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CONVERGENCE AND REMAINDER TERMS IN LINEAR RANK STATISTICS[1]

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A new approach to the asymptotic normality of simple linear rank statistics for the regression case studied earlier by Hájek (1968) is provided along with the estimation of the remainder term in the approximation to normality.

1. Introduction and summary. Let $X_1, \cdots, X_n$ be independent random variables having continuous cdf's (cumulative distribution functions) $F_1(x), \cdots, F_n(x)$ respectively. Consider a statistic $S_n = s(X_1, \cdots, X_n)$ with $ES_n = 0$ and $ES_n^2 < \infty$. Then, to prove the asymptotic normality of S_n (as $n \to \infty$), Hájek (1968) uses the method of projection which gives to the statistic S_n, the approximation of the form

$$\hat{S}_n = \sum_{j=1}^n E[S_n \mid X_j] \,. \tag{1.1}$$

Consider now the simple linear rank statistic S_n introduced by Hájek (1962, 1968)

$$S_n = \sum_{j=1}^n c_j\{\psi(R_j/n) - E[\psi(R_j/n)]\} \tag{1.2}$$

where the c's are known constants, R_j is the rank of X_j among $(X_1, \cdots, X_n)$ and $\psi(\cdot)$ is a score generating function defined on $(0, 1)$. Hájek (1962) [see also Hájek–Šidák (1967)] established the asymptotic normality of S_n in (1.2) under the assumption that the F_i are contiguous, e.g., when $F_i(x) = F(x - \Delta d_{ni})$ where Δ is the unknown parameter and the d's are the known constants. Later on Hájek (1968) studied the asymptotic normality of S_n for the general $F_i(x)$ (the noncontiguous case). Under the setup of Hájek (1962), Jurečková and Puri (1975), referred to hereafter as JP, studied the problem of determining the rate of convergence of the cdf of S_n to the limiting normal cdf and established it of order $O(N^{-\frac{1}{2}+\delta})$ for $\delta > 0$. In this paper we not only give a new approach to the asymptotic normality of S_n for the general F_i (i.e., not necessarily contiguous) but improve the results of JP in providing a sharper bound (for the general F_i's). In the passing, we may also mention that whereas JP requires ψ to have a bounded fourth derivative, here we only require the boundedness of the second

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derivative. Furthermore whereas this paper gives more explicit error bounds than the JP paper, the latter gives more information on the limiting behavior of ES_n and Var S_n.

We now introduce some notations. We define $\psi(\cdot) = 0$ outside (0, 1). Then, we can use the supremum norm

$$(1.3) \qquad ||\psi|| = \sup_{t \in (-\infty,\infty)} |\psi(t)| \, .$$

Set

$$(1.4) \qquad \rho_i = R_i/n \, , \qquad \rho_{ii} = E[\rho_i \,|\, X_i] \, , \qquad u(x) = 1 \quad \text{if} \quad x \geqq 0$$
$$\text{and} \qquad u(x) = 0 \quad \text{otherwise.}$$

Then

$$(1.5) \qquad R_i = \textstyle\sum_{j=1}^{n} u(X_i - X_j) \, .$$

In this paper, we shall deal with the following approximation of S_n.

$$(1.6) \qquad T_n = \textstyle\sum_{i=1}^{n} c_i\{\psi(\rho_{ii}) - E[\psi(\rho_{ii})] + (\rho_i - \rho_{ii})\psi'(\rho_{ii})\} \, ,$$

assuming that ψ' exists on (0, 1) and

$$(1.7) \qquad \hat{T}_n = \textstyle\sum_{j=1}^{n} E[T_n \,|\, X_j] \, .$$

Since $E[(\rho_i - \rho_{ii})\psi'(\rho_{ii})] = 0$, it follows that

$$(1.8) \qquad \hat{T}_n = \textstyle\sum_{i=1}^{n} c_i\{\psi(\rho_{ii}) - E[\psi(\rho_{ii})] + \sum_{j \neq i}^{n} E[(\rho_i - \rho_{ii})\psi'(\rho_{ii}) \,|\, X_j]\} \, .$$

