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TWO-LEVEL PARALLEL FLATS DESIGNS

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Regular 2^{n-p} designs are also known as single flat designs. Parallel flats designs (PFDs) consisting of three parallel flats (3-PFDs) are the most frequently utilized PFDs, due to their simple structure. Generalizing to f-PFD with $f > 3$ is more challenging. This paper aims to study the general theory for the f-PFD for any $f \geq 3$. We propose a method for obtaining the confounding frequency vectors for all nonequivalent f-PFDs, and to find the least G-aberration (or highest D-efficiency) f-PFD constructed from any single flat. PFDs are particularly useful for constructing nonregular fraction, split-plot or randomized block designs. We also characterize the quaternary code design series as PFDs. Finally, we show how designs constructed by concatenating regular fractions from different families may also have a parallel flats structure. Examples are given throughout to illustrate the results.

1. Introduction. Two-level regular fractional factorial designs are commonly used for screening experiments, since they are easily interpreted. For these regular fractions, factorial effects are either orthogonal or fully aliased. In contrast, a nonregular fractional factorial design is one for which some factorial effects are partially aliased. Compared with regular fractions, nonregular designs have more complicated aliasing structure, but they are more flexible in run sizes and allow estimation of more effects. For characterizing nonregular designs, Deng and Tang [11] and Tang and Deng [30] propose the minimum G-aberration and minimum G_2-aberration criteria, respectively.

Parallel flats designs (PFDs) are nonregular designs that retain some of the simplicity of regular fractional factorial designs. Connor and Young [10] first proposed PFDs, though without using this name. Srivastava and Li [29] obtained conditions for a PFD to be an orthogonal design for estimating any arbitrary set of factorial effects. Srivastava and Chopra [28] presented conditions for a PFD to be an orthogonal design with given resolution, while Liao, Iyer and Vecchia [19] provided an algorithm for constructing orthogonal two-level PFDs with user-specified resolution. A series of papers ([36], [27], [37], [26], [25]) have investigated a subset of PFDs named quaternary code (QC) designs and provide many designs having minimum G_2-aberration among all possible designs. QC designs are easily generated and retain some of the structure of regular fractional factorial designs, but QC designs are limited to run sizes of powers of 2. Recently, Jones et al. [18] and Edwards and Mee [12] presented a Kronecker product construction for PFDs. Most nonregular designs are algorithmically constructed and there is little attention to their structure. PFDs are an exception, in that they represent a broad class of nonregular designs having notable structure.

Much study of PFDs for $f > 3$ flats has focused on the construction of designs that are fully efficient for a given model. However, the run-size of such designs may be too large, so here we consider designs with smaller run sizes but with D-efficiency $< 100\%$. 3-PFDs are popular for this reason, providing efficient designs for estimating the two-factor interaction

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model, while requiring 25% fewer runs than the smallest resolution V regular fractions. Each 3-PFD is constructed by partitioning a regular fractional factorial design into four subsets defined by two factorial effect contrasts and then discarding one subset. See, for example, $n = 4$ factors with run size $N = (3/4)(2^4) = 12$ (Mee [21] Section 8.3), 7–8 factors with $N = 48$ (Addelman [1], John [15]), and 9–11 factors with $N = 96$ (John [16], Mee [20]). All these designs have simple correlation structures and perform well for estimating the two-factor interaction model. See Chai and Liao [7] for general properties of 3-PFDs.

Models estimated from PFDs have a block diagonal information matrix, with the rank of each submatrix limited by the number of flats. This structure makes f-PFDs easier to understand and analyze than general nonorthogonal designs, while retaining flexibility of run size, being available for any even number of runs. This article will greatly facilitate the construction of this important class that fills a gap between regular 2^{n-p} designs and D-optimal and other algorithmically constructed two-level designs. We present a general theory for f-PFDs for any $f \geq 3$. For any given initial regular 2^{n-p} design, we provide methods for obtaining all nonequivalent resulting f-PFDs as well as their confounding frequency vectors. We present a second method for determining the minimum G-aberration f-PFD for large p and f. We also show how to construct parallel flats split plot designs, as well as identifying parallel flats in designs constructed by concatenating arbitrary regular fractions. Examples are given to illustrate each result.

2. Definitions and preliminaries. Let D be a two-level design with N runs and n factors where each row represents a treatment combination and each column corresponds to a factor with levels ± 1. Denote the n columns of D by $d_1, \dots, d_n$. Two designs are called isomorphic if one can be obtained from the other by row permutations, column permutations and sign switches of columns (Hedayat, Sloane and Stufken [14]). For $V = \{\upsilon_1, \dots, \upsilon_q\}$, the index of q columns of D, define

$$J_q(V) = \left|\sum_{i=1}^{N} d_{i\upsilon_1} \cdots d_{i\upsilon_q}\right|,$$

where d_{ij} is the ith component of column d_j; these $J_q(V)$ are called the J-characteristics of D. If $J_q(V) = N$, these q columns in V form a complete word of length q. If $0 < J_q(V) < N$, these q columns form a partial word of length q. Following Cheng, Li and Ye [9], the aliasing index is defined as $\rho_q(V) = |J_q(V)|/N$, to characterize the degree of aliasing of these q columns in V with the intercept. For any factorial design D, not necessarily orthogonal, let f_{qj} be the frequency of q column combinations that give $J_q(V) = (N + 1 - j)$ for $j = 1, \dots, N$. Then following Deng and Tang [11], design D's confounding frequency vector (cfv) is

$$\text{cfv}(D) = \big[(f_{11}, \dots, f_{1N})_1, (f_{21}, \dots, f_{2N})_2, \dots, (f_{n1}, \dots, f_{nN})_n\big]. \tag{1}$$

Suppose that r is the smallest integer such that $\max_{|V|=r} J_r(V) > 0$, where the maximization is taken over all the size r subsets V of D. Then the generalized resolution of D is $R(D) = r + (1 - \rho)$, where $\rho = \max_{|V|=r} J_r(V)/N$.

Let $\text{cfv}_i(D)$ be the ith entry in (1) and let i^* be the smallest integer such that $\text{cfv}_{i^*}(D_1) \neq \text{cfv}_{i^*}(D_2)$. If $\text{cfv}_{i^*}(D_1) < \text{cfv}_{i^*}(D_2)$, then D_1 has less G-aberration than D_2 (Deng and Tang [11]). If no design of corresponding size has less G-aberration than D_1, then D_1 is a minimum G-aberration design. Two isomorphic designs have the same cfv, but the reverse is not true. For regular 2^{n-p} designs, minimum G-aberration is equivalent to minimum aberration (Fries and Hunter [13]).

Let $A = (a_{ij})$ denote a $p \times n$ matrix over GF[2] of rank p and c a $p \times 1$ vector with levels ± 1. All treatment combinations $x = (x_1, x_2, \dots, x_n)^T$ that satisfy the equation $A \odot x = c$

form a regular 2^{n-p} design, where $A \odot x$ is defined as the $p \times 1$ vector with the ith element to be $x_1^{a_{i1}} \cdots x_n^{a_{in}}$; this regular design is a single flat. For given A, we can have 2^p different options for the vector c, corresponding to the 2^p disjoint single flats. The full 2^n design is the concatenation of these.

Taking f distinct single flats corresponding to A, we obtain a parallel flats design with f flats (f-PFD). Such a design consists of $N = f \times 2^{n-p}$ treatment combinations. An f-PFD is determined by the pair (A, C), where $C = [c_1, c_2, \ldots, c_f]$. Meanwhile, we can also understand the concept of parallel flats design from the perspective of defining words (Liao, Iyer and Vecchia [19]). All 2^p parallel flats are 2^{n-p} designs in which the fractions are determined by the same p defining words but different sign assignment. Each choice of vector c corresponds to a sign assignment of the p defining words. All 2^p single flats are said to belong to the same family since they have the same defining words. An f-PFD is simply the concatenation of f single flats from the same family. For even f, the matrix C can sometimes be reduced, so that the design is an $(f/2)$-PFD composed of flats of size $2^{n-(p-1)}$. If an f-PFD cannot be so reduced, it is said to be of minimal form (Edwards and Mee [12]).

For any minimal form f-PFD of size $N \times n$, the values of $J_q(V) > 0$ must be in $\{N, (1 - 2/f)N, \ldots, (1 - 2(\varsigma - 1)/f)N\}$ where $\varsigma = \lfloor (f + 1)/2 \rfloor$. Redefine f_{qj} to be the frequency of q column combinations that give $J_q(V) = (1 - 2(j - 1)/f)N$ for $j = 1, \ldots, \varsigma$. The cfv of D simplifies to

$$\text{cfv}(D) = [(f_{11}, \ldots, f_{1\varsigma})_1, (f_{21}, \ldots, f_{2\varsigma})_2, \ldots, (f_{n1}, \ldots, f_{n\varsigma})_n]. \tag{2}$$

3. General results of f-PFDs for $f \geq 3$. For any given regular 2^{n-p} design, denoted as D_0, each of the 2^p distinct flats in this family corresponds to a column of a Sylvester Hadamard matrix. We describe this connection now.

Let H_{2^p} be a Sylvester-type Hadamard matrix of order 2^p, generated by the recursion $H_{2^1} = [1, 1; 1, -1]$ and

$$H_{2^p} = \begin{bmatrix} H_{2^{p-1}} & H_{2^{p-1}} \\ H_{2^{p-1}} & -H_{2^{p-1}} \end{bmatrix} \quad \text{for } p \geq 2.$$

Then H_{2^p} is symmetrical and $H_{2^p}^2 = 2^p I_{2^p}$, where I_{2^p} is the identity matrix of order 2^p. We number the rows and columns of H_{2^p} beginning with zero. Thus we write

$$H_{2^p} = [h_0, h_1, \ldots, h_{2^p-1}],$$

where h_0 is a $2^p \times 1$ vector with all elements unity, and columns $\{h_1, h_2, \ldots h_{2^{p-1}}\}$ are p basic columns. Any other column can be generated by these basic columns, such as $h_3 = h_1 * h_2$ and $h_5 = h_1 * h_4$, where $a * b = (a_1 b_1, \ldots, a_z b_z)^T$ for any $a = (a_1, \ldots, a_z)^T$, $b = (b_1, \ldots, b_z)^T$. Since H_{2^p} is symmetrical, the rows and the columns have the same defining relations.

Each of the 2^p flats corresponds to a column of H_{2^p}, in that h_i determines the sign of each word in the defining contrast subgroup for that flat; elements 1, 2,..., 2^{p-1} of h_i define the corresponding column of C. To obtain an f-PFD, we choose f flats from 2^p single flats; that is, we choose f columns from H_{2^p}. By restricting our attention to unreplicated designs, there are $2^p!/\{f!(2^p - f)!\}$ combinations to be considered. (We discuss f-PFDs with partial replication briefly in the final section.)

Without loss of generality, we take the first flat, D_0, to be $+1$ for all p defining words; that is, we choose the column h_0. Each of the remaining parallel flats corresponds to a column from $H^* = [h_1, \dots, h_{2^p-1}]$. This reduces the number of combinations to

$$t_{p,f} = (2^p - 1)!/\{(f-1)!(2^p - f)!\}.$$

Some of these choices will produce equivalent f-PFDs, with the following definition.

DEFINITION 1. Two f-PFDs are called equivalent if one f-PFD can be obtained from the other by row permutations and column sign switches.

