

PARTIAL IDENTIFIABILITY OF RESTRICTED LATENT CLASS MODELS

BY YUQI GU* AND GONGJUN XU**

Department of Statistics, University of Michigan, *yuqigu@umich.edu, **gongjun@umich.edu

Latent class models have wide applications in social and biological sciences. In many applications, prespecified restrictions are imposed on the parameter space of latent class models, through a design matrix, to reflect practitioners' assumptions about how the observed responses depend on subjects' latent traits. Though widely used in various fields, such restricted latent class models suffer from nonidentifiability due to their discreteness nature and complex structure of restrictions. This work addresses the fundamental identifiability issue of restricted latent class models by developing a general framework for strict and partial identifiability of the model parameters. Under correct model specification, the developed identifiability conditions only depend on the design matrix and are easily checkable, which provide useful practical guidelines for designing statistically valid diagnostic tests. Furthermore, the new theoretical framework is applied to establish, for the first time, identifiability of several designs from cognitive diagnosis applications.

1. Introduction and motivation. Latent class models are widely used in social and biological sciences to model unobserved discrete latent attributes. These models often assume each latent class represents a configuration of the targeted latent attributes that can explain the observed responses. In many applications, prespecified restrictions are imposed on the parameter space of the latent class model, through a design matrix. These restrictions reflect practitioners' understanding about how the responses depend on the underlying latent attributes. This paper studies such a family of *restricted* latent class models, which have been widely employed in various fields. The following are several examples.

(1) *Cognitive diagnosis in educational assessment.* Restricted latent class models play a key role in cognitive diagnosis modeling in educational and psychological assessment. Cognitive diagnosis aims to make a classification-based decision on an individual's latent attributes, based on his or her observed responses to a set of designed diagnostic items (questions). In the majority of models, the latent classes are characterized by profiles of the binary states of mastery/deficiency of the targeted ability attributes, while there are also models that allow polytomous ordinal attributes [39]. The restricted structure usually comes from the design matrix that specifies what latent attributes each item measures (e.g., [11, 22, 24, 31]). See Section 2.2 for several data examples, including the Test of English as a Foreign Language (TOEFL) (e.g., [39]) and Trends in International Mathematics and Science Study.

(2) *Psychiatric evaluation.* Restricted latent class models have also been used in psychiatric evaluation. Here, the responses are manifested symptoms and the latent classes represent the profiles of presence/absence of a set of underlying psychological or psychiatric disorders. The restricted structure results from the fact that each symptom may be shared by multiple disorders, which are specified by psychiatric diagnosis guidelines. See examples in [23, 36] and [13].

(3) *Disease etiology detection.* Another application of restricted latent class models is the diagnosis of disease etiology in epidemiology [44]. Here, the observed responses are

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imperfect measurements of subjects' biological samples, and the latent classes are the configurations of existence or nonexistence of a set of pathogens underlying a certain disease. The restricted structure naturally arises from the fact that each measurement may only target certain pathogens.

Despite the popularity of the restricted latent class models, the fundamental identifiability issue is challenging to address. Model identifiability is a prerequisite for making statistical inferences. The study of identifiability of latent class models dates back to decades ago [18, 30, 34]. For unrestricted latent class models, [21] showed the model is not identifiable in the sense that, there always exists some set of parameters, such that one can construct a different set of parameters which lead to the same distribution of the responses. Such nonidentifiability has likely impeded statisticians from looking further into this problem [2]. Due to the difficulty of establishing strict identifiability in such scenarios, [16] and [2] studied the *generic* identifiability of these models. The idea of generic identifiability is closely related to concepts in algebraic geometry and implies that the model parameters are identifiable almost everywhere in the parameter space, excluding only a Lebesgue measure zero set. Allman et al. [2] established generic identifiability results for various latent variable models, including the unrestricted latent class models.

The complex constraints of the restricted latent class models pose additional challenge to the study of model identifiability. The existing results of generic identifiability in [2] do not apply to restricted latent class models, because the restrictions imposed by the design matrix already constrain the model parameters of a restricted latent class model into a measure-zero (and hence potentially unidentifiable) subset of the parameter space of an unrestricted latent class model. To address the identifiability issue under restrictions, [45] proposed a set of sufficient conditions for identifiability of a family of restricted latent class models. However, a key assumption in [45] is that the design matrix has to satisfy a certain structural constraint and that the latent class space has to be saturated (see Section 2.3 for more details), which is often difficult to meet and may even be unrealistic in practice; see examples in [11, 12, 22, 23, 25] and many others. The same strong assumption is also imposed in [46] for identifiability of the design matrix. Therefore, the existing theory is hardly applicable to popular designs in the literature, and the previously proposed conditions may not serve as good guidelines for future test designing. Moreover, the techniques developed as in [45] for specific presumable design structure are not applicable to general designs. The fundamental identifiability issues of the restricted latent class models remain largely underexplored and call for new identifiability theory.

This paper proposes a general framework of strict and partial identifiability for restricted latent class models. Practical sufficient conditions for strict and partial identifiability are proposed and their necessity is discussed. In particular, depending on the two different types of algebraic structures of restricted latent class models, we introduce and study two useful notions of partial identifiability, respectively (see Sections 3 and 4). The established identifiability results are widely applicable in practice, by relaxing most of the constraints imposed on the design matrix. Moreover, under correct model specification, all the identifiability conditions only depend on the design matrix and are easily checkable by practitioners. We apply the new theory to several existing designs and establish identifiability under them for the first time in the literature.

The rest of the paper is organized as follows. Section 2 introduces the general model setup of restricted latent class models, including model and data examples in cognitive diagnosis applications; and then discusses the limitations of the existing studies. Sections 3 and 4 present our main identifiability results. Section 5 includes extensions of the new theory to some more complicated models. Section 6 gives a further discussion, and proofs of the theoretical results are presented in the Supplementary Material [20].

2. Model setup, examples and identifiability issues. We start with the setup for a latent class model with binary responses. Suppose there are J dichotomous items, denoted by the item set $\mathcal{S} = \{1, \dots, J\}$. For any subject, the observed variables are his/her binary responses to the J items, denoted by $\mathbf{R} = (R_1, \dots, R_J)^\top \in \{0, 1\}^J$. To model the distribution of the responses, we assume there are m latent classes existing in the population denoted by $\mathcal{A} = \{\alpha_0, \dots, \alpha_{m-1}\}$, where $m > 1$ is assumed known. For any $\alpha \in \mathcal{A}$, we use $p_\alpha = P(\mathbf{A} = \alpha)$ to denote the proportion of subjects in the population that belong to class α . Under this specification, we have $p_\alpha \in (0, 1)$ and $\sum_{\alpha \in \mathcal{A}} p_\alpha = 1$. In the application of cognitive diagnosis, a latent class α usually denotes a knowledge state characterized by a profile of mastery/deficiency of a set of latent attributes, and is represented by a binary vector (see Section 2.1).

Assume that a subject's latent class membership \mathbf{A} follows a categorical distribution with population proportion parameters $\mathbf{p} = (p_\alpha, \alpha \in \mathcal{A})$. Given a subject's latent class membership \mathbf{A} , the responses $\mathbf{R} = (R_1, \dots, R_J)$ are assumed to be conditionally independent and each R_j follows a Bernoulli distribution with the positive response probability $\theta_{j,\alpha} = P(R_j = 1 \mid \mathbf{A} = \alpha)$. This local independence is a common assumption in latent class modeling (e.g., [1, 2]). We call these $\theta_{j,\alpha}$'s as the item parameters, and write $\Theta = (\theta_{j,\alpha}; j \in \mathcal{S}, \alpha \in \mathcal{A})$, which is a $J \times m$ matrix. The rows of Θ are indexed by the J items in \mathcal{S} , and the columns by the m latent classes in \mathcal{A} . The model parameters are then characterized by \mathbf{p} and Θ .

We focus on a general family of restricted latent class models that are popularly used in various social and biological applications. Under these restricted latent class models, the item parameters in Θ are restricted by certain prespecified structures to reflect experts' understanding or hypotheses on how the responses to each diagnostic item depend on the latent classes. In particular, the restricted latent class models assume that for any item j , there exists an item-specific set of latent classes \mathcal{C}_j ; and the classes in \mathcal{C}_j share the same value of positive response probability, which is higher than those of the other latent classes. We denote such a set of latent classes by

$$(2.1) \quad \mathcal{C}_j = \left\{ \alpha \in \mathcal{A} : \theta_{j,\alpha} = \max_{\alpha^* \in \mathcal{A}} \theta_{j,\alpha^*} \right\}.$$

The latent classes in \mathcal{C}_j then correspond to those subjects who are “most capable” of giving a positive response to item j , and for each $j \in \mathcal{S}$,

$$(2.2) \quad \max_{\alpha \in \mathcal{C}_j} \theta_{j,\alpha} = \min_{\alpha \in \mathcal{C}_j} \theta_{j,\alpha} > \theta_{j,\alpha'} \quad \forall \alpha' \notin \mathcal{C}_j.$$

Additionally, it is assumed that there exists a universal “least capable” class α_0 such that $\theta_{j,\alpha} \geq \theta_{j,\alpha_0}$ for any $\alpha \in \mathcal{A}$ and $j \in \mathcal{S}$. Note that a latent class α' satisfying $\alpha' \notin \mathcal{C}_j$ and $\theta_{j,\alpha'} > \theta_{j,\alpha_0}$ can be viewed as “partially capable.”

Different restricted latent class models specify the Θ and the constraint sets \mathcal{C}_j 's differently to respect the underlying scientific assumptions. To illustrate this, we present various model examples and real data examples in Sections 2.1 and 2.2. In Section 2.3, we discuss the identifiability issue and limitations of the existing works, which call for the new identifiability theory.

2.1. *Restricted latent class models in cognitive diagnosis.* The restricted latent class models have recently gained great interests in cognitive diagnosis with applications in educational assessment, psychiatric evaluation and many other disciplines (e.g., [6, 10, 11, 31, 43]). Cognitive diagnosis is the process of arriving at a classification-based decision about an individual's latent attributes, based on the observed surrogate responses. Such diagnostic

information plays an important role in constructing efficient, focused remedial strategies for improvement in individual performance.

The restricted latent class models are important statistical tools in cognitive diagnosis to detect the presence or absence of multiple fine-grained attributes. Cognitive diagnosis models in the psychometrics literature mostly consist of binary attributes, while general diagnostic models with categorical attributes were also considered in [39]. In this work, we focus on the case of binary attributes. Specifically, consider a cognitive diagnosis test with J items designed to measure K binary latent attributes. Under the introduced model setup, a latent class α is represented by a configuration of the K latent attributes, denoted by a K -dimensional binary vector $\alpha = (\alpha_1, \dots, \alpha_K)$, where $\alpha_k \in \{0, 1\}$ denotes the deficiency or mastery of the k th attribute. A latent class α is also called an attribute profile. The latent class space \mathcal{A} is a subset of $\{0, 1\}^K$. If $\mathcal{A} = \{0, 1\}^K$, we say \mathcal{A} is *saturated*, which means the population contain subjects with all the possible configurations of attribute profiles. The universal least capable latent class α_0 corresponds to the all-zero attribute profile, that is, $\alpha_0 = (0, \dots, 0)$.

The restrictions in cognitive diagnosis models is encoded by the so-called Q -matrix [33]. A Q -matrix $Q = (q_{j,k})$ is a $J \times K$ matrix with binary entries $q_{j,k} \in \{0, 1\}$ indicating the absence or presence of the dependence of the j th item on the k th attribute. Generally, $q_{j,k} = 1$ means that item j requires the mastery of attribute k to solve and $q_{j,k} = 0$ means the opposite. The j th row vector q_j of Q , called the q -vector, gives the attribute requirements of item j . See examples of Q -matrices in Section 2.2.

