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ON OPTIMAL MEDIAN UNBIASED ESTIMATORS IN THE PRESENCE OF NUISANCE PARAMETERS

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For exponential families with density

$$x \to C(\theta, \eta)h(x)\exp[a(\theta)T(x) + \Sigma_{i=1}^{p} a_i(\theta, \eta)S_i(x)],$$
$$(\theta, \eta) \in \Theta \times H, \Theta \subset \mathbb{R},$$

a increasing and continuous, there exists for every sample size an estimator for θ which is—in the class of all median unbiased estimators—of minimal risk for any monotone loss function.

Introduction. Let $(X, \mathcal{A})$ be a measurable space and $P_{\theta,\eta}|\mathcal{A}$, $(\theta, \eta) \in \Theta \times H$ with $\Theta \subset \mathbb{R}$ a family of p-measures. We are interested in estimators for θ.

The following result of Lehmann and Scheffé (1950, page 321, Theorem 5.1) is well known: if an unbiased estimator can be expressed as a function of a complete sufficient statistic, then it is —among all unbiased estimators—of minimal risk for any convex loss function.

Since unbiasedness is not a very persuasive property, and since any realistic loss function will be bounded and therefore never convex, this result is not entirely satisfactory. It therefore seems worthwhile to show that the same tools (i.e., sufficiency and completeness) can be used to establish that under comparable conditions median unbiased estimators exist which are—among all median unbiased estimators—of minimal risk for any monotone loss function. (By a monotone loss function we mean one which assumes its minimum if the estimate agrees with the parameter, and is nondecreasing as the estimate moves away from the parameter in either direction. Convex loss functions are necessarily monotone.)

We remark that our result implies the existence of a *median unbiased* estimator of minimal monotone risk also for arbitrary monotone functions of the parameter θ, whereas *unbiased* estimators exist for one of these functions only, and convex loss functions may cease to be convex under monotone transformations.

The theorem. Let $\mathcal{B}$ denote the Borel-field of $\mathbb{R}$. In the following we shall consider *randomized estimators* for θ, which are represented by a Markov kernel $k|X \times \mathcal{B}$ with the following interpretation: After x has been observed, the estimate is determined by a random experiment, governed by the p-measure $k(x, \cdot)|\mathcal{B}$. (The consideration of randomized estimators seems to be inevitable, since for p-measures with atoms nonrandomized median unbiased estimators do not exist in general.)

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For any Markov kernel $M|X \times \mathfrak{B}$ and any p-measure $Q|\mathfrak{A}$ let $Q \circ M$ denote the p-measure over $\mathfrak{B}$ defined by $Q \circ M(B) := \int M(x, B)Q(dx)$, $B \in \mathfrak{B}$. For a measure ν and a function f, let $\nu(f)$ denote the ν-integral of f, and $\nu * f$ the induced measure, defined by $\nu * f(A) := \nu(f^{-1}A)$.

An estimator k is *median unbiased* for θ in the family $P_{\theta,\eta}$, $(\theta, \eta) \in \Theta \times H$, if for every $(\theta, \eta) \in \Theta \times H$, θ is a median of $P_{\theta,\eta} \circ k$.

THEOREM. *Let Θ be an interval, possibly infinite. Assume that $P_{\theta,\eta}$ has, with respect to some σ-finite measure μ, a density of the type*

$$C(\theta, \eta)h(x)H(T(x), \theta)G(S(x), \theta, \eta) \tag{1}$$

with $T: X \to \mathbb{R}$ and $S: X \to Y$.

To avoid technicalities we assume that $I := \{t \in \mathbb{R}: H(t, \theta) > 0\}$ is an interval independent of θ, and that $G(y, \theta, \eta) > 0$ for all $y \in Y$.

Assume that the function H, defined on $I \times \Theta$, has the following properties:

(i) *$\theta \to H(t, \theta)$ is continuous on Θ for every $t \in I$*

(ii) *$\theta \to H(t_2, \theta)/H(t_1, \theta)$ is nondecreasing on Θ for every pair $t_1, t_2 \in I$ with $t_1 < t_2$.*

Moreover, assume that the family $P_{\theta,\eta} * S$, $\eta \in H$, is complete for every $\theta \in \Theta$.

