The Annals of Statistics
1978, Vol. 6, No. 4, 755-768

LARGE SAMPLE THEORY FOR A BAYESIAN NONPARAMETRIC SURVIVAL CURVE ESTIMATOR BASED ON CENSORED SAMPLES

BY V. SUSARLA[1] AND J. VAN RYZIN[2]

University of Wisconsin—Milwaukee and University of California—Davis

Let $X_1, \cdots, X_n$ be i.i.d. F_0 and let $Y_1, \cdots, Y_n$ be independent (and independent also of $X_1, \cdots, X_n$) random variables. Then assuming that F is distributed according to a Dirichlet process with parameter α, the authors obtained the Bayes estimator $\hat{F}_\alpha$ of F under the loss function $L(F, \hat{F}) = \int (F(u) - \hat{F}(u))^2 \, dw(u)$ when $X_1, \cdots, X_n$ are censored on the right by $Y_1, \cdots, Y_n$, respectively, and when it is known whether there is censoring or not. Assuming $X_1, \cdots, X_n$ are i.i.d. F_0 and $Y_1, \cdots, Y_n$ are i.i.d. G, this paper shows that $\hat{F}_\alpha$ is mean square consistent with rate $O(n^{-1})$, almost sure consistent with rate $O(\log n/n^{\frac{1}{2}})$, and that $\{\hat{F}_\alpha(u) \mid 0 < u < T\}$, $T < \infty$, converges weakly to a Gaussian process whenever F_0 and G are continuous and that $P[X_1 > u]P[Y_1 > u] > 0$.

1. Introduction and summary. Recently attention has been drawn to the consideration of obtaining nonparametric Bayes estimates of a distribution function assuming a manageable prior (resulting in a manageable posterior distribution) on the space of distribution functions F on $R = (-\infty, \infty)$. Towards this goal, Ferguson [4] introduced a class of priors, known as Dirichlet process priors, on F which enjoy the property that the posterior distribution is again a Dirichlet process. Ferguson used this fact to obtain the Bayes estimator of the right sided cumulative distribution function F ($F(x)$ denotes the probability in (x, ∞) here and elsewhere, and this useful convention is borrowed from Efron [3]) under a weighted squared error loss function.

While treating this important problem (see, for example, Gross and Clark [6]) of estimating survival curves based on incomplete data, the authors [11] obtained the Bayes estimator of F under a weighted squared error loss function when the independent observations from F are randomly censored on the right under Dirichlet process priors of Ferguson [4]. They demonstrated that this Bayes estimator is an extension of the above mentioned Bayes estimator of Ferguson [4] and in a certain sense, also of the well-known Kaplan–Meier (KM) estimator

Received July 1976; revised June 1977.

[1] Research supported in part by United States Army under Contract No. DAAG 29-75-C-0024; by National Institute of General Medical Sciences, DHEW, Grant No. 1-RO-1-GM 23129; and by NSF Grant MCS 76-05952.

[2] Research supported in part by United States Army under Contract No. DAAG 29-75-C-0024; by National Institute of General Medical Sciences, DHEW, Grant No. 1-RO-1-GM 23129; and by National Cancer Institute, DHEW, Grant No. 1-RO-1-CA 18332.

AMS 1970 *subject classifications.* Primary 62E20; Secondary 62G05.

Key words and phrases. Survival curve estimator, Dirichlet process, censored data, weak convergence, consistency.

[7] which maximizes the likelihood of the observations. Efron [3] and in a more detailed manner, Breslow and Crowley [1] showed that the KM estimator is weakly consistent and asymptotically normal under the assumption that all the censoring random variables are i.i.d. continuous random variables.

The object of this paper is to show that our Bayes estimator has good limiting properties including mean-square consistency (m.s.c.), almost sure consistency (a.s.c.) and asymptotic normality assuming that the observations are i.i.d. with right cdf F_0 and that the censoring random variables are i.i.d. with a continuous distribution function. Efron [3] and Breslow and Crowley [1] have neither rate of convergence results for m.s.c. nor a.s.c., while we obtain rates for both m.s.c. and a.s.c. Our methods of proof, in contrast with those of Breslow and Crowley [1], involve the analysis of the expectation and the variance of the logarithm of $W_n(u)$ involved in the Bayes estimator given in (1.2).

For each fixed prior distribution involved in our Bayes estimator, we have shown that the Bayes estimator has better asymptotic properties than those established for the KM estimator by Breslow and Crowley [1]. For results of this type in parametric cases, see the bibliographies of Lindley [8] and Shapiro [9].

We now formally describe the problem along with some notation. The rest of the sections deal with various asymptotic aspects of our Bayes estimator.

Let $X_1, \cdots, X_n$ be a random sample from a right sided cdf F with $F(0) = 1$ and $Y_1, \cdots, Y_n$ be another random sample such that $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ are mutually independent. Set

$$(1.1) \qquad \delta_i = [X_i \leqq Y_i] \qquad \text{and} \qquad Z_i = \min\{X_i, Y_i\}$$

for $i = 1, \cdots, n$ and assume that $1 - F$ is a Dirichlet process with parameter measure α on the Borel σ-field $\mathscr{B}$ in $(0, \infty)$. Then the Bayes estimator of F, under the loss function $L(F, \hat{F}) = \int_0^\infty (F(u) - \hat{F}(u))^2\, dw(u)$, where w is a weight function, is shown in [11] to be

$$(1.2) \qquad \begin{aligned} \hat{F}_\alpha(u) &= \frac{\alpha(u) + N^+(u)}{\alpha(R^+) + n} \prod_{i=1}^n \left\{\frac{\alpha(Z_i^-) + N^+(Z_i) + \lambda_i}{\alpha(Z_i^-) + N^+(Z_i)}\right\}^{[\delta_i = 0, Z_i \leqq u]} \\ &= B_n(u) W_n(u) \end{aligned}$$

where $N^+(t) =$ number of observations $> t$ and $\lambda_i =$ number of observations at Z_i, $i = 1, \cdots, n$.

The main results of this paper concern the asymptotic behavior of $\hat{F}_\alpha$ under the following assumptions:

(A1) $X_1, \cdots, X_n$ are i.i.d. with right sided cdf F_0, a fixed unknown distribution on $(0, \infty)$.

(A2) $Y_1, \cdots, Y_n$ are i.i.d. with right cdf G, a fixed unknown continuous distribution on $(0, \infty)$.

Thus, while the rule under consideration is a Bayes rule, the asymptotic properties of $\hat{F}_\alpha$ are obtained in a nondecision theoretic setup. In other words, we obtain the asymptotic behavior of $\hat{F}_\alpha$ as an estimator of F_0.

Throughout we assume that for a fixed (but otherwise arbitrary) u,

(A3) $\alpha(u) = \alpha(u, \infty) > 0$.

Since it is not possible to estimate $F_0(u)$ whenever $G(u) = 0$, we assume throughout that $G(u) > 0$ without further reference.

Throughout, we point out $H^{-1}, G^{-1}, \cdots$ are not functional inverses, but stand for $1/H, 1/G, \cdots$. This notation will be used in the rest of the paper without further comment.