Let H_n, G_n and $\hat{G}_n$ be the cdf's of S_n, T_n and $\hat{T}_n$ respectively, and put

$$(1.9) \qquad \sigma_n^2 = E[S_n^2] \, , \qquad \hat{\delta}_n^2 = E[\hat{T}_n^2] \, , \qquad \Gamma_{nr}^{2r} = \frac{1}{n} \textstyle\sum_{i=1}^{n} c_i^{2r} \, , \qquad \Gamma_{nr} > 0 \, .$$

Then our theorems are the following:

THEOREM 1.1. *If ψ has a derivative on* (0, 1) *then*

$$(1.10) \qquad ||\hat{G}_n(\hat{\delta}_n \cdot) - \Phi(\cdot)|| \leqq 4C[2||\psi||^3 + ||\psi'||^3] \textstyle\sum_{i=1}^{n} |c_i|^3 \hat{\delta}_n^{-3} \, ;$$
$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \textstyle\int_{-\infty}^{x} e^{-t^2/2} \, dt$$

where C is the constant in Berry–Esseen's inequality (Zolotarev (1967) gives the approximation 0.9051). *Further,*

$$(1.11) \qquad |\hat{\delta}_n - \sigma_n| \leqq C_1(||\psi'|| + ||\psi''||)\Gamma_{n,1}$$

with an absolute constant C_1, provided ψ'' exists on (0, 1).

THEOREM 1.2. *If ψ has a second order derivative on* (0, 1), *then for any positive integers n and r such that $n^{-1}r^3 \leqq \frac{3}{8}$,*

$$(1.12) \qquad ||H_n(\hat{\delta}_n \cdot) - \Phi(\cdot)|| \leqq 4C(2||\psi||^3 + ||\psi'||^3) \textstyle\sum_{i=1}^{n} |c_i|^3 \hat{\delta}_n^{-3}$$
$$+ C_2[\hat{\delta}_n^{-1}(||\psi'|| + ||\psi''||) r \Gamma_{nr}]^{2r/(2r+1)} \, ,$$

where C_2 is an absolute constant.

REMARK. If the c_i are chosen such that $|c_i| \leqq a/n^{\frac{1}{2}}$ with constant a for all

i and n, then

$$\Gamma_{nr} \leqq a/n^{\frac{1}{2}} ,$$

and for $r = [\log n]$, $[r\Gamma_{nr}]^{2r/(2r+1)} \leqq a\sqrt[4]{e}\,(\log n)\, n^{-\frac{1}{2}}(1 + O(1/\log n))$.

Note that $\hat{\delta}_n^{-1}c_i$ is invariant and thus also $\hat{\delta}_n^{-1}\Gamma_{nr}$ is invariant under the transformation $c_i \rightarrow \gamma c_i$, $i = 1, 2, \cdots$.

2. Some lemmas.

LEMMA 2.1. *For any positive integers r and n, $2r \leqq n$, we have*

$$E[(\rho_i - \rho_{ii})^{2r}] \leqq b(r)n^{-r} \tag{2.1}$$

with

$$b(r) \leqq n^{-r} \sum_{t=1}^{r} \binom{n-1}{t} \frac{(2r)!}{(2r-2t)!}\, t^{2r-2t} \cdot 2^{-3t} \tag{2.2}$$

and for $n^{-1}r^3 \leqq \frac{3}{4}$

$$b(r) \leqq 2^{-3r} \frac{(2r)!}{r!} [1 + 8n^{-1}r^3] . \tag{2.3}$$

PROOF. By (1.4) we obtain

$$\rho_i - \rho_{ii} = \frac{1}{n} \sum_{j\neq i}^{n} [u(X_i - X_j)F_j(X_i)] .$$

By the polynomial theorem we then get

$$E[(\rho_i - \rho_{ii})^{2r}] = n^{-2r} \sum \frac{(2r)!}{s_1! \cdots s_n!} E \prod_{j\neq 1}^{n} [u(X_i - X_j) - F_j(X_i)]^{s_j} , \tag{2.4}$$
$$s_1 + \cdots + s_n = 2r .$$

