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ASSOCIATION AND PROBABILITY INEQUALITIES

BY KUMAR JOGDEO

University of Illinois

A "moving set inequality," a variant of the one considered by Anderson (1955) and Sherman (1955), is shown to yield a class of random variables whose absolute values are "associated." In particular, a model generated by "contaminated independence" forms the principal example. Further, it is proved that "concordant" functions of associated random variables are associated and then this result is applied to obtain a variety of probability inequalities related to multivariate normal and other distributions. These results generalize the ones obtained by Šidák (1967, 1968, 1971, 1973).

1. Introduction. A simple and intuitively obvious covariance inequality due to Tchebyshev (see Hardy, Littlewood, Polyà [9], pages 43–44) is somewhat neglected in the statistical literature, in spite of being a useful tool and as a result has a habit of being rediscovered (see for example [2], [3], [11] and [21]). The inequality, simply stated, asserts that if X is a random variable, (f_1, g_1) a pair of nondecreasing real functions, then

$$\text{Cov}\,[f_1(X), g_1(X)] \geqq 0\,. \tag{1.1}$$

We will be concerned with the bivariate and multivariate generalizations of (1.1). Lehmann (1966) considered a bivariate generalization and showed that *positive quadrant dependence* between (X, Y) say, is equivalent to having

$$\text{Cov}\,[f_1(X), g_1(Y)] \geqq 0\,, \tag{1.2}$$

where f_1, g_1 are nondecreasing. Further, he showed that the same covariance inequality holds between two *concordant functions* defined on independent pairs (X_i, Y_i), $i = 1, \cdots, n$ where each pair satisfies positive quadrant dependence condition. (See Section 4 for details.)

A stronger dependence condition called *association* was studied by Esary, Proschan and Walkup (1967). A k-variate random variable $\mathbf{X}$ is (or equivalently, $X_1, \cdots, X_k$ are) said to be *associated* if for every pair of nondecreasing functions f, g,

$$\text{Cov}\,[f(\mathbf{X}), g(\mathbf{X})] \geqq 0\,. \tag{1.3}$$

(Throughout the paper, a function defined on a subset of $R^k \to R$ will be said to be *nondecreasing* (nonincreasing) if it is so in each of its k arguments separately. Also $|\mathbf{x}|$ will mean $(|x_1|, \cdots, |x_k|)$, $\mathbf{x} \geqq \mathbf{y}$ will mean $x_i \geqq y_i$, $i = 1, \cdots, k$ etc.)

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The following are some of the properties which make the concept of association useful. (i) If X_i, $i = 1, \cdots, k$ are independent then they are associated. (ii) The union of independent sets of associated random variables is associated. (iii) Nondecreasing functions of associated random variables are associated. (iv) If $\mathbf{X}$ is associated, so is $-\mathbf{X}$.

An important consequence of association of $\mathbf{X}$, relevant to the present paper, is that for every $\mathbf{c}$,

$$P[\mathbf{X} \leqq \mathbf{c}] \geqq \prod_{j=1}^{m} P[X_i \leqq c_i, i \in I_j], \tag{1.4}$$

$$P[\mathbf{X} \geqq \mathbf{c}] \geqq \prod_{j=1}^{m} P[X_i \geqq c_i, i \in I_j], \tag{1.5}$$

where I_j are subsets of $I = \{1, \cdots, k\}$ whose union is I.

Inequalities of type (1.4) and (1.5) have attracted attention, particularly when $\mathbf{X}$ has a multivariate normal or t-distribution and $\mathbf{X}$ is replaced by $|\mathbf{X}|$ in (1.4) and (1.5). Dunn (1958) initiated the study in view of its applications to the so-called conservative confidence interval estimation. Šidák ([18]—[21]) has obtained a variety of results. (See [10] for a simple proof of a principal result.) In particular, he showed that for a multivariate normal random vector $\mathbf{X}$ with $E\mathbf{X} = \mathbf{0}$,

$$P[|\mathbf{X}| \leqq \mathbf{c}] \geqq \prod_{1}^{k} P[|X_i| \leqq c_i], \tag{1.6}$$

holds for every $\mathbf{c}$; however,

$$P[|\mathbf{X}| \geqq \mathbf{c}] \geqq \prod_{i=1}^{k} P[|X_i| \geqq c_i], \tag{1.7}$$

may not hold without some further restriction on the covariance structure, such as $\text{Cov}(X_i, X_j) = a_i a_j$. Das Gupta et al. (1972) considered (1.6) for elliptically contoured distributions.