2. Mean square consistency with rates. This property of $\hat{F}_\alpha$ can be studied through the corresponding one for the logarithm of $W_n(u)$ of (1.2). Under (A3), one obtains that

$$\ln W_n(u) = \sum_{i=1}^n [\delta_i = 0, Z_i \leqq u] \ln \left\{\frac{\alpha(Z_i^-) + N^+(Z_i) + 1}{\alpha(Z_i^-) + N^+(Z_i)}\right\} \tag{2.1}$$

where $\alpha(s^-) =$ limit of $\alpha(t)$ as $t \uparrow s$. Observe that (2.1) (and hence (1.2)) is well defined since $\alpha(u^-) > 0$ by (A3). This property is not enjoyed by the KM estimate which is not always well defined in the right tail. It is precisely this property of converting (1.1) into a sum by use of logarithms that allows us to obtain stronger convergence results than Breslow and Crowley [1] obtain for the KM estimator.

For dealing with the expectation and the variance (and properties based on these) of $\ln W_n$, the following decomposition which follows by a logarithmic expansion of the summands in (2.1) and the succeeding lemmas will be extremely useful. The justification for such lemmas can be given as follows: By looking at (1.2), we see that if $\hat{F}$ has to be a good estimator of F, then W_n should be a good estimator of G^{-1} since B_n obviously is a good estimator of $H\ (= F_0 G)$. Since W_n is a product of positive factors, it is clear that its properties can be studied via its logarithm. A motivation for the decomposition follows from the facts that, in general, one can neglect all the terms starting with the second in a logarithmic expansion such as $\ln W_n(u)$, and that $N^+(z_j)$ is a good estimate of $H(z_j)$. Thus we have

$$\ln W_n(u) = R_{n,1}(u) + R_{n,2}(u) + R_{n,3}(u) \tag{2.2}$$

where

$$nR_{n,1}(u) = \sum_{j=1}^n [\delta_j = 0, Z_j \leqq u] H^{-1}(Z_j)\,, \tag{2.3}$$

$$R_{n,2}(u) = \sum_{j=1}^n [\delta_j = 0, Z_j \leqq u] \sum_{l=2}^\infty l^{-1}(\alpha(Z_j^-) + 1 + N^+(Z_j))^{-l}\,, \tag{2.4}$$

and

$$nR_{n,3}(u) = \sum_{j=1}^n [\delta_j = 0, Z_j \leqq u] \times \{n[(\alpha(Z_j^-) + 1 + N^+(Z_j))]^{-1} - H^{-1}(Z_j)\}\,, \tag{2.5}$$

where

$$H = F_0 G\,. \tag{2.6}$$

In the lemmas to follow, we get bounds on the expectations of $R_{n,j}$ $(j = 1, 2, 3)$ less some functions, and bounds on their second moments or variances as required for further analysis leading to Theorems 3.1 and 3.2. In the lemmas to follow, the bounds involve constants $c_1, c_2, \cdots$ which are given in Remark 2.1 below.

LEMMA 2.1. *$E[R_{n,1}(u)] = -\ln G(u)$ and $nH^2(u)\operatorname{Var}(R_{n,1}(u)) \leqq c_1/n$.*

PROOF. Since the summands of $R_{n,1}(u)$ are identically distributed,

$$E[R_{n,1}(u)] = E[[\delta_1 = 0, Z_1 \leqq u]H^{-1}(Z_1)] = -\int_0^u F_0(t)H^{-1}(t)\,dG(t) = -\ln G(u)$$

by the definition of H in (2.6). The variance result follows since each of the i.i.d. summands in $R_{n,1}(u)$ is bounded by $H^{-1}(u)$.

LEMMA 2.2. *$E[R^2_{n,2}(u)] \leqq c_2/n^2$.*

PROOF. Since $(a_1 + \cdots + a_n)^2 \leqq n \sum_{i=1}^n a_i^2$ for any real numbers $a_1, \cdots, a_n$, and since for fixed n, the summands of $R_{n,2}(u)$ are identically distributed,

$$\begin{aligned}(2.7)\qquad n^{-2}E[R^2_{n,2}(u)] &\leqq E[[\delta_1 = 0, Z_1 \leqq u](\textstyle\sum_{l=2}^\infty l^{-1}(\alpha(Z_1^-) + 1 + N^+(Z_1))^{-l})^2] \\ &\leqq \{\textstyle\sum_{l=2}^\infty (\alpha(u^-) + 1)^{2-l}\}^2 E[(\alpha(u^-) + 1 + N^+(u))^{-4}]\end{aligned}$$

where the second inequality follows by bounding the series by $\{\sum_{l=2}^\infty (\alpha(u^-) + 1)^{2-l}\}\{\alpha(u^-) + 1 + N^+(u)\}^{-2}$ and by dropping the indicator function. The result now follows from the following inequality and (3.7):

$$\begin{aligned}E[(\alpha(u^-) + 1 + N^+(u))^{-4}] &= \textstyle\sum_{k=0}^{n-1} \binom{n-1}{k}(\alpha(u^-) + 1 + k)^{-4}H^k(u)(1 - H(u))^{n-1-k} \\ &\leqq H^{-4}(u) \textstyle\sum_{k=0}^{n-1} \binom{n+3}{k+4}H^{k+4}(u)(1 - H(u))^{n+3-(k+4)}/\binom{n+3}{4}\end{aligned}$$

where the inequality follows since $k + i \leqq i(\alpha(u^-) + k + 1)$ for $i = 1, 2, 3$, and 4.

LEMMA 2.3. (a) $|E[R_{n,3}(u)]| \leqq c_3/n$; (b) $E[R^2_{n,3}(u)] \leqq c_4/n$.

PROOF. Since the summands of $R_{n,3}(u)$ are identically distributed,

$$\begin{aligned}(2.8)\qquad E[R_{n,3}(u)] &= E[[\delta_1 = 0, Z_1 \leqq u]\{n(\alpha(Z_1^-) + 1 + N^+(Z_1)) - H^{-1}(Z_1)\}] \\ &= -\textstyle\int_0^u F_0(t)E[n(\alpha(t^-) + 1 + N^-(t))^{-1} - H^{-1}(t)]\,dG(t)\end{aligned}$$

where $N^-(t)$ is a binomial random variable with parameters $n - 1$ and $H(t)$. ($N^-(t)$ is binomial with parameters $n - 1$ and $H(t)$ instead of n and $H(t)$ since we fixed one of the z's to be equal to t.) Now observe that after some simplification,

$$\begin{aligned}&H(t)E[n(\alpha(t^-) + 1 + N^-(t))^{-1} - H^{-1}(t)] \\ (2.9)\qquad &\quad= \textstyle\sum_{k=0}^{n-1} \binom{n-1}{k}(k + 1)^{-1}\{nH(t) - \alpha(t^-) - k - 1\}H^k(t)(1 - H(t))^{n-k-1} \\ &\quad\quad+ \textstyle\sum_{k=0}^{n-1} \binom{n-1}{k}\alpha(t^-)\{k + 1 + \alpha(t^-) - nH(t)\}H^k(t)(1 - H(t))^{n-1-k} \\ &\quad\quad\times \{(k + 1)(k + 1 + \alpha(t^-))\}^{-1} = \mathrm{I} + \mathrm{II}\,.\end{aligned}$$