From Definition 1, two equivalent f-PFDs must be isomorphic while the inverse is not true, since isomorphism also allows for column permutations. For a subset of $f-1$ columns of H^*, say $s = \{h_{i_1}, \dots, h_{i_{f-1}}\}$, we have $\tilde{s} = \{h_0, s\}$, which corresponds to including the first flat, D_0. Now define the group of cosets corresponding to $\tilde{s}$ as

$$G_{\tilde{s}} = \{h_{i_j} \circ \tilde{s} : j = 0, 1, \dots, f-1\}, \quad \text{with } h_{i_j} \circ \tilde{s} = \{h_{i_j}, h_{i_j} * h_{i_1}, \dots, h_{i_j} * h_{i_{f-1}}\}, \tag{3}$$

where $i_0 = 0$. The f columns of $h_{i_j} \circ \tilde{s}$ can be obtained via multiplying the f columns of $\tilde{s}$ by h_{i_j}. Thus, $h_{i_j} \circ \tilde{s}$ can be obtained from $\tilde{s}$ by column permutations and sign switch of rows, which correspond to the row permutations and sign switches of columns of the resulting f-PFD. For any two subsets of $f-1$ columns of H^*, say s_1, s_2, if $\tilde{s}_1 = \{h_0, s_1\}$ and $\tilde{s}_2 = \{h_0, s_2\}$ belong to the same group, then they must produce equivalent f-PFDs. In turn, suppose $\tilde{s}_1$ and $\tilde{s}_2$ correspond to two equivalent f-PFDs. Then there must be a column of $\tilde{s}_1$, say ν, that can become column h_0 in $\tilde{s}_2$ by sign switches of rows. This implies that all f columns of $\tilde{s}_2$ can be obtained via multiplying $\tilde{s}_1$ by ν, respectively. Therefore, $\tilde{s}_1$ and $\tilde{s}_2$ must belong to the same group. For example, suppose $p = f = 3$; then we have

$$H_{2^p} = H_8 = [h_0,\ H^*] = \begin{array}{c} \begin{matrix} h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \end{array}.$$

As noted above, H_8 has three basic columns $\{h_1, h_2, h_4\}$ and any other column can be generated by these three basic columns such as $h_5 = h_1 * h_4$. Besides, H_8 is symmetrical, indicating that the rows and the columns have the same defining relation. For a subset of $f - 1 = 2$ columns of H^*, say $s_1 = \{h_1, h_2\}$, we now consider the group of $\tilde{s}_1 = \{h_0, h_1, h_2\}$. According to Definition 1, $G_{\tilde{s}_1} = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3\}$, where $\tilde{s}_2 = h_1 \circ \tilde{s}_1 = \{h_0, h_1, h_3\}$ and $\tilde{s}_3 = h_2 \circ \tilde{s}_1 = \{h_0, h_2, h_3\}$. Obviously, for $i = 2, 3$, the three columns of $\tilde{s}_i$ can be obtained via multiplying the three columns of $\tilde{s}_1$ by one of its columns. Thus, $\tilde{s}_i$ can be obtained from $\tilde{s}_1$ by column permutations and sign switch of rows, which correspond to the row permutations and sign switches of columns of the resulting 3-PFDs, respectively. Therefore, $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ must produce equivalent 3-PFDs. In turn, suppose $\tilde{s}$ and $\tilde{s}_1$ correspond to two equivalent 3-PFDs, where $\tilde{s} = \{h_0, s\}$ and $s = \{h_{k_1}, h_{k_2}\}$ is a subset of two columns of H^*. Then there must be a column of $\tilde{s}_1$, say ν, that can become column h_0 in $\tilde{s}$ by sign switches of rows. This implies that all three columns of $\tilde{s}$ can be obtained via multiplying the three columns of $\tilde{s}_1$ by ν. As $\nu = h_0, h_1$ or h_2, thus $\tilde{s}$ must be $\tilde{s}_1$, $\tilde{s}_2$ or $\tilde{s}_3$; that is, $\tilde{s}$ and $\tilde{s}_1$ must belong to the same group. To sum up, $G_{\tilde{s}}$ is the set of the choices of f columns of H_{2^p} (including h_0) that can produce the equivalent f-PFDs to that corresponding to $\tilde{s}$. Now we are ready to present the following theorem.

THEOREM 1. *Two f-PFDs based on different flats, $\tilde{s}_1 = \{h_0, s_1\}$ and $\tilde{s}_2 = \{h_0, s_2\}$, are equivalent f-PFDs (for every D_0) if and only if $\tilde{s}_1$ and $\tilde{s}_2$ belong to the same group.*

Let $g_{p,f}$ be the number of disjoint groups for a given p and f, according to Theorem 1. One can get all f-PFDs by including just one from each of the $g_{p,f}$ groups. From (3), for any s, a subset of $f-1$ columns of H^*, the size of $G_{\tilde{s}}$ is at most f since some of $h_{i_j} \circ \tilde{s}$ for $j = 0, 1, \ldots, f-1$ may be the same. Now we present theorems about the grouping, considering f odd and f even separately.

THEOREM 2. *If f is odd, then the size of $G_{\tilde{s}}$ is f; thus, $g_{p,f}/t_{p,f} = 1/f$ for odd f.*

Theorem 2 is confirmed by proving the following: for odd f, there does not exist $j_1 \neq j_2$ such that $h_{i_{j_1}} \circ \tilde{s} = h_{i_{j_2}} \circ \tilde{s}$. A proof is provided in the Supplementary Material [33]. As a small example, suppose $p = f = 3$. There are $t_{3,3} = 21$ choices of $\tilde{s}$, which can be partitioned into $g_{3,3} = 7$ groups of size 3:

$$\begin{aligned}
G_{\{h_0,h_1,h_2\}} &= \{\{h_0,h_1,h_2\},\{h_0,h_1,h_3\},\{h_0,h_2,h_3\}\},\\
G_{\{h_0,h_1,h_4\}} &= \{\{h_0,h_1,h_4\},\{h_0,h_1,h_5\},\{h_0,h_4,h_5\}\},\\
G_{\{h_0,h_1,h_6\}} &= \{\{h_0,h_1,h_6\},\{h_0,h_1,h_7\},\{h_0,h_6,h_7\}\},\\
G_{\{h_0,h_2,h_4\}} &= \{\{h_0,h_2,h_4\},\{h_0,h_2,h_6\},\{h_0,h_4,h_6\}\},\\
G_{\{h_0,h_2,h_5\}} &= \{\{h_0,h_2,h_5\},\{h_0,h_2,h_7\},\{h_0,h_5,h_7\}\},\\
G_{\{h_0,h_3,h_4\}} &= \{\{h_0,h_3,h_4\},\{h_0,h_3,h_7\},\{h_0,h_4,h_7\}\},\\
G_{\{h_0,h_3,h_5\}} &= \{\{h_0,h_3,h_5\},\{h_0,h_3,h_6\},\{h_0,h_5,h_6\}\}.
\end{aligned}$$

According to Theorem 1, the 3-PFDs based on the same group must be equivalent. Thus, choosing one from each group is enough to get all nonequivalent 3-PFDs.

When f is even, there are more possibilities for the group size as presented in the following theorem.

THEOREM 3. *If f is even, it can be written as $f = \lambda 2^{\mu}$, for some odd $\lambda \geq 1$ and some integer $\mu \geq 1$. Then the size of $G_{\tilde{s}}$ might be $\lambda, 2\lambda, \ldots, \lambda 2^{\mu} (= f)$. Furthermore, the group size corresponds to the reduction of the f-PFD, where if a group has size m with $m < f$, then all f-PFDs based on this group can be reduced to m-PFDs.*

See the Supplementary Material for the proof of Theorem 3. We illustrate with two examples.

EXAMPLE 1. Given $p = 4$, $f = 4$, there are $t_{4,4} = 455$ choices for $\tilde{s} = \{h_0, s\}$ with $s = \{h_{i_1}, \ldots, h_{i_3}\}$. According to the property of H_{2^p}, if $\{0, i_1, i_2, i_3\}$ form an Abelian group, then the size of $G_{\tilde{s}}$ is 1, and all the 4-PFDs based on $G_{\tilde{s}}$ can be reduced to single flats of size 2^{n-2}. By a comprehensive examination, there are 140 groups in total, with 35 groups of size 1 and 105 groups of size 4, implying 35 out of 455 choices correspond to a regular 2^{n-2}, while the remaining 420 choices correspond to 4-PFDs.

EXAMPLE 2. Given $p = 4$, $f = 6$, there are $t_{4,6} = 3003$ choices for $\tilde{s} = \{h_0, s\}$ with $s = \{h_{i_1}, \ldots, h_{i_5}\}$. According to the property of a Sylvester Hadamard matrix, if the set of indices $\{i_1, \ldots, i_5\}$ can be represented as $\{\omega_1, \omega_2, \omega_1\omega_2, \omega_3, \omega_1\omega_3\}$, where $\omega_i\omega_j$ is defined by $h_{\omega_i\omega_j} = h_{\omega_i} * h_{\omega_j}$ for $1 \leq i < j \leq 3$, then we have the following pairings:

$$\tilde{s} = h_{\omega_1} \circ \tilde{s}, \qquad h_{\omega_2} \circ \tilde{s} = h_{\omega_1\omega_2} \circ \tilde{s}, \qquad h_{\omega_3} \circ \tilde{s} = h_{\omega_1\omega_3} \circ \tilde{s}; \tag{4}$$

TABLE 1
Example 2's six $\tilde{s}$ columns transposed, paired to indicate reduction to 3 flats

h_0^T	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
h_{15}^T	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	1	−1	−1	1
h_3^T	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1
h_{12}^T	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
h_5^T	1	−1	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1
h_{10}^T	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1

thus, the size of $G_{\tilde{s}}$ is 3. Furthermore, for each of the matrices $(h_0, h_{\omega_1})^T$, $(h_{\omega_2}, h_{\omega_1\omega_2})^T$ and $(h_{\omega_3}, h_{\omega_1\omega_3})^T$, there are 8 columns with the same elements, while the other 8 columns have opposite elements. In addition, the column indices with opposite elements are identical for all three 2×16 matrices, indicating the corresponding 6-PFD based on this group can each be reduced to a 3-PFD. For example, suppose $\tilde{s} = \{h_0, s\}$ with $s = \{h_3, h_5, h_{10}, h_{12}, h_{15}\}$ as shown in Table 1; then we have $\tilde{s} = h_{15} \circ \tilde{s}$, $h_3 \circ \tilde{s} = h_{12} \circ \tilde{s}$, $h_5 \circ \tilde{s} = h_{10} \circ \tilde{s}$; that is, $\omega_1 = 15$, $\omega_2 = 3$, $\omega_3 = 5$ in (4). Each of the matrices $(h_0, h_{15})^T$, $(h_3, h_{12})^T$ and $(h_5, h_{10})^T$ has the properties just described, so these matrices corresponding to three flats from the same family. Thus, any 6-PFD based on this $\tilde{s}$ can be reduced to a 3-PFD. By comprehensive examination, there are 553 groups in total, with 105 groups of size 3 and 448 groups of size 6, implying 315 out of all 3003 choices correspond to 3-PFDs while the remaining 2688 choices correspond to 6-PFDs.

For any (p, f) let η_{κ_i} be the frequency of the groups of size κ_i; then the group size pattern (GSP) is defined as $GSP(p, f) = [\eta_{\kappa_1}, \dots, \eta_{\kappa_\iota}]$, with ι different group sizes. For Example 1, where $p = 4$, $f = 4$, there are 140 groups in total, with 35 groups of size 1 and 105 groups of size 4. Thus, $GSP(4, 4) = [35_1, 105_4]$. Similarly, for Example 2, $GSP(4, 6) = [105_3, 448_6]$. In these two examples, we obtained all groups through a complete search; now we discuss the general theory. These counts are given by the following theorem.

