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THE ASYMPTOTIC DISTRIBUTION OF THE SUPREMA OF THE STANDARDIZED EMPIRICAL PROCESSES

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The supremum of the empirical distribution function F_n centered at its expectation F and standardized by division by its standard deviation has recently been shown by Jaeschke to have asymptotically an extreme-value distribution after a second location and scale transformation depending only on the sample size n. In this paper the studentized form of the above statistic, obtained by division by the estimated standard deviation, is shown to have the same large sample behavior. This statement is equivalent to the analogous assertion for the standardized sample quantile process for the uniform distribution. The three results imply each other. The present result yields immediately confidence regions that contract to zero width in the tails. The proofs given here rest on a limit theorem by Darling and Erdös on the maxima of standardized partial sums of i.i.d. random variables. In addition, Kolmogorov's theorem is used.

0. Introduction. Let $X_1, \cdots, X_n$ be a random sample from the continuous distribution function (df) F on R^1 and let $F_n(u) := n^{-1}\sum_{i=1}^n 1(X_i < u)$ be its empirical distribution function. In order to construct confidence contours for F that contract to F in the tails, the difference

$$\bar{F}_n(u) := F_n(u) - F(u)$$

is weighted by $(F_n(u)(1 - F_n(u)))^{-\frac{1}{2}}$ which may be regarded as an estimator for $(F(u)(1 - F(u)))^{-\frac{1}{2}}$, the true inverse standard deviation of $F_n(u)$. Thus the statistic

$$\hat{V}_n(u) := n^{\frac{1}{2}}(F_n(u)(1 - F_n(u)))^{-\frac{1}{2}}\bar{F}_n(u), \qquad X_{1n} < u < X_{nn},$$

has "nearly" expectation 0 and variance 1. (X_{kn}, $k = 1, \cdots, n$ are the order statistics of $X_1, \cdots, X_n$.) $\hat{V}_n(u)$ may be considered as a studentized version of $V_n(u) := n^{\frac{1}{2}}(F(u)(1 - F(u)))^{-\frac{1}{2}}\bar{F}_n(u)$, $u \in \{t : 0 < F(t) < 1\}$. $V_n(u)$ satisfies $EV_n(u) = 0$, $EV_n^2(u) = 1$ for all u. In this paper the asymptotic df of the statistic $a_n\hat{V}_n - b_n$ where

$$\hat{V}_n := \sup_{X_{1n} < u < X_{nn}} \hat{V}_n(u),$$

and $a_n := (2 \log_2 n)^{\frac{1}{2}}$, $b_n := 2 \log_2 n + \frac{1}{2}\log_3 n$ ($\log_2 n := \log \log n$, etc.) is shown to be a (type 1) extreme value df D(see Theorem 1). The statistic $\hat{V}_n$ avoids the

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shortcoming of the Kolmogorov-Smirnov statistic which yields uniform distance of the confidence curve from F over the whole real axis. This implies that the Kolmogorov-Smirnov statistics are sensitive asymptotically only in the central range given by $\{u : (\log_2 n)^{-1} < F(u) < 1 - (\log_2 n)^{-1}\}$ as can be inferred from the theorems given below. It also results in bad power of the KS-tests against alternatives that differ only in the tails. In contrast, $a_n \hat{V}_n - b_n$ and $a_n V_n - b_n$ are sensitive only in the moderate tails given by, e.g., $\{u : n^{-1} \log n < F(u) < ((\log_2 n)\log_3 n)^{-1}\}$ and the interval symmetric to this (Proposition 1); here

$$V_n := \sup_{u\,:\,0<F(u)<1} V_n(u).$$

This defect, though, could partially be overcome by constructing confidence contours composed of curve segments of different nature (in particular, segments of lines and ellipses). Because of their intuitive appeal statistics like V_n and $\hat{V}$ have been considered for a long time (cf. the survey papers [5], Section 2.5 and [7]).

Computing algorithms have been developed even for general confidence contours (boundaries) by several authors but explicit or asymptotic probabilities seem to have been known only for straight line segments. One of the latter cases is the Rényi statistics [13], e.g.

$$\sup_{0<a\leqslant F(u)} \bar{F}_n(u)/F(u).$$

As compared with V_n or $\hat{V}_n$ this particular statistic attributes relatively heavy weight to a relatively small interval which does not move out into the tail as n gets large.

For V_n an extreme value distribution has first been obtained under the same normalization as above, namely $a_n V_n - b_n$ by Jaeschke in his dissertation [11] (Theorem 2 below). This distribution enters in his as well as in the present proof through the application of theorems by Darling and Erdös [4] (see, e.g., Theorem 2.1 below). Analogous results for $\hat{V}_n$ and the quantile process subsequently were derived by fairly straightforward methods. The proofs are given below. In [12] Jaeschke generalizes both results by admitting subintervals (Theorem 6 below) and proves them very elegantly and beautifully by also appealing to strong invariance theorems and other high-powered tools.

The proof given here proceeds by first pointing out the equivalence of Theorem 1 on the limiting df of $\hat{V}_n$ with the analogous assertion for the sample quantile process, and the latter is then actually proved. The representation of the order statistics U_{kn} from $U(0, 1)$ as ratios of two partial sums almost immediately allows application of the Darling-Erdös theorem. This theorem and Kolmogorov's theorem on sample df's are the only advanced tools that are required. Otherwise only some fairly elementary and straightforward computations are needed. In a similar fashion it has been possible to prove the equivalence of Theorem 1 with the original result of Jaeschke [11] (Theorem 2 below). In contrast, the proofs given in [12] become relatively short in particular by appealing to well-known results on the speed of convergence in embedding and invariance theorems.

The implication and interrelation of the results of this paper with strong laws (like law of the iterated logarithm type statements), with invariance principles, with other instances of probability and statistics where the extreme value df turns up are not studied although they are doubtlessly fruitful and interesting (cf. e.g. [1], [2a] and the survey paper [7]). Csörgö and Révész in [2a], Theorem 2, have generalized (1.9) below to general quantile processes. The occurrence of the extreme value distribution D in the study of crossings of stationary processes is well known (cf. e.g. [3] for some material in this context; for a particular instance cf. page 271 of this reference). The df D also occurs, naturally, when dealing with statistics of extremes. Smirnov obtains it in his 1944 paper [14], page 205, as the df of the sup-norm functional of a histogram estimator for a density to be estimated over a finite fixed interval $[a, b]$. If the number s_n of subintervals is chosen to be $\log n$ even the norming constants are very similar to the a_n, b_n used above.

Some insight regarding a "uniformly good" weight function may be gained from Theorems 6 and 7. In the latter small intervals tending towards the tails are considered. Bolshev and Smirnov have studied numerically a "uniformly good" weight function for the quantile process. However, the search for a "uniformly good" weight function may be futile in view of recent results by Berk and Jones [1a] who show that the minimum attained level statistic considered by them and which may be roughly described by the sup of probability integral transforms of empirical processes, is more efficient in the Bahadur sense than any weighted Kolmogorov statistic at every alternative. Nevertheless the authors are able to utilize Jaeschke's result to derive the asymptotic null distribution of one of their statistics (their Section 5).

1. Results.

THEOREM 1. *With the notations and assumptions stated in the introduction and if*

$$t_n \equiv T_n(t) := (2 \log_2 n)^{\frac{1}{2}}\left(1 + \frac{\log_3 n + 2t}{4 \log_2 n}\right) \qquad t \in R^1$$

then for either choice of the sign

$$P\left(\sup_{X_{1n}<u<X_{nn}}(\pm \hat{V}_n(u)) < t_n\right) \to (D(t))^2 \quad \text{as} \quad n \to \infty \tag{1.1}$$

where

$$D(t) := \exp\left(-e^{-t}/\left(2\pi^{\frac{1}{2}}\right)\right) \qquad t \in R^1 \tag{1.2}$$

is the extreme value distribution function.

In Section 2 we will derive from this

THEOREM 2 [11]. *Under the above assumptions,* (1.1) *remains true if* $\hat{V}_n$ *is replaced by* V_n.

The proof of Theorem 1 also yields the following results (see also Lemma 2.2 and Lemma 2.3).

PROPOSITION 1. *With the assumptions and notations of Theorem* 1 *there holds with* $k_n := n/((\log_2 n)(\log_3 n))$

$$P\big(\sup_{X_{\log n, n}<u<X_{k_n, n}}(\pm \hat{V}_n(u)) < t_n\big) \to D(t), \tag{1.4}$$

$$P\big(\sup_{X_{n-k_n, n}<u<X_{n-\log n, n}}(\pm \hat{V}_n(u)) < t_n\big) \to D(t), \tag{1.5}$$

$$P\big(\sup_{X_{\log n, n}<u<X_{k_n, n};\, X_{n-k_n, n}<u<X_{n-\log, n}}(\pm \hat{V}_n(u)) < t_n\big) \to (D(t))^2, \tag{1.6}$$

$$P\big(\sup_{u \notin (X_{\log n, n},\, X_{n-\log n, n})}|\hat{V}_n(u)| < t_n\big) \to 1. \tag{1.7}$$

The assertions also hold for V_n *instead of* $\hat{V}_n$.