By a rearrangement, $-nH(t)\mathrm{I} = \alpha(t^-) + (1 - H(t))^n(nH(t) - \alpha(t^-))$ while

$$\begin{aligned}\mathrm{II} &\leqq \frac{2\alpha(t^-)}{n(n + 1)} \textstyle\sum_{k=0}^{n-1} \binom{n+1}{k+2}|(k + 2 - (n + 1)H(t) + H(t) + \alpha(t^-) - 1)| \\ &\quad\times H^k(t)(1 - H(t))^{n+1-(k+2)}\end{aligned}$$

since $k + 2 \leqq 2(\alpha(t^-) + k + 1)$. By a change of variable ($k + 2 = l$) and by using the binomial moments, we can show from the above inequality that

$$\mathrm{II} \leqq \frac{2\alpha(t^-)}{H(t)} \left\{ \frac{\{H(t)(1 - H(t))\}^{\frac{1}{2}}}{n(n+1)^{\frac{1}{2}}} + \frac{H(t) + \alpha(t^-) + 1}{n(n+1)} \right\}.$$

This bound on II together with that on I, (2.8) and (2.9) give the first result since $t \leqq u$ in all the calculations after (2.8).

Using the inequality $(\sum_{i=1}^n a_i)^2 \leqq n \sum_{i=1}^n a_i^2$ and the fact that the summands of $R_{n,3}(u)$ are identically distributed for fixed n, it can be shown that $E[R_{n,3}^2(u)] \leqq -\int_0^u F_0(t)E[(n(\alpha(t^-) + N^-(t) + 1)^{-1} - H^{-1}(t))]\, dG(t)$ where $N^{-1}(t)$ is as in the proof of (a). From here on, the proof runs parallel to that in (a).

REMARK 2.1. The constants c_1, c_2, c_3, and c_4 are chosen to satisfy

$$H^2(u)c_1 = 1$$

$$\binom{n+3}{4}\alpha^2(u^-)H^4(u)c_2 \geqq n^4(1 + \alpha(u^-))^2$$

$$H^2(u)c_3 = \{1 + 7\alpha(R^+) + 2\alpha^2(R^+)\}$$

$$H^2(u)c_4 \geqq 2\{1 - H(u) + (n+2)^{-1}\{\alpha^2(R^+) + (1 - H(u))^2\}\}.$$

Two consequences of the decomposition (2.2) and the above three lemmas are given below, the first of which concerns the mean square consistency (m.s.c.) of $\ln W_n$ as an estimator of $\ln G^{-1}$ while the second one concerns the m.s.c. of $\hat{F}_\alpha$ of (1.2) as an estimator of F_0.

THEOREM 2.1. *Let $F_0(u) > 0$. Then $nE[|\ln W_n(u) - \ln G^{-1}(u)|^2]$ is bounded.*

PROOF. The proof is a direct consequence of the decomposition (3.2) and the above three lemmas.

THEOREM 2.2. *Let $F_0(u) > 0$. Then $E[(\hat{F}_\alpha(u) - F_0(u))^2] \leqq c_5/n$.*

PROOF. Recalling that B_n and W_n are defined in (2.2), we obtain by a C_r-inequality,

$$(2.10) \qquad 2^{-1}(\hat{F}_\alpha(u) - F_0(u))^2 \leqq G^{-2}(u)(B_n(u) - H(u))^2 + B_n^2(u)(W_n(u) - G^{-1}(u))^2$$

where we used the equality $H = F_0 G$. Since $nB_n(u)$ can be approximated by the binomial random variable $N^+(u)$ whose expectation is $nH(u)$ $(= nF_0(u)G(u))$, we obtain that

$$(2.11) \qquad (\alpha(R^+) + n)^2 E[(B_n(u) - H(u))^2] = (\alpha(u) - H(u)\alpha(R^+))^2 + nH(u)(1 - H(u)).$$

After writing $e^{\ln W_n(u)}$ and $e^{-\ln G(u)}$ for $W_n(u)$ and $G^{-1}(u)$, respectively, and then using the mean value theorem leads to $(W_n(u) - G^{-1}(u))^2 \leqq |\ln W_n(u) - \ln G^{-1}(u)|^2 (W_n(u) + G^{-1}(u))^2$ since $W_n(u)$ and $G^{-1}(u) \geqq 1$. Hence,

$$(2.12) \qquad E[B_n^2(u)(W_n(u) - G^{-1}(u))^2] \leqq (1 + G^{-1}(u))^2 E[|\ln W_n(u) - \ln G^{-1}(u)|^2]$$

since B_n and $B_n W_n$ are $\leqq 1$. Using the fact that $E[|A + B + C|^2] \leqq 3(E[A^2] + E[B^2] + E[C^2])$ and then using Lemmas 2.1, 2.2, and 2.3 give the result in wiew of (2.10)—(2.12).

REMARK 2.2. The bound in Theorem 2.2 is given by

$$c_5 = c_5(u) = 2\{\alpha(R^+) + n\}^{-2}\{n^2 H(u)(1 - H(u)) + n(\alpha(u) - \alpha(R^+)H(u))^2\} + 6(1 + G^{-1}(u))^2\{c_2 + c_3^2 + c_4\}.$$

3. Almost sure consistency. Looking at the estimator $\hat{F}_\alpha$ of (1.2), it is obvious that $\hat{F}_\alpha$ converges a.s. provided $W_n(u)$ does likewise. The following lemma concerning the almost sure behavior of $R_{n,1}$, $R_{n,2}$, and $R_{n,3}$ involved in $\ln W_n(u)$ of (2.2) is essential for the main result of this section.

LEMMA 3.1. *Let $F_0(u) > 0$. Then*

(a) $$|R_{n,1}(u) + \ln G(u)| = O\left(\frac{\log\log n}{n^{\frac{1}{2}}}\right) \quad \text{a.s.}$$

(b) $$|R_{n,2}(u)| = O\left(\frac{\log n}{n^{\frac{1}{2}}}\right) \quad \text{a.s.}$$

(c) $$|R_{n,3}(u)| = O\left(\frac{\log n}{n^{\frac{1}{2}}}\right) \quad \text{a.s.}$$

PROOF. (a) follows from the first part of Lemma 2.1 and the law of iterated logarithm for i.i.d. random variables. (b) follows from Lemma 2.2 and the Glivenko-Cantelli theorem since $\sum_{n=2}^{\infty} n^{-1}(\log n)^{-2} < \infty$.