THEOREM 4. *For any $f \le 2^p$, we have*

$$GSP(p, f) = \left[\frac{t_{p,f}}{f}\right]_f \quad \text{for odd } f,$$

$$GSP(p, 4) = \left[\frac{(2^p-1)(2^p-2)}{6}\right]_1, \left[\frac{(2^p-1)(2^p-2)(2^p-4)}{24}\right]_4,$$

$$GSP(p, 6) = \left[\frac{(2^p-1)(2^p-2)(2^p-4)}{24}\right]_3, \left[\frac{2^p(2^p-1)(2^p-2)(2^p-4)(2^p-8)}{720}\right]_6,$$

$$GSP(p, 8) = \left[\frac{(2^p-1)(2^p-2)(2^p-4)}{168}\right]_1, \left[\frac{(2^p-1)(2^p-2)(2^p-4)(2^p-8)}{192}\right]_4,$$
$$\left[\frac{(2^p-1)(2^p-2)(2^p-4)(2^p-8)(8^p-13*4^p+57*2^p-180)}{40320}\right]_8,$$

$$GSP(p, 10) = \left[\frac{(2^p-1)(2^p-2)(2^p-4)(2^p-6)(2^p-8)}{1920}\right]_5,$$
$$\left[\frac{2^p(2^p-1)(2^p-2)(2^p-4)(2^p-8)}{720}\right]_{10}.$$

PROOF OF THEOREM 4. Here, we give the proof of case $f = 8$; other cases can be proved with a similar argument. For $f = 8$, there are three possible group sizes, 1, 4 and 8. First, we calculate the number of groups of size 1. According to the property of H_{2^p}, group $G_{\tilde{s}}$ is of size 1 if and only if $\{0, i_1, \dots, i_7\}$ form an Abelian group. There are in total $(2^p - 1)(2^p - 2)(2^p - 4)$ options for the three basic columns; one can get the whole Abelian group from each set of 3. Since $(2^3 - 1)(2^3 - 2)(2^3 - 4) = 168$ options corresponding to the same Abelian group, there are $(2^p - 1)(2^p - 2)(2^p - 4)/168$ groups of size 1. Second, group $G_{\tilde{s}}$ is of size 4 if and only if $\{i_1, \dots, i_7\}$ can be written as $\{\omega_1, \omega_2, \omega_1\omega_2, \omega_3, \omega_1\omega_3, \omega_4, \omega_1\omega_4\}$, where h_{ω_1}, h_{ω_2}, h_{ω_3}, h_{ω_4} are four independent rows and $\{h_0, h_{\omega_1}\}$, $\{h_{\omega_2}, h_{\omega_1\omega_2}\}$, $\{h_{\omega_3}, h_{\omega_1\omega_3}\}$, $\{h_{\omega_4}, h_{\omega_1\omega_4}\}$ correspond to 4 single flats from the same family. There are in total $(2^p - 1)(2^p - 2)(2^p - 4)(2^p - 8)$ options for $(\omega_1, \omega_2, \omega_3, \omega_4)$ while $2 \times 2 \times 2 \times 3! = 48$ options corresponding to the same choice. Then there are $(2^p - 1)(2^p - 2)(2^p - 4)(2^p - 8)/48$ choices of s with group size 4, and so $(2^p - 1)(2^p - 2)(2^p - 4)(2^p - 8)/192$ groups of size 4. Finally, one can obtain the number of groups of size 8 by subtraction, since all the choices sum to $t_{p,8}$. □

Table 2 gives the values of $t_{p,f}$ (the number of all choices), $g_{p,f}$ (the number of nonequivalent f-PFDs) and the $GSP(p, f)$ (the group size pattern) for $3 \le p \le 6$, $3 \le f \le 8$. The set of $g_{p,f}$ nonequivalent f-PFDs from D_0 is invariant to reordering the columns of D_0. We now describe two symmetries in these counts.

PROPOSITION 1. *For given p, f flats and $2^p + 1 - f$ flats correspond to the same number of choices. Thus $t_{p,f} = t_{p,2^p+1-f}$.*

Proposition 1 is obvious from the formula for $t_{p,f}$.

PROPOSITION 2. *For given p, f flats and $2^p - f$ flats have the same number of nonequivalent designs. Thus, $g_{p,f} = g_{p,2^p-f}$.*

TABLE 2

Partition of all $t_{p,f}$ choices into $g_{p,f}$ equivalent groups for given p and f

p	$f=3$	$f=4$	$f=5$
3	7/21	14/35	7/35
	$[7_3]$	$[7_1\ 7_4]$	$[7_5]$
4	35/105	140/455	273/1365
	$[35_3]$	$[35_1\ 105_4]$	$[273_5]$
5	155/465	1240/4495	6293/31,465
	$[155_3]$	$[155_1\ 1085_4]$	$[6293_5]$
6	651/1953	10,416/39,711	119,133/595,665
	$[651_3]$	$[651_1\ 9765_4]$	$[119{,}133_5]$

p	$f=6$	$f=7$	$f=8$
3	7/21	1/7	1/1
	$[7_3]$	$[1_7]$	$[1_1]$
4	553/3003	715/5005	870/6435
	$[105_3\ 448_6]$	$[715_7]$	$[15_1\ 105_4\ 750_8]$
5	28,861/169,911	105,183/736,281	330,460/2,629,575
	$[1085_3\ 27{,}776_6]$	$[105{,}183_7]$	$[155_1\ 3255_4\ 327{,}050_8]$
6	1,176,357/7,028,847	9,706,503/67,945,521	69,194,232/553,270,671
	$[9765_3\ 1{,}166{,}592_6]$	$[9{,}706{,}503_7]$	$[1395_1\ 68{,}355_4\ 69{,}124{,}482_8]$

$g_{p,f}/t_{p,f}$, the $t_{p,f}$ choices can be divided into $g_{p,f}$ disjoint groups with group size pattern $GSP(p, f)$.

The complement of a choice of f single flats contains the $2^p - f$ omitted flats. Thus the number of nonequivalent f-PFDs is equal to that of nonequivalent $(2^p - f)$-PFDs. Also, any two different (2^{n-p}) flats in a family, if combined, become a single flat $2^{n-(p-1)}$ design. Since every 2-PFD can be reduced to a single flat, its complement can also be reduced.

LEMMA 1. *For any* p, *every* $(2^p - 2)$-PFD *can be reduced to a* $(2^{p-1} - 1)$-*PFD*.

From Lemma 1, any six distinct 2^{n-3} flats from the same family form a 3-PFD, any 14 distinct 2^{n-4} flats from the same family form a 7-PFD, etc.

4. Minimum G-aberration criterion. We now show how the f columns of H_{2^p} determine the J-characteristics of the PFD and present a four-step method for finding the f-PFD with minimum G-aberration for any initial design D_0.

For any $\tilde{s} = \{h_0, s\}$, where $s = \{h_{i_1}, \dots, h_{i_{f-1}}\}$, define

$$S_f(s) = 1_{2^p} + \sum_{j=1}^{f-1} h_{i_j}.$$

If we multiply $|S_f(s)|$ by 2^{n-p}, the first element equals N and the remaining values are the J-characteristics of the f-PFD corresponding to $\tilde{s}$. For simplicity, we call $S_f(s)$ and $|S_f(s)|$ the S-vector and absolute S-vector, respectively, of the f-PFD. We now present Method 1, a four-step procedure for determining the lowest aberration f-PFD based on D_0:

1. For an initial 2^{n-p} design D_0, denote as $L = (L_1, \dots, L_{2^p-1})$ the length of words in the defining contrast subgroup arranged in Yates order, $\omega_1, \omega_2, \omega_1 * \omega_2, \omega_3, \omega_1 * \omega_3, \dots$.
2. Determine $|S_f(s)|$ for a representative $\tilde{s}$ from each of the $g_{p,f}$ equivalence groups.
3. Remove duplicate absolute S-vectors to obtain a $2^p \times u_{p,f}$ matrix $S_{p,f}$ where $u_{p,f}$ is the number of the unique absolute S-vectors.
4. From $S_{p,f}$ and design D_0's L, we obtain the cfv's of all $g_{p,f}$ nonequivalent f-PFDs; the lowest G-aberration f-PFD constructed from D_0 is readily identified.

In Step 2, we consider just one representative from each group, as the f-PFDs based on the same group must be equivalent. This is verified by the following proposition.

PROPOSITION 3. *For any two subsets of* $f - 1$ *columns of* H^*, *say* s_1, s_2, *if* $\tilde{s}_1 = \{h_0, s_1\}$ *and* $\tilde{s}_2 = \{h_0, s_2\}$ *belong to the same group, then they must have the same absolute* S-*vector.*

Now we illustrate Method 1 for the case $p = 6$ and $f = 5$.

EXAMPLE 3. For any regular 2^{10-6} design, there are in total 64 disjoint single flats in its family, with each determined by the sign assignment of six defining words. For a 5-PFD, we include the first flat with $+1$ for all defining words, and consider the assignment of the other four flats from the remaining 63 flats; there are $t_{6,5} = 595{,}665$ such cases to consider, but only $g_{6,5} = 119{,}133$ nonequivalent groups of PFDs. We calculate S-vectors of all these nonequivalent PFDs and find $u_{6,5} = 119{,}133$ unique absolute S-vectors. Matching these to the length of words in Yates order of D_0, we may obtain the minimum G-aberration choice. According to Chen, Sun and Wu [8] there are four nonisomorphic resolution III 2^{10-6} designs, labeled 10-6.x for $x = 1, 2, 3, 4$. We obtain the least G-aberration 5-PFD for each of these four D_0, and show results for the best two in Table 3. The minimum aberration 5-PFD comes from design 10-6.1. When f is odd, one need not consider D_0 with resolution less than III when seeking the minimum G-aberration design, since the resulting f-PFD would have generalized resolution <3.

TABLE 3

The lowest G-aberration 5-PFD *for each of two different* 2^{10-6} D_0 *in Example* 3

D_0	10-6.1 with generator columns 3, 5, 6, 9, 14, 15
L	(3, 3, 4, 3, 4, 4, 3, 3, 4, 4, 7, 6, 5, 5, 6, 4, 5, 5, 4, 3, 6, 6, 7, 5, 4, 4, 5, 4, 5, 5, 8, 5, 4, 4, 5, 4, 5, 5, 8, 4, 5, 5, 4, 3, 6, 6, 7, 3, 4, 4, 7, 6, 5, 5, 6, 4, 7, 7, 8, 7, 8, 8, 7)
s	$\{h_7, h_{27}, h_{28}, h_{33}\}$
$\lvert S_5(s)\rvert$	(5, 1, 1, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 5, 1, 1, 1, 1, 1, 1, 3, 5, 1, 5, 1, 1, 3, 1, 1, 1, 1, 3, 1, 1, 5, 1, 1, 1, 1, 1, 1, 1, 1, 1, 5, 3, 1, 1, 1, 1, 1, 1, 5, 3, 1, 3, 1, 1, 5, 1, 1, 1, 1)
cfv	$[(0, 0, 8)_3, (0, 2, 16)_4, (4, 4, 8)_5, (2, 2, 4)_6, (0, 0, 8)_7, (1, 0, 4)_8]$
D_0	10-6.2 with generator columns 3, 5, 6, 9, 10, 13
L	(3, 3, 4, 3, 4, 4, 3, 3, 4, 4, 7, 6, 5, 5, 6, 3, 4, 6, 5, 4, 7, 5, 6, 4, 3, 5, 6, 5, 6, 4, 7, 4, 5, 3, 6, 5, 4, 6, 7, 3, 6, 4, 5, 4, 5, 7, 6, 5, 4, 4, 5, 4, 5, 5, 8, 6, 7, 7, 6, 5, 8, 8, 9)
s	$\{h_{15}, h_{22}, h_{33}, h_{58}\}$
$\lvert S_5(s)\rvert$	(5, 1, 1, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 3, 3, 3, 1, 3, 3, 3, 1, 1, 3, 1, 1, 1, 3, 1, 5, 1, 1, 1, 1, 1, 1, 3, 3, 1, 3, 3, 1, 5, 1, 1, 1, 1, 1, 3, 1, 1, 1, 3, 1, 5, 1, 1, 3, 1, 3, 3, 1, 1, 1, 3)
cfv	$[(0, 0, 9)_3, (0, 3, 13)_4, (2, 7, 6)_5, (1, 7, 4)_6, (0, 2, 5)_7, (0, 0, 3)_8, (0, 1, 0)_9]$

From Example 3, we can see that $u_{6,5} = g_{6,5}$, implying different groups correspond to different absolute S-vectors, but this is not always the case. To illustrate, suppose $p = 4$ and $f = 6$; let $\tilde{s}_1 = \{h_0, h_1, h_2, h_4, h_8, h_{15}\}$ and $\tilde{s}_2 = \{h_0, h_1, h_2, h_4, h_9, h_{14}\}$ as shown in Table 4. One can easily check that $\tilde{s}_1$ and $\tilde{s}_2$ belong to different groups, yet have the same absolute S-vector.

5. Model estimation criterion. In the previous section, we chose the best design with respect to G-aberration. This criterion, however, does not characterize all the performance of a design for model estimation. For example, dropping any treatment combination from a full 2^n, the design has high G-aberration and generalized resolution < 1. However, it supports estimation of all the factorial effects except the n-factor interaction. In this section, we study the performance of f-PFDs for estimating the two-factor interaction model

$$y = \beta_0 + \sum_{1 \le i \le n} \beta_i d_i + \sum_{1 \le i < j \le n} \beta_{ij} d_i d_j + \epsilon. \tag{5}$$

This model and models with fewer interactions (Srivastava and Li [29]) are widely used. While we focus on model (5), the results we present apply more broadly to other models.

For a parallel flats design D constructed from the single flat D_0, two orthogonal effects in D_0 remain orthogonal in D, while two completely aliased effects in D_0 may be orthogonal or partially aliased in D. Thus, for studying the estimability of D, we need to examine each alias chain of D_0, where an alias chain is the set of effects that are fully aliased. Following Block and Mee [4] and Mee [22], define the alias length pattern (*alp*) of D_0 as

$$alp = (\alpha_1, \dots, \alpha_\Lambda)$$

where α_l is the number of chains of length l ($1 \le l \le \Lambda$) for (5), and Λ is the length of the longest chain of D_0. For example, design 10-6.1 in Example 3 has one chain of length 1 (the intercept), 8 chains of length 3, 4 chains of length 4 and 3 chains of length 5; hence, $\Lambda = 5$ and $alp = (1, 0, 8, 4, 3)$.

