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STRONG ORTHOGONAL ARRAYS OF STRENGTH TWO PLUS

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Strong orthogonal arrays were recently introduced and studied in He and Tang [*Biometrika* **100** (2013) 254–260] as a class of space-filling designs for computer experiments. To enjoy the benefits of better space-filling properties, when compared to ordinary orthogonal arrays, strong orthogonal arrays need to have strength three or higher, which may require run sizes that are too large for experimenters to afford. To address this problem, we introduce a new class of arrays, called strong orthogonal arrays of strength two plus. These arrays, while being more economical than strong orthogonal arrays of strength three, still enjoy the better two-dimensional space-filling property of the latter. Among the many results we have obtained on the characterizations and constructions of strong orthogonal arrays of strength two plus, worth special mention is their intimate connection with second-order saturated designs.

1. Introduction. Computer experiments call for space-filling designs [Fang, Li and Sudjianto (2006), Santner, Williams and Notz (2003)]. One approach to finding such designs is to employ an algorithmic search based on a distance or discrepancy criterion. Though flexible, this approach is computer-intensive and quickly becomes ineffective for large problems. Design selection based on a model-dependent criterion is another good idea to use when experiments are conducted in phases. A tentative model built using the data from one phase of experimentation can then be used to guide the selection of design points for the next phase of experimentation. The most attractive approach to constructing space-filling designs is that based on orthogonal arrays or similar structures. Designs so constructed enjoy guaranteed space-filling properties. As this approach is about general theoretical constructions, computing is not an issue, unless one wants to conduct an algorithmic search on top of theoretical constructions.

The idea goes back to McKay, Beckman and Conover (1979) who introduced Latin hypercubes, which are orthogonal arrays of strength one. Owen (1992) proposed the use of randomized orthogonal arrays and Tang (1993) constructed orthogonal array based Latin hypercubes. Both classes of designs achieve t-dimensional space-filling when orthogonal arrays of strength t are used. Inspired by (t, m, s)-nets from quasi-Monte Carlo [Niederreiter (1992)], He and Tang

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(2013) introduced and studied strong orthogonal arrays (SOAs). Such arrays of strength t are more space-filling in any $g < t$ dimensions than comparable ordinary orthogonal arrays while being as space-filling in t dimensions as the latter. As Latin hypercubes, based on either class of arrays, all achieve the maximum stratification in one-dimension, and to enjoy the benefits of SOAs, they need to be of strength 3 or higher, which may result in prohibitively large run sizes for certain investigations.

We address this problem in this paper by introducing SOAs of strength 2+. This new class of arrays retains the two-dimensional space-filling property of SOAs of strength 3 while having more economical run sizes than the latter. The paper then goes on to study the characterizations and constructions of SOAs of strength 2+. Among the many results, we have obtained on SOAs of strength 2+, a surprising finding is their intimate connection with second-order saturated (SOS) designs introduced by Block and Mee (2003).

Here is a brief preview of the paper. Section 2 introduces background, gives a formal definition of SOAs of strength 2+, and then provides a general characterization of such arrays. We consider in Section 3 characterizing and constructing SOAs of strength 2+ using 2^{m-p} designs. A key result in this section is the characterization of SOAs of strength 2+ through SOS designs. Some construction results for SOS designs are also presented here. Section 4 presents similar results but from using s^{m-p} $(s > 2)$ designs. We then examine in Section 5 a generalization of SOAs of strength 2+ and their construction. Section 6 concludes the paper.

2. Defining and characterizing SOAs of strength two plus.

2.1. *Orthogonal arrays and strong orthogonal arrays.* An $n \times m$ matrix with entries from $\{0, 1, \dots, s_j - 1\}$ in the jth column is an orthogonal array of n runs for m factors, and having strength t if all possible combinations appear with the same frequency in any of its $n \times t$ submatrices. We use $\mathrm{OA}(n, m, s_1 \times \cdots \times s_m, t)$ to denote such an array. The array is symmetric if $s_1 = \cdots = s_m = s$ in which case a simpler notation $\mathrm{OA}(n, m, s, t)$ is used, and it is called asymmetric otherwise. Hedayat, Sloane and Stufken (1999) is an excellent general reference for orthogonal arrays. Many useful sources of information are also available in Dey and Mukerjee (1999) and Cheng (2014).