To prove (c), we observe that $|R_{n,3}(u)|$ is exceeded by

$$\{n^{-1} \textstyle\sum_{j=1}^{n} [\delta_j = 0, Z_j \leqq u]\} \sup_{0<t\leqq u} \left\{\left|\frac{n}{\alpha(t^-) + 1 + N^+(t)} - \frac{1}{H(t)}\right|\right\}.$$

Since the expression in the first curly brackets is bounded by unity, it is enough to show that

$$\sup_{0<t\leqq u} \left\{\left|\frac{n}{\alpha(t^-) + 1 + N^+(t)} - \frac{1}{H(t)}\right|\right\} = O\left(\frac{\log n}{n^{\frac{1}{2}}}\right) \quad \text{a.s.} \tag{3.1}$$

The left-hand side of (3.1) is bounded by

$$\frac{1}{H(u)} \frac{n}{\alpha(u^-) + 1 + N^+(u)} \sup_{0<t\leqq u} \left\{\left|\frac{N^+(t)}{n} - H(t) - \frac{1 + \alpha(t^-)}{n}\right|\right\}. \tag{3.2}$$

Now by Lemma 2 of Dvoretzky, Kiefer and Wolfowitz [2]

$$P\left[\frac{n^{\frac{1}{2}}}{\log n} \sup_{0<t\leqq u} \left\{\left|\frac{N^+(t)}{n} - H(t)\right|\right\} > \varepsilon\right] \leqq c_0 e^{-c(\log n)^2}$$

for some absolute constants c_0 and c. Hence

$$\sup_{0<t\leqq u} \left\{\left|\frac{N^+(t)}{n} - H(t)\right|\right\} = O\left(\frac{\log n}{n^{\frac{1}{2}}}\right) \quad \text{a.s.}$$

since $c(\log n)^2 \geqq 2 \log n$ for large n and $\sum_{n=1}^{\infty} n^{-2} < \infty$. This completes the proof of (3.1) and hence also of (c).

THEOREM 3.1. *Let $F_0(u) > 0$. Then*

$$\hat{F}_\alpha(u) - F_0(u) = O(\log n/n^{\frac{1}{2}}) \quad \text{a.s.}$$

PROOF. By a triangle inequality,

$$|\hat{F}_\alpha(u) - F_0(u)| \leqq G^{-1}(u)|B_n(u) - H(u)| + B_n(u)|W_n(u) - G^{-1}(u)| . \tag{3.3}$$

The first term on the right-hand side is $O(\log\log n/n^{\frac{1}{2}})$ a.s. since $|(N^+(u)/n) - H(u)| = O(\log\log n/n^{\frac{1}{2}})$ a.s. by the law of iterated logarithm.

As in the proof of Theorem 3.2, we can show that

$$B_n(u)|W_n(u) - G^{-1}(u)| \leqq (1 + G^{-1}(u))|\ln W_n(u) - \ln G^{-1}(u)| . \tag{3.4}$$

But $|\ln W_n(u) - \ln G^{-1}(u)| = O(\log n/n^{\frac{1}{2}})$ a.s. due to the decomposition (2.2), a triangle inequality, and parts (a), (b), and (c) of Lemma 3.1. Thus (3.3) and (3.4) complete the proof.

4. Weak convergence of $\hat{F}_\alpha$. In this section, we consider the weak convergence of $\{\hat{F}_\alpha(u) \mid 0 < u \leqq T\}$ where $T < \infty$. We assume throughout this section that $H(T) = F_0(T)G(T) > 0$. It is convenient sometimes to suppress the dependence of the functions and to let $\| \ \|_T$ denote the sup norm over $(0, T]$.

The discussion to follow reduces the consideration of $\{\hat{F}_\alpha(u) \mid 0 < u \leqq T\}$ to a much more accessible form. To start with, we observe that

$$n^{\frac{1}{2}}(\hat{F}_\alpha - F_0) = n^{\frac{1}{2}}(B_n - H)G^{-1} + n^{\frac{1}{2}}(W_n - G^{-1})B_n \tag{4.1}$$

and that

$$\begin{aligned} W_n - G^{-1} &= e^{\ln W_n} - e^{\ln G^{-1}} = e^{\ln G^{-1}}(e^{\ln W_n - \ln G^{-1}} - 1) \\ &= G^{-1}(\ln W_n - \ln G^{-1}) + \frac{(\ln W_n - \ln G^{-1})^2}{2} e^{c + \ln G^{-1}} \end{aligned}$$

where c is between 0 and $\ln W_n - \ln G^{-1}$. Therefore, from (4.1), we have

$$\begin{aligned} &\|n^{\frac{1}{2}}(\hat{F}_\alpha - F_0) - n^{\frac{1}{2}}G^{-1}(B_n - H) - n^{\frac{1}{2}}HG^{-1}(\ln W_n - \ln G^{-1})\|_T \\ &\quad \leqq n^{\frac{1}{2}}\|B_n - H\|_T\|\ln W_n - \ln G^{-1}\|_T\|G^{-1}\|_T + \frac{n^{\frac{1}{2}}}{2}(\|\ln W_n - \ln G^{-1}\|_T)^2 . \end{aligned} \tag{4.2}$$

The purpose of the following lemma is to show that the right-hand side of (4.2) $\rightarrow 0$ a.s. by showing that $n^\beta\|\ln W_n - \ln G^{-1}\|_T \rightarrow 0$ a.s. for any $2\beta < 1$.

LEMMA 4.1. $n^\beta\|\ln W_n - \ln G^{-1}\|_T \rightarrow 0$ a.s. *for any* $2\beta < 1$.

PROOF. With $\tilde{H}_n(u) = \int_0^u F_0(t)\, d(1 - G(t)) = P[\delta_1 = 0, Z_1 \leqq u]$ and with $n\tilde{H}_n(u) = \sum_{j=1}^n [\delta_j = 0, Z_j \leqq u]$, we have

$$\begin{aligned} &\|\ln W_n - \ln G^{-1}\|_T \\ &\quad = \|\textstyle\int_0^{\cdot} n \ln \{1 - (\alpha(s^-) + 1 + H_n(s))^{-1}\}\, d\tilde{H}_n(s) - \int_0^{\cdot} H^{-1}(s)\, d\tilde{H}(s)\|_T \\ &\quad \leqq \|\textstyle\int_0^{\cdot} n\{\alpha(s^-) + 1 + nH_n(s)\}^{-1}\, d\tilde{H}_n(s) - \int_0^{\cdot} H^{-1}(s)\, d\tilde{H}(s)\|_T \\ &\qquad + \frac{n}{(\alpha(T^-) + 1 + nH_n(T))^2} \frac{(1 + \alpha(T^-))^2}{(\alpha(T^-))^2} \end{aligned} \tag{4.3}$$

where the inequality follows by a logarithmic expansion and an obvious weakening of the series from the second term onwards. Observe that the second term in the right-hand side of (4.3) $\to 0$ a.s. at a rate $n^{-\beta}$ with $2\beta < 1$.

