)\}\,ds$$

where

(a) $h : \mathbb{R}^+ \to \mathbb{R}^+$, positive and $h(s) \searrow 0$ when $s \to +\infty$

(b) $s \to h(s)H(h(s))$ is locally integrable on $\mathbb{R}^+$ and $\int_0^t h(s)H(h(s))\,ds \to +\infty$ when $t \to +\infty$, H being a mapping from $\mathbb{R}^+$ to $\mathbb{R}^+$.

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Note that, in the discrete case, $H(u) = 1/u$ corresponds to Yamato's estimates (1971), and $H(u) = 1$ for all $u \in \mathbb{R}^+$ corresponds to the estimates studied extensively by Deheuvels (1974).

(c) The kernel K is in general assumed to be positive, bounded and $\int_R K(y)dy = 1$.

PROPOSITION 1. *Let $\{X_t\}$, $t \in \mathbb{R}^+$, be a stationary and measurable stochastic process. If the probability density f is continuous, bounded and the kernel K is bounded, then*

$$\lim_{t\to+\infty} Ef_t(x_0) = f(x_0), \qquad x_o \in \mathbb{R}.$$

PROOF. By measurability of the process and the fact that K is bounded, we have:

$$Ef_t(x_0) = 1/g(t)\int_0^t H(h(s))EK((X_s - x_0)/h(s))ds$$

where $g(t) = \int_0^t h(s)H(h(s))ds$, $t > 0$. By the construction of g, we may write:

$$Ef_t(x_0) - f(x_0) = 1/g(t)\int_0^t h(s)H(h(s))\left[E(1/h(s))K((X_s - x_0)/h(s)) - f(x_0)\right]ds.$$

By Lemma 1 of the appendix it is sufficient to show that

$$\lim_{s\to\infty}\left[E(1/h(s))K((X_s - x_0)/h(s)) - f(x_0)\right] = 0.$$

We have:

$$E(1/h(s))K((X_s - x_0)/h(s)) - f(x_0) = \int_{\mathbb{R}} K(y)\left[f(x_0 + h(s)y) - f(x_0)\right]dy.$$

For $\varepsilon > 0$, denote $\delta(\varepsilon, x_0) = \sup_{|z|\leqslant\varepsilon}|f(x_0 + z) - f(x_0)|$. By the continuity of f at x_0, and for $\hat{\varepsilon}$ small, there exists $\varepsilon = \varepsilon(\hat{\varepsilon}, x_0)$ such that

$$\delta(\varepsilon, x_0) \leqslant \hat{\varepsilon}/2.$$

For such an ε, we write:

$$\begin{aligned}
|E(1/h(s))&K((X_s - x_0)/h(x)) - f(x_0)| \\
&\leqslant \int_{\{h(s)|y|>\varepsilon\}} K(y)\left[f(x_0 + h(s)y) - f(x_0)\right]dy \\
&\quad + \int_{\{h(s)|y|\leqslant\varepsilon\}} K(y)\left[f(x_0 + h(s)y) - f(x_0)\right]dy \\
&\leqslant 2\|f\|_\infty \int_{\{|y|>\varepsilon(\ )/h(s)\}} K(y)dy + \sup_{\{|y|h(s)\leqslant\varepsilon\}}|f(x_0 + h(s)y) - f(x_0)| \\
&\leqslant 2\|f\|_\infty \int_{\{|y|>\varepsilon(\ )/h(s)\}} K(y)dy + \hat{\varepsilon}/2, \text{ where } \|f\|_\infty \text{ stands for } \sup_{\mathbb{R}} f(x).
\end{aligned}$$

Since K is integrable and $h(s) \searrow 0$ as $s \to \infty$, there exists $\hat{s} = \hat{s}(x_0, \varepsilon)$ such that:

$$s \geqslant \hat{s} \Rightarrow \int_{\{|y|>\varepsilon(x_0)/h(s)\}} K(y)dy \leqslant \hat{\varepsilon}/4\|f\|_\infty,$$

i.e., given $\hat{\varepsilon} > 0$, there exists $\hat{s}(x_0, \varepsilon)$ such that:

$$s \geqslant \hat{s} \Rightarrow |E(1/h(s))K((X_s - x_0)/h(s)) - f(x_0)| \leqslant \hat{\varepsilon}.$$

