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DISTRIBUTION AND EXPECTED VALUE OF THE RANK OF A CONCOMITANT OF AN ORDER STATISTIC[1]

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Let (X_i, Y_i) be n independent rv's having a common bivariate distribution. When the X_i are arranged in nondecreasing order as the order statistics $X_{r:n}$ $(r = 1, 2, \cdots, n)$, the Y-variate $Y_{[r:n]}$ paired with $X_{r:n}$ is termed the concomitant of the rth order statistic. The small-sample theory of the distribution and expected value of the rank $R_{r,n}$ of $Y_{[r:n]}$ is studied. In the special case of bivariate normality an illustrative table of the probability distribution of $R_{r,n}$ is given. A more extensive table of $E(R_{r,n})$ is also provided and it is found that asymptotic results require comparatively small finite-sample corrections even for modest values of n. Some applications are briefly indicated.

1. Introduction. Let (X_i, Y_i) $(i = 1, 2, \cdots, n)$ be n independent rv's having a common bivariate distribution corresponding to (X, Y). When the X_i are arranged in nondecreasing order as the order statistics $X_{r:n}$ $(r = 1, 2, \cdots, n)$, the Y-variate associated with $X_{r:n}$ may be denoted by $Y_{[r:n]}$ and termed the *concomitant of the* rth *order statistic*. These concomitants have been put to a variety of uses, recent examples including Gross (1973) and O'Connell and David (1976).

In this paper we are concerned primarily with the distribution and the expected value of the rank $R_{r:n}$ of $Y_{[r:n]}$ among the n Y_i. By way of motivation note that (X_i, Y_i) may refer to two tests taken by the ith individual A_i. We address ourselves to the following questions: If A_i has rank r in the first test, what is the probability that he will have rank s in the second test and what is his expected rank in the second test? Again, X_i may represent an observable (or phenotypic) rv, used as a basis for ranking or selection, and Y_i the true (or genotypic) rv that is really of interest.

Asymptotic results for the behavior of $R_{r,n}$ have been developed in David and Galambos (1974). Here we concentrate on the small-sample theory. In the special case when X and Y are bivariate normal an illustrative table of the probability distribution of $R_{r,n}$ is given. A more extensive table of $E(R_{r,n})$ is also provided and it is found that the asymptotic results require comparatively small finite-sample corrections even for modest values of n.

2. Probability distribution of the rank of $Y_{[r:n]}$. Let the indicator function

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$I(u)$ be defined by

(2.1) $$I(u) = 1 \quad \text{for} \quad u \geqq 0$$
$$= 0 \quad \text{otherwise.}$$

Then the rank of $Y_{[r:n]}$ is given by

(2.2) $$R_{r,n} = \sum_{i=1}^{n} I(Y_i - Y_{[r:n]}) \qquad r = 1, 2, \cdots, n\,.$$

Let X and Y have the absolutely continuous joint cdf $F(x, y)$, with pdf $f(x, y)$. Since $R_{r,n}$ is from (2.2) location and scale invariant with respect to both X and Y, we take F and f to refer to the standardized variates. Writing $r(X_i)$ for the rank of X_i among the n X's, with a similar meaning for $r(Y_i)$, we have for $r = 1, 2, \cdots, n$; $s = 1, 2, \cdots, n$

(2.3) $$\begin{aligned}\Pr\{R_{r,n} = s\} &= \sum_{i=1}^{n} \Pr\{r(Y_i) = s,\, r(X_i) = r\} \\ &= n \Pr\{r(Y_n) = s,\, r(X_n) = r\}\,,\end{aligned}$$

where the subscript is taken to be n for definiteness. The manner in which the compound event $r(Y_n) = s$, $r(X_n) = r$ can occur is best seen from the following 2 x 2 table with fixed marginals:

