*The Annals of Statistics*
2015, Vol. 43, No. 6, 2451–2483
DOI: 10.1214/15-AOS1347


# UNIFORM CHANGE POINT TESTS IN HIGH DIMENSION

BY MORITZ JIRAK[1]

*Humboldt Universität zu Berlin*

Consider $d$ dependent change point tests, each based on a CUSUM-statistic. We provide an asymptotic theory that allows us to deal with the maximum over all test statistics as both the sample size $n$ and $d$ tend to infinity. We achieve this either by a consistent bootstrap or an appropriate limit distribution. This allows for the construction of simultaneous confidence bands for dependent change point tests, and explicitly allows us to determine the location of the change both in time and coordinates in high-dimensional time series. If the underlying data has sample size greater or equal $n$ for each test, our conditions explicitly allow for the large $d$ small $n$ situation, that is, where $n/d \to 0$. The setup for the high-dimensional time series is based on a general weak dependence concept. The conditions are very flexible and include many popular multivariate linear and nonlinear models from the literature, such as ARMA, GARCH and related models. The construction of the tests is completely nonparametric, difficulties associated with parametric model selection, model fitting and parameter estimation are avoided. Among other things, the limit distribution for $\max_{1\le h\le d}\sup_{0\le t\le 1}|\mathcal{W}_{t,h} - t\mathcal{W}_{1,h}|$ is established, where $\{\mathcal{W}_{t,h}\}_{1\le h\le d}$ denotes a sequence of dependent Brownian motions. As an application, we analyze all S&P 500 companies over a period of one year.

**1. Introduction.** Modeling high-dimensional time series is a necessity in many different fields, ranging from meteorological and agricultural problems to biology, genetics, financial engineering and risk management. Particularly within the financial regulation framework, banks and insurance undertakings are required to assess and incorporate hundreds of different factors and risks. Regarding financial time series, it is well known that large panels of asset returns routinely display break points and other nonstationarities (cf. [24]). In this context, structural stability is a very important issue, since even changes in few parameters can lead to misspecified risk measures and wrong conclusions (cf. [48, 53]). The issue of structural stability also arises in many other fields, such as climatology, genetics and medicine. Hence, given a $d$-dimensional time series $\mathbf{X}_k = (X_{k,1}, \dots, X_{k,d})^\top$, there is a high interest in procedures that consistently partition the coordinates of

Received August 2014; revised April 2015.

[1]Supported by the Deutsche Forschungsgemeinschaft via FOR 1735 *Structural Inference in Statistics*: *Adaptation and Efficiency*.

*MSC2010 subject classifications.* Primary 62M10, 62G32; secondary 60F05, 60K35.

*Key words and phrases.* Change point analysis, weakly dependent high-dimensional time series, extreme value distribution, high-dimensional ARMA/GARCH, spatial econometrics.

$\{\mathbf{X}_k\}_{k\in\mathbb{Z}}$ into the two sets

$$\begin{aligned}\mathcal{S}_d &= \big\{1 \le h \le d : \{X_{k,h}\}_{k\in\mathbb{Z}} \text{ is stable}\big\},\\ \mathcal{S}_d^c &= \big\{1 \le h \le d : \{X_{k,h}\}_{k\in\mathbb{Z}} \text{ is unstable}\big\},\end{aligned} \tag{1.1}$$

such that we have the relation

$$\{1,\dots,d\} = \mathcal{S}_d \uplus \mathcal{S}_d^c.$$

The sample included in $\mathcal{S}_d$ may then be used for further inference, while the part contained in $\mathcal{S}_d^c$ requires subsequent treatment. Often, one is additionally interested in the actual time of change in each coordinate, and tests based on cumulative sums are efficient in this context. Let us denote such tests with $B_{n,h}^{\widehat{\sigma}}$, $1 \le h \le d$ for further reference; see (1.2) below for a precise definition.

In the univariate case, tests for structural stability in time series are widely available (cf. [4, 5, 7, 19, 20, 43] and the many references there), the multivariate setup, and especially the high-dimensional case are less often considered. Apart from functional data approaches (cf. [6, 26, 30]), the literature in the latter case is rather sparse compared to the univariate theory. Let us briefly mention some recent contributions in this area. In [31], the stability of panel data is considered. Using a threshold-aggregation approach, [15] study the detection of global changes (see also [27, 41]), whereas in [34], the topic of possible gain or loss in power in higher dimension is discussed. Changes in the covariance structure in a multivariate setup are addressed in [3], and an interesting connection between Dos-attacks and change point detection is explored in [40] (see also [49, 52] and therein for changes in multi-channel systems). However, to the best of my knowledge, a (thorough) treatment regarding the consistent estimation of $\mathcal{S}_d^c$, particularly in a time series framework, is lacking in the literature so far. Compared to the univariate case, handling the multivariate situation is much more complicated since breaks may or may not be present at different times in different coordinates $h$. Since it is usually unknown which coordinates $h$ have anomalies and which ones have not, determining $\mathcal{S}_d^c$ (resp., $\mathcal{S}_d$) is particularly hard if the dimension $d$ is large. The vast majority of high-dimensional change point procedures use aggregation or PCA based techniques, and are therefore inappropriate for determining $\mathcal{S}_d^c$. In this context, a natural way to measure possible deviations is to employ the statistic $T_d^{\widehat{\sigma}} = \max_{1\le h\le d} B_{n,h}^{\widehat{\sigma}}$, with coordinate-wise CUSUM-statistics

$$B_{n,h}^{\widehat{\sigma}} = \big(\widehat{\sigma}_h^2 n\big)^{-1/2} \max_{1\le k\le n}\left|\sum_{j=1}^{k} X_{j,h} - \frac{k}{n}\sum_{j=1}^{n} X_{j,h}\right|, \qquad h = 1,\dots,d. \tag{1.2}$$

Here, $\widehat{\sigma}_h^2$ is an appropriate estimator for the long-run variance, which will be more fully explained below. Control of $T_d^{\widehat{\sigma}}$ readily allows us to make inference for every single coordinate $h$. In this paper, we provide theoretic tools that allows one to

handle $T_d^{\widehat{\sigma}}$. If the random variables $X_{k_1,h_1}$ and $X_{k_2,h_2}$ become less dependent if either quantity $|k_1 - k_2|$ or $|h_1 - h_2|$ becomes large, we will show that

$$\max_{1\le h\le d} e_d\big(B_{n,h}^{\widehat{\sigma}} - f_d\big) \xrightarrow{w} \mathcal{V} \qquad \text{as both } n, d \to \infty, \tag{1.3}$$

for appropriate normalizing sequences $e_d$, $f_d$, where $\mathcal{V}$ is an extreme value distribution of Gumbel type. A general explicit connection between $n$ and $d = d_n$ is given such that (1.3) is valid for $n, d \to \infty$, allowing for $n/d \to 0$, but also for the converse where $d/n \to C \ge 0$. On the other hand, we show that the time series $\{X_{k,h}\}_{k\in\mathbb{Z}}$ may have properties such that a pivotal limit theorem like in (1.3) cannot exist. For this case, we provide bootstrap approximations, which of course work in both cases.

Studying the joint limit as $d, n \to \infty$ is a much more realistic setup than considering its sequential analogue (i.e., $\lim_{d\to\infty}\lim_{n\to\infty}\cdot$, cf. Remark 2.1 in [3] or [27]), but is also considerably harder from a mathematical point of view. In order to allow for a high flexibility in (1.3), we use a generalization of known weak dependence concepts from the univariate (multivariate) case to the high-dimensional setup, which allows for dependencies in time and space. This leads to fairly general, yet easily verifiable conditions that are valid for a large number of popular time series from the literature, including multivariate ARMA and GARCH models. Even though we only consider breaks in the mean vector, it is clear that our results are also applicable for assessing the stability of the variance or second-order structure (possibly cross-wise) up to a certain extent.

An outline of the paper can be given as follows. In Section 2, we introduce and discuss our assumptions and the main results. The aspect of concise estimation of $\mathcal{S}_d^c$ and the actual time of change within $\mathcal{S}_d^c$ is discussed in Section 3. Bootstrap procedures and their consistency are explored in Section 4. Section 5 contains a number of popular time series examples that are included in our framework. Section 6 deals with practical aspects and investigates the finite sample behavior. As a real data application, we simultaneously analyze all S&P 500 companies over the time horizon of one year. Detailed proofs are given in the supplementary material [33].