PROPOSITION 2. *Under the hypothesis of Proposition 1 and if, in addition, f is uniformly continuous, then:*

$$\lim_{t\to\infty} Ef_t(x) = f(x) \qquad \text{uniformly in } x.$$

PROOF. By virtue of Lemma 4 of the appendix, it is sufficient to verify that there exists $\psi \in \mathfrak{L}^1_{\text{loc}}[h(s)H(h(s))ds]$ such that:

$$\sup_{x \in R} |\phi_x(s)| \leqslant \psi(s).$$

We have:

$$|\phi_x(s)| \leqslant \int_{\mathbf{R}} (1/h(s)) K((y - x)/h(s)) f(y) dy + \|f\|_\infty$$
$$\leqslant 2\|f\|_\infty = \psi.$$

3. Almost sure convergence of estimates. By virtue of Proposition 1 and Proposition 2, we are led to consider the almost sure convergence of $f_t(x) - Ef_t(x)$ to 0 when $t \to \infty$.

Now assume all hypotheses of the previous paragraph. Let

$$W_x(t) = f_t(x) - Ef_t(x) = 1/g(t) \int_0^t Z_x(s) ds$$

where

$$Z_x(s) = H(h(s))\left[K((X_s - x)/h(s)) - EK((X_s - x)/h(s))\right].$$

The almost sure convergence of $W_x(t)$ is similar to the almost sure stability problem [Loève, page 487]. We shall follow the technique employed in Loève.

PROPOSITION 3. *If*:

(a) *For each* $x \in \mathbb{R}$, Var $K((X_t - x)/h(t)) = 0[H^{-2}(h(t))]$, $t \to \infty$,

(b) $g(t) \sim t^\beta$, $t \to \infty (0 < \beta < 1)$, *and for each* $x \in \mathbb{R}$, $1/t^{2\beta} \int_0^t \int_0^t \Gamma_x(s, s') ds ds' \leqslant d/t^{\gamma\beta}$ *for* t *large* $(d > 0)$, *where* Γ_x *is the covariance of the second order process* $Z_x(s)$, *with* $\gamma\beta > 2(1 - \beta)$, *then*

$$W_x(t) \to 0; \qquad t \to \infty, \quad \textit{almost surely}.$$

PROOF. $\Gamma_x(t, t) = H^2(h(t))$Var $K((X_t - x)/h(t))$. Using (a) and (b), the assertion follows from Lemma 5 and Lemma 6 of the appendix.

EXAMPLES. (*Condition a*). Note that

$$\text{(1)} \qquad \text{Var } K((X_t - x)/h(t)) \leqslant M\|f\|_\infty h(t) + \|f\|^2_\infty h^2(t)$$

where $M = \int_{\mathbb{R}} K^2(x) dx < +\infty$ (since K is integrable and bounded).

(α) $H(u) = 1, \forall u > 0$.

By the above inequality (1), it is clear that:

$$\Gamma_x(t, t) = 0(1), \qquad t \to \infty \text{ (uniformly in } x).$$

(β) $H(u) = 1/u^{\frac{1}{2}}, u > 0$.

We have $\Gamma_x(t, t) = 0(1)$, $t \to \infty$ (uniformly in x).

Additional hypothesis on the process $\{X_t\}$, $t \in \mathbb{R}^+$. To obtain condition (b) in Proposition 3, we shall assume, in addition, that the process $\{X_t\}$, $t \in \mathbb{R}^+$ is a Markov process verifying the condition G_2 of Rosenblatt [7].

Denote by P_t, $t \in \mathbb{R}^+$, the semi-group of transition operators of the stationary Markov process $\{X_t\} t \in \mathbb{R}^+$. Recall that the process $\{X_t\} t \in \mathbb{R}^+$ is said to satisfy the condition G_2 if there exists $s \in \mathbb{R}^+$ and $\alpha (0 < \alpha < 1)$ such that:

$$|P_s|_2 \leqslant \alpha$$

where

$$|P_s|_2 = \sup_{\phi \perp 1} \|P_s \phi\|_2 / \|\phi\|_2$$

with ϕ measurable and bounded, $\|\cdot\|_2$ denoting the norm in $L^2(f(x)dx)$, and $\phi \perp 1$ meaning $\int_{\mathbb{R}} \phi(x) f(x) dx = 0$. As a consequence of this condition G_2, one has (see Banon (1976))

$$\forall t \in \mathbb{R}^+, \qquad |P_t|_2 < \delta^t/\alpha \qquad \text{with} \quad 0 < \delta < 1.$$