The information matrix of any f-PFD has the block diagonal structure

$$X'X = \begin{bmatrix} X_1'X_1 & 0 & \cdots & 0 \\ 0 & X_2'X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_g'X_g \end{bmatrix},$$

where each block $X_j'X_j$ corresponds to the effects in an alias chain of D_0. Since $\operatorname{rank}(X_j) \le f$, the information matrix must be singular if the maximum chain length Λ exceeds f.

In the next subsections, we discuss some distinctives of 3-, 4- and 5-PFDs for estimating (5).

5.1. *3-PFDs*. Let D be a 3-PFD with N runs and n factors constructed from D_0, and suppose that $\Lambda \le 3$ for model (5). If all D_0's defining words of length 4 or less change sign in D, then they must be of aliasing index $1/3$ and the model is estimable. (The aliasing index of each word is its J-characteristic in D, divided by N.) For X_j corresponding to an alias chain of length 2, $|X_j^T X_j/N| = 8/9$ and the two effects in this chain are partially aliased in D with variance inflation factors, VIFs $= \operatorname{diag}(X_j^T X_j/N)^{-1}$, of 9/8. For X_j corresponding to an alias chain of length 3, $|X_j^T X_j/N| = 16/27$ and the three effects partially aliased with each other have VIF $= 1.5$. Thus, the D-efficiency (Deff) of the design, $|X'X/N|^{1/q}$, equals

$$\operatorname{Deff}(D) = \{(8/9)^{\alpha_2}(16/27)^{\alpha_3}\}^{1/q},$$

where $q = 1 + n + n(n-1)/2$ and α_i is the number of alias chains of length i for D_0. For example, John's [16] 3-PFD with $N = 96$, $n = 10$ is based on D_0 with $alp = (11, 18, 3)$, so $\operatorname{Deff}(D) = 0.9362$. This is the D-optimal 3-PFD, since no D_0 with better *alp* exists.

TABLE 4

Two subsets from different groups but having the same absolute S-vectors

	$\tilde{s}_1$															
h_0^T	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
h_1^T	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
h_2^T	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
h_4^T	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
h_8^T	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
h_{15}^T	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	1	−1	−1	1
$S_6(s_1)$	6	2	2	2	2	2	2	−2	2	2	2	−2	2	−2	−2	−2

	$\tilde{s}_2$															
h_0^T	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
h_1^T	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
h_2^T	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
h_4^T	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
h_9^T	1	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1
h_{14}^T	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1	1	1	−1	−1
$S_6(s_2)$	6	2	2	−2	2	−2	2	−2	2	2	2	2	2	2	−2	−2

5.2. 4-*PFDs*. Let D be a 4-PFD with N runs and n factors constructed from D_0, and suppose that $\Lambda \leq 4$ for model (5). For 4-PFDs, the aliasing index of any defining word must be 1, 0.5 or 0. For model (5) to be estimable, having all D_0's defining words of length ≤ 4 to change sign is necessary but not sufficient; we must add the following condition: for each alias chain of length 4, no two flats have the same signs for the six words creating this chain. This is sufficient to ensure that X_j has full column rank. We now consider the VIFs for effects in alias sets of length 2, 3 and 4, when the model is estimable.

For each alias chains of length 2, let ξ_1 be the aliasing index of the corresponding word. Then $|X_j^T X_j/N| = 1 - \xi_1^2$, and the two VIFs are the reciprocal of this determinant. The possible values of the determinant are therefore 1 and 0.75. For alias chains of length 3, if all three words creating the alias chain are of aliasing index 0, the three effects in this chain are orthogonal and $|X_j^T X_j/N| = 1$. Otherwise, one effect, say E_1, will be partially aliased with the other two effects, E_2 and E_3, while E_2 is orthogonal to E_3; then E_1 has VIF $= 2$, E_2 and E_3 have VIF $= 1.5$, and $|X_j^T X_j/N| = 0.5$. In this case, the three words creating this chain have aliasing index distribution $(2_{1/2}, 1_0)$, where η_ρ denotes there are η words of aliasing index ρ in D and we call it the aliasing index distribution of this chain. An alias chain of length 4 is a consequence of 6 words in the defining relation for D_0; when a 4-column X_j has full column rank, there are three cases: (i) if the aliasing index distribution is (6_0) then this chain is clear of aliasing, each VIF $= 1$, and $|X_j^T X_j/N| = 1$, (ii) if the aliasing index distribution is $(4_{1/2}, 2_0)$ then each VIF $= 2$ and $|X_j^T X_j/N| = 0.25$, (iii) if the aliasing index distribution is $(3_{1/2}, 3_0)$ one effect will have VIF $= 4$ (it is partially aliased with the other three effects), while the other three have VIF $= 2$. As in case (ii), $|X_j^T X_j/N| = 0.25$.

Thus, when the model can be estimated, for chains of length l for $l = 2, 3, 4$, $|X_j^T X_j/N|$ is 1 only when all the effects in this chain are orthogonal with each other; otherwise, $|X_j^T X_j/N|$ is 0.75, 0.5, 0.25, respectively. The D-efficiency of D must be

$$\mathrm{Deff}(D) = \{(0.75)^{\alpha_2 - \alpha_{20}} (0.5)^{\alpha_3 - \alpha_{30}} (0.25)^{\alpha_4 - \alpha_{40}}\}^{1/q},$$

where α_{i0} is the number of alias chains of length i that correspond to orthogonal X_j matrices in D. Clearly, the D-efficiency has lower bound $\mathrm{Deff}_0(D) = \{(0.75)^{\alpha_2}(0.5)^{\alpha_3}(0.25)^{\alpha_4}\}^{1/q}$, and each clear chain of length i for $i = 2, 3, 4$ improves the $\mathrm{Deff}_0(D)$ by a factor $1.\bar{3}^{1/q}$, $2^{1/q}$, $4^{1/q}$, respectively.

5.3. 5-*PFDs*. Let D be a 5-PFD with N runs and n factors constructed from D_0, and suppose that $\Lambda \leq 5$ for model (5). For 5-PFDs, the aliasing index of each word is either 1, 3/5 or 1/5. For model (5) to be estimable, having all D_0's defining words of length ≤ 4 to change sign is necessary but not sufficient. We must impose some additional conditions for each alias set of length 4 and 5 as follows. For each chain of length 4, there exists a subset of 4 flats for which no two flats have the same signs for all six words creating this chain. For each chain of length 5: (i) no two flats have the same signs for the ten words creating this chain, and (ii) if the aliasing index distribution is $(2_{3/5}, 8_{1/5})$, the two words with aliasing index 3/5 must share a common effect. These are sufficient to ensure that X_j has full column rank. In the Appendix, we provide the VIFs for effects in alias sets of length 2, 3, 4 and 5 for each possible aliasing index distribution.

EXAMPLE 4 (Example 3 continued). Consider again the two 2^{10-6} designs in Example 3. Designs 10-6.1 and 10-6.2 have $\Lambda = 5$ and 4, respectively; each *alp* in given in Table 5. Thus, no 4-PFDs based on 10-6.1 can estimate model (5), while some generated by 10-6.2 do. The most efficient 4-PFDs from 10-6.2 (with $s = \{h_{15}, h_{22}, h_{51}\}$) have two clear chains of length 3 and a clear chain of length 4, so D-efficiency is $\{(0.3)^3(0.25)^9\}^{1/56} = 77.11\%$. Table 5 lists

TABLE 5
4-PFD and 5-PFDs for estimating the two-factor interaction model with 10 factors

D_0	alp	4-PFD		5-PFD	
		Deff(%)	cfv	Deff(%)	cfv
10-6.1	(1,0,8,4,3)	–	–	79.18	$[(0,0,8)_3,(0,6,12)_4,\dots]$
10-6.2	(1,0,5,10,0)	77.11	$[(0,4)_3,(0,9)_4,\dots]$	85.69	$[(0,0,9)_3,(0,3,13)_4,\dots]$

−, the model cannot be estimated with any 4-PFD corresponding to D_0.

the cfv of the D-optimal design having the least G-aberration. We also list the D-optimal 5-PFD from Designs 10-6.1 and 10-6.2. Note that the D-optimal 5-PFD from 10-6.1 (with $s=\{h_7,h_{17},h_{30},h_{45}\}$) is not the minimum aberration design given in Table 3, while the D-optimal 5-PFD from 10-6.2 does correspond to the 5-PFD in Table 3.

6. Parallel flats classes. By Method 1 presented in Section 4, for any given initial 2^{n-p} design, one can obtain all nonequivalent f-PFDs and select the preferred one, provided f and p are not too large. When p and/or f is large, however, it becomes computationally infeasible to identify all $g_{p,f}$ designs. In this section, we present a method to solve this problem by introducing parallel flats classes. The proposed method lessens the computational burden for finding a satisfactory design by arranging the defining words of smallest length to positions with the smallest aliasing indices: 0 for even f, $1/f$ for odd f. This can provide useful designs without an exhaustive search of the f-PFDs for a given D_0.

For $f\ge 4$, the $u_{p,f}$ distinct $|S_f(s)|$ vectors for a given p and f can be partitioned into separate classes. We now describe how this is determined, and how it provides an alternative means for searching for the desired f-PFD.

As an initial example, note that the 5-PFDs in Table 3 have $|S_5(s)|$ vectors with differing frequencies of 5, 3 and 1. The 5-PFD from 10-6.1 has frequencies (7, 8, 48) for the sums (5, 3, 1), while the 5-PFD from 10-6.2 has frequencies (3, 20, 40). This is not a property of D_0, but rather is a property of the columns of H^*. The first 5-PFD is based on $s_1=\{h_{15},h_{22},h_{37},h_{60}\}$. These columns of H^* yield 8 replicates of a resolution IV 2^{4-1} design. In contrast, the second 5-PFD in Table 3 is based on $s_2=\{h_7,h_{11},h_{29},h_{46}\}$; these four columns produce a full 2^4 replicated four times. Taking any set of four columns from H^* will produce an orthogonal design. Besides the full 2^4 and the resolution IV fraction, there exists the resolution III 2^{4-1}. By simple counting, one can partition $t_{p,5}$ into cases generated by these three: (i) $\prod_{i=0}^{3}(2^p-2^i)/4!$ subsets of four columns form a full 2^4 (replicated, if $p>4$); (ii) $\prod_{i=0}^{2}(2^p-2^i)/4!$ subsets of four columns form a resolution IV 2^{4-1} (replicated, if $p>3$); (iii) $\prod_{i=0}^{2}(2^p-2^i)/3!$ subsets of four columns form a resolution III 2^{4-1} (replicated, if $p>3$). For $p=6$, these products are 546,840, 9765 and 39,060, respectively, and these sum to $t_{6,5}=595{,}665$. Furthermore, the five designs in each equivalence group have the same 4-column structure. Thus, $g_{6,5}$ may be similarly partitioned; the number of equivalence groups are 109,368, 1953 and 7812.

A design's row coincidence distribution can be used to determine its word length pattern (Butler [6]). For regular fractions containing the treatment combination with all factors at $+1$, the row coincidence distribution is obtained by summing the rows of D_0 (Mee [21], Appendix J). For the resolution III and IV 2^{4-1} fractions corresponding to columns {1, 2, 4, 3} and {1, 2, 4, 7} of H_8 (numbering the columns 0–7), the row coincidence distributions are $(4,0,0,0,2,-2,-2,-2)$ and $(4,0,0,0,0,0,0,-4)$, respectively. The S-vector of an f-PFD is related to the row coincidence distribution as follows. Adding a vector of

$+1$'s (i.e., the first column of H_{2^p}) to these row coincidence distributions gives the vectors $(5, 1, 1, 1, 3, -1, -1, -1)$ and $(5, 1, 1, 1, 1, 1, 1, -3)$, respectively. While these vectors differ, their absolute values have the same distribution. The $|S_5(s)|$ vectors in Table 3 are for $p = 6$, which is just 8 repeats of the vectors for $p = 3$. If we take eight replicates of each vector, we have the frequencies $(8, 8, 48)$ for $(5, 3, 1)$. This matches the frequencies in $|S_5(s)|$ for the first design in Table 3. Thus, 5-PFDs produced by four columns of H^* forming a replicated resolution IV fraction 2^{4-1} or a replicated resolution III 2^{4-1} have the same distribution of values for $|S_5(s)|$. We now show that the resolution IV and III structures produce the same set of cfv's for 5-PFDs.