	$Y_i < Y_n$	$Y_i > Y_n$	
$X_i < X_n$	k	$r - 1 - k$	$r - 1$
$X_i > X_n$	$s - 1 - k$	$n - r - s + 1 + k$	$n - r$
	$s - 1$	$n - s$	$n - 1$

Corresponding to the four cell entries write

(2.4) $$\theta_1(x, y) = \Pr\{X < x,\, Y < y\}\,, \qquad \theta_2(x, y) = \Pr\{X < x,\, Y > y\}\,,$$
$$\theta_3(x, y) = \Pr\{X > x,\, Y < y\}\,, \qquad \theta_4(x, y) = \Pr\{X > x,\, Y > y\}\,.$$

By conditioning on X_n, Y_n, we then have from (2.3)

(2.5) $$\Pr\{R_{r,n} = s\} = n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k=0}^{t} C_k \theta_1^{\,k} \theta_2^{\,r-1-k} \theta_3^{\,s-1-k} \theta_4^{\,n-r-s+1+k} f(x, y)\, dx\, dy\,,$$

where

(2.6) $$t = \min(r - 1, s - 1)\,,$$

(2.7) $$C_k(r, s, n) = \frac{(n-1)!}{k!\,(r-1-k)!\,(s-1-k)!\,(n-r-s+1+k)!}\,.$$

Equation (2.5) provides the distribution of $R_{r,n}$. However, it is quicker to obtain the following two symmetry relations directly from (2.3). Write $\pi_{rs} = \Pr\{R_{r,n} = s\}$.

Relation 1. If there exist monotone increasing transformations from X to X' and from Y to Y' such that the joint pdf $g(x', y')$ of X' and Y' is symmetric (i.e., $g(x', y') = g(y', x')$), then

(2.8) $$\pi_{rs} = \pi_{sr} \qquad r, s = 1, 2, \cdots, n\,.$$

PROOF. We have $\Pr\{r(Y_n') = s, r(X_n') = r\} = \Pr\{r(X_n') = s, r(Y_n') = r\}$ and hence that $\Pr\{r(Y_n) = s, r(X_n) = r\} = \Pr\{r(Y_n) = r, r(X_n) = s\}$ which by (2.3) gives (2.8).

RELATION 2. If $f(x, y) = f(-x, -y)$, then

$$\pi_{rs} = \pi_{n+1-r,n+1-s} . \tag{2.9}$$

PROOF. Put $Y_i^* = -Y_i$, $X_i^* = -X_i$ for $i = 1, 2, \cdots, n$. Then

$$\begin{aligned}\Pr\{r(Y_n) = s, r(X_n) = r\} &= \Pr\{r(Y_n^*) = n + 1 - s, r(X_n^*) = n + 1 - r\}\\ &= \Pr\{r(Y_n) = n + 1 - s, r(X_n) = n + 1 - r\} ,\end{aligned}$$

since the n pairs X_i^*, Y_i^* have the same joint distribution as the n pairs X_i, Y_i. This completes the proof.

3. Expected rank of $Y_{[r:n]}$. We require for $r = 1, 2, \cdots, n$

$$\begin{aligned}E(R_{r,n}) &= \sum_{s=1}^{n} s \Pr\{R_{r,n} = s\}\\ &= 1 + \sum_{u=0}^{n-1} u \Pr\{R_{r,n} = u + 1\} .\end{aligned}$$

Noting that $C_k(r, u + 1, n)$ of (2.7) may be written as

$$C_k = \binom{n-1}{r-1}\binom{r-1}{k}\binom{n-r}{u-k}$$

and setting $u = k + j$ we have from (2.5)

$$\begin{aligned}E(R_{r,n}) = 1 + n\binom{n-1}{r-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k=0}^{r-1} \sum_{j=0}^{n-r} (k + j)\\ \times \binom{r-1}{k}\binom{n-r}{j}\theta_1^{k}\theta_2^{r-1-k}\theta_3^{j}\theta_4^{n-r-j}f(x, y)\, dx\, dy .\end{aligned}$$