Let

$$\Gamma_x(t, s) = EZ_x(t) Z_x(s), \qquad x \in \mathbb{R}$$

and

$$C_x(t, s) = E\left[K((X_t - x)/h(t)) - EK((X_0 - x)/h(t))\right]$$
$$\left[K((X_s - x)/h(s)) - EK((X_0 - x)/h(s))\right].$$

It is known (Banon (1976)) that:

(2) $$\forall x \in \mathbb{R}, \qquad C_x(t, s) < c(h(t)h(s))^{\frac{1}{2}} \delta |t - s|.$$

Thus:

$$\forall x \in \mathbb{R}, \qquad \Gamma_x(t, s) < cH(h(t))H(h(s))(h(t)h(s))^{\frac{1}{2}} \delta |t - s|.$$

Since $g(t) \sim t^\beta$, we have:

if H is bounded, then:

$$\forall x \in \mathbb{R}, \qquad \int_0^t \int_0^t \Gamma_x(s, s') ds ds' \leqslant c_1 t^\beta \qquad \text{for } t \text{ large.}$$

If $H(u) \leqslant d/u^{\frac{1}{2}}$, $u > 0$, then:

$$\forall x \in \mathbb{R}, \qquad \int_0^t \int_0^t \Gamma_x(s, s') ds ds' \leqslant c_2 t \qquad \text{for } t \text{ large.}$$

Thus:

(i) H bounded and $\frac{2}{3} < \beta < 1 \Rightarrow$

$$1/t^{2\beta} \int_0^t \int_0^t \Gamma_x(s, s') ds ds' \leqslant d_s / t^{\gamma\beta} \qquad \text{for } t \text{ large,}$$

with $\gamma = 1$ and $\beta > 2(1 - \beta)$.

(ii) $H(u) \leqslant d/u^{\frac{1}{2}}$ and $\frac{3}{4} < \beta < 1 \Rightarrow$

$$1/t^{2\beta} \int_0^t \int_0^t \Gamma_x(s, s') ds ds' \leqslant d_2 / t^{\gamma\beta} \qquad \text{for } t \text{ large,}$$

with $\gamma\beta = 2\beta - 1 > 2(1 - \beta)$. Thus:

THEOREM.

(a) $\{X_t,\}$, $t \in \mathbb{R}^+$, *is a stationary measurable Markov process* (*with probability density* f *continuous, bounded*) *satisfying the condition* G_2.

(b) *The kernel* K *is a bounded probability density.*

(c) $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(s) \searrow 0$ *when* $s \to \infty$.

$H : \mathbb{R}^+ \to \mathbb{R}^+$ such that $s \to h(s)H(h(s))$ is locally integrable on $\mathbb{R}^+$ and:

$$g(t) = \int_0^t h(s)H(h(s))ds \to \infty \qquad \text{when} \quad t \to \infty.$$

(d) *If either one of the two following conditions is satisfied:*

(i) *H is bounded and $g(t) \backsim t^\beta$ with $2/3 < \beta < 1$*

(ii) *$H(u) \leqslant d/u^{\frac{1}{2}}$, $u > 0$, and $g(t) \backsim t^\beta$ with $3/4 < \beta < 1$ then:*

$$f_t(x) \to f(x), \qquad t \to \infty, \quad \textit{almost surely}.$$

COROLLARY. *Under the hypotheses of the above theorem, if, in addition, the probability density f is uniformly continuous, then*

$$f_t(x) \to f(x), \qquad t \to \infty, \quad \textit{almost surely, uniformly in } x.$$

PROOF. This assertion follows from Proposition 2 and the fact that the right-hand sides of (1) and (2) are independent of x.

REMARK. Under conditions (a), (b) and (c) of the above theorem and the assumption that f is uniformly continuous, it follows that $f_t(x) \to f(x)$, in quadratic mean, uniformly in x, if either one of the following conditions is satisfied:

(i) H is bounded

(ii) $H(u) \leqslant d/u^{\frac{1}{2}}$, $u > 0$ and $t/g^2(t) = 0(1)$, $t \to \infty$.

APPENDIX

LEMMA 1. (*Generalized Toeplitz lemma*). *Given a function $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that:*

(i) *$\forall s \in \mathbb{R}^+$, $G(\cdot, s)$ is zero at infinity.*

(ii) *$\forall t \in \mathbb{R}^+$, $G(t, \cdot)$ is locally integrable on $\mathbb{R}^+$ and there exists a positive constant c such that:*

$$\sup_{t \in \mathbb{R}^+} \int_0^t |G(t, s)| ds \leqslant c < +\infty.$$

(iii) *For t large, we have:*

$$1_{[0, y]}(x)|G(t, x)| \leqslant g_y(x),$$

$\forall y \in \mathbb{R}^+$, ($1_{[0,y]}(\cdot)$ denotes the indicator of the interval $[0, y]$), and where g_y is positive integrable on $\mathbb{R}^+$.