For the first term on the right-hand side of (4.3),

$$
\begin{aligned}
&\|\textstyle\int_0^{\bullet} n\{\alpha(s^-) + 1 + nH_n(s)\}^{-1}\, d\tilde{H}_n(s) - \int_0^{\bullet} H^{-1}(s)\, d\tilde{H}(s)\|_T \\
&\quad \leqq \|\textstyle\int_0^{\bullet} \{n\{\alpha(s^-) + 1 + nH_n(s)\}^{-1} - H^{-1}(s)\}\, d\tilde{H}_n(s)\|_T \\
&\qquad + \|\textstyle\int_0^{\bullet} H^{-1}(s)\, d(\tilde{H}_n - \tilde{H})(s)\|_T \qquad (4.4)\\
&\quad \leqq \|\textstyle\int_0^{\bullet} [n\{\alpha(s^-) + 1 + nH_n(s)\}^{-1} - H^{-1}(s)]\, d\tilde{H}_n(s)\|_T \\
&\qquad + 2H^{-1}(T)\|\tilde{H}_n - \tilde{H}\|_T
\end{aligned}
$$

where the second inequality follows by applying integration by parts to $\int_0^{\bullet} H^{-1}(s)\, d(\tilde{H}_n - \tilde{H})(s)$ and upon observing that the variation of H^{-1} on $(0, \bullet]$ is $H^{-1}(\bullet)$. By the law of iterated logarithm for i.i.d. random variables $\|n\{\alpha(\bullet^-) + 1 + nH_n(\bullet)\} - H^{-1}(\bullet)\|_T \to 0$ a.s. at a rate $O(n^{-\beta})$ with $2\beta < 1$ and by applying the argument given by Singh [10] to the random variables $[\delta_j = 0, Z_j \leqq u]$, $j = 1, \cdots, n$, we obtain that $\|\tilde{H}_n - \tilde{H}\|_T \to 0$ a.s. at a rate $O(n^{-\beta})$ with $2\beta < 1$. Consequently the right-hand side of (4.4) and hence, the right-hand side of (4.3) $\to 0$ a.s. at a rate $O(n^{-\beta})$ with $2\beta < 1$.

In view of (4.2), an easy corollary to the above lemma is

COROLLARY 4.1.

$$\|n^{\frac{1}{2}}(\hat{F}_\alpha - F_0) - n^{\frac{1}{2}}G^{-1}(B_n - H) - n^{\frac{1}{2}}HG^{-1}(\ln W_n - \ln G^{-1})\|_T \to 0$$

a.s. *at a rate* $O(n^{-\beta})$ *for any* $2\beta < 1$.

By following the method of proof of Lemma 4.1, we can also show that

LEMMA 4.2. $\|n^{\frac{1}{2}}(\ln W_n - \ln G^{-1}) - n^{\frac{1}{2}}(\int_0^{\bullet} H_n^{-1}\, d\tilde{H}_n - \int_0^{\bullet} H^{-1}\, d\tilde{H})\|_T \to 0$ a.s. *at a rate* $O(n^{-\beta})$ *with* $2\beta < 1$.

NOTE. The random integral $\int_0^{\bullet} H_n^{-1}\, d\tilde{H}_n$ could be infinity, but finite a.s. since $P[H_n(s) = 0] = P[\delta_j = 0, Z_j \leqq s \text{ for } j = 1, \cdots, n] \leqq P[Z_j \leqq T \text{ for } j = 1, \cdots, n] = (1 - H(T))^n$ for all $s \leqq T$.

Hence, by Corollary 4.1 and Lemma 4.2, we can study the weak convergence of $\{\hat{F}_\alpha(u) \mid 0 < u \leqq T\}$ through the corresponding one for

$$n^{\frac{1}{2}}G^{-1}(B_n - H) + n^{\frac{1}{2}}HG^{-1}(\textstyle\int_0^{\bullet} H_n^{-1}\, d\tilde{H}_n - \int_0^{\bullet} H^{-1}\, dH)\,. \qquad (4.5)$$

The following theorem, which is parallel to Theorem 3 of Breslow and Crowley [1], is needed in the study of the weak convergence of (4.5) or equivalently that of $\{\hat{F}_\alpha(u) \mid 0 < u \leqq T\}$.

THEOREM 4.1. *Define* $(P_n, Q_n) \in D(0, T) \times D(0, T]$ ($D(0, T]$ *as the space of functions on* $(0, T]$ *with jump discontinuities*) *by* $P_n = n^{\frac{1}{2}}(H - H_n)$ *and* $Q_n = n^{\frac{1}{2}}(\tilde{H}_n - \tilde{H})$. *Then* (P_n, Q_n) *converges weakly to a bivariate Gaussian process* (P, Q)

which has mean 0 *and a covariance structure given for* $s \leqq t$ *by*

$$\begin{aligned} &\operatorname{Cov}(P(s), P(t)) = H(t)(1 - H(s))\,, \\ (4.6) \qquad &\operatorname{Cov}(Q(s), Q(t)) = \tilde{H}(s)(1 - \tilde{H}(t))\,, \\ &\operatorname{Cov}(P(s), Q(t)) = \tilde{H}(s) - \tilde{H}(t)(1 - H(s))\,, \qquad \textit{and} \\ &\operatorname{Cov}(Q(s), P(t)) = \tilde{H}(s)H(t) \end{aligned}$$

where $H = F_0 G$ *and* $\tilde{H} = \int_0^\cdot F_0\, d(1 - G)$.

As in (7.9) of Breslow and Crowley [1], we can represent (4.5) as

$$(4.7) \qquad (4.5) = n^{\frac{1}{2}} G^{-1}(B_n - H) + A_n + B_n + R_{1,n} + R_{2,n}$$

where

$$\begin{aligned} &A_n = \int_0^\cdot P_n H^{-2}\, d\tilde{H}\,, \\ (4.8) \qquad &B_n = Q_n H^{-1} - \int_0^\cdot Q_n H^{-2}\, d(1 - H)\,, \\ &R_{1,n} = n^{-\frac{1}{2}} \int_0^\cdot P_n^2 H^{-2} H_n^{-1}\, d\tilde{H}_n\,, \qquad \text{and} \\ &R_{2,n} = \int_0^\cdot P_n H^{-1} H_n^{-1}\, d(\tilde{H}_n - \tilde{H})(u)\,. \end{aligned}$$

By the above representation for (4.5) and steps similar to Theorem 4 of Breslow and Crowley [1], we obtain the following theorem which is similar to Theorem 5 of Breslow and Crowley [1].

THEOREM 4.2. *Let* $T < \infty$ *and* $H(T) > 0$. *Let* F_0 *and* G *be continuous. Then the random function* $n^{\frac{1}{2}}(\hat{F}_\alpha - F_0)$ *on* $(0, T]$ *converges weakly to a mean* 0 *Gaussian process* $R^* = -G^{-1}P + \int_0^\cdot H^{-2}P\, d\tilde{H} + H^{-1}Q + \int_0^\cdot H^{-2}Q\, dH$ *with covariance structure given for* $s \leqq t$ *by*

$$\begin{aligned} (4.9) \qquad \operatorname{Cov}(R^*(s), R^*(t)) &= F_0(s)F_0(t)\{H^{-1}(s)(1 - H(s)) + \textstyle\int_0^s H^{-1}G^{-1}\, dG\} \\ &= F_0(s)F_0(t)\{\textstyle\int_0^s H^{-1}F_0^{-1}\, d(1 - F_0)\}\,. \end{aligned}$$

REMARK 4.1. The covariance calculations involved in (4.9) are given in the Appendix. We notice here that the right-hand side of (4.9) coincides with (7.13) of Breslow and Crowley [1].