Now $\theta_1 + \theta_2 = F_X(x)$ by (2.4). Hence from binomial-type summations such as

$$\sum_{k=0}^{r-1} k\binom{r-1}{k}\theta_1^{k}\theta_2^{r-1-k} = (r - 1)\theta_1[F_X(x)]^{r-2}$$

we obtain

$$\begin{aligned}E(R_{r,n}) = 1 + n\{\textstyle\int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} \theta_1 f(y \mid x)\, dy] f_{r-1:n-1}(x)\, dx\\ + \textstyle\int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} \theta_3 f(y \mid x)\, dy] f_{r:n-1}(x)\, dx\} ,\end{aligned} \tag{3.1}$$

where $f_{r-1:n-1}(x)$ is the pdf of $X_{r-1:n-1}$, etc.

Higher moments of $R_{r,n}$ can be obtained via the factorial moments (cf. O'Connell, 1974).

4. Numerical results when X and Y are bivariate normal. For the case

$$f(x, y) = \frac{1}{2\pi(1 - \rho^2)^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}[x^2 - 2\rho xy + y^2]\} \qquad |\rho| < 1$$

(2.5) may be simplified somewhat by setting

$$cu = y - \rho x \qquad \text{with} \quad c = |\rho|(1 - \rho^2)^{-\frac{1}{2}} .$$

This gives $\Pr\{R_{r,n} = s\}$ as

$$\pi_{rs}(\rho) = nc \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k=0}^{t} C_k\theta_1^{k}\theta_2^{r-1-k}\theta_3^{s-1-k}\theta_4^{n-r-s+1+k}\varphi(x)\varphi(cu)\, dx\, du , \tag{4.1}$$

where the θ's are now functions of x and u; e.g., for $0 < \rho < 1$

$$\theta_1(x, u) = \int_{-\infty}^{0} \varphi(t + x)\{1 - \Phi[c(t - u)]\}\, dt ,$$

φ, Φ denoting the standard normal pdf and cdf.

Relations (2.8) and (2.9) clearly hold in this case. Also results for negative ρ are given by

$$\pi_{rs}(\rho) = \pi_{r,n+1-s}(-\rho) \qquad r, s = 1, 2, \cdots, n . \tag{4.2}$$