Most likely the above assertions can be strengthened by narrowing the u-ranges. The above k_n have been so chosen for convenience.

THEOREM 3. *Under the conditions of Theorem* 1,

$$P\big(\sup_{X_{1n}<u<X_{nn}}|\hat{V}_n(u)| < t_n\big) \to (D(t))^4 = \exp\big(-2e^{-t}/\pi^{\frac{1}{2}}\big) \tag{1.8}$$

$(t \in R^1)$ *and*

$$P(\max_{k=1,\cdots,n}|U_{kn}^*| < t_n) \to (D(t))^4 \tag{1.9}$$

and

$$P\big(\sup_{X_{1n}<u<X_{nn}}|V_n(u)| < t_n\big) \to (D(t))^4 \tag{1.10}$$

where

$$U_{kn}^* := \bar{U}_{kn}(\operatorname{Var} U_{kn})^{-\frac{1}{2}},\ \bar{U}_{kn} := U_{kn} - EU_{kn}, \tag{1.11}$$
$$EU_{kn} = k(n+1)^{-1},\ \operatorname{Var} U_{kn} = k(n+1-k)(n+1)^{-2}(n+2)^{-1}$$

and $U_{1n}, \cdots, U_{nn}$ *are the order statistics of a random sample from the uniform* df *over* (0, 1).

REMARK 1. In view of Theorem 1, Theorem 3 says that crossings of the lower boundary are asymptotically independent of those of the upper boundary. Moreover, the "local" fluctuations at the lower left and right and at the upper left and right "corner" (if visualized for the rectangular df) are asymptotically independent and essentially identically distributed. These remarks may be related to the question of asymptotic independence of extremes (cf. [10, page 110]) and of high and low level crossings of stationary Gaussian processes (cf. e.g. Berman [1]). They are made more precise and proved after we have introduced some notation before Lemma 2.3. Some results of probabilistic interest connected with the Darling-Erdös theorems [4] (see Theorem 2.1 below), some of which are used in the proof of the above theorems but others not, seem worth mentioning explicitly.

THEOREM 4. *With the assumptions and notations of Theorem* 2.1 *below and with* $\varepsilon_n > 0$, $\delta_n > 0$ *the following hold*:

$$(1.12)\qquad P\left(\max_{n^{\varepsilon_n} < k < n^{1-\delta_n}} k^{-\frac{1}{2}} Z_k < t_n\right) \to D(t),$$

$$(1.13)\qquad P\left(\max_{n^{\varepsilon_n} < k < n^{1-\delta_n}} k^{-\frac{1}{2}} |Z_k| < t_n\right) \to (D(t))^2$$

$$(1.14)\qquad P\left(\max_{0 < k < n^{\varepsilon_n}} k^{-\frac{1}{2}} |Z_k| < t_n\right) \to 1$$

iff $\varepsilon_n = 0(1)$; $\delta_n = 0(1)$ (*compare also Lemma* 2.3 *below*).

This theorem specifies a qualitative remark regarding the k-range made by Darling and Erdös. Part of the theorem is stated and proved by different methods in [11] (Lemma 5.2). The proof given here consists in a simple application of a theorem by Gnedenko [8] on the class of weakly convergent linear transforms of a given sequence of random variables. By the same method one proves

THEOREM 5. *With the assumptions and notations of Theorem* 2.1 *below there holds for any* $c \in (0, 1)$ *and any positive null sequence* $0(1)$ *and with*

$$(1.15)\qquad b_n \equiv b(n) = 2 \log_2 n + \frac{1}{2} \log_3 n;$$

$$(1.16)\qquad a_n \equiv a(n) = (2 \log_2 n)^{\frac{1}{2}};$$

$$(1.17)\qquad P(a_n S_{n^c} - b_n < t) \sim P\left(\max_{n^{0(1)} < k < n^c} k^{-\frac{1}{2}} Z_k < t_n\right)$$

$$\to \exp\left(-\frac{c^2}{2\pi^{\frac{1}{2}}} e^{-t}\right) = D(t - 2 \log c)$$

and

$$(1.18)\qquad P\left(\max_{n^{0(1)} < k < n^c} k^{-\frac{1}{2}} |Z_k| < t_n\right) \to (D(t - 2 \log c))^2.$$

(This theorem obviously proves the last statement made in Theorem 4.)

An immediate consequence of the Darling-Erdös theorem is the following result which probably can be extended to more general triangular arrays of rv's:

COROLLARY 1. *Let* $\{Y_{in};\ i = 1, \cdots, n, n \in \mathbb{N}\}$ *all be identically distributed with* $EY_{11} = 0$, $EY_{11}^2 = 1$, $E|Y_{11}|^3 < \infty$ *and let them be independent within each row*(i.e., *n fixed*). *Let* $Z_n := Y_{1n} + \cdots + Y_{nn}$, $S_n := \max_{1<k<n} k^{-\frac{1}{2}} Z_k$. *Then*

$$a_n S_n - b_n \to_d D \qquad a_n S_{n,\,abs} - b_n \to_d D^2$$

$$\left(S_{n,\,abs} := \max_{1<k<n} k^{-\frac{1}{2}} |Z_k|\right).$$

Some hint as to what a "uniformly good" weight function may look like may be gained from Theorem 7 below where we consider small intervals tending towards the tails. The proof (not given in detail) depends on a recent result of Jaeschke [12] (Example 3):

THEOREM 6 [12]. *Let* $V_n(\varepsilon, \delta) := \sup_{\varepsilon \leqslant F(u) \leqslant \delta} V_n(u)$ *for* $0 \leqslant \varepsilon \leqslant \delta \leqslant 1$. *Let* $\{\varepsilon_n\}$, $\{\delta_n\}$ *be given null sequences from* $(0, 1)$ *satisfying* $\delta_n > \varepsilon_n > n^{-1}$ $(\log n)^3$, $\delta_n/\varepsilon_n \to \infty$. *Then with*

$$t_\rho(t) := \left(t + 2\log\rho + 2^{-1}\log_2\rho\right)(2\log\rho)^{-\frac{1}{2}},$$

$$\rho_n := 2^{-1}\log(\delta_n/\varepsilon_n),$$

$$P\left(V_n(\varepsilon_n, \delta_n) < t_{\rho_n}(t)\right) \to (D(t))^2 \; (t \in R^1). \tag{1.18}$$

Taking here especially $\delta_n := \varepsilon_n^{c_n}(c_n \uparrow 1$ slowly enough) it is easy to prove

THEOREM 7. *Let* $F(u) = u(0 \leqslant u \leqslant 1)$, $\varepsilon_n \to 0$,

$$\varepsilon_n > n^{-1}\log^3 n;\; 0 < c_n < 1,\; c_n \to 1 \tag{1.19}$$

such that

$$|\log(1 - c_n)| = o\left((\log_2^+ \varepsilon_n)^{\frac{1}{2}}\right) \tag{1.20}$$

where

$$\log^+ \varepsilon_n = |\log \varepsilon_n|,\; \log_2^+ \varepsilon_n = \log|\log \varepsilon_n|, \text{ etc.} \tag{1.21}$$

Then

(1.22)

$$\lim_{n\to\infty} P\left(\sup_{\varepsilon_n < u < \varepsilon_n^{c_n}} \left[\left(\frac{2n\log_2^+ u}{u(1-u)}\right)^{\frac{1}{2}} \bar{F}_n(u) - 2\log_2^+ u - 2^{-1}\log_3^+ u\right] < t - \log\frac{2}{1-c_n}\right)$$

$$= \lim_{n\to\infty} P\left((2n\log_2^+ \varepsilon_n)^{\frac{1}{2}} \sup_{\varepsilon_n < u < \varepsilon_n^{c_n}} \left[(u(1-u))^{-\frac{1}{2}} \bar{F}_n(u)\right] - 2\log_2^+ \varepsilon_n - 2^{-1}\log_3^+ \varepsilon_n < t - \log\frac{2}{1-c_n}\right)$$

$$= (D(t))^2 \; (t \in R^1).$$

In order to obtain similar results for intervals with $\delta_n - \varepsilon_n >$ const. > 0 it appears that information on the speed of the convergence in (1.19) might be useful.

Note that (1.20) implies $(1 - c_n)\log^+ \varepsilon_n \to \infty$ since $\log(1 - c_n) + \log_2^+ \varepsilon_n = (\log_2^+ \varepsilon_n)^{\frac{1}{2}}(o((\log_3^+ \varepsilon)^{-\frac{1}{2}}) + (\log_2^+ \varepsilon_n)^{\frac{1}{2}}) \to \infty$. An example for (ε_n) – , (c_n) – sequences satisfying the assumptions of the theorem is $\varepsilon_n = 1/\log n$; $c_n = 1 - (\log_3 n)^{-1}$.