Concluding remarks. There are three small sample advantages for the Bayes estimator (1.2) over the KM estimator. The first is that it is defined everywhere on the real line for any n while the KM estimator is not. Secondly, as illustrated in our paper [11], the Bayes estimator is smoother than the KM estimator. The final important advantage is that the Bayes estimator is an admissible estimator of F provided the support of α = support of $w = (0, \infty)$ under the loss function $L(F, \hat{F}) = \int_0^\infty (F(u) - \hat{F}(u))^2\, dw(u)$ and under the weak convergence topology.

The results of this paper can be extended to the case in which $Y_1, \cdots, Y_n$ are independent, but not identically distributed. The technique used to obtain this extension is different from the ones proposed here and will appear elsewhere.

APPENDIX

Covariance structure of R^* *of Theorem* 4.2. To study the covariance structure of R^*, it is convenient to study the covariance structure of

$$R = F_0^{-1}R^* = H^{-1}P + \int_0^{\cdot} H^{-2}P\, d\tilde{H} + H^{-1}Q + \int_0^{\cdot} H^{-2}Q\, dH\,.$$

We use repeatedly (4.6), integration by parts and the equalities $H = F_0 G$ and $d\tilde{H} = d(\int_0^{\cdot} F_0 d(1-G)) = -F_0 dG$. We write $\operatorname{Cov}(R(s), R(t)) = \operatorname{Var}(R(s)) + \operatorname{Cov}(R(s), R(t) - R(s))$ for $0 < s \leqq t < T$, show that $\operatorname{Var}(R(s))$ = the expression in the curly brackets of (4.9), and that $\operatorname{Cov}(R(s), R(t) - R(s)) = 0$.

Variance of $R(s)$.

(1) $$\operatorname{Var}(H^{-1}(s)P(s)) = H^{-1}(s)(1 - H(s))\,.$$

(2) $$2\operatorname{Cov}(H^{-1}(s)P(s), H^{-1}(s)Q(s)) = -2H^{-1}(s)\tilde{H}(s)\,.$$

(3) $$2\operatorname{Cov}(H^{-1}(s)P(s), \textstyle\int_0^s H^{-2}P\, d\tilde{H}) = 2H^{-1}(s)\int_0^s \frac{\operatorname{Cov}(P(u), P(s))}{H^2(u)}\, d\tilde{H}$$
$$= -2\int_0^s \frac{d\tilde{H}}{H^2} - 2\ln G(s)\,.$$

(4) $$2\operatorname{Cov}(H^{-1}(s)P(s), \textstyle\int_0^s QH^{-2}\, dH) = \frac{2}{H(s)}\int_0^s \frac{\operatorname{Cov}(P(s), Q(u))}{H^2(u)}\, dH$$
$$= +2\frac{\tilde{H}(s)}{H(s)} + 2\ln G(s)\,.$$

(5) $$\operatorname{Var}(\textstyle\int_0^s H^{-2}P\, d\tilde{H}) = 2\int_0^s\int_0^u \frac{\operatorname{Cov}(P(u), P(r))}{H^2(u)H^2(r)}\, d\tilde{H}(r)\, d\tilde{H}(u)$$
$$= 2\int_0^s \frac{1 - H(r)}{H(r)}\left\{\int_r^s \frac{d\tilde{H}(u)}{H(u)}\right\} d\tilde{H}(r)$$
$$= 2\int_0^s \frac{(\ln G(s) - \ln G(r))}{HG}\, dG - \ln^2 G(s)\,.$$

(6) $$\operatorname{Var}(H^{-1}(s)Q(s)) = \frac{\tilde{H}(s)(1 - \tilde{H}(s))}{H^2(s)}\,.$$

(7) $$\operatorname{Var}(\textstyle\int_0^s H^{-2}Q\, dH)$$
$$= 2\int_0^s\int_0^u \frac{\operatorname{Cov}(Q(u), Q(r))}{H^2(u)H^2(r)}\, dH(u)\, dH(r)$$
$$= 2\int_0^s\int_0^u \frac{\tilde{H}(r)(1 - \tilde{H}(u))}{H^2(u)H^2(r)}\, dH(r)\, dH(u)$$
$$= 2\int_0^s \frac{1 - \tilde{H}}{H^2}\left\{-\frac{\tilde{H}}{H} - \ln G\right\} dH$$
$$= \frac{(1 - \tilde{H}(s))\tilde{H}(s)}{H^2(s)} - \int_0^s \frac{d\tilde{H} - 2\tilde{H}\, d\tilde{H}}{H^2}$$
$$+ 2\left\{\frac{(1 - \tilde{H}(s))}{H(s)}\ln G(s) - \int_0^s \frac{1 - \tilde{H}}{HG}\, dG + \int_0^s \frac{\ln G}{H}\, d\tilde{H}\right\}.$$

$$2\,\mathrm{Cov}\,(\textstyle\int_0^s H^{-2}P\,d\tilde{H},\ H^{-1}(s)Q(s))$$

(8)
$$= \frac{2}{H(s)} \int_0^s \frac{\mathrm{Cov}\,(Q(s), P(u))}{H^2(u)}\, d\tilde{H}(u)$$

$$= \frac{2}{H(s)} \left\{ \int_0^s \frac{\tilde{H}}{H^2}\, d\tilde{H} - \tilde{H}(s) \int_0^s \frac{1}{H^2}\, d\tilde{H} + \tilde{H}(s) \int_0^s \frac{1}{H}\, d\tilde{H} \right\}.$$

$$2\,\mathrm{Cov}\,(H^{-1}(s)Q(s),\ \textstyle\int_0^s H^{-2}Q\,dH)$$

(9)
$$= \frac{2}{H(s)} \int_0^s \frac{\mathrm{Cov}\,(Q(u), Q(s))}{H^2(u)}\, dH(u)$$

$$= -\frac{2(1-\tilde{H}(s))\tilde{H}(s)}{H^2(s)} - \frac{2(1-\tilde{H}(s))}{\tilde{H}(s)} \ln G(s).$$

$$2\,\mathrm{Cov}\,(\textstyle\int_0^s H^{-2}P\,d\tilde{H},\ \int_0^s H^{-2}Q\,dH)$$

$$= 2 \int_0^s \int_0^u \frac{\mathrm{Cov}\,(P(u), Q(r))}{H^2(u)H^2(r)}\, dH(r)\, d\tilde{H}(u)$$

$$+ 2 \int_0^s \int_0^u \frac{\mathrm{Cov}\,(P(r), Q(u))}{H^2(u)H^2(r)}\, d\tilde{H}(r)\, dH(u)$$

(10)
$$= 2 \int_0^s \int_0^u \frac{\tilde{H}(r)H(u)}{H^2(u)H^2(r)}\, d\tilde{H}(u)\, dH(r)$$

$$+ 2 \int_0^s \int_0^u \frac{\tilde{H}(r) - \tilde{H}(u)(1-H(r))}{H^2(u)H^2(r)}\, dH(r)\, d\tilde{H}(u)$$

$$= \frac{2}{H(s)} \int_0^s \frac{\tilde{H}}{HG}\, dG + 2 \int_0^s \frac{\tilde{H}\, d\tilde{H}}{H^3} - \frac{2\tilde{H}(s)}{H(s)} \int_0^s \frac{dG}{HG} + \frac{2\tilde{H}(s)}{H(s)} \ln G(s)$$

$$- 2 \int_0^s \frac{\tilde{H}}{H^3}\, d\tilde{H} + 2 \int_0^s \frac{\tilde{H}}{H^2}\, d\tilde{H} - 2 \int_0^s \frac{(\ln G(s) - \ln G(r))}{HG}\, d\tilde{H}$$

$$+ 2 \int_0^s \frac{\ln G(s) - \ln G(r)}{HG}\, dG - 2 \int_0^s \frac{\tilde{H}}{H^2}\, d\tilde{H} + \ln^2 G(s).$$

Adding (1) through (10) and using the facts that $H = F_0 G$ and $d\tilde{H} = -F_0\,dG$ wherever necessary, we obtain the expression in the curly brackets of (4.9).