From results more generally true for any variates X and Y, respectively inde-

TABLE 1

$\pi_{rs} = \Pr\{R_{r,n} = s\}$ *as a function of* ρ *for* $n = 9$

r	s	ρ									
		.10	.20	.30	.40	.50	.60	.70	.80	.90	.95
9	9	.1407	.1746	.2133	.2576	.3087	.3686	.4404	.5306	.6564	.7510
	8	.1285	.1459	.1631	.1797	.1952	.2087	.2185	.2207	.2033	.1725
	7	.1211	.1296	.1363	.1408	.1424	.1401	.1321	.1152	.0817	.0523
	6	.1152	.1173	.1171	.1143	.1085	.0989	.0846	.0640	.0350	.0169
	5	.1100	.1069	.1015	.0938	.0836	.0706	.0546	.0357	.0149	.0053
	4	.1051	.0973	.0877	.0765	.0638	.0497	.0345	.0192	.0059	.0015
	3	.0999	.0877	.0747	.0611	.0472	.0334	.0205	.0095	.0021	.0004
	2	.0939	.0773	.0612	.0462	.0324	.0204	.0107	.0040	.0006	.0001
	1	.0856	.0635	.0451	.0300	.0183	.0097	.0041	.0011	.0001	.0000
8	8	.1220	.1334	.1458	.1602	.1779	.2014	.2355	.2907	.3992	.5129
	7	.1177	.1247	.1323	.1409	.1509	.1628	.1771	.1934	.2057	.1976
	6	.1142	.1175	.1208	.1241	.1273	.1299	.1304	.1254	.1044	.0762
	5	.1109	.1108	.1102	.1090	.1066	.1022	.0941	.0793	.0511	.0281
	4	.1078	.1042	.0999	.0946	.0877	.0785	.0656	.0477	.0233	.0094
	3	.1043	.0972	.0893	.0803	.0699	.0575	.0430	.0264	.0093	.0027
	2	.1001	.0890	.0773	.0650	.0520	.0385	.0250	.0124	.0030	.0006
	1	.0946	.0773	.0612	.0462	.0324	.0204	.0107	.0040	.0006	.0001
7	7	.1154	.1205	.1266	.1345	.1449	.1597	.1824	.2222	.3104	.4163
	6	.1133	.1165	.1205	.1259	.1331	.1428	.1561	.1743	.1964	.2002
	5	.1114	.1124	.1141	.1164	.1193	.1225	.1253	.1249	.1112	.0864
	4	.1094	.1082	.1071	.1059	.1043	.1013	.0955	.0834	.0573	.0332
	3	.1071	.1033	.0991	.0942	.0881	.0797	.0680	.0508	.0259	.0108
	2	.1043	.0972	.0893	.0803	.0699	.0576	.0430	.0264	.0094	.0027
	1	.1000	.0877	.0747	.0611	.0472	.0334	.0205	.0095	.0021	.0004
6	6	.1126	.1150	.1188	.1242	.1321	.1438	.1625	.1961	.2742	.3738
	5	.1116	.1132	.1160	.1202	.1263	.1350	.1475	.1657	.1908	.1995
	4	.1107	.1110	.1122	.1141	.1169	.1202	.1234	.1242	.1127	.0892
	3	.1095	.1081	.1071	.1060	.1043	.1013	.0955	.0834	.0573	.0332
	2	.1078	.1042	.0999	.0946	.0878	.0785	.0657	.0477	.0233	.0094
	1	.1048	.0973	.0877	.0766	.0638	.0497	.0345	.0192	.0059	.0015
5	5	.1117	.1134	.1165	.1213	.1285	.1394	.1570	.1889	.2640	.3614

N.B. Outside the range of the table, use relations (2.8), (2.9), (4.2), (4.3) and (4.4).

pendent or with Y a monotone increasing function of X, we also have

$$\pi_{rs}(0) = 1/n\,, \tag{4.3}$$

$$\begin{aligned} \pi_{rs}(1) &= 0 \quad \text{if} \quad r \neq s\,, \\ &= 1 \quad \text{if} \quad r = s\,. \end{aligned} \tag{4.4}$$

Values of π_{rs} have been computed on an IBM 360/65 from (4.1) for $n = 3, 5, 9$ and $\rho = 0.1\,(0.1)\,0.9,\ 0.95$ (O'Connell, 1974). Table 1 gives the results for $n = 9$. Among other interesting features of this table it may be observed that for small and moderate ρ-values π_{rs} is not necessarily a maximum for $r = s$; e.g., $\pi_{89}\ (= \pi_{98}) > \pi_{88}$ for $\rho \leqq 0.60$.

EXAMPLE. Suppose that the scores of candidates taking two tests are bivariate normal with $\rho = 0.8$. Out of 9 candidates taking the first (screening) test the top k are selected and given the second test. What is the smallest value of k ensuring with probability at least 0.9 that the best of the n candidates, as judged by the second test, is included among the k selected?

We require the smallest k such that

$$\pi_{99} + \pi_{89} + \cdots + \pi_{10-k,9} \geqq 0.9$$

i.e.,

$$\pi_{99} + \pi_{98} + \cdots + \pi_{9,10-k} \geqq 0.9\,.$$

Since, from the column for $\rho = 0.8$, we have

$$0.5306 + 0.2207 + 0.1152 + 0.0640 = 0.9305$$

the required value is $k = 4$.