Inspired by the notion of nets from quasi-Monte Carlo, He and Tang (2013) introduced SOAs for computer experiments. An elementary interval in base s is an interval in $[0, 1)^m$ of form $\prod_{j=1}^{m}[c_j/s^{d_j}, (c_j+1)/s^{d_j})$, where nonnegative integers c_j and d_j satisfy $0 \le c_j < s^{d_j}$. For $0 \le w \le k$, a (w, k, m)-net in base s is a set of s^k points in $[0, 1)^m$ such that every elementary interval in base s of volume s^{w-k} contains exactly s^w points [Niederreiter (1992)].

An $n \times m$ matrix with entries from $\{0, 1, \dots, s^t - 1\}$ is called an SOA of n runs, m factors, s^t levels and strength t if any subarray of g columns for any g with

$1 \le g \le t$ can be collapsed into an $\mathrm{OA}(n, g, s^{u_1} \times \cdots \times s^{u_g}, g)$ for any positive integers $u_1, \ldots, u_g$ with $u_1 + \cdots + u_g = t$, where collapsing s^t levels into s^{u_j} levels is according to $[a/s^{t-u_j}]$ for $a = 0, 1, \ldots, s^t - 1$. We use $\mathrm{SOA}(n, m, s^t, t)$ to denote such an array. As an $\mathrm{SOA}(n, m, s^t, t)$ is collapsible into an $\mathrm{OA}(n, m, s, t)$, we must have $n = \lambda s^t$ for some integer λ. This λ is called the index of the SOA.

If $\lambda = s^w$ for some integer w, then the existence of an $\mathrm{SOA}(\lambda s^t, m, s^t, t)$ is equivalent to that of a (w, k, m)-net in base s where $k = w + t$ [He and Tang (2013)]. As SOAs are defined without restricting the index to be a power of s, they provide a broader concept than (w, k, m)-nets. For more details on the relationship between the two, we refer the reader to He and Tang (2013).

From the above general definition of SOAs, an $\mathrm{SOA}(n, m, s^2, 2)$ of strength 2 is an orthogonal array of strength one in itself and becomes an orthogonal array of strength 2 if its s^2 levels are collapsed into s levels using $[a/s]$. Thus, in any two dimensions, an $\mathrm{SOA}(n, m, s^2, 2)$ achieves a stratification on an $s \times s$ grid just like an $\mathrm{OA}(n, m, s, 2)$. This means that an $\mathrm{SOA}(n, m, s^2, 2)$ is as space-filling as but no better than an $\mathrm{OA}(n, m, s, 2)$ in two-dimensions.

SOAs of strength $t \ge 3$ are more space-filling than comparable orthogonal arrays. An $\mathrm{SOA}(n, m, s^3, 3)$ of strength 3 achieves stratifications on $s^2 \times s$ and $s \times s^2$ grids in two-dimensions and $s \times s \times s$ grids in three-dimensions. Whereas an $\mathrm{OA}(n, m, s, 3)$ achieves stratifications on $s \times s \times s$ grids in three-dimensions, it only promises stratifications on $s \times s$ grids in two-dimensions. The better space-filling property in two-dimensions offered by an $\mathrm{SOA}(n, m, s^3, 3)$ is evident.

It is quite remarkable that SOAs of strength 3 can be constructed from an orthogonal array of strength 3 at almost no cost. We know that an $\mathrm{SOA}(n, m, s^3, 3)$ can be constructed from any $\mathrm{OA}(n, m + 1, s, 3)$ [He and Tang (2013)] or a semi-embeddable $\mathrm{OA}(n, m, s, 3)$ [He and Tang (2014)]. As much satisfying as these results are, SOAs of strength 3 may require run sizes that are too large for certain scientific investigations. A new class of arrays, SOAs of strength 2+, to be introduced next, is aimed at solving this problem. In a nutshell, SOAs of strength 2+ are SOAs of strength 2 that possess the two-dimensional space-filling property of SOAs of strength 3.

2.2. *Strong orthogonal arrays of strength* 2+.

DEFINITION 1. An $n \times m$ matrix with entries from $\{0, 1, \ldots, s^2 - 1\}$ is called a strong orthogonal array of strength 2+, and with n runs and m factors of s^2 levels, if any subarray of two columns can be collapsed into an $\mathrm{OA}(n, 2, s^2 \times s, 2)$ and an $\mathrm{OA}(n, 2, s \times s^2, 2)$. We denote this array by $\mathrm{SOA}(n, m, s^2, 2+)$.

Essentially, an $\mathrm{SOA}(n, m, s^2, 2+)$ is an $\mathrm{SOA}(n, m, s^2, 2)$ with a better two-dimensional space-filling property. While an $\mathrm{SOA}(n, m, s^2, 2)$ only promises stratifications on $s \times s$ grids in two-dimensions, an $\mathrm{SOA}(n, m, s^2, 2+)$ achieves stratifications on $s^2 \times s$ and $s \times s^2$ grids in two-dimensions. This property of stratifications on finer grids in two-dimensions is what makes an SOA$(n, m, s^3, 3)$ better than an OA$(n, m, s, 3)$ when it comes to constructing designs for computer experiments.