We claim that any term in this sum is equal to zero if $s_{j_0} = 1$ for some j_0. Indeed we find that the conditional expection of the product with respect to all X_j, $j \neq j_0$ is equal to 0 if $s_{j_0} = 1$. Hence we have only to regard terms with $s_j = 0$ or $\geqq 2$ for any j, and there can be at most $t \leqq r$ exponents s_j different from 0. If $s_j \geqq 2$, $j = 1, 2, \cdots, t$, $s_j = 0$ for $j > t$, $i > t$ we obtain, observing that

$$|u(X_i - X_j) - F_j(X_i)| \leqq 1$$

$$\begin{aligned} E[\textstyle\prod_{j=1}^{t} [u(X_i - X_j) - F_j(X_i)]^{s_j} &\leqq E \textstyle\prod_{j=1}^{t} [u(X_i - X_j) - F_j(X_i)]^2 \\ &= E[\textstyle\prod_{j=1}^{t} [F_j(X_i) - F_j^2(X_i)]] \leqq 4^{-t} . \end{aligned} \tag{2.5}$$

This inequality remains true for all permutations of the indices $1, \cdots, n$. Put

$$\gamma(t) = \sum_{s_1+\cdots+s_t=2r;\, s_j\geqq 2,\, j=1,\cdots,t} \frac{(2r)!}{s_1! \cdots s_t!} . \tag{2.6}$$

Since t indices out of $n - 1$ indices can be chosen in $\binom{n-1}{t}$ different ways we obtain from (2.4) through (2.6),

$$E[(\rho_i - \rho_{ii})^{2r}] \leqq n^{-2r} \sum_{t=1}^{r} \binom{n-1}{t}\gamma(t)4^{-t} . \tag{2.7}$$

We claim that

$$\gamma(t) \leqq \frac{(2r)!}{(2r-2t)!} 2^{-t} t^{2r-2t} . \tag{2.8}$$

Indeed, differentiating the identity

$$\left(\sum_{j=1}^{t} y_j\right)^{2r} = \sum_{s_1+\cdots+s_t=2r} \frac{(2r)!}{s_1! \cdots s_t!} \prod_{j=1}^{t} y_j^{s_j}$$

twice with respect to all y_j and then putting all y_j equal to 1, we obtain

$$\frac{(2r)!}{(2r-2t)!} t^{(2r-2t)} = \sum_{s_1+\cdots+s_t=2r;\, s_j \geqq 2,\, j=1\cdots t} \prod_{j=1}^{t} s_j(s_j-1) \frac{(2r)!}{s_1! \cdots s_t!} .$$

Now using (2.7) and (2.8), we get (2.1) and (2.2). We now estimate $b(r)$ further, mainly for use when n and r are large. Put $r - t = u$. Then we can write

$$b(r) \leqq 2^{-3r} \sum_{u=0}^{r-1} k(u) \tag{2.9}$$

with

$$k(u) = \frac{n^{-u}(2r)!\,(r-u)^{2u} 2^{3u}}{(r-u)!\,(2u)!} .$$

Particularly

$$k(0) = \frac{(2r)!}{r!} , \qquad k(1) < 4n^{-1} r^3 \cdot \frac{(2r)!}{r!}$$

and for $u \geqq 1$

$$\frac{k(u+1)}{k(u)} = n^{-1}\left(1 - \frac{1}{r-u}\right)^{2u} \cdot 2^3 \cdot (r-u) \frac{(r-u-1)^2}{(2u+1)(2u+2)}$$

$$< \tfrac{2}{3} n^{-1} r^3 \leqq \tfrac{1}{2} \qquad \text{for} \quad n^{-1} r^3 \leqq \tfrac{3}{4} .$$

Hence

$$b(r) \leqq 2^{-3r} \cdot \frac{(2r)!}{r!} [1 + 8n^{-1} r^3]$$

for $n^{-1} r^3 \leqq \frac{3}{4}$.