If $\phi : \mathbb{R}^+ \to \mathbb{R}$ is such that:

(a) *$\phi(\cdot)$ is zero at infinity,*

(b) *$\phi \in \mathfrak{L}^1_{loc}[g_y(x)dx]$, $\forall y \in \mathbb{R}^+$ (locally integrable with respect to the measure $g_y(x)dx$),*

then

$$t \to \int_0^t G(t, x)\phi(x)dx \qquad \textit{is zero at infinity}.$$

PROOF. Let $\varepsilon > 0$. There exists $x_0 = x_0(\varepsilon)$ such that $x \geqslant x_0 \Rightarrow |\phi(x)| < \varepsilon/c$. Suppose that (iii) is satisfied for $t \geqslant t_0$. For $t \geqslant T = \max(x_0, t_0)$, we have:

$$\begin{aligned} \left|\int_0^t G(t, x)\phi(x)dx\right| &\leqslant \int_0^t |G(t, x)||\phi(x)|dx + \int_T^t |G(t, x)||\phi(x)|dx \\ &\leqslant \int_0^T |G(t, x)||\phi(x)|dx + \varepsilon. \end{aligned}$$

By (iii) we have:

$$1_{[0,T]}(x)|G(t,x)||\phi(x)| \leqslant g_T(x)|\phi(x)| \in \mathcal{L}^1(\mathbb{R}^+).$$

The assertion follows from the dominated convergence theorem of Lebesgue (for a family of integrable functions), from (i) and letting $\varepsilon \to 0$.

Particular cases.

(α) Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be positive, locally integrable and $g(t) = \int_0^t h(s)ds \to \infty$ when $t \to \infty$. Define

$$G(t,s) = h(s)/g(t), \qquad t > 0.$$

Then:

(i) $\forall s \in \mathbb{R}^+$, $\lim_{t\to\infty} G(t,s) = 0$

(ii) $\forall t \in \mathbb{R}^+$, $G(t,s)$ is locally integrable and

$$\sup_{t\in\mathbb{R}^+} \int_0^t G(t,s)ds = 1.$$

(iii) Let t_0 be such that $t \geqslant t_0 \Rightarrow g(t) > 1$. Then for $t \geqslant t_0$, we have:

$$1_{[0,y]}(x)G(t,x) \leqslant 1_{[0,y]}(x)h(x), \qquad \forall y \in \mathbb{R}^+.$$

(β) Let $G(t,s) = 1/t$, $t > 0$, $s \in \mathbb{R}^+$. In this case $c = 1$ and $g_y(x) = 1_{[0,y]}(x)$.

LEMMA 2. *Let $\{X_t\}$, $t \in \mathbb{R}^+$, be a stationary and measurable stochastic process. If the common probability density f of the X_t's is uniformly continuous, bounded and the kernel K is bounded, then*:

$$\lim_{s\to\infty} \phi_x(s) = 0, \qquad \textit{uniformly in} \quad x.$$

where $\phi_x(s) = E(1/h(s))K((X_s - x)/h(s)) - f(x)$, $x \in \mathbb{R}$, $s \in \mathbb{R}^+$.

PROOF. With the notations at the end of the proof of Proposition 1, the uniform continuity of f implies that: given $\hat{\varepsilon}$, there exists $\varepsilon = \varepsilon(\hat{\varepsilon})$ which depends on $\hat{\varepsilon}$ but not on x, such that

$$\delta(\varepsilon, x) \leqslant \hat{\varepsilon}.$$

Thus the sets of the form $\{y : |y| > \varepsilon/h(s)\}$ do not depend on x any longer.

The following lemma is straightforward.