**2. Methodology and main results.** Throughout this paper, we use $\lesssim, \gtrsim, (\sim)$ to denote (two-sided) inequalities involving a multiplicative constant. $C$ denotes an arbitrary, absolute constant that may vary from line to line. Let $\|\cdot\|_p$ denote the $\mathbb{L}^p$-norm $\mathbb{E}[|\cdot|^p]^{1/p}$ for $p \ge 1$, and given a set $\mathcal{S}$, we write $|\mathcal{S}|$ to symbolize its cardinality. We write $\overset{d}{=}$ for equality in distribution. In the sequel, we often deal with arrays $(c_h)_{1\le h\le d}$, where $d \to \infty$ and $c_h$ may depend on $d$. We then use the abbreviations

$$\overset{*}{\inf_h} c_h = \liminf_{d\to\infty} \min_{1\le h\le d} c_h, \qquad \overset{*}{\sup_h} c_h = \limsup_{d\to\infty} \max_{1\le h\le d} c_h. \tag{2.1}$$

Let $\{\mathbf{X}_k\}_{k\in\mathbb{Z}}$ with $\mathbf{X}_k = (X_{k,1}, \dots, X_{k,d})^\top$ be a sequence of $d$-dimensional random vectors where $\mathbb{E}[\mathbf{X}_k] = \boldsymbol{\mu}_k = (\mu_{k,1}, \dots, \mu_{k,d})^\top$. The aim of this paper is to provide a simultaneous test for structural stability in $\boldsymbol{\mu}_k$, based on the observations $\mathbf{X}_1, \dots, \mathbf{X}_n$. To do so, we consider the coordinate-wise null-hypothesis

$$\mathcal{H}_{0,h} : \mu_{1,h} = \cdots = \mu_{n,h}, \qquad h = 1, \dots, d, \tag{2.2}$$

which indicates structural stability in the mean over time. Under this notion of stability, we get that $\mathcal{S}_d = \{1 \le h \le d : \mathcal{H}_{0,h} \text{ is true}\}$, that is, $\mathcal{S}_d$ denotes the set of all coordinates where $\mathcal{H}_{0,h}$ holds. We say that $\mathcal{H}_0$ is true, if $\mathcal{S}_d = \{1, \dots, d\}$. As alternative hypothesis, we specify the scenario which allows for at most one change in each coordinate of $\boldsymbol{\mu}_k$. More precisely, we assume that there exists a (usually unknown) time lag $k_h^*$, such that

$$\mathcal{H}_{A,h} : \mu_{1,h} = \cdots = \mu_{k_h^*,h} \neq \mu_{k_h^*+1,h} = \cdots = \mu_{n,h} \quad \text{for some } h = 1, \dots, d. \tag{2.3}$$

We say that the alternative $\mathcal{H}_A$ holds, if at least one $\mathcal{H}_{A,h}$ is true, and the null hypothesis $\mathcal{H}_{0,h}$ hold in all the remaining unaffected coordinates. This means there is at least one break in one coordinate $h$. Generalizations to multiple change point detection are possible, but will not be addressed here. We make the following convention. The Type I error refers to a "false alarm," that is, a break detection where there is none, and the Type II error is attributed to an unreported break. In this spirit, we then obtain $\mathcal{S}_d^c = \{1 \le h \le d : \mathcal{H}_{A,h} \text{ is true}\}$, that is, the set which consists of all coordinates where a change has occurred. In order to identify $\mathcal{S}_d^c$, we propose to use the coordinate-wise CUSUM statistic $B_{n,h}^{\widehat{\sigma}}$ defined in (1.2). We denote the whole vector of such statistics with $\mathbf{B}_{n,d}^{\widehat{\sigma}} = (B_{n,1}^{\widehat{\sigma}}, \dots, B_{n,d}^{\widehat{\sigma}})^\top$. Let

$$\mathcal{B}_h = \sup_{0\le t\le 1} |\mathcal{W}_{t,h} - t\mathcal{W}_{1,h}|, \qquad h = 1, \dots, d, \tag{2.4}$$

where $\boldsymbol{\mathcal{W}}_{t,d} = (\mathcal{W}_{t,1}, \dots, \mathcal{W}_{t,d})^\top$ is a $d$-dimensional Brownian motion, with correlations $\rho_{i,j} = \mathbb{E}[\mathcal{W}_{t,i}\mathcal{W}_{t,j}]$. If the dimension $d$ is fixed and $n \to \infty$, it is known that under quite general conditions we have weak convergence, that is,

$$\mathbf{B}_{n,d}^{\widehat{\sigma}} \xrightarrow{w} \boldsymbol{\mathcal{B}}_d,$$

where $\boldsymbol{\mathcal{B}}_d = (\mathcal{B}_1, \dots, \mathcal{B}_d)^\top$, with associated correlation matrix $\boldsymbol{\Sigma}_d = (\rho_{i,j})_{1\le i,j\le d}$. Given some mild regularity conditions for $\boldsymbol{\Sigma}_d$, we will show in Theorem A.2 in [33] that

$$\max_{1\le h\le d} e_d(\mathcal{B}_h - f_d) \xrightarrow{w} \mathcal{V}, \qquad \text{as } d \to \infty, \tag{2.5}$$

for appropriate sequences $e_d$, $f_d$, where $\mathcal{V}$ is an extreme value distribution of Gumbel type. Result (2.5) is one of the key ingredients in our proof for (1.3), and may be of independent interest. Limit theorems involving the maximum of partial sums

have played a fundamental role in statistic and probability theory for a long time (cf. [47]). Particularly the seminal contribution in [23] has stimulated much research in this area; see, for instance, [19, 20] for an account on further developments and applications, and [21] for some sharp results and a brief historic review. Related research can also be found in [38], see also the references therein.

Based on an asymptotic result like (2.5), simultaneous confidence regions can readily be constructed, we refer to (2.12) for more details. However, non-Gaussianity is often the rule rather than the exception. It is therefore of considerable interest to formulate our results in a more general manner. In the univariate case, a highly accepted model in the literature is to assume the structure $X_k = g(\varepsilon_k, \varepsilon_{k-1}, \ldots)$ for a process $\{X_k\}_{k\in\mathbb{Z}}$, where $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ is a sequence of i.i.d. random variables in some space $\mathbb{S}$ of possible infinite dimension. Let $\{\varepsilon'_k\}_{k\in\mathbb{Z}}$ be an independent copy of $\{\varepsilon_k\}_{k\in\mathbb{Z}}$. Then many well-known weak-dependence measures and concepts are based on quantifying the difference (for $p \geq 1$)

$$a_k(p) = \big\| g(\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_0, \varepsilon_{-1}, \ldots) - g\big(\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon'_0, \varepsilon_{-1}, \ldots\big)\big\|_p. \tag{2.6}$$

For example, the dependence concept in [50] is based on $a_k(p)$. In related cases, the whole past is replaced with copies; see [8, 44] and [3] for a multivariate version. We will see in Section 5 that many well-known univariate and multivariate time series such as ARMA and GARCH-models are within this framework. As is outlined, for example, in [3], such conditions have several advantages over certain mixing competitors. For instance, mixing conditions are sometimes hard to verify and may require additional smoothness assumptions (cf. [1]). A more profound discussion is given in [50]. Another advantage is that these dependence measures have a natural spatial extension which includes the univariate (multivariate) case as a special example; see, for instance, [14, 42]. More precisely, for $\{\mathbf{X}_k\}_{k\in\mathbb{Z}}$ with $\mathbf{X}_k = \{X_{k,h}\}_{h\in\mathbb{N}}$ we have the structure condition

$$X_{k,h} = g_h(\varepsilon_k, \varepsilon_{k-1}, \ldots), \qquad k \in \mathbb{Z}, h \in \mathbb{N}, \tag{2.7}$$

where $g_h$ are measurable functions. The coordinate processes $X_{k,h}$ can be viewed as projections from $\mathbb{S}$ to $\mathbb{R}$. In analogy to (2.6), for $p \geq 1$ we put [recall $\sup_h^*$ in (2.1)]

$$\begin{aligned} a_k(p) = \sup_h{}^{*} \big\| & g_h(\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_0, \varepsilon_{-1}, \ldots) \\ & - g_h\big(\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon'_0, \varepsilon_{-1}, \ldots\big)\big\|_p. \end{aligned} \tag{2.8}$$

Note that $a_k(p)$ is a *temporal* dependence measure, that is, it only measures dependence in time, and essentially does not impose any *spatial* dependence restrictions. As extreme possibly examples just consider the cases where $X_{k,h} = X_{k,h+1}$ are identical or where $\{X_{k,h}\}_{1\leq h\leq d}$ is an independent sequence for each $k \in \mathbb{Z}$. In fact, this setup is very general and contains a huge variety of popular linear and nonlinear time series models, see Section 5 for more details.