In the following, we review some popular cognitive diagnosis models and illustrate how they fall into the family of restricted latent class models. We first introduce some notation. For two vectors $a = (a_1, \dots, a_K)$, $b = (b_1, \dots, b_K)$ of the same dimension K , we write $a \succeq b$ if $a_i \geq b_i$ for all $i = 1, \dots, K$; and $a \succcurlyeq b$ if $a \succeq b$ and $a \neq b$. Denote $a - b = (a_1 - b_1, \dots, a_K - b_K)$ and $a \vee b = (\max\{a_1, b_1\}, \dots, \max\{a_K, b_K\})$. We also denote the all-zero and all-one vectors by $\mathbf{0}$ and $\mathbf{1}$, respectively.

EXAMPLE 2.1 (Conjunctive DINA and disjunctive DINO). The Deterministic Input Noisy output “And” gate (DINA) model proposed in [24] and the Deterministic Input Noisy output “Or” gate (DINO) model proposed in [36] are popular and basic diagnostic models, which adopt the conjunctive and disjunctive assumptions, respectively. Specifically, under DINA, a subject needs to master all the required attributes of an item to be “capable” of it, and mastering the attributes not required by the item will not compensate for the lack of required ones. That is, the required attributes of an item act “conjunctively” and the positive response probability is

$$\theta_{j,\alpha}^{\text{DINA}} = \begin{cases} 1 - s_j & \text{if } \alpha \succeq q_j, \\ g_j & \text{otherwise,} \end{cases}$$

where s_j is the slipping parameter, which denotes the probability that a capable subject slips the positive response, and g_j is the guessing parameter, which denotes the probability that a noncapable subject coincidentally gives the positive response by guessing. Under DINO, a subject only needs to master one of the required attributes to be “capable” of an item. That is, the required attributes of an item act “disjunctively” and

$$\theta_{j,\alpha}^{\text{DINO}} = \begin{cases} 1 - s_j & \text{if } \exists k \text{ s.t. } \alpha_k = q_{j,k} = 1, \\ g_j & \text{otherwise,} \end{cases}$$

where s_j and g_j are the slipping and guessing parameters. Both the DINA and DINO models assume $1 - s_j > g_j$ for all j .

The DINA and DINO models are restricted latent class models with appropriately defined constraint sets \mathcal{C}_j 's. Specifically, under the conjunctive DINA model, the \mathcal{C}_j defined in (2.1) takes the form of

$$(2.3) \quad \mathcal{C}_j = \{\alpha \in \mathcal{A} : \alpha \succeq \mathbf{q}_j\}, \quad j \in \mathcal{S};$$

while under the disjunctive DINO model, the \mathcal{C}_j defined in (2.1) becomes $\mathcal{C}_j = \{\alpha \in \mathcal{A} : \text{if } \exists k \text{ s.t. } \alpha_k = q_{j,k} = 1\}$ for $j \in \mathcal{S}$.

EXAMPLE 2.2 (Main-effect cognitive diagnosis models). An important family of cognitive diagnosis models assume that the $\theta_{j,\alpha}$ depends on the main effects of those attributes required by item j , but not their interactions. This family include the popular reduced Reparameterized Unified Model (reduced-RUM; [15]), Additive Cognitive Diagnosis Models (ACDM; [11]), the Linear Logistic Model (LLM; [28]) and the General Diagnostic Model (GDM; [39]). We call them the main-effect cognitive diagnosis models. In particular, under the reduced-RUM, $\theta_{j,\alpha}^{\text{RUM}} = \theta_j^+ \prod_{k=1}^K r_{j,k}^{q_{j,k}(1-\alpha_k)}$, where $\theta_j^+ = P(R_j = 1 | \alpha \succeq \mathbf{q}_j)$ represents the positive response probability of a capable subject of j , and $r_{j,k} \in (0, 1)$ is the parameter penalizing not possessing attribute k required by item j . Equivalently, the item parameter in reduced-RUM can be written as $\log \theta_{j,\alpha}^{\text{RUM}} = \beta_{j,0} + \sum_{k=1}^K \beta_{j,k}(q_{j,k}\alpha_k)$, where $\beta_{j,k} \geq 0$ for $q_{j,k} = 1$. Similarly, the ACDM assumes the parameter $\theta_{j,\alpha}$ can be written as the linear combination of the main effects of the required attributes: $\theta_{j,\alpha}^{\text{ACDM}} = \beta_{j,0} + \sum_{k=1}^K \beta_{j,k}(q_{j,k}\alpha_k)$. The LLM assumes a logistic link function with $\text{logit}(\theta_{j,\alpha}^{\text{LLM}}) = \beta_{j,0} + \sum_{k=1}^K \beta_{j,k}(q_{j,k}\alpha_k)$. These main-effect models are restricted latent class models, and under them, the \mathcal{C}_j defined in (2.1) takes the form of $\mathcal{C}_j = \{\alpha \in \mathcal{A} : \alpha \succeq \mathbf{q}_j\}$.

EXAMPLE 2.3 (All-effect cognitive diagnosis models). Another popular type of cognitive diagnosis models assume that the positive response probability depends on the main effects and the interaction effects of the required attributes of the item. We call these models the all-effect cognitive diagnosis models, of which the GDINA model [11], the log-linear cognitive diagnosis models (LCDM; [22]) and the general diagnostic model (GDM; [39]) are examples. It was recently shown in [40] and [41] that the GDINA and LCDM can be rewritten as GDMs with extended skill space. In particular, given a Q -matrix, denote the set of attributes required by item j by $\mathcal{K}_{q_j} = \{1 \leq k \leq K : q_{j,k} = 1\}$, then the item parameter under GDINA is

$$(2.4) \quad \theta_{j,\alpha}^{\text{GDINA}} = \sum_{S \subseteq \mathcal{K}_{q_j}} \beta_{j,S} \prod_{k \in S} \alpha_k,$$

where $\beta_{j,S} \geq 0$. Note that the DINA model is a submodel of the GDINA model by setting all the $\beta_{j,S}$ coefficients in (2.4), other than $\beta_{j,\emptyset}$ and $\beta_{j,\mathcal{K}_{q_j}}$, to zero. Similar to the GDINA model, the LCDM adopts the logistic link function and assumes that $\text{logit}(\theta_{j,\alpha}^{\text{LCDM}}) = \sum_{S \subseteq \mathcal{K}_{q_j}} \beta_{j,S} \prod_{k \in S} \alpha_k$. The all-effect models are restricted latent class models, and under them the \mathcal{C}_j in (2.1) also takes the form $\mathcal{C}_j = \{\alpha \in \mathcal{A} : \alpha \succeq \mathbf{q}_j\}$.

When the latent class space \mathcal{A} is saturated with $\mathcal{A} = \{0, 1\}^K$, we have $m = |\mathcal{A}| = 2^K$. In practice, however, this may not always hold. For instance, researchers may assume there exist additional restrictions on the dependence structure among the latent attributes, such as an attribute hierarchy with some attributes being the prerequisite for some others [26, 35]. A hierarchical structure among the K attributes would reduce the number of possible attribute profiles from 2^K to m ($m < 2^K$), by excluding those not respecting the hierarchy. For example, consider a diagnostic test with $K = 2$ attributes. If it is scientifically reasonable to

assume the first attribute is the prerequisite for the second one, then the latent class space is reduced to $\mathcal{A} = \{(0, 0), (1, 0), (1, 1)\}$ with $m = |\mathcal{A}| = 3$, since the attribute profile $(0, 1)$ does not respect this hierarchy. Note that as shown in [42], a cognitive diagnosis model with such a linear hierarchy can equivalently reduce to a located latent class model with $m < 2^K$ classes.

In this work, we assume the latent class space \mathcal{A} is prespecified. This would be the case when practitioners have solid scientific reasons or prior knowledge from exploratory data analysis to assume certain structure among attributes. This work aims to answer the question that for an arbitrary $\mathcal{A} \subseteq \{0, 1\}^K$, what kind of conditions would guarantee identifiability of Θ and $\boldsymbol{p} = (p_\alpha, \alpha \in \mathcal{A})$.

All the cognitive diagnosis models reviewed in Examples 2.1–2.3 are restricted latent class models. We call them the *Q-restricted latent class models*, since the C_j 's and model constraints are further determined by the Q -matrix. Moreover, we call the DINA and the DINO models the *two-parameter Q-restricted latent class models*, since each item has exactly two item parameters, and we call the main-effect and all-effect models as *multi-parameter Q-restricted latent class models*.

2.2. *Real data examples.* To further illustrate the constraints induced by the design matrix, we present several applications that utilize restricted latent class models as cognitive diagnosis modeling tools.

EXAMPLE 2.4 (TOEFL internet-based testing data). TOEFL, short for Test of English as a Foreign Language, is a standardized test to measure English language ability of nonnative speakers. Restricted latent class models have been used to analyze the TOEFL data by researchers at Educational Testing Service (ETS; e.g., [38, 39]). For instance, [39] proposed a general diagnostic model (GDM), which was used to analyze the TOEFL reading section of two parallel forms, A and B, with their Q -matrices analyzed and specified by content experts. In particular, the forms A and B contain 39 and 40 items with four latent attributes: α_1 : *Word meaning*, α_2 : *Specific information*, α_3 : *Connect information* and α_4 : *Synthesize and organize*. Table 1 gives the summary of the two Q -matrices by presenting each \boldsymbol{q} -vector's frequencies in them. For instance, the first line in Table 1 reads $(1, 0, 0, 0)$ for the row \boldsymbol{q} -vector and $(9, 9)$ for the frequencies. This means that there are nine items with \boldsymbol{q} -vector $(1, 0, 0, 0)$ in form A and nine in form B, respectively. Under the restrictions induced by the Q -matrices, the diagnostic models used to analyze the TOEFL data fall in the family of restricted latent class models.

TABLE 1
TOEFL iBT field test Q -matrix entry frequencies, Reading Forms A and B

Word meaning	Q -matrix row \boldsymbol{q} -vectors			\boldsymbol{q} -vector frequency	
	Specific information	Connect information	Synthesize and organize	Form A	Form B
1	0	0	0	9	9
0	1	0	0	8	11
1	1	0	0	1	1
0	0	1	0	10	10
1	0	1	0	0	1
0	1	1	0	2	0
0	1	0	1	1	0
0	0	1	1	7	8
1	0	1	1	1	0

EXAMPLE 2.5 (Trends in international mathematics and science study). Trends in International Mathematics and Science Study (TIMSS) is a large scale cross-country assessment, administered by the International Association for the Evaluation of Educational Achievement. TIMSS evaluates the mathematics and science abilities of fourth and eighth graders every 4 years since 1995 and covers more than 40 countries. The TIMSS data allows one to analyze trends in student progress that can provide feedback for future improvement in areas needing further instruction [25]. Researchers have used the cognitive diagnosis models to analyze the TIMSS data (e.g., [9, 25, 48]). For instance, a 43×12 Q -matrix constructed by mathematics educators and researchers was specified for the TIMSS 2003 eighth grade mathematics assessment [9]. A total number of 12 fine-grained attributes are identified, which fall in five big categories of skill domains measured by the eighth grade exam, Number, Algebra, Geometry, Measurement and Data. The Q -matrix is presented in Table 1 in the Supplementary Material [20]. Choi et al. [9] used DINA model to fit the dataset containing responses sampled from 8912 U.S. and 5309 Korean students. Main-effect and all-effect diagnostic models have also been applied to analyze the TIMSS data (e.g., [48]).

EXAMPLE 2.6 (Fraction subtraction data). The dataset contains 536 middle school students' binary responses to 20 fraction subtraction items that were designed for diagnostic assessment. The Q -matrix contains eight attributes (the 20×8 Q -matrix is presented in Table 2 in the Supplementary Material [20]). Many researchers have used various restricted latent class models to fit this dataset (e.g., [11, 12, 14, 22]).

2.3. *Concept of identifiability and issues with existing works.* Though widely used in various applications, the identifiability issue of restricted latent class models remains largely unaddressed. We next introduce the concept of identifiability and discuss the limitations of the existing theory.