LEMMA 2.2. *For any positive integers r and n*, $2r \leqq n$, *we have*

$$E(T_n - \hat{T}_n)^{2r} \leqq c(r) ||\psi||^{2r} \Gamma_{n,r}^{2r} \tag{2.10}$$

if ψ' *exists on* (0, 1), *and if* ψ'' *exists on* (0, 1)

$$E[(S_n - T_n)^{2r}] \leqq b(2r) ||\psi''||^{2r} \Gamma_{n,r}^{2r} , \tag{2.11}$$

$$E[(S_n - \hat{T}_n)^{2r}] \leqq d(r, \psi) \Gamma_{n,r}^{2r} \tag{2.12}$$

with

$$b(2r) \leqq n^{-2r} \sum_{t=1}^{2r} \binom{n-1}{t} \frac{(4r)!}{(4r-2t)!} t^{4r-2t} \cdot 2^{-3t}$$

$$c(r) \leqq 2^{2r} n^{-2r} \sum_{t=1}^{2r} \binom{n}{t} \frac{(4r)!}{(4r-2t)!} t^{4r-2t} \cdot 2^{-t}$$

$$d(r, \psi) \leqq [[b(2r)]^{1/2r} ||\psi''|| + [c(r)]^{1/2r} ||\psi'||]^{2r} .$$

Further we have the estimates

$$b(2r) \leqq 2^{-6r}\frac{(4r)!}{(2r)!}[1 + 2^6 n^{-1} r^3] \tag{2.13}$$

for $2^3 n^{-1} r^3 \leqq \frac{3}{4}$,

$$c(r) \leqq \frac{(4r)!}{(2r)!}[1 + 2^3 n^{-1} r^3] \qquad \textit{for} \quad n^{-1} r^3 \leqq \tfrac{3}{8}\,. \tag{2.14}$$

REMARK. By Stirling's approximation of the Γ-function we have

$$\frac{(4r)!}{(2r)!} \leqq 2^{6r+\frac{1}{2}} r^{2r} (\exp -2r) \exp \frac{1}{48r}\,.$$

PROOF. By (1.6) and (1.8) we get

$$T_n - \hat{T}_n = \sum_{i=1}^n c_i \{(\rho_i - \rho_{ii})\psi'(\rho_{ii}) - \sum_{j=1; j\neq i}^n E[(\rho_i - \rho_{ii})\psi'(\rho_{ii}) \mid X_j]\} \tag{2.15}$$

and for $j \neq i$

$$\begin{aligned} E[(\rho_i - \rho_{ii})\psi'(\rho_{ii}) \mid X_j] &= \frac{1}{n}\sum_{k\neq i}^n E\{[u(X_i - X_k) - F_k(X_i)]\psi'(\rho_{ii}) \mid X_j\} \\ &= \frac{1}{n} E[u(X_i - X_j) - F_j(X_i)]\psi'(\rho_{ii}) \mid X_j]\,, \end{aligned} \tag{2.16}$$

since the conditional expectations in the sum are zero for $j \neq k, i$. Now using the relation

$$(\rho_i - \rho_{ii})\psi'(\rho_{ii}) = \frac{1}{n}\sum_{j\neq i}^n [u(X_i - X_j) - F_j(X_i)]\psi'(\rho_{ii})\,,$$

and noting that

$$E[(\rho_i - \rho_{ii})\psi'(\rho_{ii}) \mid X_i] = 0$$

we obtain from (2.15)

$$T_n - \hat{T}_n = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}^n c_i V_{ij} \tag{2.17}$$

with

$$\begin{aligned} V_{ij} = {} & [u(X_i - X_j) - F_j(X_i)]\psi'(\rho_{ii}) \\ & - E\{[u(X_i - X_j) - F_j(X_i)]\psi'(\rho_{ii}) \mid X_j\}\,. \end{aligned} \tag{2.18}$$

Clearly

$$E[V_{ij} \mid X_j] = 0\,, \qquad E[V_{ij} \mid X_i] = 0\,. \tag{2.19}$$

By the polynomial theorem we get

$$\begin{aligned} E[(T_n - \hat{T}_n)^{2r}] &= n^{-2r} E[\sum_{i=1}^n \sum_{j\neq i}^n c_i V_{ij}]^{2r} \\ &= n^{-2r} \sum \frac{(2r)!}{\prod_{i=1}^n \prod_{j\neq i}^n (s_{ij}!)} E\{\prod_{i=1}^n \prod_{j\neq i}^n (c_i V_{ij})^{s_{ij}}\} \end{aligned} \tag{2.20}$$

where the sum should be taken over terms corresponding to different vector solutions $\{s_{ij}\}$, $i, j = 1, \cdots n$, $j \neq i$ of the equation

$$\sum_{i=1}^n \sum_{j\neq i}^n s_{ij} = 2r\,. \tag{2.21}$$

The expectation

$$E[\textstyle\prod_{i=1}^{n} \prod_{j\neq i}^{n} V_{ij}{}^{s_{ij}}] \tag{2.22}$$

is equal to 0 for some vector solutions of (2.21) since (2.19) holds, and we have only to regard those solutions for which the expectation (2.22) is not equal to 0.