The computation of $E(R_{r,n})$ is facilitated by noting that (3.1) may be expressed as

$$\begin{aligned} E(R_{r,n}) = 1 &+ n(n-1)\tbinom{n-2}{r-2} \textstyle\int_{-\infty}^{\infty} [\Phi(x)]^{r-2}[1-\Phi(x)]^{n-r}\theta_1(x,\cdot)\varphi(x)\,dx \\ &+ n(n-1)\tbinom{n-2}{r-1} \textstyle\int_{-\infty}^{\infty} [\Phi(x)]^{r-1}[1-\Phi(x)]^{n-r-1}\theta_3(x,\cdot)\varphi(x)\,dx\,, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \theta_1(x,\cdot) &= \textstyle\int_{-\infty}^{\infty} \theta_1(x,u)c\varphi(cu)\,du \\ &= \textstyle\int_{-\infty}^{0} [1-\Phi(2^{-\frac{1}{2}}ct)]\varphi(t+x)\,dt\,, \quad \text{etc.} \end{aligned}$$

From (2.9) we also have

$$E(R_{r,n}) = n + 1 - E(R_{n+1-r,n}) \tag{4.6}$$

and, for negative ρ, we can use by (4.2)

$$E(R_{r,n} \mid \rho) = n + 1 - E(R_{r,n} \mid -\rho)\,. \tag{4.7}$$

Table 2 effectively gives $E(R_{r,n})$ for $n = 9,\ 19$ and $\rho = 0.05\,(0.05)\,0.95,\ 0.99$ and for $n = 39$ and $\rho = 0.5\,(0.05)\,0.95,\ 0.99$. This is done by providing finite-sample correction terms, $\Delta(\rho, \lambda_r)$ to the $n = \infty$ rows containing the asymptotic expectation ratio

$$\bar{r}(\rho, \lambda_r) = \lim_{n\to\infty} E(R_{r,n}/(n+1)) \quad \text{with} \quad r/(n+1) = \lambda_r \quad \text{(constant).}$$

Values of $\bar{r}(\rho, \lambda_r)$ to 3 decimals were given in David (1973). It may be noted here

TABLE 2

Asymptotic expectation ratio $\bar{r}(\rho, \lambda_r) = \lim_{n\to\infty} E(R_{r,n}/(n+1))$ *with* $r/(n+1) = \lambda_r$, *and* $\Delta(\rho, \lambda_r) = E(R_{r,n}/(n+1)) - \bar{r}(\rho, \lambda_r)$ *for* $n = 9, 19, 39$.