For the introduced restricted latent class models, the probability mass function of the response pattern \mathbf{R} is

$$(2.5) \quad P(\mathbf{R} = \mathbf{r} \mid \Theta, \mathbf{p}) = \sum_{\alpha \in \mathcal{A}} p_{\alpha} \prod_{j=1}^J \theta_{j,\alpha}^{r_j} (1 - \theta_{j,\alpha})^{1-r_j}, \quad \mathbf{r} \in \{0, 1\}^J.$$

Following the definition of identifiability in the literature (e.g., [3]), the model parameters (Θ, \mathbf{p}) of a restricted latent class model are identifiable if for any $(\bar{\Theta}, \bar{\mathbf{p}})$ in the parameter space \mathcal{T} , there is no $(\bar{\Theta}, \bar{\mathbf{p}}) \neq (\Theta, \mathbf{p})$ such that

$$(2.6) \quad \mathbb{P}(\mathbf{R} = \mathbf{r} \mid \Theta, \mathbf{p}) = \mathbb{P}(\mathbf{R} = \mathbf{r} \mid \bar{\Theta}, \bar{\mathbf{p}}) \quad \text{for all } \mathbf{r} \in \{0, 1\}^J.$$

In the following, we also say that the model parameters are *strictly* identifiable if the above condition holds.

To establish model identifiability, a strong and often impractical assumption made by previous works is that the Q -matrix must contain at least one $K \times K$ identity submatrix I_K up to some row permutation, that is, the Q -matrix must contain all K distinct single-attribute \mathbf{q} -vectors [7, 19, 45, 47]. A Q -matrix satisfying this requirement is also said to be complete under the DINA model [8]. For general Q -restricted latent class models including the multi-parameter models, [45] requires at least two disjoint $K \times K$ identity submatrices in Q to establish identifiability. However, in practice, in the existence of a large number of fine-grained attributes and complex cognitive process, a Q -matrix rarely satisfies such requirements. For the TOEFL data in Example 2.4, in both Q -matrices, there does not exist any item that solely requires the fourth skill attribute. For the Q -matrix of the TIMSS data in Example 2.5, only three attributes (1, 7 and 8) out of twelve are measured by some single-attribute items. For

the Q -matrix in Example 2.6, there are only two attributes (2 and 7) out of eight measured by some single-attribute items. Many other examples can be found in the literature (e.g., [11, 22, 23, 25]). Moreover, another strong assumption made in existing works [19, 45] is that $\mathcal{A} = \{0, 1\}^K$, that is, $p_\alpha > 0$ for any $\alpha \in \{0, 1\}^K$, which fails when some attribute profiles are deemed impossible to exist.

Such identifiability issues of cognitive diagnosis models have long been recognized [12, 14, 29, 32, 39, 41, 49]. For instance, [39] pointed out in the study of the TOEFL data that larger numbers of skills (i.e., K) very likely pose problems with identifiability, unless the number of items per skill is “sufficiently” large. But given the complicated structure of constraints, how the number of items and the form of the design matrix influence identifiability is still an open problem in the literature.

This work addresses this open problem by developing a general theoretical framework based on a key technical tool, the *indicator matrix* Γ . Under an arbitrary restricted latent class model, we define Γ to be a $J \times m$ matrix using the sets \mathcal{C}_j 's. The Γ -matrix has the same size as the matrix Θ , with rows indexed by items in \mathcal{S} , and columns by latent classes in \mathcal{A} . The (j, α) th entry of Γ is

$$(2.7) \quad \Gamma_{j,\alpha} = I(\alpha \in \mathcal{C}_j), \quad j \in \mathcal{S}, \alpha \in \mathcal{A},$$

which is a binary indicator of whether α is “most capable” to give a positive response to j . For $\alpha \in \mathcal{A}$, denote the α th column vector of Γ by $\Gamma_{\cdot,\alpha}$. The Γ -matrix defined this way turns out to be a useful tool for developing the identifiability theory, and it helps to relax many of the existing strong assumptions, as shown later in Sections 3.1 and 4.1. Indeed, most of our identifiability conditions can be represented as requirements on the structure of Γ , since the information of which latent classes achieve the highest level of $\theta_{j,\alpha}$ of item j is what our theoretical derivations essentially rely on. Depending on two different algebraic structures of the restricted parameter spaces, we next consider two types of restricted latent class models and present their identifiability results in Sections 3 and 4, respectively.

3. Identifiability results for two-parameter models. This section considers two-parameter restricted latent class models where each item j has two item parameters, that is, $|\{\theta_{j,\alpha} : \alpha \in \mathcal{A}\}| = 2$. Specifically, a two-parameter model assumes that for each item j , the latent classes in \mathcal{C}_j share a same positive response probability, denoted by θ_j^+ , and the latent classes in the complement set $\mathcal{A} \setminus \mathcal{C}_j$ share another same positive response probability, denoted by θ_j^- . We assume $\theta_j^+ > \theta_j^-$. Note that the unique item parameters in Θ reduce to (θ^+, θ^-) , where $\theta^+ = (\theta_1^+, \dots, \theta_J^+)^T$ and $\theta^- = (\theta_1^-, \dots, \theta_J^-)^T$. The motivation for studying the two-parameter models comes from the popular DINA and DINO models in cognitive diagnosis, as introduced in Example 2.1. Moreover, the study of the two-parameter models provides insight into understanding other restricted latent class models, as they serve as submodels for many multiparameter models.

Under a two-parameter model, the Γ -matrix fully captures the model structure, in the sense that $\theta_{j,\alpha} = \theta_j^+$ if $\Gamma_{j,\alpha} = 1$ and $\theta_{j,\alpha} = \theta_j^-$ if $\Gamma_{j,\alpha} = 0$. So in this scenario, if Γ contains two identical columns, then the corresponding latent classes have the same item parameters across all items. Namely, if $\Gamma_{\cdot,\alpha} = \Gamma_{\cdot,\alpha'}$, then $\Theta_{\cdot,\alpha} = \Theta_{\cdot,\alpha'}$. Thus from an identifiability perspective, these two latent classes are equivalent and cannot be distinguished based on their observed responses. This implies that in order to distinguish the latent classes, it is necessary that each latent class in \mathcal{A} should correspond to a distinct column vector of Γ . We shall call such a Γ -matrix separable.

DEFINITION 3.1. A Γ -matrix is said to be separable, if any two column vectors of Γ are distinct. Otherwise, we say Γ is inseparable.

To see how the separability of the Γ -matrix influences model identifiability, we start with an ideal case with all the item parameters (θ^+, θ^-) known. The following proposition characterizes the importance of Γ 's separability.

PROPOSITION 3.1. *Consider a two-parameter restricted latent class model with known (θ^+, θ^-) . Then the proportion parameters \mathbf{p} are identifiable if and only if the Γ -matrix is separable.*

We use the following example as an illustration.

EXAMPLE 3.1. Consider the Q -matrix in (3.1) with $K = 2$ attributes. Under the DINA model with \mathcal{C}_j in the form of (2.3), if $\mathcal{A} = \{0, 1\}^2 = \{\alpha_0 = (0, 0), \alpha_1 = (1, 0), \alpha_2 = (0, 1), \alpha_3 = (1, 1)\}$, then $\Gamma^{(1)}$ in (3.1) represents the corresponding Γ -matrix, which is inseparable. Specifically, we can see that $\Gamma_{\cdot, \alpha_0} = \Gamma_{\cdot, \alpha_2}$ and the two classes α_0 and α_2 have the same item parameters, $\Theta_{\cdot, \alpha_0} = \Theta_{\cdot, \alpha_2} = \theta^-$. Thus α_0 and α_2 are not distinguishable and equivalently, their proportion parameters p_{α_0} and p_{α_2} are not identifiable.

$$(3.1) \quad Q = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow[\mathcal{A}=\{0,1\}^2]{\text{DINA}} \Gamma^{(1)} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\xrightarrow[\mathcal{A}=\{0,1\}^2 \setminus \{0,1\}]{\text{DINA}} \Gamma^{(2)} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, if prior knowledge shows that the first attribute is the prerequisite for the second, then \mathcal{A} reduces to $\{0, 1\}^2 \setminus \{(0, 1)\}$ and the Γ -matrix becomes $\Gamma^{(2)}$ in (3.1). The $\Gamma^{(2)}$ is separable, with each α having a distinct column vector in Γ and $\Theta_{\cdot, \alpha_0} \neq \Theta_{\cdot, \alpha_1} \neq \Theta_{\cdot, \alpha_3}$. Therefore, Proposition 3.1 gives that \mathbf{p} is identifiable in the ideal case with known Θ .

An inseparable Γ -matrix violates the necessary condition for identifying \mathbf{p} under the two-parameter models. To study the ‘‘partial’’ identifiability of \mathbf{p} when Γ is inseparable, we next define an equivalence relation ‘‘ \sim ’’ of latent classes induced by the column vectors of Γ . Specifically, we define $\alpha \sim \alpha'$ if and only if $\Gamma_{\cdot, \alpha} = \Gamma_{\cdot, \alpha'}$. Let C be the number of distinct column vectors of Γ and $\mathcal{A}_1, \dots, \mathcal{A}_C$ be the C equivalence classes under \sim . Let $\alpha_{\mathcal{A}_i}$ be a representative of \mathcal{A}_i and we write $[\alpha_{\mathcal{A}_i}] = \mathcal{A}_i$. We define the grouped population proportion parameters to be

$$(3.2) \quad v_{[\alpha_{\mathcal{A}_i}]} := \sum_{\alpha \in \mathcal{A}_i} p_{\alpha} \quad \text{for } i = 1, \dots, C,$$

and write $\mathbf{v} = (v_{[\alpha_{\mathcal{A}_1}]}, \dots, v_{[\alpha_{\mathcal{A}_C}]})^\top$. When Γ is separable, we have $C = m$, $\mathbf{v} = \mathbf{p}$ and each α represents a unique equivalence class.

The following result shows that under an inseparable Γ -matrix, though \mathbf{p} are not identifiable, the parameters \mathbf{v} are identifiable.

PROPOSITION 3.2. *Consider a two-parameter model with known (θ^+, θ^-) . When the Γ -matrix is inseparable, \mathbf{v} is identifiable. Moreover, the latent classes in the same equivalence class cannot be distinguished in the sense that for any model parameters $\mathbf{p} \neq \bar{\mathbf{p}}$, if $v_{[\alpha_{\mathcal{A}_i}]} = \bar{v}_{[\alpha_{\mathcal{A}_i}]}$, where $\bar{v}_{[\alpha_{\mathcal{A}_i}]} = \sum_{\alpha \in \mathcal{A}_i} \bar{p}_{\alpha}$ for $i = 1, \dots, C$, then $\mathbb{P}(\mathbf{R} \mid \Theta, \mathbf{p}) = \mathbb{P}(\mathbf{R} \mid \Theta, \bar{\mathbf{p}})$.*

When Γ is inseparable, Proposition 3.2 implies that even in the ideal case with known (θ^+, θ^-) , the identification of ν is the strongest identifiability result one can obtain for two-parameter restricted latent class models. This therefore motivates us to introduce the following definition of the p -partial identifiability when both (θ^+, θ^-) and p are unknown.

DEFINITION 3.2 (p -partial identifiability). For a two-parameter restricted latent class model with a given Γ -matrix, the model parameters (θ^+, θ^-, p) are said to be p -partially identifiable if $(\theta^+, \theta^-, \nu)$ are identifiable.

We point out that when the Γ -matrix is separable, the p -partial identifiability exactly becomes the strict identifiability. When Γ is inseparable, the definition of p -partial identifiability here refers to partially identifying the proportion parameters p , and strictly identifying all the item parameters. Such definition suits for the needs of cognitive diagnosis applications, by ensuring the identification of the equivalent attribute profiles of interest, and also ensuring the estimability of all item parameters so that the quality of the items can be accurately evaluated and validated.