Allowing for weak dependence in (multivariate) time series inevitably results in dealing with the long run covariances $\gamma_{i,j}$, which we formally introduce as

$$\gamma_{i,j} = \lim_{n\to\infty} n^{-1}\mathbb{E}\Bigg[\sum_{k=1}^{n}\sum_{l=1}^{n}(X_{k,i}-\mu_{k,i})(X_{l,j}-\mu_{l,j})\Bigg]. \tag{2.9}$$

We shall see (cf. [33]) that Assumption 2.1 below implies that the above limit exists and $\gamma_{i,j}$ are thus well defined. Moreover, in case of $\sigma_h^2 \overset{\text{def}}{=} \gamma_{h,h}$ we have the usual representation $\sigma_h^2 = \sum_{k\in\mathbb{Z}}\phi_{k,h}$, where $\phi_{k,h} = \mathrm{Cov}[X_{0,h}, X_{k,h}]$. If $\sigma_i, \sigma_j > 0$, we also have $\rho_{i,j} = \gamma_{i,j}/(\sigma_i\sigma_j)$. Our main temporal assumption is now as follows.

ASSUMPTION 2.1 (Temporal assumptions). Given representation (2.7) for $\{\mathbf{X}_k\}_{k\in\mathbb{Z}}$, assume that for $p > 4$ and absolute constant $\sigma^- > 0$:

(T1) $a_k(p) \lesssim k^{-\mathfrak{a}}$, with $\mathfrak{a} > 5/2$,
(T2) $\inf_h^* \sigma_h \geq \sigma^- > 0$.

Let us briefly discuss these assumptions. (T1) is a global, polynomial decay assumption on the temporal dependence. In the univariate case, $\mathfrak{a} > 1$ is possible and essentially optimal. Here, we require the slightly stronger condition $\mathfrak{a} > 5/2$, which enables us to operate in a high-dimensional context. Assumption (T2) is a nondegeneracy assumption that we require since we often normalize with $\sigma_h$ in the sequel; see, however, Remark 2.6. Note that we require Assumption 2.1 *throughout* the remainder of this paper.

Since $\sigma_h^2$ is usually unknown, we need to estimate it. The literature (cf. [11]) provides many potential candidates to estimate $\sigma_h^2$. A popular estimator is Bartlett's estimator, or more general, estimators of the form

$$\widehat{\sigma}_h^2 = \sum_{|k|\leq b_n} \omega(k/b_n)\widehat{\phi}_{k,h}, \qquad b_n \to \infty, \tag{2.10}$$

with weight function $\omega(x)$, where

$$\widehat{\phi}_{h,j} = (n-j)^{-1}\sum_{k=j+1}^{n}(X_{k,h}-\overline{X}_h)(X_{k-j,h}-\overline{X}_h),$$

and $\overline{X}_h = n^{-1}\sum_{k=1}^{n} X_{k,h}$. Setting $\omega(x) = 1$, we obtain the plain estimate (cf. [45]). For the sake of simplicity, we just consider the plain estimate for our theoretical analysis, but the results remain equally valid for other weight functions. Conditions on the possible size of the bandwidth $b_n \sim n^{\mathfrak{b}}$, $0 < \mathfrak{b} < 1$ in terms of $\mathfrak{b}$ are given below in Assumption 2.2.

In order to establish a limit theory, we also require some spatial dependence conditions. A very general way that leads to easily verifiable conditions is in terms of decay assumptions for the underlying covariance structure. This is a common

approach in the literature; see, for instance, [12, 39, 51]. In our context, it is thus natural to impose conditions on the correlations $\rho_{i,j}$. As stated in the Introduction, we consider the situation where we allow that both $n$ and $d$ jointly tend to infinity. For a more formal description, we model the dimension $d$ as $d \sim n^{\mathfrak{d}}$, $\mathfrak{d} > 0$ throughout the remainder of this paper. The necessary connection between $\mathfrak{b}, \mathfrak{d}$ and the underlying moments $p$ is now additionally collected in our *spatial* assumptions.

ASSUMPTION 2.2 (Spatial assumptions). Assume that $\mathfrak{d}, \mathfrak{b}$, $(\rho_{i,j})_{1\le i,j\le d}$, $p > 4$ satisfy the conditions below, uniformly in $d$ for absolute constants $\rho^+, C_\rho, \delta > 0$:

(S1) $0 < \mathfrak{d} < \min\{p/2 - 2, (1 - \mathfrak{b})p/2 - 1\}$,
(S2) $\sup_{i,j:|i-j|\ge 1} \rho_{i,j} \le \rho^+ < 1$,
(S3) $|\rho_{i,j}| \le C_\rho \log(|i - j| + 2)^{-2-\delta}$.

REMARK 2.3. Assumptions (S2), (S3) are only needed for establishing the asymptotic distribution in Theorem 2.5 below. Also note that the polynomial growth rate of the dimension $d = d_n$ and the polynomial decay rate of $a_k(p)$ are intimately connected. In this spirit, one may show that analogue results as presented below are valid for an exponentially growing dimension $d$, by imposing exponential decay rates on $a_k(p)$. Such results would require in addition that $\sup_h^* \mathbb{E}[e^{s_0 X_{k,h}}] < \infty$ for some $s_0 > 0$.

REMARK 2.4. For ease of exposition, we distinctly asked for $\mathfrak{d} > 0$ in (S1) to ensure that $d \to \infty$ as $n \to \infty$, which results in a minimal polynomial growth rate. However, we point out that we actually only require that $d \lesssim n^{\mathfrak{d}}$ and $d \to \infty$ as $n \to \infty$, which is slightly more general.

Assumption 2.2 only imposes mild conditions, essentially allowing for any polynomial growth rate of the dimension $d \sim n^{\mathfrak{d}}$ given sufficiently many moments. Note that high moment assumptions are common in such a context, we refer to [12, 32, 35, 51], where sometimes up to 30 moments and more are required. Also note that we only need a logarithmic decay for the correlations $\rho_{i,j}$ that is close to the best-known results in the literature in a different context (cf. [38]). We are now ready to state our first main result, which establishes the asymptotic limit distribution.

THEOREM 2.5. *Assume that $\mathcal{H}_0$ and Assumptions 2.1 and 2.2 hold. Then*

$$\lim_{n\to\infty} P\Big(\max_{1\le h\le d} B_{n,h}^{\widehat{\sigma}} \le u_d\big(e^{-x}\big)\Big) = \exp\big(-e^{-x}\big),$$

*where $u_d(e^{-x}) = x/e_d + f_d$, with $e_d = \sqrt{2\log(2d)}$, $f_d = e_d/2 - \log(3\log(2d))/e_d$.*

REMARK 2.6. Conditions (S2), (S3) are needed to exclude any pathologies. However, as is known in the literature (cf. [21, 22]), condition $\sigma_h > 0$ can be removed to some extent by more detailed arguments due to the self-normalization in $B^{\widehat{\sigma}}_{n,h}$. Moreover, let $\mathcal{I}_d \subset \{1, \dots, d\}$ be any sequence of subsets with cardinality $|\mathcal{I}_d|/d \to 0$. It is then shown in [33] that it actually suffices to have (S2), (S3) only for $i, j \in \{1, \dots, d\} \setminus \mathcal{I}_d$.