2. Tools, proofs and remarks. Before proving the main Theorem 1 (after Lemma 2.1 below) we start with some preliminaries. The main tool is the following result by Darling and Erdös:

THEOREM 2.1 [4]. *Let* $Y_1, Y_2, \cdots$ *be independent rv's with mean* 0, *variance* 1, *and uniformly bounded third absolute moment. Put* $Z_k = Y_1 + \cdots + Y_k$ *and let*

$$S_n = \max_{k=1,\cdots,n} k^{-\frac{1}{2}} Z_k.$$

Then

$$(2.1)\qquad P(S_n < t_n) \to D(t) \quad \text{as} \quad n \to \infty \ (t \in R^1) \qquad \text{and}$$
$$P\left(\max_{k=1,\dots,n} k^{-\frac{1}{2}}|Z_k| < t_n\right) \to (D(t))^2,$$

using notations of Theorem 1.

For the proof of Theorem 4 we need Theorem 2.2 below which is due to Gnedenko. We use

Definition 2.1. Let (X_n) be a sequence of rv's. Any double sequence (a_n, b_n) of real numbers with $a_n > 0$ associated with (X_n) is called *F-stabilizing* iff $a_n X_n + b_n \to_d F$ where F is some df.

Theorem 2.2 [8]. *Let F be nondegenerate. Then*

$$\{a_n X_n + b_n \to_d F \quad \text{and} \quad a_n' X_n + b_n' \to_d F\}$$

iff $a_n / a_n' \to 1$ *and* $b_n - b_n' \to 0$.

Hence the class of F-stabilizing double sequences forms an equivalence class defined by this latter property.

Definition 2.2. A stabilizing sequence *associated with* (X_n) is any double sequence (a_n, b_n) with $a_n > 0$ such that $P(a_n X_n + b_n < x) \to F(ax + b)$ with some reals a, b $(a > 0)$ and some nondegenerate F.

We recall

Definition 2.3 [9]. The df's F_1 and F_2 *belong to the same type*, if for some constants $a > 0$ and b there holds $F_2(x) = F_1(ax + b)$.

Hence the totality of df's decomposes into equivalence classes, each forming a type. Now

Theorem 2.3 ([9], page 40). *If* $F_n \to F$ *and* $F_n(a_n x + b_n) \to G(x)(x \in R^1;\ a_n > 0)$, *F, G nondegenerate, then F and G are of the same type.*

We now generalize Theorem 2.2 to convergence against df's of the same type.

Corollary 2.1. *Let F be nondegenerate and* $a, a_n > 0$. *Then*

$$\{F_n(a_n x + b_n) \to F(ax + b) \quad \text{and}$$
$$F_n(a_n' x + b_n') \to F(x)(x \in R^1)\}$$

iff

$$\frac{a_n}{a a_n'} \to 1 \text{ and } b + (b_n'/a_n') - a b_n / a_n \to 0.$$

Proof. Putting $y := ax + b \Rightarrow a_n x + b_n = (a_n/a)(y - b) + b_n$. The assertion follows from Theorem 2.2.

By Theorem 2.3 all stabilizing sequences associated with a given sequence (X_n) yield limit laws of the same type. They decompose into equivalence classes each characterized by a particular limit df which in turn is characterized by a pair of reals (a, b).

PROOF OF THEOREM 4. We have

$$m_n := n^{1+0(1)} \rightleftarrows \log m_n = e^{o(1)}\log n \rightleftarrows \log_2 m_n = \log_2 n + o(1).$$

Hence with a_n, b_n as defined in 1

$$\left(\frac{a_n'}{a_n}\right)^2 = \frac{\log_2 m_n}{\log_2 n} = 1 + o(1),\ a_n' := a_{m_n},$$

$$b_n' - b_n = 2\log_2 n + \frac{1}{2}\log_3 n + o(1) - b_n = o(1),$$

$$b_n' := b_{m_n},$$

so that the assumptions of Theorem 2.2 are satisfied.

Since by Theorem 2.1, with

$$X_n := \max_{1 \leqslant k \leqslant n} k^{-\frac{1}{2}} Z_k,$$
$$a_{m_n} X_{m_n} - b_{m_n} \to_d D$$

it follows that

$$a_n X_{m_n} - b_n \to_d D;$$

analogously

$$a_n \max_{1 \leqslant k \leqslant m_n} k^{-\frac{1}{2}}|Z_k| - b_n \to_d D^2.$$

This proves (1.12) and (1.13) except for the lower restriction on the k. That the upper bound $n^{1+o(1)}$ for the k cannot be lowered further is seen as follows.

Let for some divergent subsequence $(n_k) \subset \mathbb{N}$

$$|\log_2 m_{n_k} - \log_2 n_k| =: \phi_k > C > 0 \qquad \text{for all} \quad k;$$

then

$$|b_{n_k}' - b_{n_k}| \geqslant 2\phi_k - \frac{1}{2}\log\left(1 + \frac{\phi_k}{\log_2 n_k}\right) \geqslant 2\phi_k\left(1 - (4\log_2 n_k)^{-1}\right) \geqslant C > 0$$

for almost all k. Now Theorems 2.1 and 2.2 yield the assertion.

In order to prove (1.14) it suffices to consider $n^{o(1)} \to \infty$. We write $a(n) := a_n$, $b(n) := b_n$ and

$$M_{abs}(n) := \max_{1 \leqslant k \leqslant n} k^{-\frac{1}{2}}|Z_k|.$$

Consider the event

$$A_n := \left(a(n^{o(1)})M_{abs}(n^{o(1)}) - b(n^{o(1)}) < t\right)$$
$$= \left(a_n M_{abs}(n^{o(1)}) - b_n < -b_n + (t + b(n^{o(1)}))a_n/a(n^{o(1)}\right).$$

In

$$\log_2 n^{o(1)} = \log o(1) + \log_2 n = (\log_2 n)(1 + (\log o(1))/\log_2 n)$$

we assume without loss of generality

$$|\log o(1)| = o(\log_2 n).$$

Then $a_n/a(n^{o(1)}) = 1 - (\frac{1}{2} + o(1))|\log o(1)|/ \log_2 n \to 1$ and

$$\begin{aligned} b(n^{o(1)}) &= b_n + 2\log o(1) + \left(\frac{1}{2} + o(1)\right)(\log o(1))/\log_2 n \\ &= b_n - (2 + o(1))|\log o(1)| \end{aligned}$$

which implies

$$\begin{aligned} -b_n + (t + b(n^{o(1)}))a_n/a(n^{o(1)}) & \\ &= b_n\big(-1 + a_n/a(n^{o(1)})\big) - (2 + o(1)|\log o(1)|) \\ &= (\log_2 n)\left(\frac{1}{2} + o(1)\right)|\log o(1)|/\log_2 n - (2 + o(1))|\log o(1)| \\ &= -\big(\tfrac{3}{2} + o(1)\big)|\log o(1)| \to -\infty. \end{aligned}$$

Thus Theorem 2.1 implies $PA_n \to 1$.

Since Theorem 5 is not needed later on its proof, which runs along the same lines as the above one, is omitted.

Reformulation of the theorems of Section 1. Firstly, the probability transformation $F(X_i) = : U_i \sim U(0, 1)$ yields for $u \in (U_{1n}, U_{nn})$

$$\hat{V}_n(u) = \left(\frac{n}{F_n(u)(1 - F_n(u))}\right)^{\frac{1}{2}} \bar{F}_n(u),$$

where

$$\bar{F}_n(u) := F_n(u) - u = n^{-1}\Sigma_{i=1}^n 1(U_i < u) - u$$

and $U_{1n}, \dots, U_{nn}$ are the order statistics of the random sample $(U_1, \dots, U_n)$ on the probability space $(\Omega, \mathcal{A}, P)$. It suffices to prove the theorems for this case.

Next, the reformulation of the theorems of Section 1 in terms of the quantile process follows immediately from the fact that $u \in (U_{kn}, U_{k+1,n}]$, $k = 1, \cdots, n-1$, implies

$$\hat{V}_n(u) = n^{\frac{1}{2}} w_{kn}(kn^{-1} - u) \quad \begin{matrix} < n^{\frac{1}{2}} w_{kn}(kn^{-1} - U_{kn}) \\ > n^{\frac{1}{2}} w_{kn}(kn^{-1} - U_{k+1,n}) \end{matrix}$$

with $w_{kn} = (\frac{k}{n}(1 - \frac{k}{n}))^{\frac{1}{2}}$. Putting

$$A_{kn} := \big(kn^{-1} - t_n w_{kn} n^{-\frac{1}{2}} < U_{kn}\big) \qquad k = 1, \cdots, n-1$$

it follows

$$\hat{A}_n := (\hat{V}_n(u) < t_n \forall u \in (U_{1n}, \cdots U_{nn})) = \cap_{k \geqslant \kappa_n}^{n-1} A_{kn} \tag{2.2}$$

where $\kappa_n := t_n^2(1 + t_n^2/n)^{-1} = (2 + o(1))\log_2 n$. The smaller k can be omitted since for them the lower bound on the U_{kn} is negative (for $k \leqslant \kappa_n$ we have $A_{kn} = \Omega$). Analogously with

$$A_{kn}^* := \left((k-1)n^{-1} + t_n w_{k-1,n} n^{-\frac{1}{2}} \geqslant U_{kn}\right) \qquad k = 2, \cdots, n$$

$$A_n^* := (\hat{V}_n(u) \geqslant -t_n \ \forall u \in (U_{1n}, \cdots, U_{nn})) = \cap_{k=2}^{n+1-\kappa_n} A_{kn}^*. \tag{2.3}$$