The S-vector corresponding to the 8-run resolution III fraction for $s = \{h_1, h_2, h_4, h_3\}$ is $S_5(s) = (5, 1, 1, 1, 3, -1, -1, -1)$. If we take h_7 as basic instead of h_4 and rearrange in Yates order, we have $(h_0, h_1, h_2, h_3, h_7, h_6, h_5, h_4)$. Rearranging the elements of $S_5(s)$ accordingly, we get the absolute S-vector for the resolution IV fraction. Since the rows of H_{64} are just the repeat of these same vectors 8 times, we have shown that by rearranging the choice of basic columns, the corresponding $|S_5(s)|$ vectors for the 5-PFDs from rows forming a resolution III and resolution IV fraction are equal. This leads to our definition of parallel flats classes, which will aid the search for low aberration f-PFDs.

DEFINITION 2. Two subsets of $f - 1$ columns of H^*, say s_1, s_2, are said to be of the same parallel flats class if they have identical absolute S-vectors up to the different Yates order for H_{2^p}, that is, we are allowed to reassign the basic rows of H^*.

Any subset $s = \{h_{i_1}, \dots, h_{i_{f-1}}\}$ corresponds to a regular orthogonal array (ROA) with 2^p runs and $f - 1$ (two-level) factors, with the definition $\text{ROA}(s) = [h_{i_1}, \dots, h_{i_{f-1}}]$. According to Definition 2, the parallel flats class of s remains unchanged under column permutations and row permutations of ROA(s). It is worth noting that here the row permutations is restricted to the different Yates order of rows of H_{2^p}. Hereafter, we refer to this as Yates order row exchange. For any two subsets s_1, s_2, if ROA(s_1) and ROA(s_2) are isomorphic, then one can be obtained from the other via the column permutations and Yates order row exchange only, where the factor level sign changes is impossible as the first row of every ROA is 1^T_{f-1}. Now, we present a theorem to show that by considering all nonisomorphic ROA, we obtain all $S_f(s)$ vectors.

THEOREM 5. *Two subsets of $f - 1$ columns of H^*, say s_1, s_2, have the same S-vector up to the Yates order exchange if* ROA(s_1) *and* ROA(s_2) *are isomorphic.*

PROOF OF THEOREM 5. If ROA(s_1) and ROA(s_2) are isomorphic, then ROA(s_1) can be obtained from ROA(s_2) by column permutations and Yates order row exchange. It is obvious that column permutations do not change the S-vector. Thus, s_1 and s_2 have the same S-vector up to the Yates order exchange. □

For any two choices of $f - 1$ rows of H^*, the isomorphic ROAs correspond to the same parallel flats class. However, the reverse is not true. As we saw for $f = 5$, two nonisomorphic ROAs may also correspond to the same parallel flats class. Thus, the number of parallel flats classes is smaller than the number of nonisomorphic regular OA$(2^p, f - 1)$. We can obtain all parallel flats classes by checking all nonisomorphic regular fractions with 2^p runs and $f - 1$ factors, including equally replicated designs. Specifically, we must consider all nonisomorphic unreplicated $2^{(f-1)-(f-1-\delta)}$ designs of resolution $\geq$ III, where $\lceil \log_2(f) \rceil \leq \delta \leq \min(p, f - 1)$. For example, with $p = 6$ and $f = 5$, we considered unreplicated 4-factor designs of size 2^δ for $\delta = 3$ and 4.

TABLE 6
Parallel flats classes for f-PFD with $4 \le f \le 6$

f	Parallel flats class[1]	δ[2]	Representative s[3]	ν[4]	$(f-1)$-PFD parent	Distribution of $\|S_f(s)\|$[5]
4	4.1.4	3	1, 2, 4	6	–	$\tau_1 = 2^{p-3} - 1$ $\tau_{0.5} = 2^{p-1}$ $\tau_0 = 3 \times 2^{p-3}$
	4.2.1	2	1, 2, 3	2	–	$\tau_1 = 2^{p-2} - 1$ $\tau_0 = 3 \times 2^{p-2}$
5	5.1.5	4	1, 2, 4, 8	24	4.1.4	$\tau_1 = 2^{p-4} - 1$ $\tau_{0.6} = 5 \times 2^{p-4}$ $\tau_{0.2} = 5 \times 2^{p-3}$
	5.2.5	3	1, 2, 4, 7 1, 2, 4, 3	24 6	4.1.4, 4.2.1	$\tau_1 = 2^{p-3} - 1$ $\tau_{0.6} = 2^{p-3}$ $\tau_{0.2} = 3 \times 2^{p-2}$
6	6.1.6	5	1, 2, 4, 8, 16	120	5.1.5	$\tau_1 = 2^{p-5} - 1$ $\tau_{2/3} = 3 \times 2^{p-4}$ $\tau_{1/3} = 15 \times 2^{p-5}$ $\tau_0 = 5 \times 2^{p-4}$
	6.2.6	4	1, 2, 4, 8, 15	120	5.1.5	$\tau_1 = 2^{p-4} - 1$ $\tau_{1/3} = 15 \times 2^{p-4}$
	6.3.6	4	1, 2, 4, 8, 7 1, 2, 4, 8, 3	24 12	5.1.5, 5.2.5	$\tau_1 = 2^{p-4} - 1$ $\tau_{2/3} = 2^{p-3}$ $\tau_{1/3} = 7 \times 2^{p-4}$ $\tau_0 = 3 \times 2^{p-3}$
	6.4.3	3	1, 2, 4, 3, 5	8	5.2.5	$\tau_1 = 2^{p-3} - 1$ $\tau_{1/3} = 3 \times 2^{p-3}$ $\tau_0 = 2^{p-1}$

[1] $f.x.y$, the xth parallel flats class for f-PFD and the corresponding f-PFD can be reduced to a y-PFD.
[2] δ, the number of the basic columns in the representative ROA(s). Given p, we are restricted to classes with $\delta \le p$.
[3] Representative s, column indices for one s for each nonisomorphic ROA in this parallel flats class.
[4] ν, $(2^p - 1)(2^p - 2)\dots(2^p - 2^{\delta-1})/\nu$ is the number out of all $t_{p,f}$ cases corresponding to this parallel flats class.
[5] τ_z, the number of $z * N$ in $|S_f(s)|$.

Table 6 lists all parallel flats classes for $f = 4, 5, 6$. In the Supplementary Material, we extend Table 6, providing all parallel flats classes for $f = 7, 8, 9$. Now we present a second method to search for the lowest G-aberration f-PFD based on D_0 using parallel flats classes.

Method 2 for finding a low aberration f-PFD from D_0 consists of the following steps:

1. For an initial 2^{n-p} design D_0, obtain $L = (L_1, \dots, L_{2^p-1})$, the length of each defining word arranged by Yates order. (This is the same as in Method 1.)
2. Obtain all parallel flats classes by checking all nonisomorphic orthogonal regular fractions with 2^p runs and $f - 1$ factors. (Our tables furnish these for $f \le 9$.)
3. For each class, choose a member, say s, and assign as many of the defining words of shorter length as possible to the positions with aliasing index 0 or $1/f$, depending on whether f is even or odd, according to L and s. In this way, we can get the least G-aberration f-PFD based on this parallel flats class.

4. Choose the best one from all parallel flats classes to obtain the best f-PFD based on D_0.

We illustrate Method 2 with the following example.

EXAMPLE 5 (Example 3 continued). Let D_0 be design 10-6.1, with word length pattern (8, 18, 16, 8, 8, 5). We seek the lowest G-aberration 6-PFD based on D_0. Since $g_{6,6} > 10^6$, we use the short-cut Method 2. From Table 6, there are four classes for $f = 6$, $p = 6$, and representative members can be selected as $s_1 = \{h_1, h_2, h_4, h_8, h_{16}\}$, $s_2 = \{h_1, h_2, h_4, h_8, h_{15}\}$, $s_3 = \{h_1, h_2, h_4, h_8, h_7\}$, $s_4 = \{h_1, h_2, h_3, h_4, h_5\}$, respectively. Class 6.2.6 can be excluded since it has no J-characteristics of 0. For s_1, there are 20 J-characteristics of 0. Given design 10-6.1's L shown in Table 3, all 8 length-3 defining words can be assigned to J-characteristics of 0 under several orders, so all complete words of length 3 are removed in the resulting 6-PFD. Second, consider the defining words of length 4 under these Yates orders, and choose the orders that perform best. This is continued until we have assigned all the lower-order words. In this way, one can obtain the best 6-PFD in term of G-aberration based on the first parallel flats class 6.1.6. Taking the columns of H^* according to s_1 and the rows rearranged by taking rows 7, 11, 21, 13, 3, 54 as basic, we obtain a 6-PFD with cfv = $[(0, 0, 18)_4, (0, 8, 0)_5, \ldots]$. For more details, see the Appendix. The same operation can be applied for classes 6.3.6 and 6.4.3. Table 7 lists the Yates row order and cfv corresponding to the best 6-PFD under each of these three classes, as well as the results for taking D_0 to be Chen, Sun and Wu's [8] designs 10-6.2, 10-6.3 and 10-6.4. From Table 7, the 6-PFDs with lowest G-aberration based on these different initial designs come from different classes.

Thus, Method 2 provides an alternative way to search for low aberration f-PFDs by assigning the defining words of smaller length to positions having the lowest aliasing indices. Contrary to Method 1 in Section 4, we need not compute the absolute S-vectors for all $g_{6,6} = 1{,}176{,}357$ groups.

For $f < 9$, no two parallel flats classes have the same aliasing index distributions. However, as seen in the Supplementary Material, classes 9.11.9 and 9.12.9 have the same aliasing index distribution. The following example illustrates that the resulting 9-PFDs are indeed different.

EXAMPLE 6. We take as representative members of parallel flats classes 9.11.9 and 9.12.9 subsets $s_{11} = \{h_1, h_2, h_4, h_8, h_{16}, h_{15}, h_{19}, h_{21}\}$ and $s_{12} = \{h_1, h_2, h_4, h_8, h_{16}, h_5, h_{15}, h_{19}\}$. For $p = 5$, both classes have aliasing index distribution (0, 0, 3, 13, 15). Let D_0 be the resolution II 2^{7-5} design corresponding to columns $\{1, 2, 3, 1, 3, 2, 1\}$ of H_{32}; this D_0 has word length pattern (0, 5, 12, 7, 4, 3). Since $\Lambda = 8$, one must have at least 8 flats in order to estimate all main effects and two-factor interactions. We take $f = 9$ and so construct 36-run designs. For both s_{11} and s_{12}, all 5 defining words of length 2 and 7 of the 12 words of length 3 can be assigned to the minimum aliasing index (1/9) under several orders. The least G-aberration design from s_{12} has cfv = $[(0, 0, 0, 0, 5)_2, (0, 0, 0, 5, 7)_3, (0, 0, 1, 4, 2)_4, (0, 0, 0, 3, 1)_5, \ldots]$; this design also has the best D-efficiency for 9-PFDs from class 9.12.9 for this D_0. By contrast, the lowest G-aberration 9-PFD from class 9.11.9 has cfv = $[(0, 0, 0, 0, 5)_2, (0, 0, 0, 5, 7)_3, (0, 0, 1, 4, 2)_4, (0, 0, 2, 1, 1)_5, \ldots]$ and Deff =72.44%; a 9-PFD with higher aberration but Deff = 75.81% maximizes D-efficiency for class 9.11.9. In each case, the best design from class 9.12.9 is superior.