We say that s_{ij} gives the contribution $\frac{1}{2}s_{ij}$ to the sum (2.21) from each of the indices i and j. Hence according to this notation an index k gives the contribution

$$g(k) = \tfrac{1}{2} \textstyle\sum_{j\neq k}^{n} s_{kj} + \tfrac{1}{2} \sum_{j\neq k}^{n} s_{jk} \tag{2.23}$$

to the sum (2.21). By conditioning with respect to all $X_j, j \neq k$ we easily find that the expectation (2.22) is equal to 0 if k gives the contribution $\frac{1}{2}$ to the sum (2.21), i.e., if $s_{kj} = 1$ for exactly one index $j \neq k$, and $s_{jk} = 0$ for $j \neq k$ or if $s_{jk} = 1$ for exactly one j and $s_{kj} = 0$ for $j \neq k$.

The sum $\sum$ on the right-hand side of (2.20) can be divided into partial sums as follows. Let C be a collection of different positive integers belonging to the set $1, \cdots, 2r$, say $C = (1, 2, \cdots, t)$. Let $\sum_C$ consist of all terms in (2.20) corresponding to the vector solutions of (2.21) such that

(a) $s_{ij} = 0$ if not both i and j belong to C;

(b) for any $k \in C$ the contribution to the sum (2.21) is larger than $\frac{1}{2}$. Note that C can contain at most 2r different integers since every $k \in C$ gives at least the contribution 1 to the sum (2.21). Clearly partial sums $\sum_{C_1}$ and $\sum_{C_2}$ contain no common terms if $C_1 \neq C_2$. Consider now the expectation

$$E[\textstyle\prod_{i=1}^{t} \prod_{j\neq i}^{t} (c_i V_{ij})^{s_{ij}}]$$

where the i and j belong to the collection C. Note that s_{ij} may be equal to 0 for some pairs (i, j). By Hölder's inequality we get, using the fact that $|V_{ij}| \leqq 2||\psi'||$,

$$\begin{aligned} |E \textstyle\prod_{i=1}^{t} \prod_{j\neq i}^{t} (c_i V_{ij})^{s_{ij}}| &\leqq \textstyle\prod_{i=1}^{t} \prod_{j\neq i}^{t} |c_i|^{s_{ij}} \{E[(V_{ij})^{2r}]\}^{s_{ij}/2r} \\ &\leqq 2^{2r} ||\psi'||^{2r} \textstyle\prod_{i=1}^{t} |c_i|^{s_i} \end{aligned} \tag{2.24}$$

where

$$s_i = \textstyle\sum_{j=1}^{t} s_{ij} , \qquad \sum_{i=1}^{t} s_i = 2r . \tag{2.25}$$

The partial sum corresponding to C is then estimated by

$$\textstyle\sum_C' \dfrac{(2r)!}{\prod_{i=1}^{t} \prod_{i\neq j}^{t} (s_{ij})!} (2^{2r} ||\psi'||^{2r} \prod_{i=1}^{t} |c_i|^{s_i}) . \tag{2.26}$$