The main purpose of this paper is the following. First, (1.7) is strengthened by showing that, even with a slightly more general condition on the covariance structure, the absolute values are in fact associated. The technique used yields association for a more general model of which the multivariate normal distribution is a special case. The notion of concordance is generalized to fit the concept of association and is then applied to obtain inequalities for some distributions commonly used in multivariate analysis.

For the same model, inequalities are obtained where rectangular regions are replaced by centrally symmetric convex sets. These inequalities were obtained by Das Gupta et al. (1971) and Šidák (1973) for multivariate normal distributions by a method more involved than the present.

The basic tool used in the paper is a *moving set inequality*. This is a variant of an inequality obtained by Anderson (1955) and Sherman (1955) in which central symmetry was used. (Earlier, Fary and Redei (1949) had obtained the same inequality in another context.) Mudholkar (1966) pointed out that other concepts of symmetry could be employed to obtain more general inequalities. Recently, Rinott (1973), Eaton and Perlman (1974) and Marshall and Olkin

(1974) have employed a moving set inequality based on Schur convexity (permutation group).

In the present paper, the symmetry is expressed through sign invariance, and a vector $\mathbf{y}$ is said to majorize $\mathbf{x}$ if $|\mathbf{y}| \geqq |\mathbf{x}|$.

In Section 2, some basic inequalities are developed. Section 3 contains the main results on association while Section 4 develops theorems on concordant functions which yield the association property for several statistics.

2. Moving set inequality. It is well known that a symmetric unimodal distribution on the real line assigns maximum mass to an interval of fixed length, when the center of the interval is at the origin. The mass decreases as the interval moves away in either direction. The inequality developed here is obtained by requiring each *linear* section of the multivariate density along certain directions to be unimodal and then splicing the above phenomena together.

DEFINITION 2.1. A function $f\colon R^n \to R$ is said to be *increasing* (decreasing) *in absolute value* if f is sign invariant and $|\mathbf{v}| \geqq |\mathbf{u}|$ implies $f(\mathbf{v}) \geqq f(\mathbf{u})$ ($f(\mathbf{v}) \leqq f(\mathbf{u})$). A set S in R^n is said to be *increasing* (*decreasing*) *in absolute value* if the indicator function of S is increasing (decreasing) in absolute value.

REMARK 2.1. A probability density f is decreasing in absolute value if and only if every set $\{\mathbf{x} : f(\mathbf{x}) \geqq c\}$, $c > 0$, is decreasing in absolute value. Thus f can be considered as a mixture of uniform distributions on sets which are decreasing in absolute value.

The following is an important example of a density function decreasing in absolute value. Let $X_1, \cdots, X_n$ be independent real random variables each having symmetric unimodal distribution. Then by a characterization of Khintchine (1938) each of the distributions is a mixture of uniform distributions on symmetric intervals. Thus the joint density of $\mathbf{X} = (X_1, \cdots, X_n)$ can be viewed as a mixture of uniform distributions on n-dimensional rectangles, centered at the origin, with edges parallel to the axes. Hence the joint density is decreasing in absolute value.

THEOREM 2.1. *If f_1 and f_2 are densities in R^n, each decreasing in absolute value, then the convolution $f_1 * f_2$ is also decreasing in absolute value.*