Covariance of $R(s)$ *and* $R(t) - R(s)$. There are 16 terms in this covariance calculation which are grouped below into 5 sets of expressions. The sum of all the expressions in each group will be equal to zero, thus showing that $\mathrm{Cov}\,(R(s), R(t) - R(s)) = 0$ for $0 < s \leq t < T$.

(A1)
$$\mathrm{Cov}\left(\frac{P(s)}{H(s)},\ \frac{P(t)}{H(t)} - \frac{P(s)}{H(s)}\right) = \frac{\mathrm{Cov}\,(P(t), P(s))}{H(t)H(s)} - \frac{\mathrm{Var}\,(P(s))}{H^2(s)} = 0.$$

(A2)
$$\mathrm{Cov}\left(\frac{P(s)}{H(s)},\ \int_s^t H^{-2}P\,d\tilde{H}\right) = H^{-1}(s) \int_s^t \frac{\mathrm{Cov}\,(P(u), P(s))}{H^2(u)}\, d\tilde{H}$$

$$= \frac{1-H(s)}{H(s)} (\ln G(s) - \ln G(t)).$$

(A3) $$\operatorname{Cov}\left(\frac{P(s)}{H(s)}, \frac{Q(t)}{H(t)} - \frac{Q(s)}{H(s)}\right) = \frac{\tilde{H}(s)}{H(s)}\left\{\frac{1}{H(s)} - 1\right\} - \frac{\tilde{H}(t)(1-H(s))}{H(s)H(t)}.$$

(A4) $$\begin{aligned}\operatorname{Cov}&\left(\frac{P(s)}{H(s)}, \int_s^t H^{-2}Q\,dH\right)\\ &= \frac{1}{H(s)}\int_s^t \frac{\operatorname{Cov}(P(s), P(u))}{H^2(u)}\,dH\\ &= H^{-1}(s)\int_s^t \frac{\tilde{H}(s) - \tilde{H}(u)(1-H(s))}{H^2(u)}\,dH\\ &= \frac{\tilde{H}(s)}{H(s)}\left\{\frac{1}{H(s)} - \frac{1}{H(t)}\right\}\\ &\qquad - \frac{(1-H(s))}{H(s)}\left\{\frac{\tilde{H}(s)}{H(s)} - \frac{\tilde{H}(t)}{H(t)} + \ln G(s) - \ln G(t)\right\}.\end{aligned}$$

(B1) $$\begin{aligned}\operatorname{Cov}&\left(\int_0^s H^{-2}P\,d\tilde{H}, \frac{P(t)}{H(t)} - \frac{P(s)}{H(s)}\right)\\ &= \frac{1}{H(t)}\int_0^s \frac{\operatorname{Cov}(P(u), P(t))}{H^2(u)}\,d\tilde{H}(u) - \frac{1}{H(s)}\int_0^s \frac{\operatorname{Cov}(P(u), P(s))}{H^2(u)}\,d\tilde{H}(u)\\ &= \int_0^s \frac{1-H(u)}{H^2(u)}\,d\tilde{H}(u) - \int_0^s \frac{1-H(u)}{H^2(u)}\,d\tilde{H}(u) = 0.\end{aligned}$$

(B2) $$\operatorname{Cov}\left(\frac{Q(s)}{H(s)}, \frac{P(t)}{H(t)} - \frac{P(s)}{H(s)}\right) = \frac{\operatorname{Cov}(Q(s), P(t))}{H(s)H(t)} - \frac{\operatorname{Cov}(Q(s), P(s))}{H^2(s)} = 0.$$

(B3) $$\begin{aligned}\operatorname{Cov}&\left(\int_0^s H^{-2}Q\,dH, \frac{P(t)}{H(t)} - \frac{P(s)}{H(s)}\right)\\ &= \frac{1}{H(t)}\int_0^s \frac{\operatorname{Cov}(P(u), Q(t))}{H^2(u)}\,dH(u) - \frac{1}{H(s)}\int_0^s \frac{\operatorname{Cov}(Q(u), P(s))}{H^2(u)}\,dH(u)\\ &= 0.\end{aligned}$$

(C1) $$\begin{aligned}\operatorname{Cov}\left(\int_0^s H^{-2}P\,d\tilde{H}, \int_s^t H^{-2}P\,d\tilde{H}\right) &= \int_0^s\int_0^t \frac{\operatorname{Cov}(P(u), P(r))}{H^2(u)H^2(r)}\,d\tilde{H}(u)\,d\tilde{H}(r)\\ &= \int_0^s \frac{(1-H(r))}{H^2(r)}\,d\tilde{H}(r)\int_s^t \frac{1}{H(u)}\,d\tilde{H}(u)\\ &= \int_0^s \frac{1-H(r)}{H^2(r)}(\ln G(s) - \ln G(t))\,d\tilde{H}(r)\\ &= (\ln G(s) - \ln G(t))\int_0^s \frac{1-H(r)}{H^2(r)}\,d\tilde{H}(r).\end{aligned}$$

$$\begin{aligned}\operatorname{Cov}&\left(\int_0^s H^{-2}P\,dH, \frac{Q(t)}{P(t)} - \frac{Q(s)}{P(s)}\right)\\ &= \frac{1}{H(s)}\int_0^s \frac{\operatorname{Cov}(Q(t), P(u))}{H^2(u)} - \frac{1}{H(s)}\int_0^s \frac{\operatorname{Cov}(Q(s), P(u))}{H^2(u)}\,d\tilde{H}(u)\end{aligned}$$