EXAMPLE 1. An SOA(16, 10, 4, 2+) is given below:

$$\begin{array}{cccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 2 \\
2 & 0 & 2 & 0 & 2 & 1 & 2 & 1 & 2 & 2 \\
2 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 0 \\
0 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\
0 & 2 & 0 & 1 & 3 & 0 & 3 & 0 & 3 & 0 \\
0 & 0 & 2 & 3 & 1 & 1 & 0 & 3 & 3 & 0 \\
0 & 0 & 0 & 3 & 3 & 3 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
1 & 1 & 3 & 2 & 0 & 0 & 3 & 0 & 0 & 3 \\
1 & 3 & 1 & 0 & 2 & 1 & 0 & 3 & 0 & 3 \\
1 & 3 & 3 & 0 & 0 & 3 & 1 & 1 & 2 & 1 \\
3 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 3 & 3 \\
3 & 1 & 3 & 1 & 3 & 0 & 1 & 2 & 1 & 1 \\
3 & 3 & 1 & 3 & 1 & 1 & 2 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3
\end{array}$$

One can check that any array of two columns is collapsible into an OA$(16, 2, 4 \times 2, 2)$ and an OA$(16, 2, 2 \times 4, 2)$. An SOA(16, 7, 8, 3) enjoys the same two-dimensional space-filling property but only allows the study of 7 factors as there does not exist an SOA$(16, m, 8, 3)$ for $m \geq 8$ [He and Tang (2013), Theorem 2]. In contrast, the above SOA(16, 10, 4, 2+) is able to accommodate up to 10 factors.

SOAs of strength t can be completely characterized by generalized orthogonal arrays [He and Tang (2013)]. A similar result can also be established for SOAs of strength 2+.

PROPOSITION 1. *An* SOA$(n, m, s^2, 2+)$, *say* D, *exists if and only if there exist two arrays* A *and* B *where* $A = (a_1, \dots, a_m)$ *is an* OA$(n, m, s, 2)$ *and* $B = (b_1, \dots, b_m)$ *is an* OA$(n, m, s, 1)$ *such that* (a_j, a_k, b_k) *is an orthogonal array of strength* 3 *for any* $j \neq k$. *The three arrays are linked through* $D = sA + B$.

The proof is similar to that for the characterization of SOAs of strength t as in He and Tang (2013), and is thus omitted.

REMARK 1. In Proposition 1, array A is not required to have strength 3, but if it does, then an SOA$(n, m, s^3, 3)$ can be constructed. Dropping this requirement of strength 3 for array A is a hallmark of SOAs of strength 2+. It allows designs for more factors to be constructed, as shown in Example 1.

3. Characterization and construction using 2^{m-p} designs. This section examines how to construct SOAs of strength 2+ using regular 2^{m-p} designs. More specifically, we consider the construction of D, an SOA$(n, m, 4, 2+)$, from A and B through $D = 2A + B$ as in Proposition 1, where the columns of A and B are selected from a saturated two-level regular design. Regular two-level designs are most commonly and conveniently studied with their two levels denoted by ± 1, an approach we adopt here. When A and B have levels ± 1, they can be made to have levels 0, 1 by $(A + 1)/2$ and $(B + 1)/2$ where, for example, $A + 1$ denotes the matrix obtained by adding 1 to all elements of A. Because of this, instead of $D = 2A + B$, we should be using

$$D = 2\{(A + 1)/2\} + (B + 1)/2 = A + B/2 + 3/2 \tag{1}$$

in constructing D from A and B that have entries ± 1.

A regular saturated design S of $n = 2^k$ runs for $n - 1$ factors can be obtained by first writing down a full factorial for k factors and then adding all possible interaction columns. If we regard S as a set of $n - 1$ columns, then a subset C of m columns is a 2^{m-p} design where $p = m - k$, which is guaranteed to have resolution III or higher. The set of columns that are not in C is the complementary design of C, which we denote by $\bar{C} = S \setminus C$. Because S is regular, it has the property that $ab \in S$ for any $a, b \in S$, $a \neq b$, where ab stands for the interaction column of a and b. Block and Mee (2003) introduced the notion of second-order saturated (SOS) designs to describe those designs of which all degrees of freedom can be used to estimate main effects or two-factor interactions. In our notation, a design C is an SOS design if any $d \in \bar{C}$ can be written as $d = ab$ for some $a, b \in C$.