Note here that

$$\frac{k-1}{n} + t_n w_{k-1,n}/n^{\frac{1}{2}} < 1$$

iff

$$n + 1 - k > \kappa_n \quad \text{or} \quad k < n + 1 - \kappa_n.$$

Similarly

$$B_n := (|\hat{V}_n(u)| \leqslant t_n \ \forall u \in (U_{1n}, U_{nn})) = \hat{A}_n \cap A_n^* \tag{2.4}$$

$$= \left(\cap_{2 \leqslant k \leqslant \kappa_n} A_{kn}^*\right)\left(\cap_{n+1-\kappa_n \leqslant k \leqslant n-1} A_{kn}\right)\left(\cap_{\kappa_n < k < n+1-\kappa_n} B_{kn}\right), \tag{2.5}$$

$$B_{kn} := \left(\frac{k}{n} - t_n w_{kn}/\bar{n}^{\frac{1}{2}} \leqslant U_{kn} \leqslant \frac{k-1}{n} + t_n w_{k-1,n}/\bar{n}^{\frac{1}{2}}\right). \tag{2.6}$$

(Since $\kappa_n = (2 + o(1))\log_2 n$ we have on the k-range of the B_{kn}

$$\frac{k-1}{n} = \frac{k}{n}(1 - 1/k) = \frac{k}{n}\left(1 - o\left((\log_2 n)^{-1}\right)\right)$$

with a constant in $o(\ \ )$ independent of k and n. Thus

$$w_{k-1,n} = w_{k,n}\left(1 - o\left((\log_2 n)^{-1}\right)\right) \tag{2.7}$$

which implies $T_n(t)w_{k-1,n} \neq T_n(t + o(1))w_{kn}$ for $\kappa_n < k < n + 1 - \kappa_n$. Later on we will slightly narrow down the k-range and then equality will hold.) We shall show in several steps that various groups of quantiles are irrelevant for the limiting probabilities of $\hat{A}_n$, A_n^*, and B_n. Firstly we prove Lemma 2.1 below.

Kolmogorov's theorem states for the quantile process

$$P\left(n^{\frac{1}{2}}\max_k |\overline{U}_{kn}| < y\right) \to 1 - 2e^{-2y^2} + 2e^{-8y^2} - + \cdots =: K(y).$$

Since K is continuous, $y_n \to \infty$ implies

$$P\left(n^{\frac{1}{2}}\max_{k=1,\cdots,n} |\overline{U}_{kn}| < y_n\right) \to 1.$$

Now for a sequence of naturals $k_n < n$,

$$\omega_n := \inf_{k_n \leqslant k \leqslant n-k_n} t_n(w_{kn} \wedge w_{k-1,n}) = t_n\left(\frac{k_n-1}{n}\left(1 - \frac{k_n-1}{n}\right)\right)^{\frac{1}{2}} \to \infty$$

iff the sequence (k_n) satisfies

$$k_n \wedge (n - k_n) \geqslant \frac{n}{o(\log_2 n)} =: K_n'$$

with some $o(\ )$. Now putting $k_n = K_n'$, we have

$$B_{kn} = \left(k/(n(n+1)) - t_n w_{kn}/n^{\frac{1}{2}} \leqslant \overline{U}_{kn} \leqslant k/(n(n+1)) - \frac{1}{n} + t_n w_{k-1,n}/n^{\frac{1}{2}}\right)$$
$$\supset \left(n^{-\frac{1}{2}} - \omega_n \leqslant n^{\frac{1}{2}}\overline{U}_{kn} \leqslant -n^{-\frac{1}{2}} + \omega_n\right)$$

for $K_n' \leqslant k \leqslant n - K_n'$. It follows:

LEMMA 2.1.

$$P\left(\cap_{K_n'<k<n-K_n'} B_{kn}\right) \geqslant P\left(n^{\frac{1}{2}} \max_{K_n' \leqslant k \leqslant n-K_n'} |\overline{U}_{kn}| \leqslant \omega_n - n^{-\frac{1}{2}}\right) \to 1. \tag{2.8}$$

PROOF OF THEOREM 1. (1) Let $Y_1, Y_2 \cdots$ be i.i.d. rv's with density $e^{-y}(y \geqslant 0)$ and let $Z_k := Y_1 + \cdots + Y_k$. Then

$$(U_{1n}, \cdots, U_{nn}) =_d (Z_1/Z_{n+1}, \cdots, Z_n/Z_{n+1}).$$

Since we only prove weak convergence theorems we put $U_{kn} := Z_k/Z_{n+1}$. With

$$\overline{Z}_k := Z_k - EZ_k = Z_k - k$$

put

$$\Omega_n = \left(|\overline{Z}_{n+1}| < c_n n^{\frac{1}{2}}\right) \tag{2.9}$$

with $c_n \to \infty$ with slow speed to be specified later (obviously any particular such c_n-sequence can be replaced by a $(c_n + 0(1))$-sequence, and we may do so later on when convenient). Then

$$P\Omega_n \to 1 \qquad n \to \infty$$

by the central limit theorem.

(2) To begin with we will now prove that several restrictions of the k-range given any n large enough are possible. In step (6) we need (2.15) below to be proved now. Note

$$B_{kn} \supset \Omega_n\left(\left(1 + c_n/n^{\frac{1}{2}}\right)\left(k - t_n w_{kn} n^{\frac{1}{2}}\right) \leqslant Z_k \leqslant \left(1 - c_n/n^{\frac{1}{2}}\right)\left(k - 1 + t_n w_{k-1,n} n^{\frac{1}{2}}\right)\right)$$

$$\supset \Omega_n\left(k^{-\frac{1}{2}}|\overline{Z}_k| < u_{kn}\right), \tag{2.10}$$

$$u_{kn} := (2\log_2 n)^{\frac{1}{2}}\left(1 + \frac{\log_3 n + 2t}{4\log_2 n}\right)\left(1 - \frac{c_n}{n^{\frac{1}{2}}}\right)(1 - k^{-1})^{\frac{1}{2}}(1 - k/n)^{\frac{1}{2}} - c_n(k/n)^{\frac{1}{2}} - (1 + o(1))/k^{\frac{1}{2}}. \tag{2.11}$$

For $2 \leqslant k \leqslant \log n$ a convenient choice is $c_n = (1 + o(1))(\log_3 n)^{\frac{1}{2}}$. Then

$$u_{kn} > (2 \log_2 n)^{\frac{1}{2}}\left\{\left(1 + \frac{\log_3 n + 2t + o(1)}{4 \log_2 n}\right)2^{-\frac{1}{2}} - \left(\frac{1}{2} + o(1)\right)(\log_2 n)^{\frac{1}{2}}\right\}$$

$$= (\log_2 n)^{\frac{1}{2}}(1 + o(1))$$

(2.12) $$> T_{\log n}(1/o(1)) = T_{[\log n]}(1/o(1)) \;(k\epsilon[2, \cdots, \log n])$$

for some $o(1)\downarrow 0$ sufficiently slowly and where we use

$$T_{\log n}(s) = (2 \log_3 n)^{\frac{1}{2}}\left(1 + \frac{\log_4 n + 2s}{4 \log_3 n}\right).$$

Now Theorem 2.1 for $[\log n]$ partial sums yields

(2.13) $$\liminf_n P(\cap_{2\leqslant k\leqslant \log n} B_{kn}) \geqslant \lim_n P\big(\max_{k=2,\cdots,[\log n]} k^{-\frac{1}{2}}|\bar{Z}_k| < T_{[\log n]}(1/o(1))\big) = 1.$$

Similarly by a symmetry argument given subsequently

(2.14) $$\lim_n P(\cap_{n-\log n<k\leqslant n-1} B_{kn}) = 1$$

and hence

(2.15) $$P(\cap_{k\in[2,\log n]\cup[n-\log n, n-1]} B_{kn}) \to 1.$$

In (2.14), (2.15) the point symmetry of the quantile process w.r.t. the point

(2.16) $$\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

is exploited, i.e., with

(2.17) $$U'_{n+1-k} :\doteq 1 - U_{kn},$$

(2.18) $$(U'_{1n}, \cdots, U'_{nn}) \stackrel{d}{=} (U_{1n}, \cdots, U_{nn}).$$

Moreover, the upper and lower boundary for the U_{kn} in the definition of B_{kn} are point-symmetric w.r.t. the same point: for

(2.19) $$f(k) := \frac{k-1}{n} + t_n w_{k-1,n}/n^{\frac{1}{2}},\quad g(k) := \frac{k}{n} - t_n w_{k,n}/n^{\frac{1}{2}}$$

we have

(2.20) $$g(k) = 1 - f(n+1-k) \qquad k = 1, \cdots, n.$$

Incidentally, because of the symmetry (2.17) it suffices to prove (1.1) only for one sign (subsequently done for $+$), because we have

(2.21) $$P(\cap_{k=2}^{n} A_{kn}^{*}) = P(\cap_{k=1}^{n-1} A_{kn}).$$

(Note that no independence argument has been used here.)