In the framework of p -partial identifiability, the following Section 3.1 presents a general identifiability result, allowing \mathcal{A} to be arbitrary and Γ to be inseparable. Section 3.2 further focuses on the family of Q -restricted latent class models and discusses the necessity of the proposed conditions. Section 3.3 includes the applications of the new theory.

REMARK 3.1. For the family of two-parameter Q -restricted latent class models, the Γ -induced equivalence classes can be obtained as follows. We define two sets of attribute profiles under the conjunctive DINA and disjunctive DINO assumptions, respectively:

$$(3.3) \quad \mathcal{R}^{Q,\text{conj}} = \left\{ \alpha = \bigvee_{h \in S} \mathbf{q}_h : S \subseteq \mathcal{S} \right\}, \quad \mathcal{R}^{Q,\text{disj}} = \{ \mathbf{1} - \alpha : \alpha \in \mathcal{R}^{Q,\text{conj}} \},$$

where $\bigvee_{h \in S} \mathbf{q}_h = (\max_{h \in S} \{q_{h,1}\}, \dots, \max_{h \in S} \{q_{h,K}\})$, and $\bigvee_{h \in \emptyset} \mathbf{q}_h$ is defined to be the all-zero vector. We claim that when $\mathcal{A} = \{0, 1\}^K$, the $\mathcal{R}^{Q,\text{conj}}$ or $\mathcal{R}^{Q,\text{disj}}$ is a complete set of representatives of the conjunctive or disjunctive equivalence classes, respectively; the proof of this result is given in Section B of the Supplementary Material [20]. Moreover, for any latent class space $\mathcal{A} \subseteq \{0, 1\}^K$, define a map $f(\cdot) : \mathcal{A} \rightarrow \mathcal{R}^{Q,\text{conj}}$ (or $\mathcal{R}^{Q,\text{disj}}$) which sends each attribute pattern $\alpha \in \mathcal{A}$ to the element in $\mathcal{R}^{Q,\text{conj}}$ (or $\mathcal{R}^{Q,\text{disj}}$) equivalent to α . Then $f(\mathcal{A})$ forms a complete set of conjunctive or disjunctive representatives. A similar grouping operation in the saturated and conjunctive case was introduced in [49]. Consider Example 3.1 for an illustration. If $\mathcal{A} = \{0, 1\}^2$, $\Gamma^{(1)}$ is inseparable. The equivalence class representatives are $\mathcal{R}^{Q,\text{conj}} = \{(0, 0), (1, 0), (1, 1)\}$ by (3.3) and $\nu = (\nu_{[0,0]}, \nu_{[1,0]}, \nu_{[1,1]})$ with $\nu_{[0,0]} = p_{(0,0)} + p_{(0,1)}$, $\nu_{[1,0]} = p_{(1,0)}$, $\nu_{[1,1]} = p_{(1,1)}$. On the other hand, $\Gamma^{(2)}$ is separable with latent class space $\mathcal{A} = \mathcal{R}^{Q,\text{conj}}$. This also illustrates that a separable Γ -matrix does not necessarily correspond to a Q -matrix containing an identity submatrix I_K . Therefore, compared with existing theory that requires Q to contain at least one I_K , the Γ -matrix provides a more suitable tool than the Q -matrix for studying identifiability of Q -restricted models.

3.1. Strict and partial identifiability. This subsection presents conditions depending on the Γ -matrix that lead to the p -partial identifiability of a two-parameter restricted latent class model. We first introduce some notation. Based on the constraint sets \mathcal{C}_j 's, we categorize the entire set of items $\mathcal{S} = \{1, \dots, J\}$ into two subsets, the set of *nonbasis* items S_{non} and that of *basis* items S_{basis} as follows:

$$(3.4) \quad S_{\text{non}} = \{j : \exists h \in \mathcal{S} \setminus \{j\}, \text{ s.t. } \mathcal{C}_h \supseteq \mathcal{C}_j\} \quad \text{and} \quad S_{\text{basis}} = \mathcal{S} \setminus S_{\text{non}}.$$

By this definition, an item j is a nonbasis item if the capability of item j implies capability of some other item, and a basis item otherwise. With a slight abuse of notation, for any subset of items $S \subseteq \mathcal{S}$, denote $\mathcal{C}_S = \bigcap_{j \in S} \mathcal{C}_j$. We introduce the next definition of S -differentiable to describe the relation between an item and a set of items.

DEFINITION 3.3. For an item j and a set of items S that does not contain j , item j is said to be S -differentiable if there exist two subsets S_j^+, S_j^- of S , which are not necessarily nonempty or disjoint, such that

$$(3.5) \quad \mathcal{C}_{S_j^+} \subsetneq \mathcal{C}_{S_j^-} \quad \text{and} \quad \mathcal{C}_{S_j^-} \setminus \mathcal{C}_{S_j^+} \subseteq \mathcal{A} \setminus \mathcal{C}_j.$$

When j is S -differentiable, the set S is said to be a *separator set* of item j . An item j is S -differentiable indicates that the items in the separator set S can differentiate at least one incapable latent class of j (i.e., one latent class in $\mathcal{A} \setminus \mathcal{C}_j$) from the universal least capable class α_0 .

We need the following two conditions to establish identifiability:

(C1) *Repeated measurement condition*: For each item j , there exist two disjoint sets of items $S_j^1, S_j^2 \subset \mathcal{S} \setminus \{j\}$ such that $\mathcal{C}_j \supseteq \mathcal{C}_{S_j^1}$ and $\mathcal{C}_j \supseteq \mathcal{C}_{S_j^2}$.

(C2) *Sequentially differentiable condition*: Start with the set $S_{\text{sep}} = S_{\text{non}}$. Expand S_{sep} by including all items in $\mathcal{S} \setminus S_{\text{sep}}$ that are S_{sep} -differentiable, and repeat the expanding procedure until no items can be added to S_{sep} . The sequentially expanding procedure ends up with $S_{\text{sep}} = \mathcal{S}$.

Before presenting the formal theorem, we first give a simple illustration of how condition (C2) can be checked.

EXAMPLE 3.2. Consider the following 3×4 Γ -matrix:

$$\Gamma = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then $\mathcal{C}_1 = \{\alpha_2, \alpha_3\}$, $\mathcal{C}_2 = \{\alpha_3\}$ and $\mathcal{C}_3 = \{\alpha_1\}$. By (3.4), $S_{\text{non}} = \{2, 3\}$ and $S_{\text{basis}} = \{1\}$. To check condition (C2), we start with the separator set $S_{\text{sep}} = S_{\text{non}} = \{2, 3\}$. For basis item 1, we define $S_1^+ = \emptyset$ and $S_1^- = \{3\}$. Then $\mathcal{C}_{S_1^+} = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{C}_{S_1^-} = \{\alpha_0, \alpha_2, \alpha_3\}$, so $\mathcal{C}_{S_1^+} \setminus \mathcal{C}_{S_1^-} = \{\alpha_1\} \subseteq \mathcal{C}_1^c = \{\alpha_0, \alpha_1\}$, which means (3.5) holds for $j = 1$. Besides, $S_1^+ \cup S_1^- \subseteq S_{\text{non}}$. So by Definition 3.3, item 1 is S_{non} -differentiable. Now we can expand the separator set S_{sep} to be $S_{\text{non}} \cup \{1\} = \mathcal{S}$. So the sequentially expanding procedure described in condition (C2) ends in one step with $S_{\text{sep}} = \mathcal{S}$, and (C2) is satisfied.

THEOREM 3.1. Under the two-parameter restricted latent class models, condition (C1) is sufficient for identifiability of $(\theta^+, \theta_{\text{non}}^-)$, where $\theta_{\text{non}}^- = (\theta_j^-, j \in S_{\text{non}})$. Moreover, conditions (C1) and (C2) are sufficient for \mathbf{p} -partial identifiability of the model parameters $(\theta^+, \theta^-, \mathbf{p})$.

Theorem 3.1 presents a general identifiability result with strict identifiability being a special case. For instance, in the case of $\mathcal{A} = \{0, 1\}^K$, if the $J \times 2^K$ Γ -matrix is separable, then $\mathbf{v} = \mathbf{p}$ and the \mathbf{p} -partial identifiability in Theorem 3.1 exactly ensures strict identifiability

of all the parameters $(\theta^+, \theta^-, \mathbf{p})$. Similarly, in the case of $\mathcal{A} \subset \{0, 1\}^K$, if the $J \times |\mathcal{A}|$ Γ -matrix is separable, the \mathbf{p} -partial identifiability ensures (θ^+, θ^-) and $(p_\alpha, \alpha \in \mathcal{A})$ are strictly identifiable. Conditions (C1) and (C2) only depend on the structure of the Γ -matrix and are easily checkable. Condition (C1) implies that at least one capable class of each item is repeatedly measured by other items. Condition (C2) requires that for each basis item, at least one of its incapable classes should be differentiated from the universal least capable class through a sequential procedure. From the proof of Theorem 3.1, (C1) suffices for identifiability of $(\theta^+, \theta^-_{\text{non}})$; furthermore, the sequential procedure in condition (C2) ensures that as S_{sep} sequentially expands its size, for any item h included in S_{sep} , the parameter θ_h^- is identifiable. If (C2) holds, that is, the sequential procedure ends up with $S_{\text{sep}} = \mathcal{S}$, we have the entire θ^- identifiable, which further leads to identifiability of \mathbf{v} . The sequential statement of (C2) accurately characterizes the underlying structure of the Γ -matrix needed for identifiability. In particular, if there are no basis items, that is, $\mathcal{S} = S_{\text{non}}$, then (C2) automatically holds with zero expanding step; while if there do exist basis items and each basis item is S_{non} -differentiable, then (C2) holds with one expanding step.

The next proposition further extends the result in Theorem 3.1 to the case where the Γ -matrix may not satisfy (C1) and (C2). For any subset of items $S \subseteq \mathcal{S}$, define the S -adjusted Γ -matrix $\Gamma(S)$ as follows, which has the same size as the original Γ . Its j th row $\{\Gamma(S)\}_{j,\cdot}$ equals $\mathbf{1}_m^\top - \Gamma_{j,\cdot}$ if $j \in S$, and equals $\Gamma_{j,\cdot}$ if $j \notin S$. Here, $\mathbf{1}_m^\top$ denotes an all-one row vector of length m .

PROPOSITION 3.3. *Consider a two-parameter restricted latent class model associated with a Γ -matrix. If there exist a subset of items $S \subseteq \mathcal{S}$ such that the S -adjusted Γ -matrix $\Gamma(S)$ satisfies conditions (C1) and (C2), then the two-parameter model is \mathbf{p} -partially identifiable.*

Proposition 3.3 relaxes the conditions of Theorem 3.1, by only requiring that (C1) and (C2) can be satisfied after switching the zeros and ones for some rows of in the Γ . The identifiability conditions in Theorem 3.1 and Proposition 3.3 allow for a nonsaturated latent class space \mathcal{A} and inseparability of the Γ -matrix, which relaxes the existing identifiability conditions in the literature. Moreover, the proposed conditions (C1) and (C2) would become necessary and sufficient in certain scenarios to be discussed in the following subsection.

3.2. Results for Q -restricted latent class models. To further illustrate the result in Theorem 3.1, we focus on the two-parameter Q -restricted latent class model with a saturated latent class space $\mathcal{A} = \{0, 1\}^K$. This includes the conjunctive DINA and disjunctive DINO models in Example 2.1 as special cases. Without loss of generality, we next only consider the two-parameter conjunctive model. Nevertheless, all the \mathbf{p} -partial identifiability results presented in this subsection hold for both the conjunctive and the disjunctive models, due to the duality between them [7].