Note that $(2r)!/\prod_{i=1}^{t} \prod_{i\neq j}^{t} (s_{ij})!$ is an integer. Hence we have

$$N(t) = \textstyle\sum_C' \dfrac{(2r)!}{\prod_{i=1}^{t} \prod_{j\neq i}^{t} (s_{ij})!}$$

terms in the class C which are estimated by (2.24). Let $\mathscr{C}_t$ be the set of all terms

$$\textstyle\sum \prod_{i=1}^{n} \prod_{j\neq 1}^{n} (c_i V_{ij})^{s_{ij}}$$

in (2.23) which belong to some class C containing exactly t indices. Let $(s_1, s_2, \cdots, s_t)$ in (2.26) be given, $0 \leqq s_1 \leqq s_2 < \cdots \leqq s_t$, $\sum_{i=1}^t s_i = 2r$. Then according to the symmetry the set $\mathscr{C}_t$ contains a sum of terms, each estimated by

$$2^{2r}\|\psi'\|^{2r} \prod_{i=1}^t |c_{k_i}|^{s_i} \tag{2.27}$$

where $(k_1 \cdots k_t)$ is any combination of numbers $1, 2, \cdots, n$ to the tth class and in any order within this class. Let the number of terms in C_t for a fixed vector $(s_1, s_2, \cdots, s_t)$ as above be $n(t)$ and the sum of terms (2.27) belonging to $(s_1, s_2, \cdots, s_t)$ be $A(s_1, s_2, \cdots, s_t)$. (Note that $n(t)$ depends on $s_1, \cdots, s_t$.) Then, since $A(s_1, \cdots, s_t)$ is a symmetrical function

$$A(s_1, s_2, \cdots, s_t) = \frac{n(t)}{n!}\sum{}' 2^{2r}\|\psi'\|^{2r} \prod_{i=1}^t |c_{k_i}|^{s_i} \tag{2.28}$$

where $\sum'$ is the sum all terms belonging to all permutations of the numbers $1, 2, \cdots, n$. By Hölder's inequality we get, observing that

$$|c_{k_i}|^{s_i} = [c_{k_i}^{2r}]^{s_i/2r}, \qquad \sum_{i=1}^t \frac{s_i}{2r} = 1,$$

$$\sum{}' \prod_{i=1}^t |c_{k_i}|^{s_i} \leqq \prod_{i=1}^t \left(\sum{}' c_{k_i}^{2r}\right)^{s_i/2r} \tag{2.29}$$

and here

$$\sum{}' c_{k_i}^{2r} = \frac{n!}{n} \sum_{i=1}^n c_i^{2r}.$$

Hence we obtain by (2.28) and (2.29)

$$A(s_1, s_2, \cdots, s_t) \leqq 2^{2r}\|\psi'\|^{2r} \cdot n(t) \cdot \frac{1}{n} \sum_{i=1}^n c_i^{2r}.$$

Since $\mathscr{C}_t$ contains $\binom{n}{t}N(t)$ terms we then find that $\mathscr{C}_t$ gives at most the contribution

$$n^{-2r}2^{2r}\|\psi'\|^{2r}\binom{n}{t}N(t) \cdot \frac{1}{n} \sum_{i=1}^n c_i^{2r}$$

to the right-hand side of (2.20). Putting

$$\Gamma_{nr}^{2r} = \frac{1}{n} \sum_{i=1}^n c_i^{2r}, \qquad \Gamma_{nr} \geqq 0,$$

and regarding the sets $\mathscr{C}_t$ for $t = 1, 2, \cdots, 2r$, we obtain from (2.20) that

$$E[(T_n - \hat{T}_n)^{2r}] \leqq 2^{2r}n^{-2r}\|\psi'\|^{2r}\Gamma_{nr}^{2r} \sum_{t=1}^{2r}\binom{n}{t}N(t). \tag{2.30}$$

We estimate $N(t)$ in the following way. Consider the identity

$$\left(\sum_{i=1}^t \sum_{j=i}^t x_i x_j\right)^{2r} = \sum \frac{(2r)!}{\prod_{i=1}^t \prod_{j=1}^t (s_{ij})!} \prod_{i=1}^t \prod_{j\neq i}^t (x_i x_j)^{s_{ij}}. \tag{2.31}$$

If an index k gives the contribution $\geqq 1$ to the sum (2.21), i.e., to the sum

$$\sum_{i=1}^t \sum_{j\neq i}^t s_{ij} = 2r,$$

then the double product

$$\prod_{j=1}^{t} \prod_{j\neq i}^{t} (x_i x_j)^{s_{ij}}$$

contains x_k as factor at least in the power 2. Hence differentiating the identity twice with respect to each x_k, $k = 1, 2, \cdots, t$ and then putting all x_n equal to 1 we get the inequality

$$2^t N(t) \leqq \left\{\prod_{k=1}^{t} \frac{\partial^2}{\partial_{x_k}} \left(\sum_{i=1}^{t} \sum_{j\neq i}^{t} x_i x_j\right)^{2r}\right\}_{x_k=1,k=1,2,\cdots,t} . \tag{2.32}$$