(3) An upper inequality for the assertion (1.1), i.e.,

(2.22)

$$\limsup_{n\to\infty} P\big(\sup_{U_{1n}<u<U_{nn}} \hat{V}_n(u) < t_n\big) = \limsup_{n\to\infty} P\hat{A}_n \leqslant D^2(t),$$

can now easily be proved as follows. We have by (2.2)

$$\hat{A}_n \subset \Omega_n^c + \Omega_n \cap_{k \in K_0} A_{kn},\ K_0 := \mathbb{N}\{[1, n/\log n] \cup [n - n/\log n, n]\}.$$

For $k \in [1, n/\log n]$

$$(2.23) \qquad \begin{aligned} A_{kn} &= \left(-Z_k \leqslant \left(-k + t_n w_{kn} n^{\frac{1}{2}}\right) Z_{n+1}/n\right) \\ &\subset \left(-Z_k \leqslant \left(-k + t_n k^{\frac{1}{2}}\right) Z_{n+1}/n\right) \end{aligned}$$

$$\begin{aligned} A_{kn}\Omega_n &\subset \left(-k^{-\frac{1}{2}}\bar{Z}_k \leqslant t_n + c_n(k/n)^{\frac{1}{2}}\right) \\ &\subset \left(-k^{-\frac{1}{2}}\bar{Z}_k \leqslant (2\log_2 n)^{\frac{1}{2}}\left(1 + \frac{\log_3 n + 2t + o(1)}{4\log_2 n}\right)\right) \end{aligned}$$

if, e.g., $c_n = (\log_2 n)^{\frac{1}{2}}$. Thus

$$(2.24) \qquad \Omega_n \cap_{1 \leqslant k \leqslant n/\log n} A_{kn} \subset \cap_{k=1}^{n/\log n}\left(-k^{-\frac{1}{2}}\bar{Z}_k \leqslant T_n(t + o(1))\right).$$

Now

$$(2.25) \qquad T_n(t + o(1)) = T_{n/\log n}(t + o(1))$$

for with $\log(n/\log n) = \log n - \log_2 n = (\log n)(1 - (\log_2 n)/\log n)$,

$$\begin{aligned} \log_2(n/\log n) &= \log_2 n - (1 + o(1))(\log_2 n)/\log n \\ &= (\log_2 n)(1 - (1 + o(1))/\log n), \\ \log_3(n/\log n) &= \log_3 n + o(1), \end{aligned}$$

$$\begin{aligned} T_{n/\log n}(t + o(1)) &= (2\log_2 n)^{\frac{1}{2}}\left(1 - \left(\frac{1}{2} + o(1)\right)/\log n\right) \\ &\quad \times (1 + (\log_3 n + 2t + o(1))(1 + (1 + o(1))/\log n)/(4\log_2 n)) \\ &= T_n(t + o(1)). \end{aligned}$$

Hence by Theorem 2.1 the probability of the rhs of (2.24) tends to $D(t)$.

In order to decrease the upper bound to $D^2(t)$ a sort of asymptotic independence argument is used as follows. First, Lemma 2.1 implies

$$(2.26) \qquad P\left(\cap_{k=K_n'}^{n-K_n'} A_{kn}\right) \to 1 \qquad (K_n' = n/o(\log_2 n))$$

with an $o(\cdot)$ such that $o(\log_2 n) \to \infty$. Taking account of (2.24) we hence have to consider only large k values. We have with $Z_{n+1-k}' := Z_{n+1} - Z_k$:

$$\begin{aligned} A_{kn}\Omega_n &\subset \left(\bar{Z}_{n+1-k}' \leqslant \left(1 + c_n/n^{\frac{1}{2}}\right) t_n w_{kn} n^{\frac{1}{2}} + c_n(n-k)/n^{\frac{1}{2}}\right) \\ &\subset \left((n + 1 - k)^{-\frac{1}{2}}\bar{Z}_{n+1-k}' < T_n(t + o(1)) + c_n((n-k)/n)^{\frac{1}{2}}\right) \end{aligned}$$

and with $c_n = (\log_2 n)^{\frac{1}{2}}$ and $n - k = o(n(\log_2 n)^{-2})$:

$$A_{n+1-k,n}\Omega_n \subset \left(k^{-\frac{1}{2}}\bar{Z}_k' \leqslant T_n(t + o(1))\right).$$

Now, up to an asymptotically negligible event (indicated by $\subset_{n\to\infty}$) and with (2.24),

$$\text{(2.27)} \qquad \Omega_n \cap_{k=1}^{n-1} A_{kn} \subset_{n\to\infty} \Omega_n\left(\cap_{k=1}^{n/\log n} A_{kn}\right)\left(\cap_{k \geqslant n-n/\log n}^{n-1} A_{kn}\right)$$

$$\subset \left(\cap_{k=1}^{n/\log n}\left(-k^{-\frac{1}{2}}\bar{Z}_k \leqslant T_n(t + o(1))\right)\right)\left(\cap_{k=1}^{n/\log n}\left(k^{-\frac{1}{2}}\bar{Z}_k' \leqslant T_n(t + o(1))\right)\right).$$

For $k < n/2$, Z_k and Z_k' are independent. Hence the probability of the very last event tends to $D^2(t)$.

(4) In order to prove

$$\liminf_n P\hat{A} \geqslant D^2(t)$$

and thus to prove (1.1) we split up the k-range into five ranges. The first range is $K_n' := K_n := n(\log_2 n)^{-1} \log_4 n$ up to $n - K_n$ upon which Kolmogorov's theorem will be applied: by (2.26)

$$\text{(2.28)} \qquad P\left(\cap_{k=K_n}^{n-K_n} A_{kn}\right) \to 1.$$

The second range is $k_n := n(\log_2 n)^{-1}(\log_3 n)^{-1}$ up to K_n upon which the Darling-Erdös theorem will be applied, yielding events of asymptotic probability one. The third range is $k = 1, \cdots, k_n$ upon which again Darling-Erdös will be applied yielding the probability $D(t)$. The remaining two ranges are essentially symmetrical about $n/2$ to the two last ones.

In order to handle the second range, (k_n, K_n), let k_n, K_n be chosen as above, let Ω_n as given in (2.9) and $\Omega_{1n} := (|\bar{Z}_{k_n}| < k_n^{\frac{1}{2}} \log_3 n)$, thus $P\Omega_{1n} \to 1$.

For $k \in (k_n, K_n)$ we can replace the B_{kn} defined in (2.6) by simpler and smaller events. We need

$$\begin{aligned}
k - 1 + t_n w_{k-1,n} n^{\frac{1}{2}} &> k - 1 + t_n w_{kn}(1 - 1/k) n^{\frac{1}{2}} \\
&\geqslant k + (2k \log_2 n)^{\frac{1}{2}}\left\{\left(1 + \frac{\log_3 n + 2t}{4 \log_2 n}\right)\left(1 - \frac{(\log_2 n)\log_3 n}{n}\right)\right. \\
&\qquad \left.\left(1 - \frac{\log_4 n}{\log_2 n}\right) - (2k \log_2 n)^{-\frac{1}{2}}\right\} \\
&> k + (2k \log_2 n)^{\frac{1}{2}}\left(1 + \frac{\log_3 n + 2t + o(1) - 4 \log_4 n}{4 \log_2 n}\right) \\
&> k + k^{\frac{1}{2}} T_n(-5 \log_4 n)
\end{aligned}$$

for all n large enough, any t fixed. Then, we obtain the following if $c_n \to \infty$,