PROOF. Sign invariance of $f_1 * f_2$ can be verified rather easily. To check the decreasing property first assume f_1 and f_2 to be indicators of sets C_1 and C_2, both decreasing in absolute value. It follows that

$$f_1 * f_2(\mathbf{y}) = \int_{C_2+\mathbf{y}} I[C_1; \mathbf{x}]\, d\mathbf{x}\,, \tag{2.1}$$

where I denotes the indicator. To see that the right side of (2.1) is decreasing in absolute value, let $\mathbf{u} = (u_1, 0, 0, \cdots, 0)$ and $\mathbf{v} = (v_1, 0, 0, \cdots, 0)$ where $|v_1| > |u_1|$. The right side of (2.1) can be viewed as the volume of the intersection of C_1 and the set obtained by translating C_2 by $\mathbf{y}$. Due to the decreasing property of the sets C_1, C_2 it is clear that the volume obtained when $\mathbf{y}$ is replaced by $\mathbf{u}$ is more than the corresponding volume when $\mathbf{y}$ is replaced by $\mathbf{v}$. The sign invariance and the decreasing property of C_1 and translated C_2, with respect to remaining $(n-1)$ coordinates, are still preserved and the process can be repeated to yield the desired property.

In view of Remark 2.1 it follows that in general, $f_1 * f_2$ can be considered as a mixture of convolutions of uniform distributions on sets which are decreasing in absolute value. Thus the proof extends to this case.

DEFINITION 2.2. Let $\mathbf{X}$, $\mathbf{Y}$ be n-variate random variables. The distribution of $\mathbf{Y}$ is said to be stochastically larger than that of $\mathbf{X}$ (or $\mathbf{Y}$ is stochastically larger than $\mathbf{X}$) *in absolute value* if for every g, nondecreasing in absolute value (see Definition (2.1)),

$$Eg(\mathbf{Y}) \geqq Eg(\mathbf{X}) .$$

THEOREM 2.2. *Suppose the density f of an n-variate random variable $\mathbf{X}$ is decreasing in absolute value and $\mathbf{v}$, $\mathbf{u}$ is a pair of vectors such that $|\mathbf{v}| \geqq |\mathbf{u}|$. Then $\mathbf{X}+\mathbf{v}$ is stochastically larger than $\mathbf{X}+\mathbf{u}$ in absolute value.*

PROOF. It suffices to prove the assertion for the case where g in Definition 2.2 is assumed to be an indicator function. Simply apply Theorem 2.1 with $f_1 = f$, $f_2 = 1 - g$.

3. Contaminated independence model and association. Let $\mathbf{Z}$ be an n-variate random variable with independent components, each having a symmetric unimoidal distribution. (By Remark 2.1, the density of $\mathbf{Z}$ is decreasing in absolute value.) Suppose that, due to certain experimental conditions, $\mathbf{Z}$ is *contaminated* and what can be observed is

$$\mathbf{X} = \mathbf{Z} + \mathbf{U} , \tag{3.1}$$

where $\mathbf{Z}$ and $\mathbf{U}$ are independent. In particular, if $\mathbf{U} = (\alpha_1 W, \alpha_2 W, \cdots, \alpha_n W)$ where W is a real random variable then it follows that

$$\operatorname{Cov}(X_i, X_j) = \lambda_i \lambda_j , \qquad i \neq j , \tag{3.2}$$

where λ_i equals α_i times the standard deviation of W. Note that if $\mathbf{X}$ has a multivariate normal distribution then the converse holds; that is, if the covariance structure satisfies (3.2) and if $\operatorname{Var}(X_i) \geqq \lambda_i^2$, then $\mathbf{X}$ has a representation given by (3.1) where $\mathbf{U} = \boldsymbol{\alpha} W$.

Šidák (1971) considered the normal case with (3.2) and proved the inequality (1.7). In a subsequent paper [21], he proved similar inequalities for convex sets. The following theorem generalizes these results in several respects. A condition weaker than (3.2) is assumed while a stronger conclusion is derived, namely that $|\mathbf{X}|$ is in fact associated. Further, the assumption of normality is removed. The same technique is then shown to yield the inequalities (3.4) and (3.5) for centrally symmetric convex sets when $\mathbf{U}$ is of the special form $\boldsymbol{\alpha} W$.