In Section 6, we give a brief account on the implications and relevance of the necessary assumptions for real data sets. A problem that can appear in practical applications is the rate of convergence to extreme value distributions, see Section 6 for details. One way out are bootstrap methods. We first present a (comparatively) fast and easy to implement method for a parametric bootstrap. To this end, let $\{Z_{k,h}\}_{k\in\mathbb{Z},h\in\mathbb{N}}$ be a standard Gaussian IID sequence. Denote with

$$\begin{aligned} B^Z_{n,h} &= \frac{1}{\sqrt{n}} \max_{1\le k\le n} \Big| \sum_{j=1}^{k} Z_{j,h} - \frac{k}{n} \sum_{j=1}^{n} Z_{j,h} \Big| \quad \text{and} \\ T^Z_d &= \max_{1\le h\le d} B^Z_{n,h} \quad \text{and recall} \quad T^{\widehat{\sigma}}_d = \max_{1\le h\le d} B^{\widehat{\sigma}}_{n,h}. \end{aligned} \tag{2.11}$$

Next, we introduce the exact quantile $u^Z_d(z)$, defined as

$$P\big(B^Z_{n,h} \le u^Z_d(z)\big) = 1 - \frac{z}{d}.$$

It then comes as no surprise that we have the following result.

PROPOSITION 2.7. *Grant the assumptions of Theorem 2.5. Then*

$$\sup_{x\in\mathbb{R}} \big| P\big(T^Z_d \le u^Z_d(e^{-x})\big) - P\big(T^{\widehat{\sigma}}_d \le u^Z_d(e^{-x})\big) \big| = \mathcal{O}(1) \qquad \text{as } n \to \infty.$$

We thus obtain a very simple bootstrap method, which just requires the generation of i.i.d. Gaussian random variables. Note that unlike to $u_d(z)$, the quantiles $u^Z_d(z)$ are highly nonlinear, which seems to make them less attractive. In practice though, it turns out that $u^Z_d(z)$ often yields much better results than $u_d(z)$, also for dependent time series. For more details and empirical results, see Section 6. Based on Theorem 2.5 and Proposition 2.7, we can construct asymptotic honest $1-\alpha$ confidence regions $\widehat{\mathcal{S}}_d(\alpha)$ and $\widehat{\mathcal{S}}^Z_d(\alpha)$ via

$$\begin{aligned} \widehat{\mathcal{S}}_d(\alpha) &= \big\{1 \le h \le d : B^{\widehat{\sigma}}_{n,h} \le x_\alpha/e_d + f_d\big\}, \qquad x_\alpha = -\log\big(-\log(1-\alpha)\big), \\ \widehat{\mathcal{S}}^Z_d(\alpha) &= \big\{1 \le h \le d : B^{\widehat{\sigma}}_{n,h} \le u^Z_d(z_\alpha)\big\}, \qquad z_\alpha = d\big(1-(1-\alpha)^{1/d}\big). \end{aligned} \tag{2.12}$$

Let us now turn to the important question when we have less spatial structure. As is demonstrated in Example 5.6, a pivotal limit result like in Theorem 2.5 cannot exist if we drop condition (S3). Fortunately, things do not go totally wrong. Our

next result essentially implies that the "rate" (resp., normalization) $u_d(\cdot)$ in Theorem 2.5 acts as an upper bound, even under considerably less assumptions. This is important for statistical applications, since we remain in control of the Type I error.

THEOREM 2.8. *Assume that $\mathcal{H}_0$, Assumption 2.1 and (S1) hold. Then*

$$\lim_{n\to\infty} P\Big(\max_{1\le h\le d} B_{n,h}^{\widehat{\sigma}} \le u_d\big(e^{-x}\big)\Big) \ge 1 - e^{-x},$$

*where $u_d(\cdot)$ is as in Theorem 2.5.*

A useful implication of Theorem 2.8 is that under less assumptions, $\widehat{\mathcal{S}}_d(\alpha)$ and $\widehat{\mathcal{S}}_d^Z(\alpha)$ can be modified to also supply (asymptotic) honest confidence sets. Indeed, using the power series of $\log(1-\alpha)$, we obtain that

$$1 - \alpha \ge 1 - \exp(-z_\alpha) = 1 - \alpha - \frac{\alpha^2}{2} + \mathcal{O}(\alpha^3).$$

In particular, if we select $a = 1 - \exp(-\alpha)$, then we can construct the confidence sets $\widehat{\mathcal{S}}_d(a)$, $\widehat{\mathcal{S}}_d^Z(a)$, which according to Theorem 2.8 at least have nominal level $\alpha$, since we have [with $x_a = -\log(-\log(1-a))$]

$$1 - \exp(-x_a) = 1 + \log(1-a) = 1 - \alpha.$$

Hence, the resulting confidence regions might be too large, but never too small, which implies that the Type I error of the null-hypothesis $\mathcal{H}_0$ remains controlled. Note, however, that such a modification is more conservative, and thus results in a loss in power. Some further properties of $\widehat{\mathcal{S}}_d(\alpha)$, $\widehat{\mathcal{S}}_d^Z(\alpha)$ and their behavior under the alternative hypothesis $\mathcal{H}_A$ are the topic of Section 3. Another option to construct confidence regions if Theorem 2.5 fails to hold is bootstrapping. In the context of dependent data, blockwise bootstrap procedures are a possible way out. This topic is more fully explored in Section 4.

**3. Estimating the location of change and general consistency of long run variance estimation.** We first make the following convention. We say that an estimator $\widetilde{\mathcal{S}}_d$ is consistent, if

$$\lim_n P\big(|\widetilde{\mathcal{S}}_d \triangle \mathcal{S}_d| = 0\big) = 1,$$

where $\widetilde{\mathcal{S}}_d \triangle \mathcal{S}_d$ stands for the symmetric difference of the sets $\widetilde{\mathcal{S}}_d$ and $\mathcal{S}_d$. Note that trivially any consistent estimator $\widetilde{\mathcal{S}}_d$ gives a consistent estimator $\widetilde{\mathcal{S}}_d^c$ for the complement set. For further analysis, we assume that under $\mathcal{H}_A$ the times of change depend on $n$. This is a common assumption in the literature, and one way to guarantee this is by demanding that $k_h^* = \lfloor \tau_h n \rfloor$ for $\tau_h \in (0,1)$. If there is no change in

coordinate $h$, we set $\tau_h = 1$. Another important quantity is the actual minimal size of the change, which we denote with

$$\Delta\mu = \min_{h \in \mathcal{S}_d^c} \Delta\mu_h \qquad \text{where } \Delta\mu_h = |\mu_{k_h^*,h} - \mu_{k_h^*+1,h}|. \tag{3.1}$$

We assume throughout that $\Delta\mu$ is a monotone decreasing sequence, and express the direct connection to $\mathcal{H}_A$ through the notation $\mathcal{H}_A^{(\Delta\mu)}$. This means that under $\mathcal{H}_A^{(\Delta\mu)}$, the minimal size of change is $\Delta\mu$. Suppose now that $h \in \mathcal{S}_d^c$. Then elementary calculations show that

$$B_{n,h}^{\widehat{\sigma}} \geq \widehat{\sigma}_h^{-1} \sqrt{n} \Delta\mu_h \tau_h (1 - \tau_h)\big(1 - \mathcal{O}(1)\big) - \overline{B}_{n,h}^{\widehat{\sigma}}, \tag{3.2}$$

where

$$\begin{aligned} \overline{B}_{n,h}^{\widehat{\sigma}} &= \frac{1}{\widehat{\sigma}_h \sqrt{n}} \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} U_{j,h} - \frac{k}{n} \sum_{j=1}^{n} U_{j,h} \right|, \\ U_{j,h} &= X_{j,h} - \mathbb{E}[X_{j,h}]. \end{aligned} \tag{3.3}$$