$c_n = o((\log_4 n)^{\frac{1}{2}})$,

$$B_{kn} \supset \Omega_n\left(\left(1 + \frac{c_n}{n^{\frac{1}{2}}}\right)\left(k - k^{\frac{1}{2}} T_n(-5 \log_4 n)\right) < Z_k < \left(1 - \frac{c_n}{n^{\frac{1}{2}}}\right)\right)$$
$$\times \left(k + k^{\frac{1}{2}} T_n(-5 \log_4 n)\right)$$
$$\supset \Omega_n\left(|\bar{Z}_k| < k^{\frac{1}{2}} T_n(-6 \log_4 n)\right)$$
$$\supset \Omega_n \Omega_{1n}\left(|\bar{Z}_k - \bar{Z}_{k_n}| < k^{\frac{1}{2}} T_n(-6 \log_4 n) - k_n^{\frac{1}{2}} \log_3 n\right)$$
$$\supset \Omega_n \Omega_{1n}\left((k - k_n)^{-\frac{1}{2}} |\bar{Z}_k - \bar{Z}_{k_n}| < w'_{kn}\right),$$

where w'_{kn} is a constant satisfying

$$w'_{kn} > \left(\frac{k}{k - k_n}\right)^{\frac{1}{2}} (T_n(-6 \log_4 n) - \log_3 n)$$
$$> \left(1 + \left(\frac{1}{2} + o(1)\right)(\log_3 n)^{-1}(\log_4 n)^{-1}\right)(2 \log_2 n)^{\frac{1}{2}}$$
$$\times \left(1 - (\log_3 n)(\log_2 n)^{-\frac{1}{2}}\right)$$
$$= (2 \log_2 n)^{\frac{1}{2}}\left(1 + \left(\frac{1}{2} + o(1)\right)(\log_3 n)^{-1}(\log_4 n)^{-1}\right)$$
$$> T_n(\log_3 n) > T_N(\log_3 n),$$
$$N := [K_n - k_n],$$

for $(\frac{1}{2} + o(1))(\log_3 n)^{-1}(\log_4 n)^{-1} > (\log_3 n)(2 \log_2 n)^{-1}$ and $T_n(x)$ is isotonic in n for $\log_2 n > 50$, say, and $0 < x < \log_2 n$, as may be seen as follows. Put $(2 \log_2 n)^{\frac{1}{2}} = y$ so that

$$T_n(x) = y + \left(x - \frac{\log 2}{2} + \log y\right)/y,$$

(2.29) $$\frac{\partial}{\partial y} T_{n(y)}(x) = 1 - \left(\log y + x - \frac{\log 2}{2} - 1\right)y^{-2} > 0.$$

Hence by Theorem 2.1

$$\liminf_{n\to\infty} P\left(\cap_{k=k_n}^{K_n} B_{kn}\right) \geqslant \lim_{N\to\infty} P\left(\max_{j=1,\ldots,N} j^{-\frac{1}{2}}|\bar{Z}_j| < T_N(\log_3 n)\right) = 1.$$

By a symmetry argument like (2.17) the same relation holds for $\cap_{k=n-K_n}^{n-k_n} B_{kn}$. Hence (2.8) has been sharpened to

(2.30) $$P\left(\cap_{k=k_n}^{n-k_n} B_{kn}\right) \to 1.$$

(5) The third range (and the one symmetrical to it) yields the interesting figure. It is dealt with by

LEMMA 2.2.

(2.31) $$\lim_n P(\cap_{k=1}^{k_n} A_{kn}) = \lim_n P(\cap_{k \geqslant n-k_n}^{n} A_{kn}) = D(t)$$

for $k_n = n(\log_2 n)^{-1}(\log_3 n)^{-1}$.

PROOF. On account of step (3) of this proof we only have to show that lim $\inf_n$ of the lhs probabilities is not smaller than $D(t)$. Analogously to (2.23)ff. and using $c_n = o(\log_4 n)$, say, in the definition of Ω_n we have for $k > t_n^2(1 + t_n^2/n)^{-1} = (2 + o(1)) \log_2 n$ (cf. (2.2a))

(2.32) $$A_{kn} \supset \Omega_n(-k^{-\frac{1}{2}}\bar{Z}_k \leqslant x_{kn}).$$

Here with $o(\cdot)$-sequences that are possibly negative

$$\begin{aligned} x_{kn} &:= t_n(1 - k/n)^{\frac{1}{2}} - (k/n)^{\frac{1}{2}} o(\log_4 n) \\ &> t_n(1 - k_n/n) - (k_n/n)^{\frac{1}{2}} o(\log_4 n) \\ &\geqslant (2 \log_2 n)^{\frac{1}{2}} \left\{ \left(1 + \frac{\log_3 n + 2t}{4 \log_2 n}\right)\left(1 - (\log_2 n)^{-1} \right.\right. \\ &\qquad \left.\left. \times (\log_3 n)^{-1}\right) - o\left((\log_2 n)^{-1}\right) \right\} \\ &= (2 \log_2 n)^{\frac{1}{2}} \left(1 + \frac{\log_3 n + 2t + o(1)}{4 \log_2 n}\right) \\ &> T_{kn}(t + o(1)), \end{aligned}$$

the latter according to (2.25) a fortiori. The assertion for $n - k_n < k \leqslant n$ follows from symmetry considerations already employed.

(6) The asymptotic independence argument already used in (2.27)ff. yields

(2.33) $$\liminf_n P(\cap_{k \in \{1, \cdots, k_n\}\{n-k_n, \cdots, n\}} A_{kn}) \geqslant D^2(t),$$

and recalling (2.30),

$$\liminf_n P(\hat{A}_n) \geqslant D^2(t).$$

Together with (2.22), the assertion (1.1) of Theorem 1 with the plus sign is established. The other sign has been dealt with by the remark following (2.20).

PROOF OF THEOREM 3. In order to prove (1.8) it suffices to show according to (2.4) that

$$P(\cap_{\log n < k < k_n} B_{kn}) \to D^2(t) \quad \left(t \in R^1;\ k_n := \frac{n}{(\log_2 n)\log_3 n}\right).$$

Here the restrictions of the k-range permitted by (2.15) and (2.30) and the symmetry and independence argument already used in (2.27) are taken into account. We first prove

(2.34) $$\limsup_{n\to\infty} P(\cap_{\log n < k < k_n} B_{kn}) \leqslant D^2(t)$$

by noting that from the definition (2.6) of the B_{kn} and under employment of the notations in the proof of Theorem 1 it follows

$$B_{kn} \subset \left(n^{-1}Z_{n+1}\left(k - t_n k^{\frac{1}{2}}\right) \leqslant Z_k \leqslant n^{-1}Z_{n+1}\left(k + t_n k^{\frac{1}{2}}\right)\right),$$

$$B_{kn}\Omega_n \subset \left(\left(1 - c_n n^{-\frac{1}{2}}\right)\left(k - t_n k^{\frac{1}{2}}\right) \leqslant Z_k \leqslant \left(1 + c_n n^{-\frac{1}{2}}\right)\left(k + t_n k^{\frac{1}{2}}\right).\right)$$

Choosing $c_n = \log_4 n$ we have $(1 \pm c_n n^{-\frac{1}{2}})t_n = T_n(t + o(1))$. Denoting $Z^*_k := k^{-\frac{1}{2}}\overline{Z}_k$ one gets for $k < k_n$

$$\begin{aligned} B_{kn}\Omega_n &\subset \left(|Z_k^*| < T_n(t + o(1)) + c_n(k/n)^{\frac{1}{2}}\right) \\ &\subset \Bigg(|Z_k^*| < (2\log_2 n)^{\frac{1}{2}}\Bigg(1 + \frac{\log_3 n + 2t + o(1)}{4\log_2 n} \\ &\quad + (\log_2 n)^{-1}0\left((\log_3 n)^{-\frac{1}{2}}\log_4 n\right)\Bigg)\Bigg) \\ &= \left(|Z_k^*| < T_n(t + o(1))\right) = \left(|Z_k^*| < T_{k_n}(t + o(1))\right), \end{aligned}$$

the latter according to (2.25) a fortiori. Now (2.34) follows from (1.13) taken for k_n partial sums.

In order to prove

$$\liminf_{n\to\infty} P\left(\cap_{\log n<k<k_n} B_{kn}\right) \geqslant D^2(t) \tag{2.35}$$

note

$$w_{kn} \geqslant (k/n)^{\frac{1}{2}}(1 - k/n) \geqslant (k/n)^{\frac{1}{2}}\left(1 - ((\log_2 n)\log_3 n)^{-1}\right)$$

for the k in question. Hence

$$\begin{aligned} t_n w_{kn} &\geqslant ((k/n)2\log_2 n)^{\frac{1}{2}}\left(1 + \frac{\log_3 n + 2t}{4\log_2 n} - o(1/\log_2 n)\right) \\ &= (k/n)^{\frac{1}{2}} T_n(t + o(1)) \end{aligned}$$

and

$$\begin{aligned} t_n w_{k-1,n} &> (1 - 1/k)^{\frac{1}{2}} t_n w_{kn} \geqslant (1 - 1/\log n)(k/n)^{\frac{1}{2}} T_n(t + o(1)) \\ &= (k/n)^{\frac{1}{2}} T_n(t + o(1)). \end{aligned}$$

Thus

$$\begin{aligned} B_{kn} &\supset \Omega_n\Big(\left(1 + c_n n^{-\frac{1}{2}}\right)\left(k - k^{\frac{1}{2}}T_n(t + o(1))\right) < Z_k \\ &\qquad < \left(1 - c_n n^{-\frac{1}{2}}\right)\left(k + k^{\frac{1}{2}}T_n(t + o(1))\right)\Big) \\ &= \Omega_n\left(|Z_k^*| < T_n(t + o(1)) - c_n(k/n)^{\frac{1}{2}}\right) \\ &= \Omega_n\left(|Z_k^*| < T_{k_n}(t + o(1))\right) \end{aligned}$$

which implies (2.35) again by (1.13).