Then the median unbiased estimator k_0 defined by (16) (*see* (12) *and* (14)) *is maximally concentrated in the following sense: for each $(\theta, \eta) \in \Theta \times H$, let $u \to L_{\theta,\eta}(u)$ be a loss function which assumes its minimal value for $u = \theta$, and is nondecreasing as u moves away from θ in either direction. ($L_{\theta,\eta}$ is not necessarily bounded from above.) If k is any median unbiased estimator for θ, then*

$$P_{\theta,\eta} \circ k_0(L_{\theta,\eta}) \leqslant P_{\theta,\eta} \circ k(L_{\theta,\eta}) \quad \textit{for all} \quad (\theta, \eta) \in \Theta \times H.$$

For the case without nuisance parameter, i.e., for families P_θ with μ-density $C(\theta)h(x)H(T(x), \theta)$, a corresponding result was obtained by Lehmann (1959, page 83) under the additional assumption that the distribution $P_\theta * T$ is nonatomic, and by Pfanzagl (1970, page 33, Theorem 1.12) for the general case. For the special case of an exponential family a related result was obtained independently by Brown, Cohen and Strawderman (1976, page 719, Corollary 4.1).

REMARK 1. The estimator k_0 is not necessarily strict (i.e., the measure $k_0((T(x), S(x)), \cdot)|\mathfrak{B}$ may assign positive probability to $\Theta^c - \Theta$ for some $x \in X$). If Θ is an interval with boundary points a, b, we have $k_0((T(x), S(x)), [a, b]) = 1$ for all $x \in X$.

REMARK 2. The most important application of the theorem is to exponential families with μ-density

$$C(\theta, \eta)h(x)\exp\left[a(\theta)T(x) + \Sigma_{i=1}^{p} a_i(\theta, \eta)S_i(x)\right], \tag{2}$$

which fulfill the assumptions of the theorem for *every sample size* if $\theta \to a(\theta)$ is increasing and continuous, and if $\{(a_1(\theta, \eta), \cdots, a_p(\theta, \eta)): \eta \in H\}$ has a non-empty interior for every $\theta \in \Theta$. (For the last condition see Lehmann (1959), page 132, Theorem 1.)

PROOF OF THE THEOREM. Since S is sufficient for the family $P_{\theta,\eta}$, $\eta \in H$, for each fixed $\theta \in \Theta$, the nuisance parameter disappears if we consider the family of conditional distributions, given S. Since these families have monotone likelihood ratios, a randomized estimator $k(T, y)$ exists which is optimal median unbiased on the partition $S = y$. Hence $k(T, S)$ is an estimator which is median unbiased on the whole for all values of the nuisance parameter. To prove its optimality, one has to show that an arbitrary median unbiased estimator is also median unbiased on the partition $S = y$, hence inferior on the partition $S = y$ for all y, hence inferior on the whole.

(i) The first part of the proof collects a few auxiliary results on the conditional distribution, given S.

Let $\nu|\mathcal{A}$ be the measure defined by $\nu(A) := \int_A h(x)\mu(dx)$. We remark that ν is σ-finite, and that $P_{\theta,\eta}$ has ν-density

$$x \to C(\theta, \eta)H(T(x), \theta)G(S(x), \theta, \eta). \tag{3}$$

Since T is real valued, there exists a regular conditional probability of $\nu * T|\mathcal{B}$, given S, i.e., a Markov kernel $\bar{\nu}|Y \times \mathcal{B}$ such that for all $B \in \mathcal{B}$ and all $D \in \mathcal{D} := \{D \subset Y: S^{-1}D \in \mathcal{A}\}$

$$\nu\big(T^{-1}(B) \cap S^{-1}(D)\big) = \int_D \bar{\nu}(y, B)\nu * S(dy). \tag{4}$$

(See, e.g., Breiman (1968), page 78, Theorem 4.30. Breiman's theorem refers to the case that ν is a p-measure. The proof for an arbitrary σ-finite measure ν is the same.) (4) implies in particular

$$\nu * T(B) = \int \bar{\nu}(y, B)\nu * S(dy). \tag{5}$$

With

$$H_0(y, \theta) := \int H(t, \theta)\bar{\nu}(y, dt) \tag{6}$$

we obtain that

$$y \to C(\theta, \eta)H_0(y, \theta)G(y, \theta, \eta) \tag{7}$$

is a density of $P_{\theta,\eta} * S$ with respect to $\nu * S$.