We introduce the following definitions adapted from Section 3.1. Under the conjunctive model assumption with \mathcal{C}_j taking the form of (2.3), the non-basis and basis items defined earlier in (3.4) can be equivalently expressed in terms of the \mathbf{q} -vectors as follows:

$$(3.6) \quad S_{\text{non}} = \{j : \exists h \in \mathcal{S} \setminus \{j\} \text{ s.t. } \mathbf{q}_h \leq \mathbf{q}_j\} \quad \text{and} \quad S_{\text{basis}} = \mathcal{S} \setminus S_{\text{non}}.$$

Moreover, item j is set S -differentiable if there exist $S^+, S^- \subseteq S$ such that

$$(3.7) \quad \mathbf{0} \not\leq \bigvee_{h \in S^+} \mathbf{q}_h - \bigvee_{h \in S^-} \mathbf{q}_h \leq \mathbf{q}_j.$$

In addition, conditions (C1) and (C2) are equivalent to:

(C1*) *Repeated measurement condition:* For each $j \in \mathcal{S}$, there exist two disjoint item sets $S_j^1, S_j^2 \subseteq \mathcal{S} \setminus \{j\}$ such that $\mathbf{q}_j \preceq \bigvee_{h \in S_j^1} \mathbf{q}_h$ and $\mathbf{q}_j \preceq \bigvee_{h \in S_j^2} \mathbf{q}_h$.

(C2*) *Sequentially differentiable condition:* The same as condition (C2), but using definition (3.7) of \mathcal{S} -differentiable regarding the \mathbf{q} -vectors.

Following Theorem 3.1, the next corollary shows that the derived conditions on the Q -matrix suffice for the \mathbf{p} -partial identifiability of both the conjunctive and disjunctive two-parameter models.

COROLLARY 3.1. *Under the two-parameter Q -restricted latent class models, assuming $v_{[\alpha]} > 0$ for any equivalence class $[\alpha]$, (C1*) and (C2*) are sufficient for the \mathbf{p} -partial identifiability of $(\boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \mathbf{p})$.*

We use the following example as an illustration of the identifiability result; see also real data examples in Section 3.3.

EXAMPLE 3.3. Under the DINA model, consider the following Q -matrix:

$$(3.8) \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

This Q -matrix lacks the single-attribute item $(0, 0, 1)$, and the corresponding Γ -matrix under $\mathcal{A} = \{0, 1\}^3$ is inseparable. In this case, we have the following 7 equivalence classes $\{[0, 0, 0], [1, 0, 0], [0, 1, 0], [1, 1, 0], [0, 1, 1], [1, 0, 1], [1, 1, 1]\}$, with the equivalence class $[0, 0, 0]$ containing attribute profiles $(0, 0, 0)$ and $(0, 0, 1)$, while each of the other equivalence classes contains one attribute profile. Following the definition in (3.6), items 1 and 2 are basis items, and items 3, 4 and 5 are nonbasis items. For all the five items, condition (C1*) is satisfied by taking $(S_1^1, S_1^2) = (\{3\}, \{5\})$, $(S_2^1, S_2^2) = (\{3\}, \{4\})$, $(S_3^1, S_3^2) = (\{1, 4\}, \{2, 5\})$, $(S_4^1, S_4^2) = (\{3\}, \{2, 5\})$, and $(S_5^1, S_5^2) = (\{3\}, \{1, 4\})$. In addition, condition (C2*) is also satisfied since the basis items 1 and 2 are $(S_1^+ \cup S_1^-)$ - and $(S_2^+ \cup S_2^-)$ -differentiable, respectively, where $(S_1^+, S_1^-) = (\{3\}, \{4\})$ and $(S_2^+, S_2^-) = (\{3\}, \{5\})$. By Corollary 3.1, the DINA model parameters are \mathbf{p} -partially identifiable.

As shown above, conditions (C1*) and (C2*) are sufficient conditions to ensure \mathbf{p} -partial identifiability. In the following, we discuss the necessity of (C1*) and (C2*) and further provide procedures to establish identifiability in certain cases when these conditions fail to hold.

For a general Q -matrix, condition (C1*) implies that each attribute is required by at least three items. In the next theorem, we show that it is necessary for each attribute to be required by at least two items; in particular, if some attribute is required by only two items, the identifiability conclusion would depend on the structure of the \mathbf{q} -vectors of those two items.

THEOREM 3.2 (Discussion of (C1*)). *Consider a two-parameter Q -restricted latent class model:*

(a) *If some attribute is required by only one item, then the model is not \mathbf{p} -partially identifiable.*

(b) *If some attribute is required by only two items, without loss of generality, suppose the first attribute is required by the first two items and the Q -matrix takes the following form:*

$$(3.9) \quad Q = \begin{pmatrix} 1 & \mathbf{v}_1^\top \\ 1 & \mathbf{v}_2^\top \\ \mathbf{0} & Q' \end{pmatrix}_{J \times K},$$

where Q' is a $(J - 2) \times (K - 1)$ sub-matrix of Q and $\mathbf{v}_1, \mathbf{v}_2$ are $(K - 1)$ -dimensional vectors.

(b.1) *If $\mathbf{v}_1 = \mathbf{0}$ or $\mathbf{v}_2 = \mathbf{0}$, the model is not \mathbf{p} -partially identifiable.*

(b.2) *If $\mathbf{v}_1 \neq \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$, the model is \mathbf{p} -partially identifiable if the sub-matrix Q' satisfies conditions (C1*) and (C2*), and either (a) or (b) below holds for $i = 1$ and 2: (a) There exists some $j \geq 3$ such that $\mathbf{q}_{j,2:K} \not\subseteq \mathbf{v}_i$; (b) There does not exist any $j \geq 3$ such that $\mathbf{q}_{j,2:K} \not\subseteq \mathbf{v}_i$, and among the attributes required by \mathbf{v}_i , there exists at least one attribute k that is not required by every item $j \in \{3, \dots, J\}$.*

Theorem 3.2 characterizes the different situations when condition (C1*) fails to hold for some attribute, and provides sufficient conditions for identifiability when the Q -matrix falls in the scenario (B). In addition, the result in Theorem 3.2 can be easily extended to the case where there are multiple attributes that are required by only two items.

The next theorem discusses the necessity of Condition (C2*) and states that if there exists some basis item that does not have any separator set, then the model parameters are not \mathbf{p} -partially identifiable.

THEOREM 3.3 (Discussion of (C2*)). *Under the two-parameter Q -restricted models, the condition that each basis item j is $(\mathcal{S} \setminus \{j\})$ -differentiable, is necessary for the \mathbf{p} -partial identifiability.*

Furthermore, under the two-parameter Q -restricted models with a separable Γ -matrix and a saturated latent class space \mathcal{A} , the following theorem shows conditions (C1*) and (C2*) are exactly the minimal requirement for strict identifiability of the model.

THEOREM 3.4 (Result on the necessary and sufficient condition). *Under the two-parameter Q -restricted models, if \mathcal{A} is saturated and Γ is separable, then conditions (C1*) and (C2*) are necessary and sufficient for the strict identifiability of $(\boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \mathbf{p})$.*

Under the assumptions of Theorem 3.4, conditions (C1*) and (C2*) are equivalent to the following explicit conditions on the structure of the Q -matrix: (C1') Each attribute is required by at least three items; (C2') With Q in the form $Q = (I_K^\top, (Q')^\top)^\top$, any two different columns of the submatrix Q' are distinct. Please see the proof of Theorem 3.4 for details.

3.3. Applications. One important implication of the established identifiability theory is the consistent estimability of the model parameters. Consider a sample of size N and denote the i th subject's multivariate binary responses by $\mathbf{R}_i = (R_{i,1}, \dots, R_{i,J})^\top$. Assume $\mathbf{R}_1, \dots, \mathbf{R}_N$ identically and independently follow the categorical distribution with the probability mass function (2.5). The likelihood based on the sample can be written as $L(\boldsymbol{\Theta}, \mathbf{p} \mid \mathbf{R}_1, \dots, \mathbf{R}_N) = \prod_{i=1}^N \mathbb{P}(\mathbf{R} = \mathbf{R}_i \mid \boldsymbol{\Theta}, \mathbf{p})$. We denote the true parameters by $(\boldsymbol{\Theta}^0, \mathbf{p}^0)$ and the maximum likelihood estimators (MLE) by $(\hat{\boldsymbol{\Theta}}, \hat{\mathbf{p}})$, which may not be unique. We further define the corresponding parameters \mathbf{v}^0 and $\hat{\mathbf{v}}$ as in (3.2). We have the following conclusion on the estimability of a two-parameter model.

PROPOSITION 3.4. *If a two-parameter model is \mathbf{p} -partially identifiable, then $(\widehat{\Theta}, \widehat{\mathbf{v}}) \rightarrow (\Theta^0, \mathbf{v}^0)$ almost surely as $N \rightarrow \infty$. In addition, if Γ -matrix is also separable, then $(\widehat{\Theta}, \widehat{\mathbf{p}}) \rightarrow (\Theta^0, \mathbf{p}^0)$ almost surely. On the other hand, if Γ -matrix is inseparable, \mathbf{p} can not be consistently estimated.*

With the consistency result, we can directly establish the asymptotic normality of $(\widehat{\Theta}, \widehat{\mathbf{v}})$ when the model is \mathbf{p} -partially identifiable, following a standard argument of asymptotic statistics [37].

We next apply the newly developed theory to the data examples introduced in Section 2.2, and establish the \mathbf{p} -partial identifiability of the two-parameter restricted latent class model under the Q -matrices.

For the TOEFL iBT data introduced in Example 2.4, the two-parameter restricted latent class models associated with the Q -matrices corresponding to reading forms A and B, denoted by Q_A and Q_B , respectively, are both \mathbf{p} -partially identifiable. Specifically, under the conjunctive DINA model, the Q_A and Q_B specified in Table 1 induce 14 and 12 equivalence classes of attribute profiles, respectively, for which the sets of representatives are $\mathcal{R}^{Q_A} = \{0, 1\}^4 \setminus \{(0, 0, 0, 1), (1, 0, 0, 1)\}$ and $\mathcal{R}^{Q_B} = \{0, 1\}^4 \setminus \{(0, 0, 0, 1), (1, 0, 0, 1), (0, 1, 0, 1), (1, 1, 0, 1)\}$. The \mathcal{R}^{Q_A} and \mathcal{R}^{Q_B} are calculated following the procedure introduced in Remark 3.1. It is straightforward to check that for both Q_A and Q_B , condition (C1*) holds and there is no basis item, which further implies the satisfaction of condition (C2*). Therefore, Corollary 3.1 gives the \mathbf{p} -partial identifiability of the two-parameter models associated with both Q_A and Q_B . Furthermore, Proposition 3.4 implies the consistent estimability of $(\theta^+, \theta^-, \mathbf{v})$. In particular, the proportion parameters of the equivalence classes $\mathbf{v} = (v_{[\alpha]}, \alpha \in \mathcal{R}^{Q_A})$ can be consistently estimated, while for those attribute profiles in a same equivalent class, their proportions cannot be consistently estimated. For instance, under Q_A , attribute patterns $\alpha^* = (0, 0, 0, 1)$ and $\alpha^{**} = (0, 0, 0, 0)$ share the same equivalent class; so p_{α^*} and $p_{\alpha^{**}}$ are not estimable, and it is only possible and meaningful to estimate $v_{[\alpha^*]} = p_{\alpha^*} + p_{\alpha^{**}}$.

Other than the TOEFL data, our new results in Section 3.2 also guarantee the \mathbf{p} -partial identifiability of two-parameter models associated with the $Q_{43 \times 12}$ for the TIMSS data, and the $Q_{20 \times 8}$ for the fraction subtraction data. The details of checking our conditions for $Q_{43 \times 12}$ and $Q_{20 \times 8}$ are included in Section A of the Supplementary Material [20].

4. Identifiability results for multiparameter models. This section considers multiparameter restricted latent class models where each item j allows for more than two item parameters, that is, $|\{\theta_{j,\alpha} : \alpha \in \mathcal{A}\}| \geq 2$. In a multiparameter model, those latent classes in \mathcal{C}_j still have the same level of positive response probability, according to the definition of \mathcal{C}_j in (2.1); however, the classes in $\mathcal{A} \setminus \mathcal{C}_j$ can have multiple levels of positive response probabilities, depending on the extents of their “partial” capability of item j . Examples of multiparameter models include the main-effect and the all-effect models introduced in Examples 2.2 and 2.3, respectively.