THEOREM 1. *If an SOA of strength* 2+ *is to be constructed through* (1) *where the columns of A and B are selected from S, a saturated regular design, then it is necessary and sufficient that $\bar{A}$ is an SOS design.*

PROOF. Let $D = A + B/2 + 3/2$ in (1) be an SOA of strength 2+. We need to show that $\bar{A}$ is SOS. By Proposition 1, $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_m)$ have the property that (a_j, a_k, b_k) is of strength 3 for any $j \neq k$, which is impossible if $b_k = a_k$ or $b_k = a_j$. This implies that $b_k \in \bar{A}$ for any k. It is also clear that if $a_k b_k = a_j$, then (a_j, a_k, b_k) cannot have strength 3. This shows that $a_k b_k \in \bar{A}$. Let $b'_k = a_k b_k \in \bar{A}$. We then have $a_k = b_k b'_k$ where $b_k, b'_k \in \bar{A}$, proving that $\bar{A}$ is SOS.

Now suppose $\bar{A}$ is SOS. Then $a_j = b_j b'_j$ for some $b_j, b'_j \in \bar{A}$. Choose $B = (b_1, \dots, b_m)$. Since $a_j b_j = b'_j \in \bar{A}$, we have that $a_j b_j \neq a_k$ for any k, which means that (a_j, b_j, a_k) must have strength 3. Invoking Proposition 1, we conclude that D obtained from (1) is an SOA of strength 2+. $\square$

The proof of the sufficiency part shows that an SOA$(2^k, m, 4, 2+)$ can be constructed from an SOS design C as follows:

Step 1. Take $A = \bar{C}$. Write $A = (a_1, \dots, a_m)$.

Step 2. Since C is an SOS design, we must have $a_j = b_j b'_j$ for some $b_j, b'_j \in C$. Take $B = (b_1, \ldots, b_m)$.

Step 3. Obtain D, an SOA$(2^k, m, 4, 2+)$, using equation (1).

EXAMPLE 2. Let (a_1, a_2, a_3, a_4) be a full 2^4 factorial in 16 runs. Then

$$S = \{a_1, a_2, a_3, a_4, a_1a_2, a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_3a_4,$$
$$a_1a_2a_3, a_1a_2a_4, a_1a_3a_4, a_2a_3a_4, a_1a_2a_3a_4\}$$

is a saturated design of 16 runs in 15 factors. One can check that $C = \{a_1, a_2, a_3, a_4, a_1a_2a_3a_4\}$ is an SOS design. According to the construction method above, matrix A and B can be obtained as follows:

$$A = \bar{C} = \{a_1a_2, a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_3a_4, a_1a_2a_3, a_1a_2a_4, a_1a_3a_4, a_2a_3a_4\}$$

and one choice for B is $B = (a_1, a_1, a_1, a_2, a_2, a_3, a_4, a_3, a_2, a_1)$. Then $D = A + B/2 + 3/2$ is an SOA(16, 10, 4, 2+), one version of which has been displayed in Example 1.

We now present some constructions of SOS designs. The saturated design S of $n = 2^k$ runs for $n - 1$ factors is based on k independent columns, which we now denote by $a_1, \ldots, a_{k_1}, b_1, \ldots, b_{k_2}$ where $k = k_1 + k_2$. Further let P be a subset of S consisting of $a_1, \ldots, a_{k_1}$ and all their interaction columns, and similarly Q be a subset of S consisting of $b_1, \ldots, b_{k_2}$ and all their interaction columns. To avoid the trivial cases, we let $k_1 \geq 2$ and $k_2 \geq 2$ for the rest of the section. Consider the following four design constructions:

(i) $C_1 = P \cup Q$,
(ii) $C_2 = (P \setminus \{a_1\}) \cup (Q \setminus \{b_1\}) \cup \{a_1b_1\}$,
(iii) $C_3 = (P \setminus \{a_1\}) \cup (a_1 Q)$,
(iv) $C_4 = (b_1 P) \cup (a_1 Q \setminus \{a_1b_1\})$.

PROPOSITION 2. *Designs C_1, C_2, C_3 and C_4 given above are all SOS designs.*

The proof of Proposition 2 is a bit tedious but quite straightforward, and thus omitted.

Let m_k be the largest m for an SOA$(2^k, m, 4, 2+)$ based on regular designs to exist. Then the following result can be established.