Due to Theorem 2.5, we can control $\max_{1 \leq h \leq d} \overline{B}_{n,h}^{\widehat{\sigma}}$ as long as $\widehat{\sigma}_h$ behaves "reasonably" under $\mathcal{H}_A^{(\Delta\mu)}$. If this is indeed the case, then we can expect from (3.2) that $B_{n,h}^{\widehat{\sigma}}$ becomes large, and thus detect a change in coordinate $h$ using the confidence sets $\widehat{\mathcal{S}}_d(\alpha)$ or $\widehat{\mathcal{S}}_d^Z(\alpha)$ in (2.12). Unfortunately though, $\widehat{\sigma}_h$ may not at all behave reasonably and can cause the problem of "none monotone power" (i.e., the power can decrease when the alternative gets farther away from the null); see, for instance, [18]. One way to overcome this problem is to use self-normalization, as proposed in [46]. Here, we propose a different method that will lead to no loss in power. To this end, we first discuss the estimation of the possible time of change $\tau_h$ for each affected coordinate. For this, we propose the following estimates. Pick any fixed $0 < \mathfrak{t} \leq 1/2$, preferably small, and consider

$$\widehat{\tau}_h(\mathfrak{t}) = \operatorname*{argmax}_{t \in (\mathfrak{t}, 1-\mathfrak{t})} \left( n^{-1/2} \left| \sum_{j=1}^{\lceil nt \rceil} X_{j,h} - \frac{\lceil nt \rceil}{n} \sum_{j=1}^{n} X_{j,h} \right| \right), \qquad h = 1, \ldots, d. \tag{3.4}$$

Note that $\widehat{\sigma}_h$ is not included in the above definition. Of course, one would like to select $\mathfrak{t}$ such that

$$\mathfrak{t} < \inf_{h \in \mathcal{S}_d^c} \tau_h \leq \sup_{h \in \mathcal{S}_d^c} \tau_h < 1 - \mathfrak{t}. \tag{3.5}$$

In the sequel, we put $\widehat{\tau}_h = \widehat{\tau}_h(\mathfrak{t})$ to lighten the notation if the dependence on $\mathfrak{t}$ is of no relevance. Bounds for uniform deviation probabilities for $\widehat{\tau}_h(\mathfrak{t})$ follow from the next result.

THEOREM 3.1. *Assume that $\mathcal{H}_A^{(\Delta_\mu)}$ is valid and Assumption 2.1 holds. If we have in addition (3.5) for some $0 < \mathfrak{t} \le 1/2$, then*

$$P\Big(\max_{h\in\mathcal{S}_d^c}|\widehat{\tau}_h(\mathfrak{t}) - \tau_h| \ge x\Big) \lesssim |\mathcal{S}_d^c|(xn\log n)^{-p/2+2},$$

*where we require that $x \ge C_a \frac{\log n}{(\Delta\mu)^2 n}$, $C_a > 0$ sufficiently large.*

REMARK 3.2. The above constant $C_a$ depends on $\mathfrak{t}$ and the sequence $(a_j(p))_{j\in\mathbb{N}}$, and thus also implicitly on the long run variances $(\sigma_h)_{1\le h\le d}$. Assumption $\mathfrak{t} > 0$ and (T1) ensure that $C_a < \infty$, uniformly in $d$.

Since $|\mathcal{S}_d^c| \le d$, we get the following uniform consistency result.

COROLLARY 3.3. *Grant the assumptions of Theorem 3.1. If in addition*

$$\limsup_{n\to\infty} \frac{\log n}{(\Delta\mu)^2 n} = 0 \quad \textit{and} \quad \mathfrak{d} < p/2 - 2,$$

*then*

$$\max_{h\in\mathcal{S}_d^c}|\widehat{\tau}_h(\mathfrak{t}) - \tau_h| = \mathcal{O}_P(1).$$

Armed with Corollary 3.3, we construct the following simple estimators $\widehat{\sigma}_h^*$ for $\sigma_h$:

- Choose a constant $0 < B_\tau < 1$, and use the estimator $\widehat{\tau}_h(\mathfrak{t})$ in every coordinate to split the sample into $\mathcal{T}_h^- = \{k \le B_\tau \widehat{\tau}_h(\mathfrak{t})n\}$ and $\mathcal{T}_h^+ = \{n - B_\tau(1 - \widehat{\tau}_h(\mathfrak{t}))n < k \le n\}$.
- Use the usual estimator $\widehat{\sigma}_h$ to construct the estimators $\widehat{\sigma}_h^-$ and $\widehat{\sigma}_h^+$ based on the samples $\mathcal{T}_h^-$ and $\mathcal{T}_h^+$. The final estimator $\widehat{\sigma}_h^*$ is then obtained by the convex combination

$$(\widehat{\sigma}_h^*)^2 = \widehat{\tau}_h(\mathfrak{t})(\widehat{\sigma}_h^-)^2 + (1 - \widehat{\tau}_h(\mathfrak{t}))(\widehat{\sigma}_h^+)^2, \qquad 1 \le h \le d.$$

REMARK 3.4. As was pointed out by a reviewer, possible alternatives are

$$\widehat{\sigma}_h^{\min} = \min\{\widehat{\sigma}_h^-, \widehat{\sigma}_h^+\}, \qquad \widehat{\sigma}_h^{\max} = \max\{\widehat{\sigma}_h^-, \widehat{\sigma}_h^+\} \quad \text{and}$$

$$(\widehat{\sigma}_h^{\text{mean}})^2 = ((\widehat{\sigma}_h^-)^2 + (\widehat{\sigma}_h^+)^2)/2.$$

Another alternative is $\widehat{\sigma}_h^\diamond = \widehat{\sigma}_h^-$ if $|\mathcal{T}^-| \ge |\mathcal{T}^+|$ and $\widehat{\sigma}_h^\diamond = \widehat{\sigma}_h^+$ otherwise. Note that we have the relation

$$\widehat{\sigma}_h^{\min} \le \widehat{\sigma}_h^*, \widehat{\sigma}_h^\diamond, \widehat{\sigma}_h^{\text{mean}}, \widehat{\sigma}_h^-, \widehat{\sigma}_h^+ \le \widehat{\sigma}_h^{\max}.$$

As follows from the result below, all estimators are consistent, both under $\mathcal{H}_0$ and $\mathcal{H}_A^{(\Delta_\mu)}$, and thus yield the correct limit distribution. From a practical perspective though, $\widehat{\sigma}_h^{\min}$ leads to a more liberal test with more power, whereas $\widehat{\sigma}_h^{\max}$ leads to a more conservative test.

The following result establishes the desired properties of the above variance estimators.

PROPOSITION 3.5. *Theorems 2.5 and 2.8 remain valid if we replace $\widehat{\sigma}_h$ with either $\widehat{\sigma}_h^{\min}, \widehat{\sigma}_h^*, \widehat{\sigma}_h^\diamond, \widehat{\sigma}_h^{\mathrm{mean}}, \widehat{\sigma}_h^-, \widehat{\sigma}_h^+$ or $\widehat{\sigma}_h^{\max}$. Moreover, if Corollary 3.3 holds, these estimates are consistent under $\mathcal{H}_A^{(\Delta_\mu)}$.*

Now that we have settled the problem of the long run variance estimation, we may return to our original problem of determining $\mathcal{S}_d$, which is now an easy task. In fact, combining Proposition 3.5 with the lower bound in (3.2) and Corollary 3.3, we immediately get the following result.

PROPOSITION 3.6. *Let $\alpha = \alpha_n \to 0$ such that $\alpha_n \gtrsim n^{-1}$. Assume in addition that (3.5) holds for some $0 < \mathfrak{t}_0 \le 1/2$, and that*

$$(\Delta\mu)^2 \ge \frac{C\sigma_+^2 \log d}{n\mathfrak{t}_0^2(1-\mathfrak{t}_0)^2}, \qquad C > 1 \quad \text{and} \quad \mathfrak{d} < p/2 - 2, \tag{3.6}$$

*where $\sigma_+^2 = \sup_h^* \sigma_h^2$. Then $\widehat{\mathcal{S}}_d(\alpha_n)$, and hence also $\widehat{\mathcal{S}}_d^c(\alpha_n)$ are consistent if we either use $\widehat{\sigma}_h^{\min}, \widehat{\sigma}_h^*, \widehat{\sigma}_h^\diamond, \widehat{\sigma}_h^{\mathrm{mean}}, \widehat{\sigma}_h^-, \widehat{\sigma}_h^+$ or $\widehat{\sigma}_h^{\max}$.*

Note that Proposition 3.6 is only valid if we use estimators $\widehat{\tau}_h(\mathfrak{t})$ with $\mathfrak{t} < \mathfrak{t}_0$. Combining Corollary 3.3 with Proposition 3.6 we obtain the following corollary, which marks the final result of this section.