We would like to point out that the Γ -matrix defined in (2.7) still provides a useful technical tool for studying identifiability of multiparameter models, despite the fact that the entry $\Gamma_{j,\alpha}$ only indicates whether α belongs to the most-capable-set \mathcal{C}_j and it does not summarize all the structural assumptions in multiparameter models.

On the one hand, similar to the two-parameter case, under a multi-parameter model, the separability of the Γ -matrix is still *necessary* for the *strict* identifiability of (Θ, \mathbf{p}) . This is because a two-parameter model, such as DINA, can be viewed as a submodel of a multiparameter model, such as GDINA or GDM, by constraining certain parameters in the multiparameter model to zero. So in order to ensure identifiability of all possible model parameters

in the parameter space of a multiparameter model, Proposition 3.1 implies the Γ must be separable.

On the other hand, when the Γ -matrix is inseparable and contains identical columns, the item parameter vectors in the matrix Θ may still be distinct. This is because under the general constraints (2.2), when $\Gamma_{j,\alpha} = 0$ under a multiparameter model, α could be either least capable or partially capable of item j , and hence the latent classes in the set $\mathcal{A} \setminus \mathcal{C}_j = \{\alpha : \Gamma_{j,\alpha} = 0\}$ can still have different positive response probabilities, as shown in Examples 2.2 and 2.3. Such a difference from the two-parameter models makes the \mathbf{p} -partial identifiability theory developed in Section 3 not applicable to multiparameter models. To study identifiability of multiparameter models when Γ is inseparable, we therefore need an alternative partial identifiability notion and technique. We use the next example to illustrate this and show how the separable requirement of the Γ -matrix in Proposition 3.1 could be relaxed under multiparameter models.

EXAMPLE 4.1. Consider the Q -matrix in (3.1). Under a two-parameter conjunctive restricted latent class model, we have shown attribute profiles $\alpha_0 = (0, 0)$ and $\alpha_2 = (0, 1)$ are not distinguishable. However, a multi-parameter model models the main effect of each required attribute for an item. Consider the main-effect model with the identity link function as introduced in Example 2.2 (the ACDM), one has $\Theta_{\cdot,\alpha_0} = (\beta_{1,0}, \beta_{2,0})^\top$ and $\Theta_{\cdot,\alpha_2} = (\beta_{1,0}, \beta_{2,0} + \beta_{2,2})^\top$; then $\Theta_{\cdot,\alpha_0} \neq \Theta_{\cdot,\alpha_2}$ as long as $\beta_{2,2} \neq 0$. When this inequality constraint $\beta_{2,2} \neq 0$ holds, $\Theta_{\cdot,\alpha_0} \neq \Theta_{\cdot,\alpha_2}$ despite that $\Gamma_{\cdot,\alpha_0} = \Gamma_{\cdot,\alpha_2}$. In such scenarios, the grouping operation of the proportion parameters introduced in Section 3 is not appropriate, and one needs to treat these two latent classes α_0 and α_2 separately. Consider any possible Θ for which the inequality constraint $\beta_{2,2} \neq 0$ does not hold, then all such Θ indeed fall into a subset of the parameter space \mathcal{T} with smaller dimension than \mathcal{T} , characterized by $\mathcal{V} = \{(\Theta, \mathbf{p}) : \beta_{2,2} = 0\}$. This implies that for almost all valid model parameters (Θ, \mathbf{p}) in \mathcal{T} , except a Lebesgue measure zero set \mathcal{V} , the Θ satisfy $\Theta_{\cdot,\alpha_0} \neq \Theta_{\cdot,\alpha_1}$. This observation naturally leads to the following notion of generic identifiability.

Motivated by Example 4.1, when the Γ -matrix is inseparable, we shall study the *generic identifiability* of the restricted latent class model. Let \mathcal{T} denote the restricted parameter space of (Θ, \mathbf{p}) under the general constraints (2.2), and let d denote the number of free parameters in (Θ, \mathbf{p}) , so \mathcal{T} is of full dimension in \mathbb{R}^d . Generic identifiability means that identifiability holds for almost all points except a subset of \mathcal{T} that has Lebesgue measure zero. Generic identifiability is closely related to the concept of algebraic variety in algebraic geometry. Following the definition in [2], an algebraic variety \mathcal{V} is defined as the simultaneous zero-set of a finite collection of multivariate polynomials $\{f_i\}_{i=1}^n \subseteq \mathbb{R}[x_1, x_2, \dots, x_d]$, $\mathcal{V} = \mathcal{V}(f_1, \dots, f_n) = \{\mathbf{x} \in \mathbb{R}^d \mid f_i(\mathbf{x}) = 0, 1 \leq i \leq n\}$. An algebraic variety \mathcal{V} is all of \mathbb{R}^d only when all the polynomials defining it are zero polynomials; otherwise, \mathcal{V} is called a *proper subvariety* and is of dimension less than d , hence necessarily of Lebesgue measure zero in \mathbb{R}^d . The same argument holds when \mathbb{R}^d is replaced by the parameter space $\mathcal{T} \subseteq \mathbb{R}^d$ that has full dimension in \mathbb{R}^d . We next present the definition of generic identifiability for restricted latent class models.

DEFINITION 4.1 (Generic identifiability). A restricted latent class model is said to be generically identifiable on the parameter space \mathcal{T} , if (Θ, \mathbf{p}) are strictly identifiable on $\mathcal{T} \setminus \mathcal{V}$ where \mathcal{V} is a proper algebraic subvariety of \mathcal{T} .

Generic identifiability could be viewed as some “partial” identification of model parameters in the sense that, the nonidentifiable parameters fall in a subset of the parameter space

that can be characterized as solutions to some nonzero polynomial equations. As can be seen from the form of (2.2), the constraints on the parameter space introduced by the Γ -matrix already force the parameters fall into a proper algebraic subvariety of the unrestricted parameter space, so previous results established in [2] for unrestricted latent class models do not apply to the models considered in this work.

REMARK 4.1. Under multiparameter models, it is still possible that two latent classes α and α' always have the same positive response probabilities, that is, $\Theta_{\cdot, \alpha} = \Theta_{\cdot, \alpha'}$ and α, α' are not distinguishable even generically. In this case, one could have p -partial identifiability of the model. However, this happens only when $\Gamma_{\cdot, \alpha} = \Gamma_{\cdot, \alpha'} = \mathbf{1}$; moreover, under Q -restricted models, this happens only if the Q -matrix contains an all-zero column, which is a trivial case with a redundant column in Q . Under such a Q -matrix, we can simply remove these all-zero columns and study the (generic) identifiability under the reduced Q -matrix. Therefore, without loss of generality, in the following discussion we assume the Q -matrix does not contain any all-zero column such that $\Theta_{\cdot, \alpha} = \Theta_{\cdot, \alpha'}$ would not happen.

Based on the above discussions, to study identifiability of multiparameter restricted latent class models, we consider two situations in Section 4.1: first, when the Γ -matrix is separable, we study the *strict* identifiability of model parameters; second, when the Γ -matrix is inseparable, we study the *generic* identifiability of model parameters. Furthermore, in Section 4.2 we present sufficient conditions for generic identifiability of the family of Q -restricted latent class models, and discuss the necessity of the proposed conditions.

4.1. *Strict and generic identifiability.* First, consider the case where the Γ -matrix is separable. For a subset of items S , denote the corresponding $|S| \times m$ indicator matrix by $\Gamma^S = (\Gamma_{j, \alpha}, j \in S, \alpha \in \mathcal{A})$, which is a submatrix of the previously defined Γ -matrix. We say α succeeds α' with respect to S and denote it by $\alpha \succeq_S \alpha'$, if $\Gamma_{j, \alpha} \geq \Gamma_{j, \alpha'}$ for any $j \in S$; this means α is at least as capable as α' of items in set S . With this definition, any subset of items S induces a partial order “ \succeq_S ” on the set of latent classes \mathcal{A} . When two sets S_1 and S_2 induce the same partial order on \mathcal{A} , that is, for any α' and $\alpha \in \mathcal{A}$, $\alpha' \succeq_{S_1} \alpha$ if and only if $\alpha' \succeq_{S_2} \alpha$, we write “ \succeq_{S_1} ” = “ \succeq_{S_2} ”. The following theorem gives conditions that lead to strict identifiability of multiparameter restricted latent class models.

THEOREM 4.1. *For a multiparameter restricted latent class model, if the Γ -matrix satisfies the following conditions, then the parameters (Θ, p) are strictly identifiable:*

(C3) *There exist two disjoint item sets S_1 and S_2 , such that Γ^{S_i} is separable for $i = 1, 2$ and “ \succeq_{S_1} ” = “ \succeq_{S_2} ”.*

(C4) $\Gamma_{\cdot, \alpha}^{(S_1 \cup S_2)^c} \neq \Gamma_{\cdot, \alpha'}^{(S_1 \cup S_2)^c}$ for any α, α' such that $\alpha' \succeq_{S_i} \alpha$ for $i = 1$ or 2 .

Condition (C3) implies the entire Γ -matrix is separable, and it requires two disjoint sets of items S_1 and S_2 to have enough information to distinguish the latent classes, and it serves as a *repeated measurement condition* for the identifiability of multiparameter restricted latent class models. Condition (C4) states that, for those pairs of latent classes α and α' such that α is more capable than α' uniformly on either S_1 or S_2 , the remaining items in $(S_1 \cup S_2)^c$ should differentiate α and α' by their column vectors in $\Gamma^{(S_1 \cup S_2)^c}$.

Strict identifiability can be achieved with a relaxation of condition (C4) together with a stronger version of condition (C3). Before presenting this result, we define a latent class α as a basis latent class under an item set S , if there does not exist $\alpha' \in \mathcal{A}$ such that $\alpha' \preceq_S \alpha$. Denote the set of all basis latent classes under S by \mathcal{B}_S . Then “ \succeq_{S_1} ” = “ \succeq_{S_2} ” implies $\mathcal{B}_{S_1} = \mathcal{B}_{S_2}$.

PROPOSITION 4.1. *Under a multiparameter restricted latent class model, if the Γ -matrix satisfies the following conditions, then (Θ, \mathbf{p}) are identifiable:*

(C3*) *There exist two disjoint item sets S_1 and S_2 , such that Γ^{S_i} is separable for $i = 1, 2$ and “ \succeq_{S_1} ” = “ \succeq_{S_2} ”. Moreover, for any $j \in S_1 \cup S_2$, there exists $\alpha \in \mathcal{B}_{S_1}$ such that $\Gamma_{j,\alpha} = 1$.*

(C4*) *$\Gamma_{\cdot, \alpha}^{(S_1 \cup S_2)^c} \neq \Gamma_{\cdot, \alpha_0}^{(S_1 \cup S_2)^c}$ for any $\alpha \in \mathcal{B}_{S_1}$ and $\alpha \neq \alpha_0$, where α_0 is the universal least capable class.*

REMARK 4.2. Theorem 4.1 and Proposition 4.1 show the trade-off between the conditions on the separable submatrices part of Γ and on the remaining part. They establish identifiability for a wide range of restricted latent class models, with the Γ -matrix ranging in the spectrum of different extents of inseparability. Specifically, for a Q -restricted latent class model that lacks many single-attribute items, (C3) is easier to satisfy than (C3*) and Theorem 4.1 would be more applicable; while for a Q -restricted model that lacks few single-attribute items, Proposition 4.1 would become more applicable as (C4*) imposes a weaker condition on the set $(S_1 \cup S_2)^c$.