TABLE 7

Basic rows for reordering H^* and cfv for lowest G-aberration 6-PFD *from each class*

Class	D_0 = 10-6.1	D_0 = 10-6.2	D_0 = 10-6.3	D_0 = 10-6.4
6.1.6	{7 11 21 13 3 54}√	{7 11 21 13 22 55}	{7 11 22 13 28 51}√	{7 11 21 32 13 22}
	$[(0, 0, 18)_4, (0, 8, 0)_5, \dots]$	$[(0, 0, 16)_4, (0, 8, 0)_5, \dots]$	$[(0, 0, 15)_4, (0, 8, 0)_5, \dots]$	$[(1, 0, 15)_4, (0, 6, 0)_5, \dots]$
6.3.6	{9 10 13 11 17 44}	{9 10 13 29 46 1}√	{9 10 13 11 27 44}	{9 10 13 19 11 46}
	$[(1, 0, 17)_4, (0, 4, 0)_5, \dots]$	$[(0, 0, 16)_4, (0, 6, 0)_5, \dots]$	$[(1, 0, 14)_4, (0, 4, 0)_5, \dots]$	$[(1, 0, 15)_4, (0, 4, 0)_5, \dots]$
6.4.3	{1 3 5 13 20 35}	{1 3 5 13 23 34}	{1 3 5 9 19 39}	{1 3 5 14 21 39}√
	$[(2, 0, 16)_4, (0, 0, 0)_5, \dots]$	$[(1, 0, 15)_4, (0, 0, 0)_5, \dots]$	$[(1, 0, 14)_4, (0, 0, 0)_5, \dots]$	$[(1, 0, 15)_4, (0, 0, 0)_5, \dots]$

√ marks the best 6-PFD for each D_0 among all classes. Class 6.2.6 is ignored, since it cannot remove any words.

7. Quaternary code construction. Xu and Wong [36] present a series of QC designs D of size $N = 4^b$ and $n \le 4^b - 2^b$ having generalized resolution at least 3.5. Each design for even n is constructed from a quaternary linear code as follows. Let $G = (\gamma_1, \dots, \gamma_{n/2})$ be a $b \times n/2$ matrix over $Z_4 = \{0, 1, 2, 3\} (\bmod 4)$. All possible linear combinations of the rows in G over Z_4 form a quaternary linear code, denoted by the $4^b \times n/2$ matrix Q. To obtain a two-level design in n factors, apply the Gray map

$$0 \to (-1, -1), \qquad 1 \to (-1, 1), \qquad 2 \to (1, 1), \qquad 3 \to (1, -1).$$

That is, each element in Z_4 is replaced with a pair from -1 and 1. Designs with an odd number of factors are obtained by omitting one column. From these designs, a series with $N = 2^{2b+1}$ is constructed having the form $[D\ D;\ D\ -D]$. We now show that both of these series produce 2^b-PFDs.

THEOREM 6. *Designs with 4^b treatment combinations constructed from a quaternary linear code are parallel flats designs consisting of 2^b flats of size 2^b, where the $n - b$ defining words are $\{d_1^{\varrho_{1j}} d_3^{\varrho_{2j}} \cdots d_{2b-1}^{\varrho_{bj}} d_j : j = 2, 4, \dots, 2b, 2b+1, \dots, n\}$ with $\varrho_{ij} = \gamma_{i\lfloor (j+1)/2 \rfloor}$ for $i = 1, 2, \dots, b$. Here, $\gamma_{i\lfloor (j+1)/2 \rfloor}$ represents the ith component of column $\gamma_{\lfloor (j+1)/2 \rfloor}$. The defining word associated with d_j is taken to be*

$$\begin{cases} (-1)^{\lfloor 2^{-1} \sum_{i=1}^{b} \varrho_{ij} \varphi_{il} \rfloor} & \text{if } j \text{ is odd}, \\ (-1)^{\lfloor 2^{-1} (1 + \sum_{i=1}^{b} \varrho_{ij} \varphi_{il}) \rfloor} & \text{if } j \text{ is even}, \end{cases}$$

with φ_{il} equal to 0 or 1, and defined by $l - 1 = \sum_{i=1}^{b} \varphi_{il} 2^{b-i}$ in the lth flat ($l = 1, \dots, 2^b$).

COROLLARY 1. *The design $[D\ D;\ D\ -D]$ consists of the same number of flats as D.*

Now we illustrate Theorem 6 for the case $b = 4$ and $n = 16$.

EXAMPLE 7. Consider the 4×8 matrix from Xu and Wong's [36] Example 2:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 \end{bmatrix}.$$

All linear combinations of the rows of G over Z_4 form a 256×8 quaternary linear code Q. Then a 256×16 design $D = (d_1, \dots, d_{16})$ can be obtained via the the Gray map from Q. As shown in Xu [35], D is of generalized resolution 6.5 and projectivity 7, where a two-level design is said to have projectivity χ if any χ-factor projection contains a complete 2^χ factorial design, possibly with some points replicated, and χ is the largest integer having that property (Box and Tyssedal [5]). This is a very useful nonregular design since every regular design of this size has poorer resolution and projectivity.

Now, we study this QC design from the perspective of a PFD. According to Theorem 6, D is a 16-PFD defined by

$$C = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1
\end{bmatrix}
\begin{matrix}
d_1 d_2 \\ d_3 d_4 \\ d_5 d_6 \\ d_7 d_8 \\ d_3 d_5 d_7 d_9 \\ d_3 d_5 d_7 d_{10} \\ d_1 d_3 d_7 d_{11} \\ d_1 d_3 d_7 d_{12} \\ d_1 d_3 d_5 d_{13} \\ d_1 d_3 d_5 d_{14} \\ d_1 d_5 d_7 d_{15} \\ d_1 d_5 d_7 d_{16}
\end{matrix}.$$

Each of the 16 flats is a resolution II, 2^{16-12}, where the four basic factors are $\{d_1, d_3, d_5, d_7\}$ and the 12 generators are $\{d_j = \pm d_1^{\varrho_{1j}} d_3^{\varrho_{2j}} d_5^{\varrho_{3j}} d_7^{\varrho_{4j}} : j = 2, 4, 6, 8, 9, \dots, 16\}$, with $\varrho_{ij} = \gamma_{i\lfloor (j+1)/2 \rfloor}$ for $i = 1, 2, 3, 4$. Consider $j = 2$; since $\lfloor (j+1)/2 \rfloor = 1$, we look to $\gamma_1 = [1, 0, 0, 0]^T$, the first column of G. Thus, $[\varrho_{12}, \varrho_{22}, \varrho_{32}, \varrho_{42}] = \gamma_1^T = [1, 0, 0, 0]$ so $d_2 = \pm d_1$. This implies that d_2 appears in the length-2 defining word $d_1 d_2$. We list all 12 defining words in the right-hand side of C. Furthermore, the sign of the defining word associated with d_j ($j = 2, 4, 6, 8, 9, \dots, 16$) and flat l ($l = 1, \dots, 2^4$) is

$$\begin{cases} (-1)^{\lfloor 2^{-1}(\varrho_{1j}\varphi_{1l} + \varrho_{2j}\varphi_{2l} + \varrho_{3j}\varphi_{3l} + \varrho_{4j}\varphi_{4l}) \rfloor} & \text{if } j \text{ is odd}, \\ (-1)^{\lfloor 2^{-1}(1 + \varrho_{1j}\varphi_{1l} + \varrho_{2j}\varphi_{2l} + \varrho_{3j}\varphi_{3l} + \varrho_{4j}\varphi_{4l}) \rfloor} & \text{if } j \text{ is even}, \end{cases}$$

with $l - 1 = \sum_{i=1}^{4} \varphi_{il} 2^{b-i}$. Note that φ_{il} is the coefficient of 2^{b-i} when $l - 1$ is converted to a binary number. For example, $[\varphi_{12}, \varphi_{22}, \varphi_{32}, \varphi_{42}] = [0, 0, 0, 1]$ for $l = 2$ as $1 = 0 * 2^3 + 0 * 2^2 + 0 * 2^1 + 1 * 2^0$. For $j = 2$, the defining word $d_1 d_2$'s sign is

$$(-1)^{\lfloor 2^{-1}(\varrho_{12}\varphi_{1l} + \varrho_{22}\varphi_{2l} + \varrho_{32}\varphi_{3l} + \varrho_{42}\varphi_{4l}) \rfloor} = (-1)^{\lfloor 2^{-1}(1 + \varphi_{1l}) \rfloor}$$

in the lth flat for $l = 1, \dots, 2^4$. The coefficient of 2^3 is 0 for any element in $0 \le l - 1 \le 7$ and 1 for $8 \le l - 1 \le 15$, $\varphi_{1l} = 0$ and $\lfloor 2^{-1}(1 + \varphi_{1l}) \rfloor = 0$ for $l = 1, \dots, 8$ while $\varphi_{1l} = 1$ and $\lfloor 2^{-1}(1 + \varphi_{1l}) \rfloor = 1$ for $l = 9, \dots, 16$. This defines the first row of C above.

The first $b = 4$ rows of C correspond to the length 2 defining words, while the remaining words are length 4. Thus, each flat is an even design; that is, it contains eight fold-over pairs of runs. The last 4 columns of G each involve three odd values; for instance, $\gamma_8 = [1, 2, 1, 3]^T$ so the generators for d_{15} and d_{16} are defined by $\pm d_1 d_5 d_7$. Since a different entry is even in $\gamma_5, \dots, \gamma_8$, all four three-factor interactions are used as generators. Interestingly, if one drops the last row of G, the 3×8 matrix generates Xu and Wong's [36] recommended 64-run, 16-factor design. Thus, for some cases, there is a nesting of G matrices for good QC designs as b increases.

Quaternary code designs of size $N = 4^2$ are of class 4.1.4 or 4.2.1, that is, either a 4-PFD or a regular fraction. QC designs of size $N = 4^3$ are from class 8.2.8, 8.8.8, 8.11.4 or 8.12.1. QC designs of size 4^4 fall into 14 different classes. While it is prohibitive to identify all classes for 16-PFDs, the Supplementary Material lists the 14 classes for 256-run QC designs. Therein, classes 16.Q7.16 and 16.Q8.8 have the same aliasing index distribution and so do classes 16.Q9.16 and 16.Q10.8. Example 7 is of class 16.Q8.8 (see Table S.2 of the Supplementary Material); once reduced, it is class 8.2.8.

Xu and Wong [36] present QC designs with $N = 256$ and $n \le 64$, with generalized resolution of 4 for the designs with 31 or more factors. They acknowledge the limitations of their forward selection method, as they were unable to obtain any resolution 4 QC design with more than 68 factors. However, following their Lemma 1, for any QC design with $N = 4^b$, there are 4^{b-1} eligible columns of G with an odd number of odd values. A resolution 4 QC design with $n = N/2$ factors is produced from G consisting of these 4^{b-1} columns. This produces as even design, that is, $D = [H_n; -H_n]$, where H_n is an order n Hadamard matrix. The Supplementary Material gives the results for $b = 4$: the 4×64 matrix G and code to generate H_{128}. Larger QC designs with the maximal number of factors can be constructed similarly. For example, the $N = 4^5$, $n = 512$ QC design is constructed taking G as the 5×256 matrix having all eligible columns with 1, 3 or 5 odd entries.

8. Constructing split-plot parallel flats designs. Split-plot designs are widely used in industrial applications when there are some processing factors that are difficult, expensive or time-consuming to change from one level to another. In such situations, a useful strategy is to implement some or all runs with a specific level combination of such hard-to-change factors in succession. We call the hard-to-change factors whole-plot factors and the others subplot factors. One can find more general theory about split-plot designs in Addelman [2], Wooding [34] and Bingham and Sitter [3]. In this section, we provide methods for constructing D-efficient split-plot designs for model (5), making use of parallel flats designs.

Note that when taking the initial regular design D_0 to be a resolution I fraction, $n_w \geq 1$ factors are constant within a flat. The resulting f-PFD can be run as a split-plot design by randomizing the order of the flats and then randomizing the runs within a flat. The n_w factors whose level remains unchanged in each flat are the whole-plot factors while the other $n_s = n - n_w$ factors are the subplot factors. From Section 3, all $g_{p,f}$ nonequivalent D can be obtained and from which we can identify the D-optimal one. (See Appendix C for enumerating Resolution I and Resolution II regular fractions.) This is a straightforward method for constructing D-optimal split-plot design when p is not too large. However, it becomes computationally infeasible to identify all $g_{p,f}$ nonequivalent designs when p and/or f is large. Now we present a method to solve this problem by constructing the whole-plot and subplot designs separately.

Method 3 is for constructing a design with f whole plots of size 2^{n_s-p}.

1. Obtain a D-optimal two-level design of size f for the n_w whole plot factors.
2. Let $D_{0,s}$ be an initial 2^{n_s-p} design of the subplot factors with resolution $\geq$ II. Then according to Section 3, obtain all $g_{p,f}$ nonequivalent f-PFDs.
3. For each one of the $g_{p,f}$ nonequivalent f-PFDs, assign one whole-plot treatment combination to the h_0 flat of the subplot design, and assign the remaining $f-1$ treatment combinations to flats in $(f-1)!$ different ways. Each produces a split-plot design with n_w whole-plot factors, n_s subplot factors and $f2^{n_s-p}$ runs.
4. Choose the best one from all $(f-1)!g_{p,f}$ split-plot designs for estimating model (5) by a comprehensive examination.