The right-hand side, however, is at most equal to

$$\left\{\prod_{k=1}^{t} \frac{\partial^2}{\partial x_k} \left(\left(\sum_{i=1}^{t} x_i\right)^{4r}\right)\right\}_{x_k=1,k=1,\cdots,t} = \frac{(4r)!}{(4r-2t)!}\, t^{4r-2t} . \tag{2.33}$$

Combining (2.30), (2.32) and (2.33), we get

$$E[(T_n - \hat{T}_n)^{2r}] \leqq c(r)\|\psi'\|^{2r}\Gamma_{nr}^{2r}$$

with

$$c(r) = 2^{2r} n^{-2r} \sum_{t=1}^{2r} \binom{n}{t} \frac{(4r)!}{(4r-2t)!}\, t^{4r-2t} \cdot 2^{-t}$$

$$\Gamma_{nr}^{2r} = \frac{1}{n} \sum_{i=1}^{n} |c_i|^{2r} .$$

We estimate $c(r)$ exactly in the same way as we have estimated $b(r)$ in Lemma 2.1 and then obtain for $u = 2r - t$

$$c(r) \leqq \sum_{u=0}^{r-1} k(u)$$

with

$$k(u) = n^{-u} \frac{(4r)!}{(2u)!\,(2r-u)!} (2r-u)^{2u} \cdot 2^u .$$

Hence

$$k(0) = \frac{(4r)!}{(2r)!} , \qquad k(1) < n^{-1} \cdot (2r)^3 \frac{(4r)!}{(2r)!}$$

and for $u \geqq 1$

$$\frac{k(u+1)}{k(u)} \leqq \tfrac{4}{3} n^{-1} r^3 \leqq \tfrac{1}{2} \qquad \text{for} \quad n^{-1} r^3 \leqq \tfrac{3}{8} .$$

Hence for $n^{-1} r^3 \leqq \frac{3}{8}$

$$c(r) \leqq \frac{(4r)!}{(2r)!} [1 + 8n^{-1} r^3] .$$

Thus we have proved (2.13) and (2.14) of the lemma.

It follows by the definition of T_n that

$$S_n - T_n = \sum_{i=1}^{n} c_i[\xi_i - E(\xi_i)]$$

with

$$|\xi_i| \leqq \tfrac{1}{2}(\rho_i - \rho_{ii})^2 \|\psi''\| .$$

Hence

$$E[(S_n - T_n)^{2r}] \leqq n^{2r-1} \textstyle\sum_{i=1}^n c_i^{2r} E[(\xi_i - E\xi_i)^{2r}]$$

and by Lemma 2.1

$$E[(\xi_i - E(\xi_i))^{2r}] \leqq 2^{2r} E[\xi_i^{2r}] \leqq \|\psi''\|^{2r} E[(\rho_i - \rho_{ii})^{4r}] \leqq n^{-2r} b(2r) \|\psi''\|^{2r} .$$

Thus we get (2.11)

$$E[(S_n - T_n)^{2r}] \leqq b(2r) \Gamma_{nr} .$$

By Minkovski's inequality we obtain (2.12) from (2.10) and (2.11)

$$E^{1/2r}[(S_n - \hat{T}_n)^{2r}] \leqq E^{1/2r}[(S_n - T_n)^{2r}] + E^{1/2r}[(T_n - \hat{T}_n]^{2r}] .$$

Lemma 2.3. $\hat{T}_n = \sum_{j=1}^n \hat{T}_n^{(j)}$ *with independent random variables*

$$\text{(i)} \qquad \hat{T}_n^{(j)} = c_j\{\psi(\rho_{jj}) - E[\psi(\rho_{jj})]\} + \frac{1}{n} \textstyle\sum_{i \neq j}^n c_i [E(u(X_i - X_j) - F_j(X_i)) \psi'(\rho_{ii}) \mid X_j] .$$