REMARK 4.3. Theorem 4.1 and Proposition 4.1 extend the existing work [45]. Compared with the identifiability result in [45] that requires two copies of the identity submatrix I_K to be included in the Q -matrix, in the special case with $\mathcal{A} = \{0, 1\}^K$, the proposed conditions (C3*) and (C4*) reduce to the conditions in [45]. Furthermore, in general cases of an unsaturated latent class space with $|\mathcal{A}| < 2^K$, the conditions in Theorem 4.1 and Proposition 4.1 impose much weaker requirements than those in [45], because a Q -matrix lacking some single-attribute items may suffice for a separable Γ -matrix and further suffice for strict identifiability under the conditions in this paper.

Next, we consider the case where the multiparameter restricted latent class model is associated with an inseparable Γ -matrix, which violates condition (C3). We study the generic identifiability of the model parameters.

THEOREM 4.2. *Consider a multiparameter restricted latent class model. If there exist two disjoint item sets S_1 and S_2 , such that altering some entries of zero to one in $\Gamma^{S_1 \cup S_2}$ can yield a $\tilde{\Gamma}^{S_1 \cup S_2}$ that satisfies condition (C3); and that the $\Gamma^{(S_1 \cup S_2)^c}$ satisfies condition (C4), then the model parameters (Θ, \mathbf{p}) under the original Γ -matrix are generically identifiable.*

Theorem 4.2 is established based on the theoretical development of Theorem 4.1. By relaxing the condition (C3) and allowing Γ to be inseparable, we may not have strict identifiability, as discussed in Example 4.1. We use the following example to further illustrate the results of Theorems 4.1–4.2.

EXAMPLE 4.2. For a multiparameter restricted latent class model, if $\Gamma = ((\Gamma^{\text{sub}})^\top, (\Gamma^{\text{sub}})^\top, (\Gamma^{\text{sub}})^\top)^\top$ contains three copies of the following Γ^{sub} , then (C3) and (C4) are satisfied and (Θ, \mathbf{p}) under Γ are strictly identifiable:

$$\Gamma^{\text{sub}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \Gamma^{S_1} = \begin{pmatrix} 0 & \mathbf{0} & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma^{S_2} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{0} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Instead, consider $\Gamma_{\text{new}} = ((\Gamma^{S_1})^\top, (\Gamma^{S_2})^\top, (\Gamma^{\text{sub}})^\top)^\top$ with two submatrices in the forms of Γ^{S_1} and Γ^{S_2} above, then neither of Γ^{S_i} is separable. But by changing the (1, 2)th entry of Γ^{S_1} and (2, 3)th entry of Γ^{S_2} from zero to one, the resulting $\tilde{\Gamma}^{S_1}$ and $\tilde{\Gamma}^{S_2}$ are separable, so the conditions of Theorem 4.2 are satisfied and (Θ, \mathbf{p}) under Γ_{new} are generically identifiable.

4.2. *Results for Q-restricted latent class models.* In this subsection, we characterize how the Q -matrix impacts the identifiability of multiparameter models. Similar to Section 3.2, we consider the case $\mathcal{A} = \{0, 1\}^K$. For strict identifiability, the result of either Theorem 4.1 or Proposition 4.1 implies the result of Theorem 1 in [45], as discussed in Remark 4.3. Our next result gives a flexible structural condition on Q that leads to generic identifiability.

THEOREM 4.3. *Under a multiparameter Q-restricted latent class model, if the Q-matrix satisfies the following conditions, then the model parameters are generically identifiable, up to label swapping among those latent classes that have identical column vectors in Γ .*

(C5) Q contains two $K \times K$ submatrices Q_1, Q_2 , such that for $i = 1, 2$,

$$(4.1) \quad Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q' \end{pmatrix}_{J \times K}; \quad Q_i = \begin{pmatrix} 1 & * & \dots & * \\ * & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 1 \end{pmatrix}_{K \times K}, \quad i = 1, 2,$$

where each “*” can be either zero or one.

(C6) With the Q -matrix taking the form of (4.1), in the submatrix Q' each attribute is required by at least one item.

The above identifiability result does not require the Q to contain an identity submatrix I_K and provides a flexible new condition for generic identifiability that are satisfied by various Q -matrix structures; see examples in Section 4.3. Under a multiparameter restricted latent class model with all entries of the Q -matrix being ones, conditions (C5) and (C6) in Theorem 4.3 equivalently reduce to $J \geq 2K + 1$, which is consistent with the result in [2] for unrestricted latent class models.

Next, we discuss the necessity of the proposed sufficient conditions for generic identifiability. Conditions (C5) and (C6) imply that each attribute is required by at least three items. The next theorem shows that it is necessary for each attribute to be required by at least two items.

THEOREM 4.4. *Consider a multiparameter Q-restricted latent class model:*

(a) *If some attribute is required by only one item, then the model is not generically identifiable.*

(b) *If some attribute is required by only two items, without loss of generality assume Q takes the following form:*

$$(4.2) \quad Q = \begin{pmatrix} 1 & \mathbf{v}_1^\top \\ 1 & \mathbf{v}_2^\top \\ 0 & Q' \end{pmatrix},$$

then as long as $\mathbf{v}_1 \vee \mathbf{v}_2 \neq \mathbf{1}_{K-1}$ and the submatrix Q' satisfies conditions (C5) and (C6), then the model parameters (Θ, \mathbf{p}) are generically identifiable, up to label swapping among those latent classes that have identical column vectors in Γ .

REMARK 4.4. As a notion of partial identification of model parameters, generic identifiability does not imply strict identifiability. For instance, if the Q -matrix is in the form of (4.2) and $\mathbf{v}_i = \mathbf{0}$ for $i = 1$ and 2 , then the model is not strictly identifiable, but generic identifiability can still hold as stated in Theorem 4.4. This is also an analogue to the situations discussed in Theorem 3.2 for two-parameter restricted latent class models. Based on Theorems 4.3 and 4.4, we would recommend practitioners in diagnostic test designs to ensure each attribute is measured by at least three items.

4.3. *Applications.* Similar to the discussion in Section 3.3, our results of generic identifiability also lead to the estimability of the model parameters.

PROPOSITION 4.2. *Suppose a restricted latent class model is generically identifiable on the parameter space \mathcal{T} with a measure-zero nonidentifiable set \mathcal{V} . If the true parameters $(\Theta^0, \mathbf{p}^0) \in \mathcal{T} \setminus \mathcal{V}$, then $(\hat{\Theta}, \hat{\mathbf{p}}) \rightarrow (\Theta^0, \mathbf{p}^0)$ almost surely as $N \rightarrow \infty$.*

We apply the new theory of generic identifiability to the designs introduced in Section 2.2, and establish generic identifiability of the multi-parameter restricted latent class models. Consider the TOEFL iBT data. Both Q -matrices corresponding to TOEFL reading forms A and B can be transformed into the form of (4.1) through some row rearrangements, with the corresponding Q' requiring each attribute at least once. Therefore, both Q -matrices satisfy conditions (C5) and (C6) and any multiparameter Q -restricted models associated with them are generically identifiable and estimable. Our results in this section also guarantee the generic identifiability of multiparameter models associated with the $Q_{43 \times 12}$ for the TIMSS data, and the $Q_{20 \times 8}$ for the fraction subtraction data; please see Section A in the Supplementary Material [20] for details of checking the conditions.

5. Extensions to more complex models. In this section, we extend our identifiability theory to some more complicated latent variable models.

5.1. *Mixed-items restricted latent class models.* Our identifiability theory based on Γ directly applies to the case of mixed types of items, where the J items can conform to different models, including two-parameter conjunctive, two-parameter disjunctive or multiparameter.

First, consider the *two-parameter-mixed* restricted latent class model, where each item is either two-parameter conjunctive or disjunctive. Let $I(\cdot)$ denote the binary indicator function. For any Q -matrix and latent class space \mathcal{A} , denote the Γ -matrix under the two-parameter conjunctive model by $\Gamma^{\text{conj}}(Q, \mathcal{A})$ with the (j, α) th entry being $I(\alpha \succeq \mathbf{q}_j)$, and denote the Γ -matrix under the two-parameter disjunctive model by $\Gamma^{\text{disj}}(Q, \mathcal{A})$ with the (j, α) th entry being $I(\exists k \text{ s.t. } \alpha_k = q_{j,k} = 1)$. These definitions can be easily deduced from (2.3) and the sentence after it. The following is a corollary of Theorem 3.1.

COROLLARY 5.1. *Consider a two-parameter-mixed restricted latent class model with $Q = (Q_{\text{disj}}^\top, Q_{\text{conj}}^\top)^\top$, where Q_{disj} and Q_{conj} correspond to disjunctive and conjunctive items, respectively. If the following condition (E1) holds, then $(\theta^+, \theta^-, \mathbf{p})$ are \mathbf{p} -partially identifiable.*

(E1) *The $J \times |\mathcal{A}|$ matrix $\Gamma = (\Gamma^{\text{disj}}(Q_{\text{disj}}, \mathcal{A})^\top, \Gamma^{\text{conj}}(Q_{\text{conj}}, \mathcal{A})^\top)^\top$ satisfies conditions (C1) and (C2) in Theorem 3.1.*

In particular, if $\mathcal{A} = \{0, 1\}^K$ and the Γ defined in (E1) is separable, then $(\theta^+, \theta^-, \mathbf{p})$ are strictly identifiable.

One implication of Corollary 5.1 is that when a diagnostic test contains both conjunctive and disjunctive items, the underlying Q -matrix does not need to include a submatrix I_K for $(\theta^+, \theta^-, \mathbf{p})$ to be strictly identifiable. This is in contrary to the case of a purely conjunctive or purely disjunctive two-parameter model, where this requirement is indeed necessary [19, 47]. The following application of Corollary 5.1 illustrates this point.

EXAMPLE 5.1. Consider a diagnostic test with 4 conjunctive items and 2 disjunctive items with the following Q -matrix:

$$Q = \begin{pmatrix} Q_{4 \times 2}^{\text{conj}} \\ Q_{2 \times 2}^{\text{disj}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ \hline 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \Gamma = \begin{matrix} & (0, 0) & (0, 1) & (1, 0) & (1, 1) \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

Then if $\mathcal{A} = \{0, 1\}^2$, the corresponding Γ -matrix as shown above is separable, and conditions (C1*) and (C2*) are satisfied. So $\theta^+ = (\theta_1^+, \dots, \theta_6^+)^\top$, $\theta^- = (\theta_1^-, \dots, \theta_6^-)^\top$ and $\mathbf{p} = (p(0,0), p(0,1), p(1,0), p(1,1))^\top$ are strictly identifiable, despite that Q does not contain a submatrix I_2 .

If there exist both two-parameter items and multiparameter items in the model, we have the following identifiability result, the part (a) of which directly results from Theorem 4.1 and Proposition 4.1. Please see Section D in the Supplementary Material [20] for details.

COROLLARY 5.2. Assume $Q = (Q_{\text{disj}}^\top, Q_{\text{conj}}^\top, Q_{\text{mult}}^\top)^\top$ where Q_{disj} , Q_{conj} and Q_{mult} correspond to the two-parameter disjunctive, two-parameter conjunctive, and multiparameter items, respectively.

(a) If $\Gamma = (\Gamma^{\text{disj}}(Q_{\text{disj}}, \mathcal{A})^\top, \Gamma^{\text{conj}}(Q_{\text{conj}}, \mathcal{A})^\top, \Gamma^{\text{conj}}(Q_{\text{mult}}, \mathcal{A})^\top)^\top$ satisfies conditions (C3) and (C4) in Theorem 4.1; or conditions (C3*) and (C4*) in Proposition 4.1, then (Θ, \mathbf{p}) are strictly identifiable.

(b) If Γ satisfies condition (E2) in Section D of the Supplementary Material [20], then (Θ, \mathbf{p}) are generically identifiable.