Statement (1.9) is essentially a reformulation of (1.8) as is seen from (2.6) since in (1.11) we have

$$U_{kn}^* = (U_{kn} - k/(n+1))\big(k(n+1-k)(n+1)^{-2}(n+2)^{-1}\big)^{-\frac{1}{2}}$$

and for $k \wedge (n-k) \to \infty$, $k = o(n(\log_2 n)^{-\frac{1}{2}})$,

$$\frac{k}{n+1} - t_n\left(\frac{k(n+1-k)}{(n+1)^2(n+2)}\right)^{\frac{1}{2}} = \frac{k}{n} - T_n(t + o(1))w_{kn}/n^{\frac{1}{2}}.$$

Finally (1.10) will be proved below.

PROOF OF THEOREM 2 AND THEOREM 3, (1.10). Due to the last part of Proposition 1 and due to Lemma 2.3 below it suffices to prove the convergence to $D(t)$ in case $F(u) = u$ for the expressions

$$P(V_n(u) < t_n\ u \in I_n)$$

$$\underset{(>)}{\leqslant} P\left(\hat{V}_n(u) < t_n\left(\frac{1-u}{1-F_n(u)}\right)^{\frac{1}{2}} \right. \tag{2.36}$$

$$\left. \times\left(1 \underset{(-)}{+} (nF_n(u))^{-\frac{1}{2}}\sup_u|\hat{V}_n(u)|\right)^{\frac{1}{2}} \quad \forall u \in I_n\right),$$

where

$$I_n := \left(\frac{\log n}{n} \vee U_{1n'}((\log_2 n)\log_3 n)^{-1}\right).$$

In order to evaluate the rhs probability the event may and will be intersected with

$$\Omega_n := \left(\sup_u|\bar{F}_n(u)| < \frac{1}{o(n^{\frac{1}{2}})}\right),\ P\Omega_n \to 1$$

where $o(n^{\frac{1}{2}}) > n^{\frac{1}{3}}$, say, and with

$$\Omega_{n1} := (\sup_u|\hat{V}_n(u)| < T_n(1/o(1))),\ P\Omega_{n1} \to 1, \quad \text{any} \quad o(1) > 0$$

(where the claimed convergences hold due to Kolmogorov's theorem respectively to Theorem 1). Now uniformly in u, $u \leqslant o(1)$ on Ω_n,

$$\left(\frac{1-u}{1-F_n(u)}\right)^{\frac{1}{2}} \leqslant \left(1 - (1 + o(1))\bar{F}_n(u)\right)^{-\frac{1}{2}} \leqslant 1 + 1/o(n^{\frac{1}{2}}). \tag{2.37}$$

We shall construct a third sequence of events Ω_{n2} with $P\Omega_{n2} \to 1$. For $u \in I_n$ we have

$$(nF_n(u))^{-\frac{1}{2}} \leqslant \left(nF_n\left(\frac{\log n}{n}\right)\right)^{-\frac{1}{2}}.$$

Now the central limit theorem can be applied to $F_n(\log n/n)$: putting $p_n := \log n/n$, we have

$$\sigma_n^2 := \operatorname{Var}(nF_n(p_n)) = np_n(1-p_n) = (1+o(1))\log n \to \infty$$

and hence

$$n\bar{F}_n(p_n)/\sigma_n \to_d N(0, 1).$$

It follows that

$$P(nF_n(p_n) > np_n - \sigma_n/o(1)) \\ = P(nF_n(p_n) > (1 + o(1))\log n) \to 1$$

for all negative $o(1)$ converging not too fast to 0. Hence

$$\Omega_{n2} := \left((nF_n(u))^{-\frac{1}{2}} \leqslant (1 + o(1))(\log n)^{-\frac{1}{2}} \forall u \in I_n\right)$$

satisfies $P\Omega_{n2} \to 1$.

Now for $u \in I_n$ and under restriction to $\Omega_n \cap \Omega_{n1} \cap \Omega_{n2}$

$$(nF_n(u))^{-\frac{1}{2}} \sup_u |\hat{V}_n(u)| < (1 + o(1))(\log n)^{-\frac{1}{2}} \\ \times T_n(1/o(1)) = \left(\frac{2\log_2 n}{\log n}\right)^{\frac{1}{2}}(1 + o(1)) \tag{2.38}$$

and finally, with (2.37),

$$t_n\left(\frac{1-u}{1-F_n(u)}\right)^{\frac{1}{2}}\left(1 \underset{(-)}{+} (nF_n(u))^{-\frac{1}{2}} \sup_u |\hat{V}_n(u)|\right)^{\frac{1}{2}} \\ \underset{(>)}{<} (2\log_2 n)^{\frac{1}{2}}\left(1 + \frac{\log_3 n + 2t + o(1)}{4\log_2 n}\right)\left(1 \underset{(-)}{+} \left(\frac{1}{2} + o(1)\right)\left(\frac{2\log_2 n}{\log n}\right)^{\frac{1}{2}}\right) \\ = T_n(t + o(1)).$$

It now follows that

$$P(V_n(u) < t_n \forall u \in I_n) = P(\hat{V}_n(u) < T_n(t + o(1))\forall u \in I_n) + o(1) \to D(t)$$

by Proposition 1 (first part), if we there replace $U_{\log n, n}$ by $(1 + o(1))n^{-1}\log n$ and $U_{k_n, n}$ by $(1 + o(1))n^{-1}k_n$ which can be done. The analogous statement is true for the u-range $(1 - o(1/\log_2 n), 1 - (\log n)/n)$ and for $|V_n(u)|$ instead of $V_n(u)$. An asymptotic independence type argument like (2.33) in the proof of Theorem 1 and as in the proof of Theorem 3, first half, together with Lemma 2.3 below, yields all the assertions.

Proof of Remark 1. With the notions just introduced we have

$$PB_n = P(\hat{A}_n A_n^*) = D^4(t) + o(1),$$

the latter by (1.8) of Theorem 3. On the other hand by Theorem 1 $\lim_n P\hat{A}_n = \lim_n PA_n^* = D^2(t)$ so that we may say that $\hat{A}_n$ and A_n^* are asymptotically independent for any fixed t. Moreover, in the proof of Theorem 1 it will be shown that

$$P(\cap_{1\leqslant k<k_n} A_{kn}) + o(1) = P(\cap_{n-k_n<k\leqslant n} A_{kn}) + o(1) = \\ P(\cap_{1\leqslant k<k_n} A_{kn}^*) + o(1) = P(\cap_{n-k_n<k\leqslant n} A_{kn}^*) + o(1) = D(t).$$

Together with the assertions of Theorem 1 this implies that all four of these intersections are asymptotically independent. Lemma 2.3 below concerns the last relation of Proposition 1 for $V_n(\cdot)$ instead of $\hat{V}_n(\cdot)$. It is related to Lemma 6.1 of [11] for the Poisson process and is identical with the sufficiency half of Corollary 2 of [12] but the proof given here requires only Darling-Erdös' theorem.

LEMMA 2.3. *Let $d_n > 0$ be real. Then*

$$P(\sup_{0<u<n^{d_n-1}}|V_n(u)| < t_n) \to 1, \qquad t \in R^1; \tag{2.39}$$

if $d_n \to 0$ by symmetry the analogous assertion follows for $1 - n^{d_n-1} < u < 1$.

The necessity of the condition can also be proven with the tools used below, but it is not needed. Both directions are proved in [12] with the help of more advanced tools.

PROOF.

(1) W.l.o.g. let $k_n := n^{o(1)} > t_n^{10}$. We shall show that it suffices to prove

$$PA_n \to 1 \quad \text{where} \quad A_n := \left(\sup_{0<u<U_{2k_n,n}}|V_n(u)| < t_n\right).$$

With $\Omega_n := (\bar{Z}_{n+1} > n^{\frac{1}{2}}/o(1))$, $P\Omega_n \to 1$, $o(1)$ being a positive null sequence converging slowly enough, and with U_{kn} replaced by Z_k/Z_n we have

$$P(nU_{2k_n,n} > k_n) \geqslant P\left(\Omega_n\left(\bar{Z}_{2k_n} > -k_n(1 + o(1))\right)\right) \to 1$$

by the central limit theorem. Now

$$P(\sup_{0<u<n^{o(1)-1}}|V_n(u)| < t_n) \geqslant P\left((nU_{2k_n,n} > k_n)A_n\right) = o(1) + PA_n.$$

Since $2k_n$ is of the same type $n^{o(1)}$ as k_n we henceforth write k_n instead of $2k_n$. Moreover, since on the u-range in question $(1 - u)^{\frac{1}{2}}t_n \geqslant (1 - n^{o(1)-1})t_n > t_n' := T_n(t')$ with some $t' < t$ we also have

$$A_n \supset A_n' := \left(|\bar{F}_n(u)| < t_n'(u/n)^{\frac{1}{2}} \ \forall u \in (0, U_{k_n,n})\right)$$

and it suffices to show $PA_n' \to 1$. We shall write here and more often subsequently t_n instead of t_n' since an alteration of t is inessential.