LEMMA 3. *If $G_x(t,s)$ is a family of positive, measurable functions which depend on the parameter $x \in \mathbb{R}$, and satisfy the conditions*:

(i) *There exists a positive integrable function F (on $\mathbb{R}^+$) such that*:

$$G_x(t,s) \leqslant F(s), \qquad \forall x \in \mathbb{R}, \qquad \forall t \in \mathbb{R}^+.$$

(ii) $\lim_{t\to\infty} G_x(t,s) = 0$, *uniformly in x.*

Then:

$$\lim_{t\to\infty} \int_{\mathbb{R}^+} G_x(t,s)ds = 0, \qquad \textit{uniformly in} \quad x.$$

LEMMA 4. *If f is uniformly continuous and*

$$\sup_{x\in\mathbb{R}} |\phi_x(s)| \leqslant \psi(s)$$

where $\psi \in \mathcal{L}^1_{loc}(h(s)H(h(s))ds)$ *then*

$$\lim_{t\to\infty} 1/g(t)\int_0^t h(s)H(h(s))\phi_x(s)ds = 0, \qquad \textit{uniformly in} \quad x.$$

PROOF. The proof is similar to that of Lemma 1. Since

$$(h(s)H(h(s))/g(t))|\phi_x(s)| \leq h(s)H(h(s))\psi(s) \in \mathcal{L}^1_{loc}(\mathbb{R}^+)$$

for large t, and

$$\lim_{t\to\infty}(h(s)H(h(s))/g(t))|\phi_x(s)| = 0, \qquad \text{uniformly in} \quad x,$$

the assertion follows from Lemma 2 and Lemma 3.

LEMMA 5. *Let* $Z(t)$, $t \in \mathbb{R}^+$, *be a measurable second order process* (*with* $EZ(t) = 0$, $\forall t \in \mathbb{R}^+$). *Let* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be such that*:

$$g(t) > 0 \quad \textit{for} \quad t > 0 \quad \textit{and} \quad \lim_{t\to\infty} g(t) = +\infty.$$

If:

(i) $g(t) \sim t^\beta$ *at infinity*, $0 < \beta < 1$;

(ii) *there exists a positive constant c such that for large t, we have*: $\Gamma(t, t) \leq c$, *where* Γ *denotes the covariance function of* $Z(t)$, $t \in \mathbb{R}^+$,
then: $\forall a$ *such that* $0 < a < \frac{1}{2}(1 - \beta)$

$$W(t) - W(m^a) \to 0, \qquad t \to \infty, \quad \textit{almost surely},$$

where

$$W(t) = 1/g(t)\int_0^t Z(s)ds$$

and $\{m^a\}_{m\in\mathbb{N}}$ *is a sequence of positive real numbers.*

PROOF. For $m^a \leq t < (m + 1)^a$,

$$(g(t)/g(m^a))W(t) - W(m^a) = 1/g(m^a)\int_{m^a}^t Z(s)ds = Y(m^a, t).$$

Let

$$V(m^a) = \sup_{m^a\leq t<(m+1)^a}|Y(m^a, t)|.$$

Since

$$V(m^a) \leq 1/g(m^a)\int_{m^a}^{(m+1)^a}|Z(s)|ds,$$

$$E|V(m^a)|^2 \leq \left[1/\left(g(m^a)\int_{m^a}^{(m+1)^a}[\Gamma(s, s)\right]^{\frac{1}{2}}ds^2.$$

By (i) and (ii) we have:

$$\begin{aligned}\Sigma_{m=1}^{\infty}E|V(m^a)|^2 &\leq \Sigma_{m=1}^{\infty}(c/g^2(m^a))|(m + 1)^a - m^a|^2 \\ &\sim \Sigma_{m=1}^{\infty}(c'/m^{2a\beta+2(1-a)}) \\ &< +\infty, \qquad \text{since} \quad 2a\beta + 2(1 - \beta) > 1.\end{aligned}$$

LEMMA 6. *Under the hypotheses of Lemma 5 and if, in addition,*
(*iii*) *For large t, we have*:

$$1/t^{2\beta}\int_0^t\int_0^t\Gamma(s, s')dsds' \leq d/t^{\gamma\beta}, \qquad d > 0$$

with $\gamma\beta > 2(1-\beta)$, *then*:

$$W(t) \to 0, \qquad t \to \infty, \quad \textit{almost surely}.$$

PROOF. By Lemma 5, choose a such that $1/\gamma\beta < a < 1/2(1-\beta)$. Our assertion will be proved if $W(m^a) \to 0$, $m \to \infty$, almost surely. But:

$$\Sigma_{m=1}^{\infty} E|W(m^a)|^2 = \Sigma_{m=1}^{\infty} 1/g^2(m^a) \int_0^{m^a}\int_0^{m^a} \Gamma(s, s')\,ds\,ds'$$

$$\sim \Sigma_{m=1}^{\infty} 1/m^{\alpha\beta\gamma} < +\infty, \qquad \text{since} \quad a > 1/\gamma\beta.$$

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