THEOREM 2. *We have that*

$$2^k - 2^{[k/2]} - 2^{k-[k/2]} + 2 \leq m_k \leq 2^k - 1 - M(k),$$

where $M(k)$ is the largest m for a resolution V design to exist.

PROOF. Consider an SOS design of 2^k runs with the smallest number m'_k of factors. By simply counting degrees of freedom, we obtain $m'_k \geq M(k)$, which

TABLE 1
Maximum numbers of factors for SOAs of strength 3 and 2+

k	$n = 2^k$	h_k (strength 3)	m_k (strength 2+)
4	16	7	10*
5	32	15	22*
6	64	31	50*
7	128	63	106
8	256	127	226

implies that $m_k \le 2^k - 1 - M(k)$ as $m_k + m'_k = 2^k - 1$ due to Theorem 1. Constructions (ii), (iii) and (iv) all give SOS designs for $2^{k_1} + 2^{k_2} - 3$ factors. This leads to $m'_k \le 2^{k_1} + 2^{k_2} - 3$. Minimizing the right-hand side of the above inequality for given k, we obtain $m'_k \le 2^{[k/2]} + 2^{k-[k/2]} - 3$. Thus, $m_k \ge 2^k - 2^{[k/2]} - 2^{k-[k/2]} + 2$. □

Recall that an SOA$(2^k, m, 4, 2+)$ enjoys the same two-dimensional space-filling property as an SOA$(2^k, m, 8, 3)$. The maximum number h_k of factors for an SOA$(2^k, h_k, 8, 3)$ to exist is $h_k = 2^{k-1} - 1$ [He and Tang (2013), Theorem 2]. In contrast, the maximum number m_k of factors for an SOA$(2^k, m_k, 4, 2+)$ is at least $2^k - 2^{[k/2]} - 2^{k-[k/2]} + 2$, which is substantially larger than $h_k = 2^{k-1} - 1$ for all nontrivial cases $k \ge 4$. Table 1 provides a comparison.

The m_k values in Table 1 are all from the lower bound in Theorem 2. Those entries with a * are in fact exact, which is obvious for $k = 4$ and can be easily checked using the complete catalog of Chen, Sun and Wu (1993) for $k = 5$. For $k = 6$, the exactness of $m_6 = 50$ has been verified through a combination of theoretical arguments and computer search.

4. Characterization and construction using s^{m-p} designs. This section studies the characterization and construction of SOA$(n, m, s^2, 2+)$ using regular s^{m-p} designs where $s \ge 3$ is a prime power. Let GF$(s) = \{w_0 = 0, w_1 = 1, w_2, \dots, w_{s-1}\}$ denote a Galois field of order s. The Rao–Hamming construction produces a linear orthogonal array OA$(s^k, (s^k - 1)/(s - 1), s, 2)$ for any $k \ge 2$, which is a regular saturated design of s^k runs and s levels for $(s^k - 1)/(s - 1)$ factors. This saturated design can also be obtained by first writing down a full s^k factorial for k factors and adding all possible interaction columns.

If we use $e_1, \dots, e_k$ to denote the k independent columns, then any linear combination $u_1e_1 + \cdots + u_ke_k$, where $u_j \in$ GF(s) that are not all zero, is an interaction column. Two linear combinations $u_1e_1 + \cdots + u_ke_k$ and $u'_1e_1 + \cdots + u'_ke_k$ actually represent the same interaction if $(u_1, \dots, u_k) = w(u'_1, \dots, u'_k)$ for some $w \in$ GF(s). One way to make sure to generate all different interaction columns is to use the $(u_1, \dots, u_k)$'s with the first nonzero u_j equal to 1. The set of all

$(s^k - 1)/(s - 1)$ columns, denoted by S, is in fact a projective geometry. For any two columns $a, b \in S$, there are $s - 1$ distinct interaction columns $a + w_j b$ where $j = 1, \ldots, s - 1$. In the language of projective geometry, the $s + 1$ columns $a, b, a + w_1 b, \ldots, a + w_{s-1} b$ form a line.

We consider constructing D, an SOA$(s^k, m, s^2, 2+)$ through

$$D = sA + B \tag{2}$$

using $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_m)$ with the columns a_j's and b_k's selected from S, the saturated design. Because A and B, when their columns are selected from S, have entries from $\mathrm{GF}(s) = \{w_0, w_1, \ldots, w_{s-1}\}$, one should convert these symbols to $\{0, 1, \ldots, s - 1\}$ before applying (2). No conversion is necessary if s is a prime.