ρ	n	λ_r								
		.55	.60	.65	.70	.75	.80	.85	.90	.95
.05	∞	.5018	.5036	.5055	.5075	.5096	.5119	.5147	.5181	.5233
	9		−.0001		−.0002		−.0001		+.0007	
	19	−.0001	−.0001	−.0001	−.0001	−.0001	−.0001	+.0001	+0.003	+.0014
.10	∞	.5037	.5073	.5110	.5149	.5192	.5239	.5294	.5363	.5465
	9		−.0003		−.0004		−.0002		+.0014	
	19	−.0002	−.0002	−.0002	−.0002	−.0002	−.0001	+.0001	+.0007	+.0029
.15	∞	.5055	.5109	.5165	.5224	.5289	.5359	.5441	.5545	.5698
	9		−.0003		−.0006		−.0003		+.0021	
	19	−.0002	−.0003	−.0003	−.0003	−.0003	−.0001	+.0002	+.0011	+.0042
.20	∞	.5073	.5146	.5221	.5300	.5385	.5480	.5590	.5727	.5930
	9		−.0005		−.0007		−.0003		+.0027	
	19	−.0003	−.0003	−.0004	−.0004	−.0003	−.0002	+.0003	+.0014	+.0055
.25	∞	.5092	.5183	.5277	.5376	.5483	.5601	.5739	.5911	.6162
	9		−.0006		−.0008		−.0004		+.0032	
	19	−.0003	−.0003	−.0003	−.0004	−.0004	−.0002	+.0004	+.0018	+.0067
.30	∞	.5110	.5220	.5334	.5454	.5582	.5725	.5890	.6095	.6394
	9		−.0007		−.0010		−.0005		+.0035	
	19	−.0003	−.0004	−.0004	−.0005	−.0004	−.0001	+.0005	+.0021	+.0076
.35	∞	.5129	.5258	.5392	.5533	.5683	.5850	.6043	.6282	.6626
	9		−.0007		−.0011		−.0006		−.0038	
	19	−.0003	−.0004	−.0004	−.0005	−.0004	−.0002	+.0005	+.0023	+.0084
.40	∞	.5148	.5298	.5452	.5614	.5787	.5978	.6199	.6470	.6859
	9		−.0008		−.0013		−.0008		+.0038	
	19	−.0002	−.0004	−.0005	−.0005	−.0004	−.0002	+.0005	+.0024	+.0086
.45	∞	.5168	.5338	.5513	.5697	.5894	.6110	.6358	.6661	.7091
	9		−.0009		−.00015		−.0011		+.0036	
	19	−.0002	−.0004	−.0005	−.0006	−.0005	−.0003	+.0005	+.0024	+.0089
.50	∞	.5189	.5380	.5577	.5784	.6004	.6245	.6520	.6856	.7324
	9		−.0001		−.0017		−.0015		+.0080	
	19	−.0002	−.0004	−.0006	−.0006	−.0006	−.0004	+.0003	+.0022	+.0086
	39	−.0001	−.0001	−.0002	−.0002	−.0002	+.0000	+.0004	+.0014	+.0048
.55	∞	.5211	.5425	.5644	.5874	.6118	.6385	.6688	.7053	.7558
	9		−.0011		−.0020		−.0020		+.0021	
	19	−.0003	−.0005	−.0006	−.0008	−.0008	−.0006	+.0000	+.0019	+.0079
	39	−.0001	−.0001	−.0002	−.0003	−.0002	−.0001	+.0003	+.0013	+.0045
.60	∞	.5234	.5471	.5714	.5968	.6238	.6530	.6860	.7255	.7790
	9		−.0013		−.0024		−.0027		+.0008	
	19	−.0003	−.0005	−.0008	−.0010	−.0010	−.0009	−.0004	−.0013	+.0068
	39	−.0001	−.0002	−.0003	−.0003	−.0004	−.0003	+.0007	+.0010	+.0040

(cont.)

TABLE 2 (continued)