We add several remarks regarding Method 3:

- Step 1 may be repeated with other nonisomorphic D-optimal designs for the whole-plot factors.
- In step 2, the initial design for the subplot factors is a 2^{n_s-p} design of resolution $\geq$ II. The enumeration of resolution II designs is presented in Appendix C.
- If $g_{p,f}$ is very large and some of the $g_{p,f}$ nonequivalent f-PFDs have the same cfv, one may reduce the number of subplot designs considered in Step 3 by removing f-PFDs with duplicate cfv's to obtain $v_{p,f}$ candidate designs for the split-plot factors, where $v_{p,f}$ is the number of the unique cfv's based on $D_{0,s}$.
- Using Method 1, there are $g_{p+n_w,f}$ cases to be considered for a given D_0. With Method 3, we reduce it to $(f-1)!g_{p,k}$ (or $(f-1)!v_{p,k}$) by considering whole-plot and subplot designs separately.

We illustrate Method 3 for a split-plot design having 3 whole-plot factors and 7 split-plot factors in 8 whole plots of size 8.

EXAMPLE 8. With Method 1, let D_0 be a 2^{10-7} design of resolution I, where 3 factors (W_1, W_2, W_3) have unchanged level and the remaining 7 factors ($S_1, \dots, S_7$) form the 2^{7-4}_{III} design. Then there are $g_{7,8} = 1.1170e^{10}$ nonequivalent 8-PFDs to be examined, which is computationally infeasible. Consider Method 3 instead. First, specify the 2^3 design as the design for the whole-plot factors. Each whole plot consists of a resolution III, 2^{7-4} design in the 7 split plot-factors. There are $g_{4,8} = 870$ nonequivalent 8-PFDs based on $D_{0,s}$, the initial design just for the split-plot factors, and they produce $v_{4,8} = 24$ unique cfv's. Now for each of the 24 8-PFDs with unique cfv's, assign the level combination (1, 1, 1) to the first flat, and consider all possible cases of assigning the other seven combinations into the remaining flats. This produces $7!v_{4,8} = 120{,}960$ eligible split-plot designs. By a comprehensive search, the 64 treatment combinations defined by

$$C = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ S_1S_2S_4 \\ S_1S_3S_5 \\ S_2S_3S_6 \\ S_1S_2S_3S_7 \end{matrix}$$

is a D-efficient 8-PFD for estimating model (5). Since no regular 2^{10-4} design can estimate (5), the literature does not list a split-plot design of this size. However, there are algorithms for generating split plot designs. JMP, implementing the algorithm of Jones and Goos [17], produced a design with D-efficiency only 0.33% higher than our 8-PFD. This illustrates the usefulness of the parallel flats designs for constructing split-plot designs.

9. Designs composed of flats from different families. The split plot design in the previous section can also be constructed by combining two 2^{10-5} fractions. Flats 1, 6, 7, 8 form a regular 2^{10-5} with defining words $\{W_1, W_2W_3S_1S_2S_4, W_2W_3S_1S_3S_5, W_3S_2S_3S_6, W_2S_1S_2S_3S_7\}$, while flats 2 – 5 have defining words $\{-W_1, W_2S_1S_2S_4, -W_2S_1S_3S_5, W_2W_3S_2S_3S_6, S_1S_2S_3S_7\}$. Each 2^{10-5} has a defining contrast subgroup of size 32, but these subgroups share the following independent words: W_1, $S_2S_3S_4S_5$ and $W_2W_3S_1S_6S_7$. Thus, the combined group has $5 + 5 - 3 = 7$ independent words, so the 64-run design can be partitioned into eight 2^{10-7} fractions from the same family.

The parallel flats structure for any design constructed by concatenating two 2^{n-p} regular fractions is readily understood in terms of the size of the combined defining contrast subgroup; for our example that size is 2^7.

THEOREM 7. *Let 2^o be the size of the combined defining contrast subgroup for two regular 2^{n-p} fractions. Note that $o \leq n$. If $o < n$, then the concatenated design consists of $f = 2^{1+o-p}$ parallel flats of size 2^{n-o}.*

Pajak [23] and Pajak and Addelman [24] present series of designs for estimating model (5) based on concatenating regular 2^{n-p} designs from different families. Recently, Vazquez and Xu [32] proposed a new class of nonregular designs by concatenating f 2^{n-p} designs from different families formed by permuting a subset of basic factors. They provided many strength-3 nonregular designs of this class with large run sizes. Based on the same D_0, let the resulting designs from Vazquez and Xu [32] and this article be D_{VX} and D, respectively. According to Theorems 1 and 2 in Vazquez and Xu [32], D_{VX} has π complete words, and $f(2^p - \pi - 1)$ partial words with aliasing index of $1/f$, where π is the number of generating words of specific types for D_0. By our method, D have $2^p - 1$ words with aliasing index in $\{f/f, (f-2)/f, \dots, (f - 2\lfloor f/2 \rfloor)/f\}$. Note that when f is even, $(f - 2\lfloor f/2 \rfloor)/f = 0$, indicating that some complete words in D_0 can be removed in D. For instance, in Example 3, take D_0 as design 10-6.1, then all eight words of length 3 can be removed in D. In this way, we can get strength-t PFDs based on D_0 with lower strength, which is not the case for the concatenated designs from Vazquez and Xu [32]; that is, if a strength-t design D_{VX} is needed, then the underlying design D_0 must be of strength t or higher. In addition, the concatenated designs from Vazquez and Xu [32] may have repeat runs while our PFDs do not (unless a flat is repeated). Thus, PFDs with distinct flats are more suitable for computer experiments where the simulations are deterministic.

In some cases, D_{VX} will be an $f2^i$-PFD for some $i \geq 1$, as presented in the following theorem.

THEOREM 8. *Let D_0 be a regular 2^{n-p} design, and $D_1, \dots, D_{f-1}$ be $f-1$ isomorphic copies by applying linear permutation l_i to the factors in a subset of basic factors (whose cardinality is a prime number) for $i = 1, 2, \dots, f-1$. Given $Z_b = \{1, \dots, b\}$, for an integer u, a linear permutation l_u over Z_b is the permutation such that $l_u(x) = (x - 1 + u)(\operatorname{mod} b) + 1$, for $x \in Z_b$. Then the concatenated design $D_{VX} = (D_0^T, D_1^T, \dots, D_{f-1}^T)^T$, is a $f2^{o-p}$-PFD, where each flat is a 2^{n-o} design and 2^o is the size of the group constructed from all $f(2^p - 1)$ words of $D_0, D_1, \dots$ and D_{f-1}.*

We illustrate with a small example.

EXAMPLE 9. Given $n = 9$, $p = 4$, $f = 3$, let D_0 be the minimum aberration 2_{IV}^{9-4} design with basic factors 1, 2, 3, 4 and 5, and generators $6 = 123$, $7 = 124$, $8 = 125$ and $9 = 1345$ (provided by the supplemental material of Vazquez and Xu [32]).

The defining relation of D_0 is

$$
\begin{aligned}
I &= 1236 = 1247 = 1258 = 13459 \\
&= 3467 = 3568 = 24569 = 4578 = 23579 = 23489 \\
&= 12345678 = 15679 = 14689 = 13789 = 26789.
\end{aligned}
\tag{6}
$$

Consider D_1, an isomorphic copy of D_0 formed by applying the linear permutation l_1 to the set of basic factors. The permutation transforms $1 \to 2$, $2 \to 3$, $3 \to 4$, $4 \to 5$ and $5 \to 1$. The defining relation of D_1 is

$$
\begin{aligned}
I &= 2346 = 2357 = 1238 = 12459 \\
&= 4567 = 1468 = 13569 = 1578 = 13479 = 34589 \\
&= 12345678 = 12679 = 25689 = 24789 = 36789.
\end{aligned}
\tag{7}
$$

Let D_2 be the second isomorphic copy formed by applying l_2 to the set of basic factors, where l_2 maps $1 \to 3$, $2 \to 4$, $3 \to 5$, $4 \to 1$ and $5 \to 2$. The defining relation of D_2 is

$$
\begin{aligned}
I &= 3456 = 1347 = 2348 = 12{,}359 \\
&= 1567 = 2568 = 12469 = 1278 = 24579 = 14589 \\
&= 12345678 = 23679 = 13689 = 35789 = 46789.
\end{aligned}
\tag{8}
$$

We can see that each of the defining relations (6), (7) and (8) can be generated by the first four independent words, and these three defining relations have 12345678 in common. Furthermore, all these 43 unique words can be generated by the following eight independent words (i.e., $o = 8$): $w_1 = 1236$, $w_2 = 1247$, $w_3 = 1258$, $w_4 = 13{,}459$, $w_5 = 2346$, $w_6 = 2357$, $w_7 = 12459$ and $w_8 = 12359$. For example, $1238 = w_1 w_2 w_3 w_5 w_6$, $3456 = w_1 w_2 w_4 w_6 w_7$, $1347 = w_2 w_4 w_7$ and $2348 = w_2 w_3 w_6$. In this way, each of D_0, D_1 and D_2 can be divided into 16 2^{9-8} parallel flats according to the signs of these eight words. Specifically, design D_0 can be divided into 16 2^{9-8} design by the 16 combinations

$$
\{(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8) : w_1 = w_2 = w_3 = w_4 = I, w_5, w_6, w_7, w_8 = \pm I\},
$$

indicating D_0 is a 16-PFD defined by

$$C_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{matrix} .$$

D_1 can be divided into 16 2^{9-8} design by the 16 combinations

$$\{(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8) : w_5 = w_6 = w_7 = w_1 w_2 w_3 w_5 w_6 = I\},$$

that is, D_1 is a 16-PFD defined by

$$C_1 = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{matrix} .$$

D_2 can be divided into 16 2^{9-8} design by the 16 combinations

$$\{(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8) : w_8 = w_1 w_2 w_4 w_6 w_7 = w_2 w_4 w_7 = w_2 w_3 w_6 = I\},$$

that is, D_2 is a 16-PFD defined by

$$C_2 = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{matrix} .$$

Obviously, all these 48 2^{9-8} are from the same family, so $D_{VX} = (D_0^T, D_1^T, D_2^T)^T$, is a 48-PFD defined by $C = (C_0, C_1, C_2)$. It is easily checked that there is one column $(1, 1, 1, 1, 1, 1, 1, 1)^T$ with frequency 3, and three columns $(1, -1, -1, -1, 1, 1, 1, 1)^T$, $(1, 1, 1, 1, -1, 1, 1, 1)^T$ and $(1, 1, 1, 1, 1, 1, 1, -1)^T$ with frequency 2 in C. This means that one flat is repeated three times and three flats are repeated twice in the resulting 48-PFD.

The 48-PFD above has one complete word, 12345678. Let D_{-1} denote the design obtained by reversing the sign of factor 8 in D_1. By concatenating D_0, D_{-1} and D_2, we obtain the 9-factor 96-run design of strength 3 shown in their Table 1. For this design 12345678 becomes a partial word with aliasing index 1/3 (see Table S.10 in the supplemental material of Vazquez and Xu [32]). D_{-1} is a 16-PFD with

$$C_{-1} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{matrix} ,$$

where C_{-1} is obtained by switching the signs of the rows of C_1 corresponding to words containing factor 8 in D_1. Correspondingly, the resulting design $D'_{VX} = (D_0^T, D_{-1}^T, D_2^T)^T$ is a 48-PFD defined by $C' = (C_0, C_{-1}, C_2)$. It is easily checked that there are two columns $(1, 1, 1, 1, 1, 1, 1, 1)^T$ and $(1, 1, 1, 1, -1, 1, 1, 1)^T$ with frequency 2 in C', indicating that two flats repeat twice.

10. Summary and future work. In this article, we present a theory enumerating all f-PFD for any $f \geq 3$. Two methods were given to search for the best f-PFD, given an initial design. The first method is exhaustive in its search, which works efficiently for the cases $f \leq 10$ with $p = 3, 4$, $f \leq 8$ with $p = 5$, $f \leq 5$ with $p = 6$, $f \leq 4$ with $p = 7$ or 8. For other cases, when $g_{p,f}$ is too large to list all possible S-vectors, we can search within each parallel flats class, as discussed in Section 6, for an assignment that minimizes the aliasing index for all the shortest words of D_0's defining relation. These methods are feasible, easy to implement and useful in both theoretical research and practical applications. We leave for future work creation of an algorithm that automates Method 2.