5.2. *Restricted latent class models with categorical responses.* We next study restricted latent class models with multiple levels of responses per item, that is, categorical responses, instead of binary responses considered in previous sections. These models have been considered in [27, 39] and [4]. We consider the setting in [4]. Suppose for each item j out of the J items in a diagnostic test, there are L_j categories of responses. For each item j and each category of response $l \in \{0, \dots, L_j - 1\}$, there are a set of positive response parameters of the latent classes $\theta_j^{(l)} = \{\theta_{j,\alpha}^{(l)} : \alpha \in \mathcal{A}\}$ with $\theta_j^{(0)} = \mathbf{1} - \sum_{l>0} \theta_j^{(l)}$. Further, for each item j , the \mathbf{q} -vector \mathbf{q}_j constrains the vector $\theta_j^{(l)}$ based on (2.2) for each category $l \in \{1, \dots, L_j - 1\}$ independently, other than the basic level $l = 0$. Namely, for any $j \in \mathcal{S}$,

$$\max_{\alpha \in \mathcal{C}_j} \theta_{j,\alpha}^{(l)} = \min_{\alpha \in \mathcal{C}_j} \theta_{j,\alpha}^{(l)} > \theta_{j,\alpha'}^{(l)} \quad \forall l \in \{1, \dots, L_j - 1\} \text{ and } \forall \alpha' \notin \mathcal{C}_j.$$

We collect all the model parameters in $(\Theta^{\text{cat}}, \mathbf{p})$ with $\Theta^{\text{cat}} = \{\theta_j^{(l)} : j = 1, \dots, J; l = 0, \dots, L_j - 1\}$. Then we have the following identifiability result.

PROPOSITION 5.1. For a given Q -matrix, consider the following cases:

- (a) If for any $j \in \mathcal{S}$ and $l \in \{1, \dots, L_j\}$, item parameters $\{\theta_{j,\alpha}^l, \alpha \in \mathcal{A}\}$ follow the two-parameter assumption, and Q satisfies (C1*) and (C2*) in Corollary 3.1, then $(\Theta^{cat}, \mathbf{p})$ are p -partially identifiable.
- (b) If for any $j \in \mathcal{S}$ and $l \in \{1, \dots, L_j\}$, item parameters $\{\theta_{j,\alpha}^l, \alpha \in \mathcal{A}\}$ follow the multiparameter assumption, and Q satisfies conditions (C5) and (C6) in Theorem 4.3, then $(\Theta^{cat}, \mathbf{p})$ are generically identifiable.

5.3. *Deep restricted Boltzmann machines.* Restricted latent class models share great similarities with Restricted Boltzmann Machines (RBM) [17]. We use a simple example to illustrate how the RBM architecture can be used as a special restricted latent class model for cognitive diagnosis. The RBM on the right panel of Figure 1 consists of two latent layers $\alpha^{(1)}$ and $\alpha^{(2)}$ and one observed layer \mathbf{R} . In a diagnostic test, the \mathbf{R} represents multivariate binary responses to test items, the first latent layer $\alpha^{(1)}$ represents the fine-grained binary skill attributes measured by the items, while the second binary latent layer $\alpha^{(2)}$ helps to model the dependence among $\alpha^{(1)}$ and may be interpreted as more general skill domains. Denote the lengths of vectors $\mathbf{R}, \alpha^{(1)}$ and $\alpha^{(2)}$ by J, K_1 and K_2 . Under RBM assumptions, the probability distribution of all the observed and latent variables is

$$(5.1) \quad \mathbb{P}(\mathbf{R}, \alpha^{(1)}, \alpha^{(2)}) = \frac{1}{Z} \exp(-\mathbf{R}^\top \mathbf{W}^Q \alpha^{(1)} - (\alpha^{(1)})^\top \mathbf{U} \alpha^{(2)}),$$

where Z is the normalization constant, and \mathbf{W}^Q, \mathbf{U} are parameter matrices, of size $J \times K_1$ and $K_1 \times K_2$, respectively. We drop the bias terms in the above energy function without loss of generality [17]. We can impose a Q -matrix of size $J \times K_1$ to restrict the parameters \mathbf{W}^Q in (5.1). Specifically, Q specifies which entries of $\mathbf{W}^Q = (w_{j,k})$ are zero, that is, $w_{j,k} = 0$ if $q_{j,k} = 0$. The form of Q underlying the \mathbf{W}^Q in Figure 1 is on the left panel of the figure.

We call \mathbf{W}^Q the *item parameters* of a RBM, since these parameters relate to the observed responses to items; and call a RBM with a Q -matrix structure an *item-parameter-restricted RBM*. Then an item-parameter-restricted RBM can be viewed as a multiparameter main-effect restricted latent class model, with $\alpha^{(1)}$ belonging to the latent class space $\{0, 1\}^{K_1}$. The next proposition establishes identifiability of the item parameters \mathbf{W}^Q .

PROPOSITION 5.2. For a given Q -matrix, consider the following cases:

- (a) If there is no sparsity structure in \mathbf{W}^Q (i.e., $Q = \mathbf{1}_{J \times K}$), then as long as $J \geq 2K_1 + 1$, the item parameters \mathbf{W}^Q are generically identifiable.
- (b) If the Q -matrix satisfies the sufficient conditions for strict or generic identifiability in Section 4, then \mathbf{W}^Q are strictly or generically identifiable, respectively.

Proposition 5.2 establishes identifiability of the item parameters \mathbf{W}^Q , which provides the theoretical guarantee in the application of item calibration to assess the quality of the items. It would also be interesting to further investigate identifiability of other parameters besides the item parameters in a deep restricted Boltzmann machine, which we leave for future study.

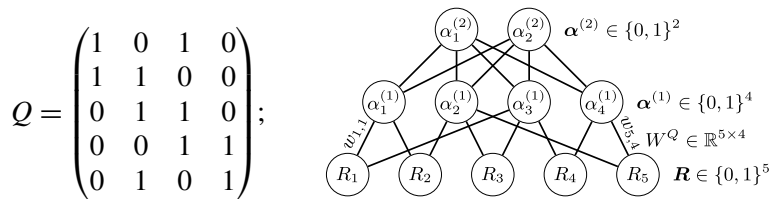


FIG. 1. (Deep) Restricted Boltzmann machine.

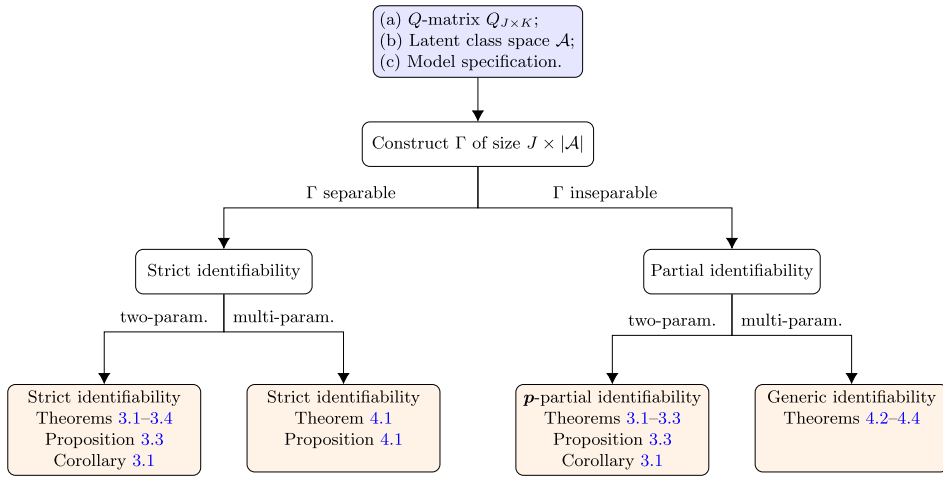


FIG. 2. Flowchart of the results in Sections 3 and 4.

6. Discussion. This paper proposes a general framework of strict and partial identifiability of restricted latent class models.

We provide a flowchart in Figure 2 to summarize our main theoretical results in Sections 3 and 4. The flowchart illustrates how to apply the new theory in cognitive diagnosis. Specifically, given the specification of the Q -matrix, the latent class space $\mathcal{A} \subseteq \{0, 1\}^K$, and the diagnostic model assumptions, one can construct the corresponding $J \times |\mathcal{A}|$ Γ -matrix based on the \mathcal{C}_j 's defined in (2.1). Then in the case of a separable Γ -matrix, if the model is two-parameter, the p -partial identifiability exactly reduces to strict identifiability and one can use results in Section 3 to establish strict identifiability; and if the model is multiparameter, one can use results Theorem 4.1 and Proposition 4.1 in Section 4.1 for strict identifiability. On the other hand, if the Γ -matrix is inseparable, depending on whether the model is two-parameter or multiparameter, one can use the results in Section 3.2 or those in Section 4 to check whether the model is p -partially identifiable or generically identifiable, respectively. Note that in the special case of $\mathcal{A} = \{0, 1\}^K$, the Γ -matrix with 2^K columns is separable if and only if the Q -matrix contains an identity submatrix I_K , a key condition assumed in previous works (e.g., [45, 46]). Hence, this work not only largely relaxes these existing conditions for strict identifiability by allowing more flexible attribute structures with an arbitrary \mathcal{A} , but also provides the first study on partial identifiability when the Q -matrix does not include an I_K (the Γ -matrix is inseparable). We give easily-checkable identifiability conditions to ensure estimability of the model parameters, and these conditions serve as practical guidelines for designing statistically valid diagnostic tests.

We point out that the strict identifiability results in Section 4.1 (Theorem 4.1 and Proposition 4.1) apply to the general family of restricted latent class models satisfying constraints (2.2), including not only multiparameter but also two-parameter models; on the other hand, since these results are established under the general constraints (2.2), their conditions are stronger than those in Section 3 under two-parameter models. In contrast, the generic identifiability results in Sections 4.1 and 4.2 (Theorems 4.2–4.4) only apply to multiparameter models. This is because under generic identifiability, the nonidentifiable measure-zero subset of a multiparameter model's parameter space (such as GDINA), could still contain the parameter space corresponding to a two-parameter submodel (such as DINA), making these generic identifiability results not applicable to two-parameter models. Nevertheless, generic identifiability is a general concept not just restricted to the multi-parameter models. An interesting future direction to study is the generic identifiability of two-parameter models under

the introduced p -partial identifiability framework; that is, one can study what conditions lead to the generic identifiability of $(\theta^+, \theta^-, \nu)$. We also point that a multiparameter model can also be p -partially identifiable, as discussed in Remark 4.1.

For the p -partial identifiability and generic identifiability results in Sections 3–5, we assume that the model specification for each item, the design matrix and latent class space \mathcal{A} are available as prior knowledge. In practice, there can be scenarios where not all of such information is available. As pointed out by one reviewer, in applications of cognitive diagnostic modeling, both the advances in modeling capacity and computing flexibility, and the recent real-data examples provide ground for adopting a model with mixed type of items, which are determined in a data-driven way. To this end, our strict identifiability results in Section 4.1 and those in Section 5.1 for mixed-items models can be applied to assess identifiability *a posteriori*. In practice, when deciding which model to adopt, one can use the response data to determine the number of latent classes and determine which diagnostic model an item conforms to. For instance, one may employ the popular information criteria such as AIC and BIC to perform model selection; or one may first fit a general cognitive diagnostic model, such as GDINA or GDM, then use the Wald test to determine which submodel an item follows [11]. Alternatively, one may use a penalized likelihood method [46] or Bayesian method [5] to directly estimate the structure of the item parameters for each item; such structure informs the model specification of the item. For the selected candidate models, we would recommend further applying our identifiability theory to assess their identifiability and validity. The general theoretical framework developed in this paper would be a useful tool to develop the identifiability and estimability conditions for learning the item-level model structure and the population-level latent class space \mathcal{A} . This is an interesting and important direction that we plan to pursue in the future.

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SUPPLEMENTARY MATERIAL

Supplement to “Partial identifiability of restricted latent class models” (DOI: 10.1214/19-AOS1878SUPP; .pdf). The supplementary material contains the proofs of main results and some technical lemmas.

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