THEOREM 3. *If D, an* SOA$(s^k, m, s^2, 2+)$, *is to be constructed using* (2) *from A and B whose columns are selected from S, then it is necessary and sufficient that for each $a_j \in A$, there exist s columns $b_{j1}, \ldots, b_{js}$ in the complementary design $\bar{A} = S \setminus A$ such that the $s + 1$ columns $a_j, b_{j1}, \ldots, b_{js}$ form a line.*

PROOF. Suppose D is an SOA$(s^k, m, s^2, 2+)$. Then (a_k, a_j, b_j) where $k \neq j$ has strength 3, which implies that $b_j \in \bar{A}$ for any j. That (a_k, a_j, b_j) has strength 3 also implies that $a_j + w_1 b_j, \ldots, a_j + w_{s-1} b_j$ must be all in $\bar{A}$. These s columns $b_j, a_j + w_1 b_j, \ldots, a_j + w_{s-1} b_j$ that are all in $\bar{A}$, together with a_j, form a line.

On the other hand, suppose that $s+1$ columns $a_j, b_{j1}, \ldots, b_{js}$ form a line where $b_{j1}, \ldots, b_{js}$ are all in $\bar{A}$. If we take $B = (b_1, \ldots, b_m)$ with $b_j = b_{j1}$, then it is easy to see that (a_k, a_j, b_j) must be of strength 3. This shows that D obtained from (2) is an SOA$(s^k, m, s^2, 2+)$. $\square$

Once we have a regular design A with the property described in Theorem 3, then D, an SOA$(s^k, m, s^2, 2+)$ can be constructed as follows. Let $A = (a_1, \ldots, a_m)$ and $B = (b_{11}, \ldots, b_{m1})$. If s is a prime, obtain $D = sA + B$ directly. If s is a prime power, first replace w_j by j in both A and B for $j = 0, 1, \ldots, s-1$ and then obtain $D = sA + B$.

We next present a construction result for an SOA$(s^k, m, s^2, 2+)$ through the construction of A with the property as stated in Theorem 3. To save space, an interaction column $u_1 e_1 + \cdots u_k e_k$ is represented by $e_1^{u_1} \cdots e_k^{u_k}$, which is not uncommon in standard design textbooks.

EXAMPLE 3. For $s = k = 3$, the saturated design S has 27 runs for 13 factors. The 13 columns are given by

$$S = \{e_1, e_2, e_3, e_1e_2, e_1e_2^2, e_1e_3, e_1e_3^2, e_2e_3, e_2e_3^2, e_1e_2e_3, e_1e_2^2e_3, e_1e_2e_3^2, e_1e_2^2e_3^2\}.$$

It can be verified that

$$A = \{e_1e_2^2, e_1e_3^2, e_2e_3^2, e_1e_2^2e_3, e_1e_2e_3^2, e_1e_2^2e_3^2\}$$

has the property required by Theorem 3. One choice for B is $B = (e_1, e_1, e_2, e_2, e_3, e_1)$. Then we obtain $D = 3A + B$ which is an SOA(27, 6, 9, 2+).

Example 3 is an illustration of a general method of constructing an SOA(s^k, m, s^2, 2+) we now present. Let A select all interaction columns $e_1^{u_1} \cdots e_k^{u_k}$ such that at least one u_j equals w_{s-1}. Since the first nonzero u_j is set at 1, the set A excludes the selection of columns $e_j^{w_{s-1}}$.

THEOREM 4. *For any $k \geq 3$ and any prime power $s \geq 3$, an* SOA(s^k, m, s^2, 2+) *where $m = (s^k - 1)/(s - 1) - ((s - 1)^k - 1)/(s - 2)$ can be constructed.*

PROOF. A simple counting shows that A contains $m = (s^k - 1)/(s - 1) - ((s - 1)^k - 1)/(s - 2)$ columns. What remains to be done is to verify that A defined above satisfies the condition in Theorem 3. For ease of presentation, we do this for $k = 3$. For $k > 3$, the same idea applies. An element in A must be of one of the following three kinds: (i) $e_1 e_2^{w_{s-1}}$, (ii) $e_1 e_2^{w_i} e_3^{w_{s-1}}$ where $1 \leq i \leq s - 2$ and (iii) $e_1 e_2^{w_{s-1}} e_3^{w_{s-1}}$. In case (i), the s columns $e_1, e_2, e_1 e_2, \ldots, e_1 e_2^{w_{s-2}}$ are all in $\bar{A}$, and together with $e_1 e_2^{w_{s-1}}$, they form a line. For case (ii), the $s + 1$ columns $e_1 e_2^{w_i}, e_3, e_1 e_2^{w_i} e_3, \ldots, e_1 e_2^{w_i} e_3^{w_{s-1}}$ form a line where all except $e_1 e_2^{w_i} e_3^{w_{s-1}}$ are in $\bar{A}$. For case (iii), the $s + 1$ columns $e_1, e_2 e_3, e_1 e_2 e_3, \ldots, e_1 (e_2 e_3)^{w_{s-1}}$ form a line where all except $e_1 e_2^{w_{s-1}} e_3^{w_{s-1}}$ are in $\bar{A}$. $\square$