ρ	n	λ_r								
		.55	.60	.65	.70	.75	.80	.85	.90	.95
.65	∞	.5258	.5520	.5789	.6068	.6363	.6682	.7039	.7461	.8022
	9		−.0015		−.0029		−.0035		−.0009	
	19	−.0030	−.0006	−.0009	−.0012	−.0014	−.0013	−.0009	+.0005	+.0052
	39	−.0010	−.0002	−.0004	−.0005	−.0005	−.0005	−.0002	+.0006	+.0033
.70	∞	.5285	.5573	.5867	.6173	.6494	.6840	.7223	.7671	.8252
	9		−.0018		−.0034		−.0045		+.0025	
	19	−.0004	−.0008	−.0011	−.0015	−.0017	−.0018	−.0016	−.0005	+.0033
	39	−.0001	−.0003	−.0005	−.0006	−.0007	−.0007	−.0006	−.0001	+.0023
.75	∞	.5313	.5628	.5951	.6284	.6633	.7006	.7415	.7885	.8480
	9		−.0020		−.0040		−.0056		+.0033	
	19	−.0004	−.0009	−.0013	−.0018	−.0021	−.0024	−.0023	−.0017	+.0009
	39	−.0002	−.0004	−.0006	−.0010	−.0010	−.0011	−.0010	−.0006	+.0011
.80	∞	.5343	.5689	.6041	.6404	.6781	.7181	.7614	.8103	.8702
	9		−.0023		−.0046		−.0067		+.0077	
	19	−.0005	−.0010	−.0016	−.0021	−.0026	−.0030	−.0032	−.0030	−.0016
	39	−.0002	−.0004	−.0007	−.0009	−.0012	−.0015	−.0015	−.0013	−.0002
.85	∞	.5376	.5754	.6139	.6533	.6939	.7366	.7821	.8324	.8919
	9		−.0026		−.0053		−.0079		−.0105	
	19	−.0006	−.0012	−.0018	−.0024	−.0030	−.0036	−.0041	−.0044	−.0032
	39	−.0002	−.0005	−.0008	−.0011	−.0014	−.0017	−.0019	−.0020	−.0017
.90	∞	.5412	.5827	.6246	.6673	.7110	.7563	.8038	.8549	.9127
	9		−.0027		−.0057		−.0089		−.0127	
	19	−.0006	−.0012	−.0019	−.0027	−.0034	−.0041	−.0049	−.0057	−.0065
	39	−.0003	−.0006	−.0009	−.0012	−.0016	−.0020	−.0024	−.0027	−.0029
.95	∞	.5453	.5908	.6365	.6827	.7296	.7773	.8264	.8774	.9321
	9		−.0027		−.0057		−.0090		−.0134	
	19	−.0006	−.0012	−.0019	−.0027	−.0034	−.0043	−.0052	−.0062	−.0077
	39	−.0003	−.0006	−.0009	−.0013	−.0017	−.0021	−.0025	−.0030	−.0036
.99	∞	.5490	.5980	.6472	.6964	.7458	.7953	.8452	.8955	.9469
	9		−.0017		−.0037		−.0059		−.0091	
	19	−.0004	−.0008	−.0013	−.0018	−.0023	−.0028	−.0035	−.0042	−.0055
	39	−.0002	−.0004	−.0006	−.0009	−.0011	−.0014	−.0017	−.0021	−.0026

N.B. For $\lambda_r < 0.5$ and for $\rho < 0$ use relations (4.6) and (4.7).

that by a transformation of variables the integral expression for $\bar{r}(\rho, \lambda_r)$ previously derived can be simplified to

$$\bar{r}(\rho, \lambda_r) = \Phi\left(\frac{\rho\Phi^{-1}(\lambda_r)}{(2-\rho^2)^{\frac{1}{2}}}\right). \tag{4.8}$$

Likewise the formula for the limiting cdf of $R_{r,n}$ developed in David and Galambos (1974) reduces, for any constant z ($0 \leqq z \leqq 1$), to

$$\lim_{n\to\infty} \Pr\{R_{r,n} \leqq nz\} = \Phi\left(\frac{\Phi^{-1}(z) - \rho\Phi^{-1}(\lambda_r)}{(1-\rho^2)^{\frac{1}{2}}}\right). \tag{4.9}$$

As an example of how to use Table 2 consider $r = 8$, $n = 9$, $\rho = 0.75$. Then $\lambda_r = 0.8$ and

$$E(R_{8,9}) = 10\,(0.7006 - 0.0056) = 6.95\ .$$

Since the correction terms in Table 2 are quite small, even for $n = 9$, the table and (4.8) are useful for a wide range of sample sizes. Similarly (4.9) provides an approximation to the exact cdf of $R_{r,n}$ but the approximation remains quite rough, at least for $n = 9$, in spite of attempted continuity corrections.

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Added in proof. It has recently come to our notice that what we call "concomitants of order statistics" are termed "induced order statistics" by Bhattacharya (1974) and Sen (1976). The emphasis of their work is, however, quite different from ours.

BHATTACHARYA, P. K. (1974). Convergence of sample paths of normalized sums of induced order statistics. *Ann. Statist.* **2** 1034–1039.

SEN, P. K. (1976). A note on invariance principles for induced order statistics. *Ann. Probability* **4** 474–479.