THEOREM 3.1. *Let $\mathbf{Z} = (Z_1, \cdots, Z_n)$ be an n-vector with independent real* *components, each having a symmetric unimodal distribution. Suppose*

$$\mathbf{X} = \mathbf{Z} + \mathbf{U}$$

where $\mathbf{U}$ *is independent of* $\mathbf{Z}$ *and* $|\mathbf{U}|$ *is associated. Then* $|\mathbf{X}|$ *is associated.*

PROOF. Let f and g be a pair of real nondecreasing functions defined on R^n. Observe that

$$\text{(3.3)} \qquad \operatorname{Cov}[f(|\mathbf{X}|), g(|\mathbf{X}|)] = E\{\operatorname{Cov}[f(|\mathbf{X}|), g(|\mathbf{X}|)] \mid \mathbf{U}\} + \operatorname{Cov}\{E[f(|\mathbf{X}|) \mid \mathbf{U}], E[g(|\mathbf{X}|) \mid \mathbf{U}]\}.$$

When $\mathbf{U}$ is given, the random variables $X_1, \cdots, X_n$ are independent. Hence, conditionally $|X_1|, \cdots, |X_n|$ are associated and the first term on the right side of (3.3) is nonnegative.

To show that the same is true for the second term, note that $\mathbf{X}$ is the same as $\mathbf{Z} + \mathbf{U}$ where $\mathbf{Z}$ has a density decreasing in absolute value. However, by Theorem 2.2, the conditional expectations in this term are nondecreasing functions of $|\mathbf{U}|$. Since by our assumption, $|\mathbf{U}|$ is associated, the covariance of these functions is nonnegative and the proof is complete.

Theorem 3.1 has some similarity with a result stated by Marshall and Olkin (1974) (Section 5.1). However, here we are dealing with absolute values so that the conditions and techniques are different. The following three corollaries of Theorem 3.1 are related to some recent results (references will be cited subsequently).

COROLLARY 3.1. *In Theorem* 3.1 *one may assume that* $\mathbf{U} = (\alpha_1 V, \cdots, \alpha_k V, \alpha_{k+1} W, \cdots, \alpha_n W)$ *where* (V, W) *has a bivariate normal distribution centered at* 0.

PROOF. From Theorem 3.1 it follows that $|V|$, $|W|$ are associated. Hence $|\mathbf{U}|$ is associated. (The result does not extend to trivariate normal since it follows from an example in Šidák (1971) that the absolute values may not be associated.)

REMARK 3.1. The result stated above is more general than the one given by Šidák (1971), who assumed $k = n$ and derived the probability inequalities (1.7). In fact, it is not necessary that V and W have expected values 0. The pair $(|V|, |W|)$ can be shown to be associated when the product of their expected values is related to their covariance in a certain way. This slight improvement is omitted to make the assertion simple. However, the reader may obtain this version in view of the following corollary, which is an easy consequence of Theorem 3.1.

COROLLARY 3.2. *In Theorem* 3.1 *it may be assumed that* $\mathbf{U} = \boldsymbol{\alpha} W$, *where* $\boldsymbol{\alpha}$ *is an arbitrary but fixed n-vector and* W *is an arbitrary real random variable.*

REMARK 3.2. The random variable W need not have expected value 0. In Corollary 3.2 note that $EX_i = \alpha_i EW$, so the quantities $EX_i EX_j$ are proportional to $\operatorname{Cov}(X_i, X_j)$. In the normal case, inequalities under such conditions on the expected values were proved by Das Gupta et al. (1971). It should be noted that the present method removes the assumption of normality.

REMARK 3.3. The following probability inequalities (3.4) and (3.5) for convex sets were shown by Khatri (1967) for the multivariate normal distribution with mean **0** and special covariance structure, and with some other conditions on the means by Das Gupta et al. (1971). However, they can be easily extended to the contaminated independence model of Theorem 3.1. Let $\mathbf{Y}_i$, $i = 1, \cdots, k$ be vectors obtained by regrouping $X_1, \cdots, X_n$, such that $\mathbf{Y}_i$ has p_i components, where $\sum p_i = n$. Let $C_1, \cdots, C_k$ be centrally symmetric convex sets in $R^{p_1}, \cdots, R^{p_k}$. If **U** in Theorem 3.1 is of the form $\boldsymbol{\alpha}W$ then by conditioning on W and applying the moving set inequality of Anderson (1955) and (1.1), it can be seen that

$$P[\mathbf{Y}_i \in C_i, i = 1, \cdots, k] \geqq \prod_{i=1}^k P[\mathbf{Y}_i \in C_i] \tag{3.4}$$

and

$$P[Y_i \in \bar{C}_i, i = 1, \cdots, k] \geqq \prod_{i=1}^k P[\mathbf{Y}_i \in \bar{C}_i] . \tag{3.5}$$