For $k = 3$, the SOA(s^3, m, s^2, 2+) constructed in Theorem 4 has $m = 2s$ factors whereas an SOA(s^3, m, s^3, 3) for $m = s + 1$ can be constructed [He and Tang (2014), Proposition 2]. For $k = 4$, the SOA(s^4, m, s^2, 2+) from Theorem 4 has $m = 3s^2 - s + 1$ factors whereas He and Tang (2014) allows the construction of an SOA(s^4, m, s^3, 3) with $m = s^2 + 1$ factors. The advantage of SOAs of strength 2+ over SOAs of strength 3 is evident. Table 2 provides some numerical comparisons of the numbers of factors for the two classes of arrays.

TABLE 2
A comparison of the number m' of factors for SOA(s^k, m', s^3, 3) *in He and Tang (2014) and the number m'' of factors for* SOA(s^k, m'', s^2, 2+) *from Theorem 4*

k	s	$n = s^k$	m'	m''
3	3	27	4	6
3	4	64	5	8
3	5	125	6	10
4	3	81	10	25
4	4	256	17	45
4	5	625	26	71

5. Generalization and further results. The ideas in Definition 1 and Proposition 1 can be easily made more general. Now let $A = (a_1, \ldots, a_m)$ be an $\mathrm{OA}(n, m, s_1 \times \cdots \times s_m, 2)$ and $B = (b_1, \ldots, b_m)$ be an $\mathrm{OA}(n, m, \alpha_1 \times \cdots \times \alpha_m, 1)$ such that (a_j, a_k, b_k) is an $\mathrm{OA}(n, 3, s_j \times s_k \times \alpha_k, 3)$ for any $j \neq k$. Then $D = (d_1, \ldots, d_m)$ with $d_j = \alpha_j a_j + b_j$ is an $\mathrm{OA}(n, m, (\alpha_1 s_1) \times \cdots \times (\alpha_m s_m), 1)$ of which any two columns (d_j, d_k) can be collapsed into an $\mathrm{OA}(n, 2, (\alpha_j s_j) \times s_k, 2)$ and an $\mathrm{OA}(n, 2, s_j \times (\alpha_k s_k), 2)$. This is a generalization of an $\mathrm{SOA}(n, m, s^2, 2+)$.

Of special interest is when $s_1 = \cdots = s_m = s$ and $\alpha_1 = \cdots = \alpha_m = \alpha$, in which case we obtain a symmetric D. We use $\mathrm{SOA}_\alpha(n, m, \alpha s, 2+)$ to denote such an array. Note that there is a need to make α explicit in the notation. Below, we will give a construction of such $\mathrm{SOA}_\alpha(n, m, \alpha s, 2+)$'s. Though simple, the construction provides some very interesting space-filling designs that are not possible to obtain from the results in Sections 3 and 4.

Let $A_0, A_1, \ldots, A_{\alpha-1}$ all be $\mathrm{OA}(n_1, m, s, 2)$'s, which can be but are not necessarily the same. Then create an array by appending one column to the juxtaposition of $A_0, A_1, \ldots, A_{\alpha-1}$ as follows:

$$A^* = \begin{pmatrix} 0\cdots 0 & 1\cdots 1 & \cdots & \alpha-1\cdots\alpha-1 \\ A_0^T & A_1^T & \cdots & A_{\alpha-1}^T \end{pmatrix}^T. \tag{3}$$

This array has $n = \alpha n_1$ runs with α levels in the first column and s levels in the rest of columns. Write $A^* = (a_0, a_1, \ldots, a_m)$ where $a_0, a_1, \ldots, a_m$ denote the columns of A^* in (3). Now define $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_m)$ where $b_j = a_0$ for $j = 1, \ldots, m$. Clearly, we have that (a_j, a_k, b_k) is an $\mathrm{OA}(n, 3, s \times s \times \alpha, 3)$ for any $j \neq k$. This leads to the following result.

THEOREM 5. *Design D obtained from $D = \alpha A + B$ with A and B constructed above is an* $\mathrm{SOA}_\alpha(n, m, \alpha s, 2+)$.