For technical reasons we need a few results on H_0.

Since $\{t \in \mathbb{R}: H(t, \theta) > 0\}$ does not depend on θ, the same holds true for $Y_0 := \{y \in Y: H_0(y, \theta) > 0\}$.

Since $\bar{\nu}$ is a Markov kernel, it follows from (6) that $y \to H_0(y, \theta)$ is $\mathcal{D}$-measurable for every $\theta \in \Theta$.

Moreover, $\theta \to H_0(y, \theta)$ is continuous for $\nu * S$ – a.a. $y \in Y$. Since (7) is the density of a p-measure and $G(y, \theta, \eta) > 0$ for all $y \in Y$, there exists a $\nu * S$-null set N_θ such that $H_0(y, \theta) < \infty$ for $y \notin N_\theta$. Let $N_1 := \cup \{N_\theta: \theta \in \Theta_1\}$, where Θ_1 is a countable dense subset of Θ. Because of assumption (i) continuity of $H_0(y, \cdot)$

follows from (6) by the bounded convergence theorem: for $\tau', \tau'' \in \Theta_1$ with $\tau' < \tau''$ and $t_0 \in I$ we obtain from (ii) for every $\theta \in (\tau', \tau'')$

$$0 \leqslant H(t, \theta) \leqslant H(t, \tau') \frac{H(t_0, \theta)}{H(t_0, \tau')} \qquad \text{for} \quad t \leqslant t_0$$

$$\leqslant H(t, \tau'') \frac{H(t_0, \theta)}{H(t_0, \tau'')} \qquad \text{for} \quad t \geqslant t_0.$$

Since $\theta \to H(t_0, \theta)$ is continuous and $t \to H(t, \tau)$ is $\bar{\nu}(y, \cdot)$-integrable for $\tau \in \Theta_1$ and $y \in Y_* := Y_0 - N_1$, this entails the existence of a $\bar{\nu}(y, \cdot)$-integrable bound of $t \to H(t, \theta)$ which holds uniformly for $\theta \in (\tau', \tau'')$, if $y \in Y_*$. Hence $\theta \to H_0(y, \theta)$ is continuous on Θ° for $y \in Y_*$.

It is easy to check that

(8) $$M_\theta(y, B) := H_0(y, \theta)^{-1} \int_B H(t, \theta) \bar{\nu}(y, dt) \qquad y \in Y_0, B \in \mathfrak{B}$$

$$:= \bar{\nu}(y, B) \qquad y \notin Y_0, B \in \mathfrak{B}$$

defines a Markov kernel $M_\theta | Y \times \mathfrak{B}$, and that this Markov kernel is a regular conditional probability of $P_{\theta,\eta} * T$, given S, and that $\theta \to M_\theta(y, B)$ is continuous on Θ° for every $y \in Y_*$, $B \in \mathfrak{B}$.

For later use we remark that for any measurable function f: $\mathbb{R} \times Y \to \mathbb{R}$

(9) $$\int f(T(x), S(x)) P_{\theta,\eta}(dx) = \int \left(\int f(t, y) M_\theta(y, dt) \right) P_{\theta,\eta} * S(dy).$$

Relation (8) makes explicit what was obvious from the beginning: that versions of the conditional probability of $P_{\theta,\eta} * T$, given S, exist which do not depend on η, because for each $\theta \in \Theta$ the statistic S is sufficient for the family $P_{\theta,\eta}$, $\eta \in H$. We need, however, the explicit representation (8) which shows that for each $y \in Y$ the family $M_\theta(y, \cdot) | \mathfrak{B}$, $\theta \in \Theta$, has monotone likelihood ratios, a consequence of assumption (ii).

(ii) Now we construct a randomized estimator which is optimal median unbiased under the condition $S = y$, and combine the estimators obtained for different partitions $S = y$ to an estimator on the whole.