Further,

$$\text{(ii)} \qquad \textstyle\sum_{j=1}^n [E|\hat{T}_n^{(j)}|^3] \leqq 4[2\|\psi\|^3 + \|\psi'\|^3\} \textstyle\sum_{j=1}^n |c_i|^3 .$$

Proof. We get the representation (i) by (2.16). Using well-known inequalities

$$|(a + b)^3| \leqq 4[|a|^3 + |b|^3] , \qquad |(\textstyle\sum_{i=1}^n a_i)^3| \leqq n^2 \textstyle\sum_{i=1}^n |a_i|^3$$

we obtain

$$E[|\hat{T}_n^{(j)}|^3] \leqq 4|c_j|^3 E[|[\psi(\rho_{jj})] - E\psi(\rho_{jj})|^3] + \frac{4}{n} \textstyle\sum_{i \neq j}^n |c_i|^3 \|\psi'\|^3 .$$

Here

$$E[|\psi(\rho_{jj}) - E[\psi(\rho_{jj})]|^3] \leqq 2\|\psi\| E(\psi(\rho_{jj}) - E(\psi(\rho_{jj}))^2 .$$

Thus we get (ii).

3. Proofs of the theorems.

(a) Proof of Theorem 1.1. (1.10) follows from Berry–Esseen's inequality and Lemma 2.3 and (1.11) from Lemma 2.2 (2.12).

(b) Proof of Theorem 1.2. For $h > 0$ we get

$$\text{(3.1)} \qquad \begin{aligned} P[S_n \leqq \hat{\delta}_n x] &\leqq P(S_n \leqq \hat{\delta}_n x, |S_n - \hat{T}_n| < h\hat{\delta}_n) + P[|S_n - \hat{T}_n| \geqq h\delta_n] \\ &\leqq P[\hat{T}_n \leqq \hat{\delta}_n(x + h)] + P[|S_n - \hat{T}_n| \geqq h\hat{\delta}_n] . \end{aligned}$$

Applying Theorem 1.1 we get

$$\text{(3.2)} \qquad P[\hat{T}_n \leqq \hat{\delta}_n(x + h)] \leqq \Phi(x + h) + 4C(2\|\psi\|^3 + \|\psi'\|^3) \cdot \textstyle\sum_{i=1}^n |c_i|^3 \hat{\delta}_n^{-3} .$$

Here

$$\text{(3.3)} \qquad \Phi(x + h) \leqq \Phi(x) + \|\Phi'(x)\| = \Phi(x) + \frac{h}{(2\pi)^{\frac{1}{2}}} .$$

By Chebyshev's inequality and the inequality (2.12) of Lemma 2.2 we get

$$\text{(3.4)} \qquad P[|S_n - \hat{T}_n| \geqq h\hat{\delta}_n] \leqq d(r, \psi) \Gamma_{nr}^{2r} (h\hat{\delta}_n)^{-2r} .$$

Now we choose n such that

$$\frac{h}{(2\pi)^{\frac{1}{2}}} = d(r, \psi)\Gamma_{nr}^{2r}(h\hat{\delta}_n)^{-2r} ,$$

i.e.,

$$h = [(2\pi)^{\frac{1}{2}} d(r, \psi)\hat{\delta}_n^{-2r}\Gamma_{nr}^{2r}]^{1/(2r+1)} . \tag{3.5}$$

It follows by Lemma 2.2, (2.12), (2.13) and (2.14), and the remark made after Lemma 2.2 that for $n^{-1}r^3 \leqq \frac{3}{8}$

$$[d(r, \psi)]^{1/2r} \leqq C'r(||\psi'|| + ||\psi''||)$$

with an absolute constant C'. Then it follows by (3.4) and (3.5) that

$$\frac{h}{(2\pi)^{\frac{1}{2}}} + d(r, \psi)\Gamma_{nr}^{2r}(h\hat{\delta}_n)^{-2r} \leqq C_2[\hat{\delta}_n^{-1}(||\psi'|| + ||\psi''||)r\Gamma_{nr}]^{2r/(2r+1)} .$$

By (3.1)—(3.6) we get the inequality (1.12) in one direction. It follows for the other direction in the same way.

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