(C2)
$$= \frac{1}{H(t)} \int_0^s \frac{\tilde{H}(u) - \tilde{H}(t)(1 - H(u))}{H^2(u)} d\tilde{H}(u)$$
$$- \frac{1}{H(s)} \int_0^s \frac{\tilde{H}(u) - \tilde{H}(s)(1 - H(u))}{H^2(u)} d\tilde{H}(u) \frac{1}{H(t)} \int_0^s \frac{\tilde{H}}{H^2} d\tilde{H}$$
$$- \frac{\tilde{H}(t)}{H(t)} \int_0^s \frac{1 - H}{H^2} d\tilde{H} - \frac{1}{H(s)} \int_0^s \frac{\tilde{H}}{H^2} d\tilde{H} + \frac{\tilde{H}}{H} \int_0^s \frac{1 - H}{H^2} d\tilde{H} .$$

$$\operatorname{Cov}\left(\int_0^s H^{-2}P\, d\tilde{H}, \int_s^t H^{-2}Q\, dH\right)$$
$$= \int_0^s \int_s^t \frac{\operatorname{Cov}(Q(r), P(u))}{H^2(r)H^2(u)} dH(r)\, d\tilde{H}(u)$$
(C3)
$$= \int_0^s \int_s^t \frac{\tilde{H}(u) - \tilde{H}(r)(1 - H(u))}{H^2(u)H^2(r)} dH(r)\, d\tilde{H}(u)$$
$$= \left\{\frac{1}{H(s)} - \frac{1}{H(t)}\right\} \int_0^s \frac{\tilde{H}\, d\tilde{H}}{H^2}$$
$$+ \left\{\frac{\tilde{H}(t)}{H(t)} - \frac{\tilde{H}(s)}{H(s)} + \ln G(t) - \ln G(s)\right\} \int_0^s \frac{1 - H}{H^2} d\tilde{H} .$$

(D1)
$$\operatorname{Cov}\left(H^{-1}(s)Q(s), \int_s^t H^{-2}P\, d\tilde{H}\right) = H^{-1}(s) \int_s^t \frac{\operatorname{Cov}(Q(s), P(u))}{H^2(u)} d\tilde{H}(u)$$
$$= \frac{\tilde{H}(s)}{H(s)} (\ln G(s) - \ln G(t)) .$$

(D2)
$$\operatorname{Cov}\left(H^{-1}(s)Q(s), \frac{Q(t)}{H(t)} - \frac{Q(s)}{H(s)}\right) = \frac{\operatorname{Cov}(Q(s), Q(t))}{H(s)H(t)} - \frac{\operatorname{Cov}(Q(s), Q(s))}{H^2(s)}$$
$$= \frac{\tilde{H}(s)(1 - \tilde{H}(t))}{H(s)H(t)} - \frac{\tilde{H}(s)(1 - \tilde{H}(s))}{H^2(s)} .$$

$$\operatorname{Cov}\left(\frac{Q(s)}{H(s)}, \int_s^t H^{-2}Q\, dH\right)$$
(D3)
$$= \frac{1}{H(s)} \int_0^t \frac{\operatorname{Cov}(Q(s), Q(u))}{H^2(u)} dH(u)$$
$$= \frac{\tilde{H}(s)}{H(s)} \int_s^t \frac{1 - \tilde{H}}{H^2} dH$$
$$= \frac{\tilde{H}(s)}{H(s)} \left\{\frac{1}{H(s)} - \frac{1}{H(t)} + \frac{\tilde{H}(t)}{H(t)} - \frac{\tilde{H}(s)}{H(s)} + \ln G(t) - \ln G(s)\right\} .$$

$$\operatorname{Cov}\left(\int_0^s H^{-2}Q\, dH, \int_t^s H^{-2}P\, d\tilde{H}\right)$$
$$= \int_0^s \int_s^t \frac{\operatorname{Cov}(P(r), Q(u))}{H^2(r)H^2(u)} d\tilde{H}(r)\, dH(u)$$
(E1)
$$= \int_0^s \int_s^t \frac{H(r)\tilde{H}(u)}{H^2(r)H^2(u)} d\tilde{H}(r)\, dH(u)$$
$$= -\{-\ln G(t) + \ln G(s)\} \left\{\frac{\tilde{H}(s)}{H(s)} + \ln G(s)\right\} .$$

$$\operatorname{Cov}\left(\int_0^s H^{-2}Q\,dH,\ \frac{Q(t)}{H(t)} - \frac{Q(s)}{H(s)}\right)$$

$$\text{(E2)} \qquad = \frac{1}{H(t)} \int_0^s \frac{\operatorname{Cov}(Q(t), Q(u))}{H^2(u)}\,dH(u) - \frac{1}{H(s)} \int_0^s \frac{\operatorname{Cov}(Q(s), Q(u))}{H^2(u)}\,dH(u)$$

$$= \frac{1-\tilde{H}(t)}{H(t)} \int_0^s \frac{\tilde{H}\,dH}{H^2} - \frac{1-\tilde{H}(s)}{H} \int_0^s \frac{\tilde{H}}{H^2}\,dH\,.$$

$$\operatorname{Cov}\left(\int_0^s H^{-2}Q\,dH,\ \int_s^t H^{-2}Q\,dH\right)$$

$$= \int_0^s \int_s^t \frac{\operatorname{Cov}(Q(r), Q(u))}{H^2(u)H^2(r)}\,dH(u)\,dH(r)$$

$$\text{(E3)} \qquad = \int_0^s \left(\int_s^t \frac{1-\tilde{H}(u)}{H^2(u)}\,dH(u)\right) \frac{\tilde{H}(r)}{H(r)}\,dH(r)$$

$$= \left\{\frac{1}{H(s)} - \frac{1}{H(t)}\right\} \int_0^s \frac{\tilde{H}}{H^2}\,dH + \left(\int_0^s \frac{\tilde{H}}{H^2}\,dH\right)$$

$$\times \left\{\frac{\tilde{H}(t)}{H(t)} - \frac{\tilde{H}(s)}{H(s)} + \ln G(t) - \ln G(s)\right\}.$$

REFERENCES

[1] Breslow, N. and Crowley, J. (1974). A large sample study of the life table and product limit estimates under random censorship. *Ann. Statist.* **2** 437–453.

[2] Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1956). Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. *Ann. Math. Statist.* **27** 642–669.

[3] Efron, B. (1967). The two sample problem with censored data. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **4** 831–853, Univ. of California Press.

[4] Ferguson, T. S. (1973). A Bayesian analysis of some nonparametric problems. *Ann. Statist.* **1** 209–230.

[5] Ferguson, T. S. (1974). Prior distributions on spaces of probability measures. *Ann. Statist.* **2** 615–629.

[6] Gross, Alan J. and Clark, Virginia A. (1975). *Survival Distributions: Reliability Applications in the Biomedical Sciences.* Wiley, New York.

[7] Kaplan, E. L. and Meier, P. (1958). Non-parametric estimation from incomplete observations. *J. Amer. Statist. Assoc.* **53** 457–481.

[8] Lindley, D. V. (1972). *Bayesian Statistics, A Review.* Regional Conf. Series in Applied Math., SIAM, Philadelphia.

[9] Shapiro, C. P. (1975). Bayesian classification: Asymptotic results. *Ann. Statist.* **2** 763–774.

[10] Singh, Radhey S. (1975). On the Glivenko-Cantelli theorem for weighted empiricals based on independent random variables. *Ann. Probability* **3** 371–374.

[11] Susarla, V. and Van Ryzin, J. (1976). Nonparametric Bayesian estimation of survival curves from incomplete observations. *J. Amer. Statist. Assoc.* **61** 897–902.

Department of Mathematics
University of Wisconsin—Milwaukee
Milwaukee, Wisconsin 53701

Department of Mathematics
University of California—Davis
Davis, California 95616