Since $M_\theta(y, \cdot) | \mathfrak{B}$, $\theta \in \Theta$, has monotone likelihood ratios for each $y \in Y$, we obtain from Theorem 1.12 in Pfanzagl (1970), page 33, the existence of a median unbiased estimator $k_y | \mathbb{R} \times \mathfrak{B}$ such that $M_\theta(y, \cdot) \circ k_y$ is of minimal monotone risk among all median unbiased estimators for the family $M_\theta(y, \cdot)$, $\theta \in \Theta$. Since we have to combine the estimators for different y to an estimator on the whole, certain measurability questions occur, the solution of which requires knowing k_y explicitly.

If the distribution function of $M_\theta(y, \cdot) | \mathfrak{B}$ is continuous and increasing for every $\theta \in \Theta$, $y \in Y$, then there exists $c(\theta, y)$ such that $M_\theta(y, (-\infty, c(\theta, y)]) = \frac{1}{2}$, and, since the family $M_\theta(y, \cdot)$, $\theta \in \Theta$, has m.l.r., the function $\theta \to c(\theta, y)$ is increasing. If $m(\cdot, y)$ is the inverse function (i.e., $m(c(\theta, y), y) \equiv \theta$), then $t \to m(t, y)$ is the desired median unbiased estimator. In the general case, more care is needed in defining this estimator. In particular: if $M_\theta(y, \cdot)$ has atoms, then randomization is needed to achieve median unbiasedness.

In the general case, an estimator k_y can be defined as follows (see Pfanzagl (1970), 1.8, page 32/3): let

$$(10) \qquad c(\theta, y) := \inf\{t \in \mathbb{R}: M_\theta(y, (-\infty, t]) \geqslant \tfrac{1}{2}\}$$

and

$$(11) \qquad \begin{aligned} F_y(t, \theta) &:= 1 && \text{if } t < c(\theta, y) \\ &:= r_{(\theta, y)} && \text{if } t = c(\theta, y) \\ &:= 0 && \text{if } t > c(\theta, y) \end{aligned}$$

where

$$\begin{aligned} r_{(\theta, y)} &= 0 \quad \text{if } M_\theta(y, \{c(\theta, y)\}) = 0 \\ &= \left[\tfrac{1}{2} - M_\theta(y, (-\infty, c(\theta, y)))\right] / M_\theta(y, \{c(\theta, y)\}) \quad \text{otherwise.} \end{aligned}$$

Let $y \in Y*$ be fixed. If Θ has boundary points a and b we define

$$\begin{aligned} G_y(t, \theta) &:= 0 && \text{for } \theta \leqslant a \\ &= F(t, \theta) && \text{for } \theta \in (a, b) \\ &= \sup\{F(t, \tau): \tau < b\} && \text{for } \theta = b \\ &= 1 && \text{for } \theta > b. \end{aligned}$$

The function $\theta \to G_y(t, \theta)$ is nondecreasing and continuous from below (since $\theta \to c(\theta, y)$ has these properties on Θ^0). Hence there exists a Markov kernel $k_y | \mathbb{R} \times \mathfrak{B}$ such that

$$(12) \qquad k_y(t, (-\infty, \theta)) = G_y(t, \theta) \qquad \text{for all } t, \theta \in \mathbb{R}.$$

We have

$$k_y(t, \theta^c) = 1.$$

Similarly as in Pfanzagl (1970), Theorem 1.12, one can show that k_y is median unbiased. For a more precise formulation of this property, let

$$B'_\theta := \{\tau \in \Theta^c, \tau \leqslant \theta\} \quad \text{and} \quad B''_\theta := \{\tau \in \Theta^c, \tau \geqslant \theta\}.$$

By B_θ we mean either B'_θ or B''_θ. Then we have for all $y \in Y_*$

$$(13) \qquad M_\theta(y, \cdot) \circ k_y(B_\theta) \geqslant \tfrac{1}{2} \quad t \quad \text{for all} \quad \theta \in \Theta$$

and

$$(13') \qquad M_\theta(y, \cdot) \circ k_y(B_\theta) = \tfrac{1}{2} \qquad \text{for all} \quad \theta \in \Theta^\circ, \qquad \text{the interior of} \quad \Theta.$$