REMARK 3.4. Since Theorem 2.2 holds for distributions which possess densities decreasing in absolute value, it is natural to ask whether the independence assumption in Theorem 3.1 can be replaced by this more general one. If true, this general version would imply that $|X_i|$, $i = 1, \cdots$, n, are associated whenever **X** has density decreasing in absolute value. However, for $n = 2$ when the distribution is uniform on a sign invariant convex set, $|X_1|$ and $|X_2|$ are always *negatively regression dependent*. To see this, note that the conditional distribution of X_2 given $|X_1| = z$, say, is such that for every $a > 0$,

$$P[|X_2| \leqq a \mid |X_1| = z]$$

is nondecreasing in z for $z > 0$, and in fact, unless the convex set is a rectangle, there exists $a > 0$ such that for some values of z, the above probability is *strictly increasing*.

This example brings out an interesting feature of the uncorrelated bivariate normal distribution which can be viewed as a *mixture* of uniform distributions on sign invariant convex sets, namely, ellipses with axes along the coordinate axes. Each pair of absolute values corresponding to the uniform distribution on such an ellipse is *strictly negatively* dependent. However, the mixture so generated has independent components.

4. Association for concordant functions and applications. Esary, Proschan and Walkup (1967) showed several applications of the two key results for the associated random variables; namely, that the union of independent sets of associated random variables is associated and that nondecreasing functions of associated random variables are associated. In several applications however, one wants to consider functions which are nondecreasing in some arguments and nonincreasing in others. In order to motivative the result in this direction, first we state the following theorem due to Lehmann (1966).

Let f, g be a pair of functions $R^n \rightarrow R$. The pair is said to be concordant if

f and g are monotone in each argument and the direction of the monotonicity for ith argument, for $i = 1, \cdots, n$, is same for both functions (that is, if f is nonincreasing in ith argument, so is g). Now suppose (X_i, Y_i), $i = 1, \cdots, n$ are independent pairs, each pair being positively quadrant dependent, that is, for every real a, b,

$$P[X_i > a, Y_i > b] \geqq P[X_i > a]P[Y_i > b] . \tag{4.1}$$

Then for a concordant pair f, g

$$\operatorname{Cov}[f(X_1, \cdots, X_n), g(Y_1, \cdots, Y_n)] \geqq 0 . \tag{4.2}$$

The type of extension considered in the following requires a more general definition of concordance.

DEFINITION 4.1. A set of m functions $h_1, h_2, \cdots, h_m$ each defined on $R^{kn} \to R$, is said to be k-concordant if all the functions are monotone in each of the kn arguments and the direction of the monotonicity is the same for each block of k-arguments, $jk + 1, jk + 2, \cdots, (j + 1)k$, where $0 \leqq j \leqq (n - 1)$.

For example, $h_1, \cdots, h_m$, simultaneously, may be nondecreasing for the first k arguments, nonincreasing for the next k arguments and so on. The pair in (4.2) is 2-concordant since f, g considered as functions of $X_1, Y_1, X_2, \cdots, X_n, Y_n$ do satisfy the requirement. In the next theorem, a condition of association is used in place of the (weaker) condition given by (4.1), and a stronger proposition is derived. The main difference is that Lehmann's theorem is applicable only for bivariate distributions while the present version is valid for a multivariate distribution. The next theorem paves the way for a more general theorem which will follow.

THEOREM 4.1. *Let $\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_n$ be independent k-vectors such that every $\mathbf{U}_i$ is associated. If f and g is a pair of k-concordant functions then*

$$\operatorname{Cov}[f(\mathbf{U}_1, \cdots, \mathbf{U}_n), g(\mathbf{U}_1, \cdots, \mathbf{U}_n)] \geqq 0 .$$