COROLLARY 3.7. *Grant the conditions of Proposition 3.6. Then*

$$\max_{h \in \widehat{\mathcal{S}}_d^c(\alpha_n)} \big|\widehat{\tau}_h(\mathfrak{t}) - \tau_h\big| = \mathcal{O}_P(1).$$

**4. Bootstrap under $\mathcal{H}_\mathcal{A}$.** As was mentioned before, another option to construct confidence regions if Theorem 2.5 fails to hold is bootstrapping. In the context of dependent data, blockwise bootstrap procedures are proposed in the literature. One of the main problems we face here in this context is the possibility of change points, and thus a "naive" block bootstrap can go severely wrong. In the univariate or multivariate case, possible way outs can be found in [2] and [36]. Here, we employ a different approach that uses the same idea as in Section 3, which resulted in consistent long run variance estimators under both $\mathcal{H}_0$ and $\mathcal{H}_A$. This is outlined in detail in the next section below. In Section 4.2, we show how this approach can be modified and simplified. In particular, as a somewhat surprising result, we end up having a simple "naive" block bootstrap that is consistent.

4.1. *Bootstrap* I. Introduce the following notation. For a probability measure $P$ and a $\sigma$-algebra $\mathcal{G}$, we denote with $P_{|\mathcal{G}}$ the conditional probability with respect to $\mathcal{G}$. Moreover, we denote with $\mathcal{X} = \sigma(\mathbf{X}_1, \dots, \mathbf{X}_n)$ the $\sigma$-algebra generated by the underlying sample. Pick $0 < \mathfrak{t} \le 1/2$, recall that $U_{j,h} = X_{j,h} - \mathbb{E}[X_{j,h}]$ and in

analogy to $\overline{B}^{\widehat{\sigma}}_{n,h}$ denote with

$$\overline{B}^{\widehat{\sigma}^*}_{n,h}(\mathrm{t}) = \frac{1}{\widehat{\sigma}^*_h\sqrt{n}} \sup_{\mathrm{t}\le t\le 1-\mathrm{t}} \left|\overline{S}_{n,h}(t) - \frac{\lceil nt\rceil}{n}\overline{S}_{n,h}(1)\right|, \qquad 1\le h\le d,$$

$$\overline{S}_{n,h}(t) = \sum_{j=1}^{\lceil nt\rceil} U_{j,h}, \qquad \overline{T}^{\widehat{\sigma}^*}_d(\mathrm{t}) = \max_{1\le h\le d} \overline{B}^{\widehat{\sigma}^*}_{n,h}(\mathrm{t}).$$

The objective of this section is to obtain an approximation in the sense of

$$\sup_{x\in\mathbb{R}} \left| P_{|\mathcal{X}}\big(\widehat{T}^{\widehat{s}}_{d,L}(\mathrm{t})\le x\big) - P\big(\overline{T}^{\widehat{\sigma}^*}_d(\mathrm{t})\le x\big)\right| = \mathcal{O}_P\big(n^{-C}\big), \qquad C>0,$$

where $\widehat{T}^{\widehat{s}}_{d,L}(\mathrm{t})$ is an appropriately bootstrapped version. To this end, let $K, L$ such that $n = KL$. In the sequel, $K$ will denote the size of the blocks, and correspondingly $L$ the number of blocks. For simplicity, we always assume that $K, L\in\mathbb{N}$, which has no impact on the results. Consider the following block bounds:

$$\begin{aligned} \widehat{L}^-_h &= \sup\big\{l\in\mathbb{N} : lK + K/2 \le \widehat{\tau}_h(\mathrm{t})n\big\},\\ \widehat{L}^+_h &= \inf\big\{l\in\mathbb{N} : lK - K/2 \ge \widehat{\tau}_h(\mathrm{t})n\big\}, \end{aligned} \tag{4.1}$$

where $\widehat{\tau}_h(\mathrm{t})$ is as in (3.4). These estimated limits will allow us to "filter" the contaminated blocks, and thus allow for a consistent bootstrap procedure. For the actual construction, consider the mean estimates

$$\overline{X}^-_h = \frac{1}{K\widehat{L}^-_h}\sum_{j=1}^{K\widehat{L}^-_h} X_{j,h}, \qquad \overline{X}^+_h = \frac{1}{K(L-\widehat{L}^+_h)}\sum_{j=K\widehat{L}^+_h+1}^{KL} X_{j,h},$$

and introduce the random variables

$$\widehat{X}_{j,h} = \begin{cases} X_{j,h} - \overline{X}^-_h, & \text{if } j\le K\widehat{L}^-_h,\\ 0, & \text{if } K\widehat{L}^-_h < j\le K\widehat{L}^+_h,\\ X_{j,h} - \overline{X}^+_h, & \text{if } j > K\widehat{L}^+_h, \end{cases} \tag{4.2}$$

and the block variables

$$\widehat{V}_{l,h}(k) = \sum_{j=(l-1)K+1}^{lK} \widehat{X}_{j,h}\mathbf{1}(j\le k), \qquad 1\le l\le L, 1\le h\le d. \tag{4.3}$$

Note the presence of the indicator function $\mathbf{1}(j\le k)$ in $\widehat{V}_{l,h}(k)$, which will allow us to take the maximum within the individual blocks; see below.

Based on $\widehat{V}_{l,h}(\cdot)$, we now have several options for the construction of a bootstrap, which are among others:

(i) Multiplier bootstrap.

(ii) Sampling with replacement.
(iii) Sampling with no replacement.
(iv) Mixed versions: (i) + (ii) or (i) + (iii).

In the sequel, we establish results for (i) and (iv), see also [33] for extensions. For $1 \le l \le L$, consider the random variables $\pi(l)$ which take values in the set $\mathcal{L} = \{1, \dots, L\}$, and denote with $\pi = \sigma(\pi(1), \dots, \pi(L))$ the corresponding $\sigma$-algebra. The random variables $\pi(l)$ select the blocks $\widehat{V}_{l,h}(\cdot)$. Depending on the desired choice of bootstrap, we have that:

(M) $\pi(l) = l$ for $l \in \mathcal{L}$ (deterministic, multiplier bootstrap),
(SR) $\pi(l)$ are IID and uniformly distributed over $\mathcal{L}$ (sampling with replacement),
(SNR) $\pi(1), \dots, \pi(L)$ results from a permutation of $\mathcal{L}$ (sampling with no replacement).

Let $\xi_1, \dots, \xi_L$ be a sequence of IID standard Gaussian random variables. We then consider the overall statistic $\widehat{T}^{\widehat{s}}_{d,L}(\mathfrak{t})$, defined as

$$\widehat{T}^{\widehat{s}}_{d,L}(\mathfrak{t}) = \max_{1\le h\le d} \frac{\max_{\lfloor n\mathfrak{t}\rfloor \le k \le n-\lfloor n\mathfrak{t}\rfloor}}{\widehat{s}_{h\mathcal{X},\pi}\sqrt{n}} \left| \sum_{l=1}^{L} \xi_l \widehat{V}_{\pi(l),h}(k) - \frac{k}{n} \sum_{l=1}^{L} \xi_l \widehat{V}_{\pi(l),h}(n) \right|, \tag{4.4}$$

where

$$\widehat{s}^2_{h|\mathcal{X},\pi} = \frac{1}{KL} \sum_{l=1}^{L} \xi_l^2 \widehat{V}^2_{\pi(l)}(n), \qquad 1 \le h \le d, \tag{4.5}$$

denotes the conditional long run variance estimator, which acts as a replacement for $(\widehat{\sigma}^*_h)^2$. Note that one may also set $\xi_l^2 = 1$ in the definition of $\widehat{s}^2_{h|\mathcal{X},\pi}$. Also note in particular that the maximum (in time) is taken over $\lfloor n\mathfrak{t}\rfloor \le k \le n - \lfloor n\mathfrak{t}\rfloor$ in $\widehat{T}^{\widehat{s}}_{d,L}(\mathfrak{t})$. Subject to a specific sample, our bootstrap procedure is now the following.