EXAMPLE 4. Using $\alpha = 2$ and an $\mathrm{OA}(9, 4, 3, 2)$ as both A_0 and A_1, we obtain an $\mathrm{SOA}_2(18, 4, 6, 2+)$ (transposed here):

0	0	0	2	2	2	4	4	4	1	1	1	3	3	3	5	5	5
0	2	4	0	2	4	0	2	4	1	3	5	1	3	5	1	3	5
0	2	4	2	4	0	4	0	2	1	3	5	3	5	1	5	1	3
0	2	4	4	0	2	2	4	0	1	3	5	5	1	3	3	5	1

This array achieves stratifications on 6×3 and 3×6 grids in all two-dimensions.

REMARK 2. The above construction method works just well if $A_0, A_1, \ldots, A_{\alpha-1}$ are all $\mathrm{OA}(n_1, m, s_1 \times \cdots \times s_m, 2)$. Then we will obtain an $\mathrm{SOA}_\alpha(n, m, (\alpha s_1) \times \cdots \times (\alpha s_m), 2+)$, a natural notation to use for such an array.

Two families of $\mathrm{SOA}_\alpha(n, m, \alpha s, 2+)$s can be readily obtained. Using $\mathrm{OA}(s^k, m, s, 2)$ from Rao–Hamming construction, we obtain an $\mathrm{SOA}_\alpha(\alpha s^k, m, \alpha s, 2+)$

TABLE 3
Some $\mathrm{SOA}_\alpha(n, m, \alpha s, 2+)s$

α	s	n	m	$\mathbf{OA}(n_1, m, s, 2)$	$\mathbf{SOA}_\alpha(n, m, \alpha s, 2+)$
2	3	18	4	OA(9, 4, 3, 2)	$\mathrm{SOA}_2(18, 4, 6, 2+)$
2	3	36	7	OA(18, 7, 3, 2)	$\mathrm{SOA}_2(36, 7, 6, 2+)$
2	3	54	13	OA(27, 13, 3, 2)	$\mathrm{SOA}_2(54, 13, 6, 2+)$
2	4	32	5	OA(16, 5, 4, 2)	$\mathrm{SOA}_2(32, 5, 8, 2+)$
2	5	50	6	OA(25, 6, 5, 2)	$\mathrm{SOA}_2(50, 6, 10, 2+)$
3	4	48	5	OA(16, 5, 4, 2)	$\mathrm{SOA}_3(48, 5, 12, 2+)$
3	5	75	6	OA(25, 6, 5, 2)	$\mathrm{SOA}_3(75, 6, 15, 2+)$

where $m = (s^k - 1)/(s - 1)$. The second family is to use $\mathrm{OA}(2s^k, m, s, 2)$ from Addelman–Kempthorne construction, which gives an $\mathrm{SOA}_\alpha(2\alpha s^k, m, \alpha s, 2+)$ where $m = 2(s^k - 1)/(s - 1) - 1$. Table 3 lists some of SOA_αs along with the corresponding orthogonal arrays.

6. Conclusions and further work. This paper introduces and constructs a new class of arrays, namely SOAs of strength 2+. These arrays share the same two-dimensional space-filling property as comparable SOAs of strength 3 but do so in a more economical fashion. The same simple device as constructing OA-based Latin hypercubes can be used to turn an SOA of strength 2+ into a Latin hypercube. It would be interesting to study if Latin hypercubes so constructed can be made orthogonal. Recent work on orthogonal Latin hypercubes includes Sun, Liu and Lin (2009) and Georgiou and Efthimiou (2014).

Regular SOS designs are equivalent to 1-saturating sets in projective geometry and the dual codes of linear codes with covering radius 2 in coding theory. We are currently studying these results from projective geometry and coding theory, and will report our findings in a future paper. Noteworthy is a sophisticated yet quite ingenious construction of Gabidulin, Davydov and Tombak (1991), which can be used to improve the lower bound in Theorem 2 for large run sizes. The same future paper will also document our investigation into the use of nonregular designs to construct SOAs of strength 2+. An excellent review paper on nonregular designs is Xu, Phoa and Wong (2009). Although nonregular designs may not help improve the bounds in Theorem 2, they are advantageous due to their flexible run sizes. For example, one result from this investigation will allow the construction of an SOA of strength 2+ with 48 runs for 34 factors.

Finally, we briefly mention a potential application of SOAs of strength 2+ to numerical integration. As SOAs of strength 2+ are more space-filling in two-dimensional margins than SOAs of strength 2, they should do a better job in integrating the bivariate interaction terms in the functional ANOVA decomposition of an integrand.

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