So far, we have a separate estimator for each $y \in Y_*$. It is, however, easy to see that the function $(t, y) \to k_y(t, B)$ is measurable for every $B \in \mathfrak{B}$. For every $r \in \mathbb{R}$ we have (see (10)) $c(\theta, y) > r$ iff $M_\theta(y, (-\infty, r]) < \frac{1}{2}$. Since $y \to M_\theta(y, (-\infty, r])$ is measurable, this implies that $y \to c(\theta, y)$ is measurable. Hence, by (11), $(t, y) \to F_y(t, \theta)$ is measurable. By (12), $(t, y) \to k_y(t, B)$ is, therefore, measurable for every $B = \{\tau \in \Theta: \tau < \theta\}$. Since the class of all $B \in \mathfrak{B}$ with $(t, y) \to k_y(t, B)$ measurable is a σ-field, this implies that $(t, y) \to k_y(t, B)$ is measurable for every $B \in \mathfrak{B}$.

Finally, we define k_y for $y \notin Y_*$ by k_{y_0}, say, with $y_0 \in Y_*$ arbitrary. Since Y_* is measurable, $(t, y) \to k_y(t, B)$ is measurable on $\mathbb{R} \times Y$. Hence we may define a

Markov kernel $k_0|(\mathbb{R} \times Y) \times \mathfrak{B}$ by

(14) $$k_0((t, y), B) := k_y(t, B).$$

Using (9) we obtain from (13)

(15) $$(P_{\theta,\eta} * (T, S)) \circ k_0(B_\theta) = \int\left(\int k_y(t, B) M_\theta(y, dt)\right) P_{\theta,\eta} * S(dy) \geq \tfrac{1}{2} \quad \text{for all} \quad \theta \in \Theta$$

and

$$(P_{\theta,\eta} * (T, S)) \circ k_0(B_\theta) = \int\left(\int k_y(t, B) M_\theta(y, dt)\right) P_{\theta,\eta} * S(dy) = \tfrac{1}{2} \quad \text{for} \quad \theta \in \Theta^\circ.$$

Hence the estimator defined by

(16) $$x \to k_0((T(x), S(x)), \cdot)|\mathfrak{B}$$

is median unbiased for all $(\theta, \eta) \in \Theta \times H$.

(iii) It remains to be shown that the estimator k_0 is of minimal monotone risk among all median unbiased estimators.

To this aim let $k|X \times \mathfrak{B}$ be any other median unbiased estimator. The basic idea of the proof is to replace $x \to k(x, B)$ for each $B \in \mathfrak{B}$ by a conditional expectation, given (T, S), not depending on (θ, η) (notice that (T, S) are sufficient by the factorization theorem), thus defining a function $\bar{k}|(\mathbb{R} \times Y) \times \mathfrak{B}$, and to use completeness of the family $P_{\theta,\eta} * S$, $\eta \in H$, to establish that for each $y \in Y$ the function $t \to \bar{k}((t, y), \cdot)|\mathfrak{B}$ is median unbiased for the family $M_\theta(y, \cdot)$, $\theta \in \Theta$. Since $t \to k_0((t, y), \cdot)$ is optimal for this family, the optimality of k_0 for the family $P_{\theta,\eta}$, $(\theta, \eta) \in \Theta \times H$, follows by integration with respect to $P_{\theta,\eta} * S$.

In carrying through this idea, we meet the following obstacle. Median unbiasedness of k means

(17) $$P_{\theta,\eta} \circ k(B_\theta) \geq \tfrac{1}{2} \quad \text{for all} \quad (\theta, \eta) \in \Theta \times H.$$

If for some $\theta \in \Theta$,

(18) $$P_{\theta,\eta} \circ k(B_\theta) = \tfrac{1}{2} \quad \text{for all} \quad \eta \in H,$$

we obtain from (9) for all $\eta \in H$

$$\int\left(\int \bar{k}((t, y), B_\theta) M_\theta(y, dt)\right) P_{\theta,\eta} * S(dy) = \int \bar{k}((t, y), B_\theta) P_{\theta,\eta} * (T, S)(dt, dy) = \int k(x, B_\theta) P_{\theta,\eta}(dx) = \tfrac{1}{2},$$

so that completeness of $P_{\theta,\eta} * S$, $\eta \in H$, implies

(19) $$\int \bar{k}((t, y), B_\theta) M_\theta(y, dt) = \tfrac{1}{2} \quad \text{for} \quad P_{\theta_0,\eta_0} * S - \text{a.a.}\, y \in Y,$$

where $(\theta_0, \eta_0) \in \Theta \times H$ is arbitrarily fixed.