ALGORITHM 4.1 (Bootstrap algorithm I).

*Step* 1. Pick $0 < \mathfrak{t} \le 1/2$, preferably small, compute $\widehat{\tau}_h(\mathfrak{t})$ for $1 \le h \le d$ and select either (M), (SR) or (SNR). Set $m = 1$.
*Step* 2. Generate $\{\pi(l)\}_{1\le l\le L}$ according to step 1.
*Step* 3. Generate IID $\xi_1, \dots, \xi_L$ with standard Gaussian distribution.
*Step* 4. Calculate the value of $\widehat{T}^{\widehat{s}}_{d,L}(\mathfrak{t})$ and set $T_m = \widehat{T}^{\widehat{s}}_{d,L}(\mathfrak{t})$.
*Step* 5. Go to step 2 and set $m = m + 1$.

REMARK 4.2. As was noted by a reviewer, the definition of $\widehat{T}^{\widehat{s}}_{d,L}(\mathfrak{t})$ in (4.4) implies that both (M) and (SNR) give an identical procedure.

Stopping Algorithm 4.1 at $m = M$, we have obtained a Monte Carlo vector $\mathbf{T}_M = (T_1, \dots, T_M)^\top$. For stating consistency results of quantile estimates based on $\mathbf{T}_M$, it is convenient to parameterize the number of blocks $L$ as $L = L_n \sim n^{\mathfrak{l}}$, where $0 < \mathfrak{l} < 1$. Note that this implies $K \sim n^{1-\mathfrak{l}}$. The required connection between $\mathfrak{a}, \mathfrak{d}, \mathfrak{l}$ and $p$ is stated in our main assumption below, which can be considered as mirror conditions for the spatial Assumptions 2.2.

ASSUMPTION 4.3 (Bootstrap assumptions). Assume that $\mathfrak{a}, \mathfrak{d}, \mathfrak{l}$ and $p > 8$ satisfy:

(B1) $\mathfrak{d} < \min\{p(1-\mathfrak{l})(2\mathfrak{a}-1)(\mathfrak{a}-1)/\mathfrak{a} - 8\mathfrak{l}, 2\mathfrak{l}(p-4)\}/8$,
(B2) for some absolute constant $C > 0$

$$|\mathcal{S}_d^c|(K \log n)^{-p/2+2} \lesssim n^{-C} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log n}{K(\Delta\mu)^2} = 0.$$

We are now ready to state the main result of this section, which enables us to establish consistency of the above bootstrap procedure.

THEOREM 4.4. *Grant Assumptions 2.1, 4.3 and one of (M), (SR) or (SNR). Assume in addition that (3.5) holds for some $0 < \mathfrak{t}_0 \le 1/2$, and that (S1) is valid. Then for any $0 < \mathfrak{t} < \mathfrak{t}_0$*

$$\sup_{x\in\mathbb{R}} \big| P_{|\mathcal{X}}\big(\widehat{T}_{d,L}^{\widehat{s}}(\mathfrak{t}) \le x\big) - P\big(\overline{T}_d^{\widehat{\sigma}^*}(\mathfrak{t}) \le x\big)\big| = \mathcal{O}_P(n^{-C}), \qquad C > 0.$$

Let us briefly elaborate on the underlying conditions. Assumption 2.1 is our usual temporal and nondegeneracy condition. (B1) and (S1) provide the necessary relation between $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{l}$ and the moments $p$. Finally, (B2) and (3.5) are necessary to control possible change points. Overall, these are rather mild assumptions. A special point is condition $\mathfrak{t} > 0$. It is a purely technical condition that is required for the proof. One may argue heuristically that in fact one can also set $\mathfrak{t} = 0$ in Theorem 4.4, a rigorous argument appears to be rather technical and lengthy though, and was therefore not pursued. In the simulations in Section 6.3.1, we set $\mathfrak{t} = 0$, and this does not seem to cause any trouble.

Denote with $\widehat{z}_{\alpha,L}(\mathfrak{t})$ the (conditional) $1-\alpha$-quantile of $\widehat{T}_{d,L}^{\widehat{s}}(\mathfrak{t})$, that is,

$$\widehat{z}_{\alpha,L}(\mathfrak{t}) = \inf\big\{x : P_{|\mathcal{X}}\big(\widehat{T}_{d,L}^{\widehat{s}}(\mathfrak{t}) \le x\big) \ge 1-\alpha\big\}, \qquad \alpha \in (0,1). \tag{4.6}$$

It then follows from Theorem 4.4 that

$$\big| P_{|\mathcal{X}}\big(\overline{T}_d^{\diamond,\widehat{\sigma}^*}(\mathfrak{t}) \le \widehat{z}_{\alpha,L}(\mathfrak{t})\big) - (1-\alpha)\big| = \mathcal{O}_p(n^{-C}), \qquad C > 0,$$

where $\overline{T}_d^{\diamond,\widehat{\sigma}^*}(\mathfrak{t})$ is a copy of $\overline{T}_d^{\widehat{\sigma}^*}(\mathfrak{t})$, independent of $\mathcal{X}$. Standard empirical process theory (cf. [47]) now implies that we are able to consistently estimate $\widehat{z}_{\alpha,L}(\mathfrak{t})$

based on $\mathbf{T}_M$ for large enough $M = M_n$. In particular, in analogy to $\widehat{\mathcal{S}}_d^Z(\alpha)$, the bootstrapped confidence set $\widehat{\mathcal{S}}_d(\alpha, L, \mathfrak{t}, M)$ can be constructed via

$$\widehat{\mathcal{S}}_d(\alpha, L, \mathfrak{t}, M) = \big\{1 \le h \le d : B_{n,h}^{\widehat{\sigma}^*}(\mathfrak{t}) \le \widehat{z}_{\alpha,L}(\mathfrak{t}, M)\big\}, \tag{4.7}$$

where $\widehat{z}_{\alpha,L}(\mathfrak{t}, M)$ is a consistent estimate of $\widehat{z}_{\alpha,L}(\mathfrak{t})$ based on $\mathbf{T}_M$. Corresponding empirical examples are given in Section 6.3.2.

4.2. *Bootstrap* II. Let us step back for a moment to reconsider our original testing, respectively, estimation problem, that is, construct an estimator for the stable set $\mathcal{S}_d$. We now make the following observation. Recall that from the discussion in Section 3 we can expect that with high probability

$$\min_{h \in \mathcal{S}_d^c} B_{n,h}^{\widehat{\sigma}^*} > C\sqrt{\log d} \qquad \text{as } n \to \infty,$$

for $C > 0$ large enough, if the change in mean is sufficiently strong. Hence, in order to control the error of estimation for $\widehat{\mathcal{S}}_d(\alpha)$ or $\widehat{\mathcal{S}}_d(\alpha, L, \mathfrak{t})$, we only need to control $\max_{h \in \mathcal{S}_d} B_{n,h}^{\widehat{\sigma}^*}$. This has interesting consequences for a bootstrap method, as we will now explain. Recall from Section 3 that we needed to modify $\widehat{\sigma}_h$ to $\widehat{\sigma}_h^*$ to avoid the problem of inconsistent variance estimation. Here, we will actually exploit this problem to our advantage. More precisely, we construct (conditional) variance estimators $\widetilde{s}_{h|\mathcal{X}}$ that explode for $h \in \mathcal{S}_d^c$ sufficiently fast, that is, we have with high probability

$$\widetilde{s}_{|\mathcal{X}} \gtrsim \sqrt{K}\,\Delta\mu_h \tau_h(1-\tau_h) \qquad \text{as } n \to \infty, \tag{4.8}$$

where $\Delta\mu_h$ is given in (3.1). Consequently, we can expect that

$$\widehat{T}_{d,L}^{\widetilde{s}}(\mathfrak{t}) = \max_{h \in \mathcal{S}_d} \frac{\max_{\lfloor n\mathfrak{t} \rfloor \le k \le n - \lfloor n\mathfrak{t} \rfloor}}{\widetilde{s}_{h|\mathcal{X}}\sqrt{n}} \left| \sum_{l=1}^{L} \xi_l \widehat{V}_{l,h}(k) - \frac{k}{n} \sum_{l=1}^{L} \xi_l \widehat{V}_{l,h}(n) \right| + \mathcal{O}_P(1)$$