Several authors have advocated using parallel flats designs with partial replication, in order to provide a pure error estimate of the error variance; see the recent paper by Tsai and Liao [31] and the references therein. Earlier, we calculated the number of possible f-PFDs, $t_{p,f}$, assuming that the f flats were distinct. One can relax that requirement so that the methods proposed in this article cover cases where a specified number of flats are repeated. Most simply, suppose the flat corresponding to h_0 appears $r + 1$ times ($r > 0$), while $f - 1$ other flats each appear once. This produces a $f.r$-PFD with $f.r = f + r$. Note that the number of combinations remains unchanged; that is, $t_{p,f.r} = t_{p,f}$ independent of r. One must adjust the S-vector as follows: $S_{f.r}(s) = S_f(s) + r * 1_{2^p-1}$. This partial replication provides $r2^{n-p}$ degrees of freedom for estimating the error variance. We leave further details to the reader.

Another research direction worth pursuing is to consider constructing a parallel flats design by augmenting an arbitrary initial design. The task would be to specify the size of the flats (i.e., specify p), and then determine the minimal number of runs to create a PFD.

APPENDIX A: VIFS FOR 5-PFDS

For a 5-PFD D, we present the VIFs for effects in alias sets of length 2, 3, 4 and 5, when the model (5) is estimable. Table A.1 lists the different VIF values, according to the aliasing index distribution of the defining words for each alias chain from D_0. Table A.1 also lists the value of $|X_j^T X_j|^{1/\lambda}$, where λ is the number of columns of X_j; the D-efficiency of D is the geometric mean of these numbers. The case IDs are λ.x, where λ denotes the size of the alias set and x is the number of aliasing indices of 3/5. In two cases, described below, the VIFs depend on the pattern of aliasing indices, not only on their frequency. It is interesting to note that the determinants are equal for several different aliasing index distribution cases. Surprisingly, case 5.5, which has the most aliasing indices of 3/5 has the third best VIFs for alias sets of size 5.

For both $\lambda = 4$ and 5, there is one aliasing index distribution where the VIFs depend on the pattern of the correlations in the $X_j^T X_j/N$ matrix. Let E_i ($i = 1, \dots, \lambda$) denote the effects corresponding to the λ columns of X_j. Cases 4.3a and 4.3b are differentiated as follows. If one effect E_i appears in all three words with aliasing index 3/5, then we have case 4.3b and E_i has VIF $= 35/8$. Cases 5.4a and 5.4b are differentiated as follows. If one effect E_i appears in all four words with aliasing index 3/5, then we have case 5.4b and E_i has VIF $= 10$.

APPENDIX B: DETAILS FOR EXAMPLE 5

With $p = 6$, there are $\prod_{i=0}^{5}(64 - 2^i) = 20.16$ billion different Yates row orders for class 6.1.6. Design 10-6.1 has eight length-3 words appearing in positions 1, 2, 4, 7, 8, 20, 44, 48

TABLE A.1
Block efficiencies and VIF*s for* 5*-PFDs*

Length of chain: λ	Case	Aliasing index distribution	$\lvert X_j^T X_j/N\rvert^{1/\lambda}$	VIF distribution
2	2.0	$(1_{1/5})$	0.979796	$(2_{25/24})$
	2.1	$(1_{3/5})$	0.800000	$(2_{25/16})$
3	3.0	$(3_{1/5})$	0.964057	$(3_{15/14})$
	3.1	$(1_{3/5}, 2_{1/5})$	0.800000	$(2_{15/8}, 1_{5/4})$
	3.2	$(2_{3/5}, 1_{1/5})$	0.726848	$(1_{5/2}, 2_{5/3})$
4	4.0	$(6_{1/5})$	0.951366	$(4_{35/32})$
	4.1	$(1_{3/5}, 5_{1/5})$	0.800000	$(2_{35/16}, 2_{5/4})$
	4.2	$(2_{3/5}, 4_{1/5})$	0.672717	$(1_{35/8}, 2_{5/2}, 1_{15/8})$
	4.3a	$(3_{3/5}, 3_{1/5})$	0.672717	$(2_{5/2}, 2_{15/8})$
	4.3b	$(3_{3/5}, 3_{1/5})$	0.672717	$(1_{35/8}, 3_{15/8})$
5	5.0	$(10_{1/5})$	0.940863	$(5_{10/9})$
	5.1	$(1_{3/5}, 9_{1/5})$	0.800000	$(2_{5/2}, 3_{5/4})$
	5.2	$(2_{3/5}, 8_{1/5})$	0.606287	$(1_{10}, 2_5, 2_{5/2})$
	5.3	$(3_{3/5}, 7_{1/5})$	0.606287	$(2_5, 3_{5/2})$
	5.4a	$(4_{3/5}, 6_{1/5})$	0.606287	$(1_5, 4_{5/2})$
	5.4b	$(4_{3/5}, 6_{1/5})$	0.606287	$(1_{10}, 4_{5/2})$
	5.5	$(5_{3/5}, 5_{1/5})$	0.606287	$(5_{5/2})$

f_v denotes the number of effects, f, with VIF $= v$.

of L. Fixing rows 7, 11, 21 as the first three basic rows defines the Yates order for the first 7 rows as {7, 11, 12, 21, 18, 30, 25}. This eliminates the first four words of length 3 and reduces the number of choices to $\prod_{i=3}^{5}(64 - 2^i) = 86{,}016$. Rows 13, 3, 54 for the next basic rows produced the lowest G-aberration of all these remaining orders. The resulting 6-PFD has cfv $= [(0, 0, 18)_4, (0, 8, 0)_5, (1, 0, 7)_6, (0, 4, 0)_7, (0, 0, 5)_8]$ and C matrix

$$C = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \begin{matrix} F_1F_2F_5 \\ F_1F_3F_6 \\ F_2F_3F_7 \\ F_1F_4F_8 \\ F_2F_3F_4F_9 \\ F_1F_2F_3F_4F_{10} \end{matrix}$$

Taking several other choices for the first 3 rows resulted in designs with identical or worse cfv.

APPENDIX C: CONSTRUCTION METHOD OF TWO-LEVEL FRACTIONAL FACTORIAL DESIGNS WITH SMALL RUNS

Chen, Sun and Wu [8] proposed an algorithm for constructing complete sets of regular designs with resolution $R \geq$ III. In this section, we generalize the algorithm to obtain complete sets of regular designs with resolution $R =$ II and $R =$ I. In this way, we can obtain all nonequivalent parallel flats designs for any fixed size by considering all regular designs with resolution $R \geq$ I for D_0.

C.1. Basic idea. Two designs are isomorphic if one can be obtained from the other by row permutations, column permutations and sign switches of columns (Hedayat, Sloane and Stufken [14]). Any regular 2^{n-p} design with $R \geq$ III is isomorphic to a subset of the columns of $H_{2^{n-p}}$; we take the $b = n - p$ basic columns with indices $1, 2, \dots, 2^{n-p-1}$ and p additional columns. When considering designs with $R = \mathrm{II}$, two or more columns are identical. Thus, for $R \geq \mathrm{II}$, there are $(2^b - 1)^p$ cases to be considered. Let $D_{n,p}$ denote the complete set of nonisomorphic, unreplicated 2^{n-p} designs with resolution $\geq \mathrm{II}$. We apply a sequential construction method that does not need to traverse all $(2^b - 1)^p$ cases to obtain $D_{n,p}$. Given $D_{n-1,p-1}$, we construct $D_{n,p}$ by assigning the additional factor to any column of H_{2^b} (besides h_0). There are $2^{n-p} - 1$ ways to assign this factor. Therefore, we obtain a class of designs, denoted by $\tilde{D}_{n,p}$ with cardinality

$$\{\#\text{ of designs in } D_{n-1,p-1}\} \times (2^{n-p} - 1).$$

Clearly, $\tilde{D}_{n,p} \supset D_{n,p}$. However, some designs in $\tilde{D}_{n,p}$ are isomorphic, so we need the following isomorphism check.

C.2. Isomorphism check. To identify isomorphic regular designs, we can first partition all designs into different categories according to their word length patterns with the definition $\mathrm{wlp} = (a_2, a_3, \dots)$, where a_i is the number of length i words in the defining contrast subgroup. Designs with different word length patterns are obviously nonisomorphic. Therefore, we only need to examine the isomorphism of designs with the same word length pattern.

Here, we use an isomorphism check similar to that proposed in Chen, Sun and Wu [8]. The difference is that we take designs with duplicate columns into account while they do not. Consider the two 2^{5-3} designs $D_1 = \{h_1, h_2, h_1h_2, h_1h_2, h_1h_2\}$ and $D_2 = \{h_1, h_2, h_1, h_1, h_1h_2\}$. Each of these designs has $\mathrm{wlp} = (3, 3, 0, 1)$. To check isomorphism between D_1 and D_2, we apply the following 4-step algorithm:

1. Select two independent columns from D_2, say, $\{h_1h_2, h_2\}$. There are three choices.
2. Select a relabeling map from $\{h_1h_2, h_2\}$ to $\{h_1^*, h_2^*\}$, that is, $h_1^* = h_1h_2$, $h_2^* = h_2$. There are two choices.
3. Write the remaining columns $\{h_1, h_1, h_1\}$ in D_2 as interactions of $\{h_1^*, h_2^*\}$, that is, $h_1 = h_1^*h_2^*$. Then D_2 can be written as $\{h_1^*, h_2^*, h_1^*h_2^*, h_1^*h_2^*, h_1^*h_2^*\}$.
4. Compare the new representation of D_2 with that of D_1. If they match, they are isomorphic, and the process stops. Otherwise, return to step 2 and try another map of $\{h_1^*, h_2^*\}$. When all the relabeling maps are exhausted, return to step 1 and find another two independent columns.

With this algorithm, a map will be found eventually if two designs are isomorphic. If there is no such map, then the two designs are nonisomorphic. Using this algorithm, we find all 2^{n-p} designs with $R \geq 2$. For designs of size 2, there is only one independent column, h_1, and each $D_{n,p}$ consists of a single member. For designs of size 4 and 8, see the Supplementary Material, where Table S.2 (S.3) lists the 4-run (8-run) designs in $D_{n,p}$ for $n = 3, \dots, 11$ ($n = 4, \dots, 11$).

Now consider the set of nonisomorphic 2^{n-p} designs with resolution $R = \mathrm{I}$, denoted by $D_{n,p}^{\mathrm{I}}$. Any regular design of $R = \mathrm{I}$ becomes a design of $R \geq \mathrm{II}$ after removing every column that equals h_0. Thus, $D_{n,p}^{\mathrm{I}}$ can be obtained by adding column h_0 once to every design in $D_{n-1,p-1}$, twice to every design in $D_{n-2,p-2}, \dots, p$ times to $D_{n-p,0}$. Note that $D_{n-p,0}$ contains just one design, the full 2^b. Obviously, all these designs are nonisomorphic with each other. Thus, $D_{n,p}^{\mathrm{I}}$ has cardinality

$$\sum_{i=1}^{p} \{\#\text{ of designs in } D_{n-i,p-i}\}.$$

For example, there are $16\ (= 6 + 4 + 3 + 2 + 1)$ nonisomorphic 2^{7-5} designs with resolution I.

In summary, the generalized method for getting complete sets of regular designs with resolution $R \geq$ I is efficient. And it ensures the acquisition of complete nonequivalent parallel flat design for any fixed size by taking the initial regular design to be all these regular designs with resolution $R \geq$ I.

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SUPPLEMENTARY MATERIAL

Supplementary Material to "Two-level parallel flats designs" (DOI: 10.1214/21-AOS2071SUPP; .pdf). Table S.1 lists all parallel flats classes for $f = 7, 8, 9$, extending the parallel flats classes in Table 6. Table S.2 lists the 12 possible parallel flats classes for QC designs of 256 runs. Table S.3 lists the 4-run designs with resolution $R \geq$ II for $n = 3, \ldots, 11$ while Table S.4 lists the 8-run designs with resolution $R \geq$ II for $n = 4, \ldots, 11$. S5, S6 and S7 provide the proofs of Theorems 2, 3 and 6, respectively, while S8 provides the 4×64 matrix G and code to generate H_{128} in Section 7.

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