An inequality like (17) cannot be used in the same way. However, (17) implies $P_{\theta,\eta} \circ k(B_\theta) = \frac{1}{2}$ for all but countably many values of θ, and therefore completeness can be used as indicated above.

This basic idea can be carried through as follows. Since $P_{\theta,\eta} \ll P_{\theta_0,\eta_0}$, we have $P_{\theta,\eta} \circ k \ll P_{\theta_0,\eta_0} \circ k$ for any Markov kernel k. If $P_{\theta,\eta} \circ k(B'_\theta) \geq \frac{1}{2}$ and

$P_{\theta,\eta} \circ k(B_\theta'') \geq \frac{1}{2}$ and $P_{\theta,\eta} \circ k(B_\theta') + P_{\theta,\eta} \circ k(B_\theta'') > 1$, then $P_{\theta,\eta} \circ k\{\theta\} > 0$ and therefore $P_{\theta_0,\eta_0} \circ k\{\theta\} > 0$. Let $\Theta_0 := \{\theta \in \Theta : P_{\theta_0,\eta_0} \circ k\{\theta\} > 0\}$. Since $P_{\theta_0,\eta_0} \circ k$ is finite, Θ_0 is at most countable.

By definition of Θ_0 relation (18) and therefore relation (19) holds true for every $\theta \in \Theta - \Theta_0$. Let $N_\theta \in \mathfrak{D}$ denote the exceptional $P_{\theta_0,\eta_0} * S$-null set occurring in (19). Let $\Theta_2 \subset \Theta^\circ - \Theta_0$ be a countable set which is dense in Θ, and let $N_2 := \cup \{N_\theta : \theta \in \Theta_2\}$.

To prove the optimality of k_0 we proceed similarly as in the proof of Theorem 1.12(b) in Pfanzagl (1970), page 34. Let $y \notin N_2$ and $\tau \in \Theta_2$ be fixed. Then by (13′), (14) and (19),

$$\int \bar{k}((t,y), B_\tau'')M_\tau(y, dt) = \int k_0((t,y), B_\tau'')M_\tau(y, dt). \tag{20}$$

By definition of k_0 (see (11), (12) and (14)),

$$\begin{aligned} k_0((t,y), B_\tau'') &= 0 \qquad t < c(\tau, y) \\ &= 1 \qquad t > c(\tau, y). \end{aligned}$$

Hence, by Lehmann (1959), Theorem 2(ii), page 68

$$\begin{aligned} \int \bar{k}((t,y), B_\tau'')M_\theta(y, dt) &\leq \int k_0((t,y), B_\tau'')M_\theta(y, dt) \qquad \text{for} \quad \theta > \tau \\ &\geq \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad \theta < \tau. \end{aligned} \tag{21}$$

Let now $\theta \in \Theta$ be fixed. Since (21) holds true for all $\tau \in \Theta_2$, and since Θ_2 is dense in Θ, it holds true for all $\tau \in \Theta$, except $\tau = \inf\Theta$, if this element belongs to Θ. For this element, (21) is, however, trivially true.

So far, $y \notin N_2$ was fixed. Since N_2 is a $P_{\theta,\eta} * S$-null set, integration over y with respect to $P_{\theta,\eta} * S$ yields (see 9)

$$\begin{aligned} P_{\theta,\eta} \circ k(B_\tau'') &\leq \big(P_{\theta,\eta} * (T, S)\big) \circ k_0(B_\tau'') \qquad \text{for} \quad \theta > \tau \\ &\geq \qquad\qquad\qquad\qquad\qquad \text{for} \quad \theta < \tau. \end{aligned} \tag{22}$$

From this, the assertion follows easily.

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