(2) We solve the inequality in the definition of A_n' for u. From $u^2 - 2uF_n(u) + F_n^2(u) < t_n^2u/n$ we get

$$|u - F_n(u) - t_n^2(2n)^{-1}| < (2n)^{-1}t_n^2(1 + 4nF_n(u)t_n^{-2})^{\frac{1}{2}}.$$

It suffices to consider the upper inequality arising from this only at $U_{kn}-$ and the lower one at $U_{kn}+$ $(k = 1, \cdots, k_n)$. Thus

$$\begin{aligned} A_n' &\supset \cap_{k=1}^{k_n}\left(k - 2^{-1}t_n^2\left\{(1 + 4kt_n^{-2})^{\frac{1}{2}} - 1\right\} < nU_{kn} < k + k^{\frac{1}{2}}t_n\right) \\ &\supset \Omega_n \cap_k\left(\left(k - 2^{-1}t_n^2\left\{(1 + 4kt_n^{-2})^{\frac{1}{2}} - 1\right\}\right)\left(1 + 1/o(n^{\frac{1}{2}})\right) < Z_k \right. \\ &\qquad\qquad \left. < \left(k + k^{\frac{1}{2}}t_n\right)\left(1 - 1/o(n^{\frac{1}{2}})\right)\right) \\ &\supset \Omega_n \cap_k\left(\left(k/o(n^{\frac{1}{2}})\right) - 2^{-1}t_n^2\left\{(1 + 4kt_n^{-2})^{\frac{1}{2}} - 1\right\} < \bar{Z}_k \right. \\ &\qquad\qquad \left. < -\left(k/o(n^{\frac{1}{2}})\right) + k^{\frac{1}{2}}t_n'\right). \end{aligned} \tag{2.40}$$

Here the factors $1 \pm 1/o(n^{\frac{1}{2}})$ can be absorbed in

$$t_n := (2 \log_2 n)^{\frac{1}{2}}\big(1 + (4 \log_2 n)^{-1}(2t + \log_3 n)\big)$$

to yield some t_n' since $1/o(n^{\frac{1}{2}}) = o((\log_2 n)^{-1})$. We shall henceforth again drop the dash at t_n'. Similarly the term $k/o(n^{\frac{1}{2}}) < n^{o(1)-\frac{1}{2}}/o(1) = o((\log_2 n)^2)$ if the $o(1)$-sequence in Ω_n is chosen suitably, so that this term can also be absorbed in $2^{-1}t_n^2\{\cdot \cdot \cdot\}$ in the lower bound. Summarizing, we want to show

$$\text{(2.41)} \qquad PA_n' \geqslant o(1) + P\Big(\cap_{k=1}^{k_n}\big(-2^{-1}t_n^2\big\{(1 + 4kt_n^{-2})^{\frac{1}{2}} - 1\big\}k^{\frac{1}{2}} < Z_k^*\big)\Big) $$
$$< t_n)\big) \to 1 \ \big(Z_k^* := k^{-\frac{1}{2}}(Z_k - k)\big).$$

(3) Using

$$\log_2 n = \log_2 k_n + |\log o(1)| = (\log_2 k_n) \times (1 + |\log o(1)|/\log_2 k_n)$$

we have

$$\begin{aligned} t_n^2 &> (2 \log_2 n)\big(1 + (2t + \log_3 n)(2 \log_2 n)^{-1}\big) \\ &= 2 \log_2 n + \log_3 n + 2t \\ &> 2 \log_2 k_n + \log_3 k_n + |\log o(1)| + 2t \qquad \text{(2.42)} \\ &= (2 \log_2 k_n)\big(1 + (2t + |\log o(1)| + \log_3 k_n)(2 \log k_n)^{-1}\big) \\ &> T_{k_n}^2(1/o(1)) \end{aligned}$$

where the argument of the T function can be chosen to tend to $+\infty$. Now (2.41) can be proved immediately for the upper inequalities applying Theorem 2.1 with n replaced by k_n :

$$P\big(\max_{k=1,\cdots,k_n} Z_k^* < t_n\big) > P\big(\max_k Z_k^* < T_{k_n}(1/o(1)\big) \to 1.$$

(4) The equivalent assertion for the lower inequalities in (2.40) resp. (2.41) is more complicated and requires splitting the k-range up into five sections. Firstly, for $k = 1, \cdot \cdot \cdot , o(t_n)$, observing $(1 + x)^{\frac{1}{2}} > 1 + x/2 - x^2/8$ for $x > 0$, we get

$$\begin{aligned} &P\Big(\cap_{k=1}^{o(t_n)}\Big(k - 2^{-1}t_n^2\big\{(1 + 4kt_n^{-2})^{\frac{1}{2}} - 1\big\} < nU_{kn}\Big)\Big) \\ &\qquad \geqslant P\big(\cap_{k=1}^{o(t_n)}(k^2t_n^{-2} < nU_{kn})\big) \geqslant P(o(1) < nU_{1n}) \\ &\qquad = P(o(1) < Z_1) \to 1. \end{aligned}$$

Secondly, by a slightly sharper analysis and putting

$$\text{(2.43)} \qquad A_{kn} := \big(-2^{-1}k^{-\frac{1}{2}}t_n^2\big\{(1 + 4kt_n^{-2})^{\frac{1}{2}} - 1\big\} < Z_k^*\big),$$

one has starting from (2.41)

$$\text{(2.44)} \qquad \cap_{k=o(t_n)}^{3t_n^2/4} A_{kn} \supset \cap_k\big(-k^{\frac{1}{2}}(1 - kt_n^{-2}) < Z_k^*\big) \supset \cap_k\big(-o\big(t_n^{\frac{1}{2}}\big) < Z_k^*\big)$$

since $k^{\frac{1}{2}}(1 - kt_n^{-2})$ assumes its minimum on the k-range in question at the left endpoint. But now $o(t_n^{\frac{1}{2}}) > (1/o(1))(\log_2(3t_n^2/4))^{\frac{1}{2}} > T_{3t_n^2/4}(1/o(1))$ with suitable positive $o(1)$ – sequences, and again by Theorem 2.1 with $3t_n^2/4$ instead of n the probabilities of the events (2.44) are seen to tend to one.

Thirdly, for $3t_n^2/4 < k < t_n^3$ we have in (2.43) $(1 + 4kt_n^{-2})^{\frac{1}{2}} - 1 \geqslant 1$, resulting in

$$A_{kn} \supset \left(-2^{-1}k^{-\frac{1}{2}}t_n^2 < Z_k^*\right) \supset \left(-2^{-1}t_n^{\frac{1}{2}} < Z_k^*\right)$$
$$\supset \left(-T_{t_n^3}(1/o(1)) < Z_k^*\right)$$

since $\log_2 t_n^3 = o(t_n)$. Theorem 2.1 with n replaced by t_n^3 thus yields

$$P\left(\cap_{k=3t_n^2/4}^{t_n^3} A_{kn}\right) \geqslant P\left(\inf_{1 \leqslant k \leqslant t_n^3} Z_k^* > -T_{t_n^3}(1/o(1)\right) \to 1.$$

On the fourth range, $t_n^3 \leqslant k \leqslant t_n^{10}$, we have in (2.43)

$$(2.45) \quad (1 + 4kt_n^{-2})^{\frac{1}{2}} - 1 > 2k^{\frac{1}{2}}t_n^{-1}\left(1 - 2^{-1}t_n k^{-\frac{1}{2}}\right) > 2k^{\frac{1}{2}}t_n^{-1}\left(1 - t_n^{-\frac{1}{2}}\right).$$

Hence

$$A_{kn} \supset \left(-t_n\left(1 - t_n^{-\frac{1}{2}}\right) < Z_k^*\right) \supset \left(-T_{t_n^{10}}(1/o(1)) < Z_k^*\right)$$

since $T_{t_n^{10}}(1/o(1)) = 0(\log_2 t_n) = o(t_n) = o(t_n(1 - t_n^{-\frac{1}{2}}))$. Theorem 2.1 now yields

$$P\left(\cap_{k=t_n^3}^{t_n^{10}} A_{kn}\right) \to 1.$$

On the fifth range, $t_n^{10} \leqslant k \leqslant n^{o(1)}$, we have instead of (2.45)

$$\left(1 + 4kt_n^{-2}\right)^{\frac{1}{2}} - 1 > 2k^{\frac{1}{2}}t_n^{-1}\left(1 - t_n^{-4}\right)$$
$$= 2k^{\frac{1}{2}}t_n^{-1}\left(1 - 0\left((\log_2 n)^{-2}\right)\right) > 2k^{\frac{1}{2}}T_n^{-1}(t + o(1)).$$

By this and taking the extreme inequalities in (2.42) we have

$$A_{kn} \supset (-t_n < Z_k^*) \supset \left(-T_{k_n}(1/o(1)) < Z_k^*\right);$$

finally

$$P\left(\cap_{k=1}^{k_n} A_{kn}\right) \geqslant P\left(\inf_{1 \leqslant k \leqslant k_n} Z_k^* > T_{k_n}(1/o(1))\right) \to 1,$$

which completes the proof of the lemma.

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