PROOF. For $n = 1$, the assertion is equivalent to the definition of association. We exhibit extension to $n = 2$, while general extension from n to $n + 1$ is similar and is omitted. Writing

$$\operatorname{Cov}[f(\mathbf{U}_1, \mathbf{U}_2), g(\mathbf{U}_1, \mathbf{U}_2)] = E\{\operatorname{Cov}[f, g \mid \mathbf{U}_2]\} + \operatorname{Cov}[E\{f \mid \mathbf{U}_2\}, E\{g \mid \mathbf{U}_2\}] , \tag{4.3}$$

it follows that the first term is nonnegative since $\mathbf{U}_2$ is independent of $\mathbf{U}_1$ which is associated. Now f and g are either both nondecreasing in all components of $\mathbf{U}_2$ or both nonincreasing, and hence the same is true of expected values. Since $\mathbf{U}_2$ is associated, it follows that the second term of (4.3) is nonnegative and the proof is complete.

The next theorem forms the basic tool for generating associated statistics via concordant functions defined on independent sets of associated random variables.

THEOREM 4.2. *Suppose $h_1, \cdots, h_m$ are k-concordant functions defined for n k-tuples, $\mathbf{U}_1, \cdots, \mathbf{U}_n$ are independent, and every $\mathbf{U}_i$ is associated. Then $h_i(\mathbf{U}_1, \cdots, \mathbf{U}_n)$, $i = 1, \cdots, m$, are associated.*

PROOF. To make writing easier let

$$h_i(\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_n) = V_i\,, \qquad i = 1, \cdots, m\,.$$

Let p and q be a pair of two nondecreasing functions on $R^m \to R$. The theorem will be proved when it is shown that

$$\text{Cov}\,[p(V_1, \cdots, V_m), q(V_1, \cdots, V_m)] \geqq 0\,. \tag{4.4}$$

Since each V_i is a function of $\mathbf{U}_1, \cdots, \mathbf{U}_n$ one may write

$$p(V_1, \cdots, V_m) = p^*(\mathbf{U}_1, \cdots, \mathbf{U}_n) \tag{4.5}$$

and

$$q(V_1, \cdots, V_m) = q^*(\mathbf{U}_1, \cdots, \mathbf{U}_n)\,. \tag{4.6}$$

It is readily seen, however, that p^*, q^* is a pair of k-concordant functions, so (4.4) follows from Theorem 4.1.

5. Applications.

(i) *Multivariate t-distribution.* Let $\mathbf{X}_i = (X_{i1}, \cdots, X_{ik})$, $i = 0, 1, \cdots, n$, be $(n+1)$ independent k-vectors, each satisfying conditions of either of the Corollaries 3.1 or 3.2, so that each $|\mathbf{X}_i|$ is associated. Suppose

$$t_j^2 = \frac{X_{0j}^2}{\sum_{i=1}^n X_{ij}^2/n}\,, \qquad j = 1, \cdots, k\,.$$

Then it is clear that t_j^2 are k-concordant and Theorem 4.2 implies that the t_j^2, or equivalently the $|t_j|$, are associated. By introducing the normality assumption this implies the association of the absolute values of a t-vector under certain conditions on correlation coefficients as given in Corollaries 3.1 and 3.2. This strengthens the result of Šidák (1971) which provided probability inequalities under more stringent assumptions. In addition, inequalities (1.4) and (1.5) hold for $|t_j|$ since these are associated.

(ii) Suppose the $\mathbf{X}_i$ satisfy the same conditions as in (i). Let T be a $k \times k$ matrix with elements

$$T_{ij} = \sum_{l=1}^n X_{li} X_{lj}\,, \qquad i, j = 1, \cdots, k\,.$$

Then by Theorem 4.2 again the diagonal elements T_{jj} are associated. In particular, if $\mathbf{X}_i$ are assumed to be identically normally distributed with mean $\mathbf{0}$ and correlation structure (3.2), then T is a Wishart matrix whose diagonal elements are associated. This allows one to form conservative simultaneous upper or lower confidence bounds for variances by utilizing partial knowledge about correlation coefficients.

(iii) If one is interested in obtaining simultaneous upper or lower confidence bounds for the ratio of the variance of the corresponding components of two multivariate populations satisfying the conditions of Corollaries 3.1 or 3.2, one can again apply Theorem 4.2 to F ratios to facilitate conservative bounds. In general a variety of functions which are monotone in absolute values would produce associated statistics.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 61801