[see (4.13) below for $\widehat{T}_{d,L}^{\widetilde{s}}(\mathfrak{t})$]. From Theorem 4.4, we then essentially get that

$$\widehat{T}_{d,L}^{\widetilde{s}}(\mathfrak{t}) \overset{d}{=} \max_{h \in \mathcal{S}_d} B_{n,h}^{\widehat{\sigma}^*} + \mathcal{O}_P(1). \tag{4.9}$$

In other words, $\widehat{T}_{d,L}^{\widetilde{s}}(\mathfrak{t})$ automatically *adapts* to the number of unaffected coordinates and, therefore, allows for a better control of the Type I and II errors. Note, however, that this will only have a significant impact if

$$|\mathcal{S}_d| / |\mathcal{S}_d^c| \to 0 \qquad \text{as } d \to \infty. \tag{4.10}$$

Implementation of this idea will lead to the bootstrap procedure Algorithm 4.5. However, even more is possible. Exploiting the explosions another time, we will see that one may entirely skip estimation of $\widehat{\tau}_h(\mathfrak{t})$, by using a "naive" bootstrap method. This will lead to Algorithm 4.7.

To simplify the following exposition, we only concentrate on multiplier bootstrap procedures in the remainder of this section. Consider the overall mean estimator $\overline{X}_h$ and the "centered" random variables $\widetilde{X}_{j,k}$

$$\overline{X}_h = \frac{1}{n}\sum_{j=1}^{n} X_{j,h}, \qquad \widetilde{X}_{j,h} = X_{j,h} - \overline{X}_h, \tag{4.11}$$

and the block variables

$$\widetilde{V}_{l,h}(k) = \sum_{j=(l-1)K+1}^{lK} \widetilde{X}_{j,h}\mathbf{1}(j \le k), \qquad 1 \le l \le L, 1 \le h \le d.$$

We then construct the (conditional) long run variance estimator $\widetilde{s}^2_{h|\mathcal{X}}$ as

$$\widetilde{s}^2_{h|\mathcal{X}} = \frac{1}{KL}\sum_{l=1}^{L} \xi_l^2 \widetilde{V}^2_{l,h}(n), \tag{4.12}$$

where $\xi_1, \ldots, \xi_L$ is a sequence of IID standard Gaussian random variables. Next, pick any $0 < \mathfrak{t} \le 1/2$. In analogy to $\widehat{T}^{\widehat{s}}_{d,L}(\mathfrak{t})$, we then consider the overall statistic $\widehat{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})$, defined as

$$\widehat{T}^{\widetilde{s}}_{d,L}(\mathfrak{t}) = \max_{1\le h\le d} \frac{\max_{\lfloor n\mathfrak{t}\rfloor \le k \le n-\lfloor n\mathfrak{t}\rfloor}}{\widetilde{s}_{h|\mathcal{X}}\sqrt{n}} \left| \sum_{l=1}^{L} \xi_l \widehat{V}_{l,h}(k) - \frac{k}{n}\sum_{l=1}^{L} \xi_l \widehat{V}_{l,h}(n) \right|. \tag{4.13}$$

Note that in comparison to $\widehat{T}^{\widehat{s}}_{d,L}(\mathfrak{t})$, we have replaced $\widehat{s}_{h|\mathcal{X},\pi}$ with $\widetilde{s}_{h|\mathcal{X}}$. Subject to a specific sample, our bootstrap procedure is now the following.

ALGORITHM 4.5 (Bootstrap algorithm II).

*Step* 1. Pick $0 < \mathfrak{t} \le 1/2$, preferably small, compute $\widehat{\tau}_h(\mathfrak{t})$ for $1 \le h \le d$ and $\widetilde{s}_{h|\mathcal{X}}$. Set $m = 1$.

*Step* 2. Generate IID $\xi_1, \ldots, \xi_L$ with standard Gaussian distribution.

*Step* 3. Calculate the value of $\widehat{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})$ and set $T_m = \widehat{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})$.

*Step* 4. Go to step 2 and set $m = m + 1$.

By the discussion after Theorem 4.4, the following result allows us to establish the consistency of the bootstrap procedure in Algorithm 4.5.

THEOREM 4.6. *Grant Assumptions* 2.1, 4.3. *Assume that* (3.5) *holds for some* $0 < \mathfrak{t}_0 \le 1/2$, *and that* (S1) *is valid. Then for any* $0 < \mathfrak{t} < \mathfrak{t}_0$

$$\sup_{x\in\mathbb{R}} \left| P_{|\mathcal{X}}\big(\widehat{T}^{\widetilde{s}}_{d,L}(\mathfrak{t}) \le x\big) - P\Big(\max_{h\in\mathcal{S}_d} B^{\widehat{\sigma}^*}_{n,h}(\mathfrak{t}) \le x\Big)\right| = o_P(1).$$

As a next step, we now show how one can entirely remove the estimation of $\widehat{\tau}_h(\mathfrak{t})$. The central idea is that one can show

$$\max_{h\in\mathcal{S}_d^c}\frac{1}{\widetilde{s}_{h|\mathcal{X}}\sqrt{n}}\max_{\lfloor n\mathfrak{t}\rfloor\le k\le n-\lfloor n\mathfrak{t}\rfloor}\Big|\sum_{l=1}^{L}\xi_l\widetilde{V}_{l,h}(k)-\frac{k}{n}\sum_{l=1}^{L}\xi_l\widetilde{V}_{l,h}(n)\Big|=\mathcal{O}_P(1).$$

Thus, roughly speaking, the "explosions" in both $\widetilde{s}_{h|\mathcal{X}}$ and $\xi_l\widetilde{V}_{l,h}(\cdot)$ cancel. Hence, we automatically obtain the desired relation

$$\begin{aligned}\widetilde{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})&=\max_{h\in\mathcal{S}_d}\frac{1}{\widetilde{s}_{h|\mathcal{X}}\sqrt{n}}\max_{\lfloor n\mathfrak{t}\rfloor\le k\le n-\lfloor n\mathfrak{t}\rfloor}\Big|\sum_{l=1}^{L}\xi_l\widetilde{V}_{l,h}(k)-\frac{k}{n}\sum_{l=1}^{L}\xi_l\widetilde{V}_{l,h}(n)\Big|+\mathcal{O}_P(1)\\&\stackrel{d}{=}\max_{h\in\mathcal{S}_d}B^{\widehat{\sigma}^*}_{n,h}+\mathcal{O}_P(1),\end{aligned}$$

where

$$\widetilde{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})=\max_{1\le h\le d}\frac{1}{\widetilde{s}_{h|\mathcal{X}}\sqrt{n}}\max_{\lfloor n\mathfrak{t}\rfloor\le k\le n-\lfloor n\mathfrak{t}\rfloor}\Big|\sum_{l=1}^{L}\xi_l\widetilde{V}_{l,h}(k)-\frac{k}{n}\sum_{l=1}^{L}\xi_l\widetilde{V}_{l,h}(n)\Big|.$$

Note that, in comparison to $\widehat{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})$, we have replaced $\widehat{V}_{l,h}(\cdot)$ with $\widetilde{V}_{l,h}(\cdot)$. Subject to a specific sample, our bootstrap procedure is now the following.

ALGORITHM 4.7 (Bootstrap algorithm III).

*Step* 1. Pick $0<\mathfrak{t}\le 1/2$, preferably small, compute $\widetilde{s}_{h|\mathcal{X}}$ and set $m=1$.
*Step* 2. Generate IID $\xi_1,\dots,\xi_L$ with standard Gaussian distribution.
*Step* 3. Calculate the value of $\widetilde{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})$ and set $T_m=\widetilde{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})$.
*Step* 4. Go to step 2 and set $m=m+1$.

As before in Theorems 4.4 and 4.6, the following result allows us to conclude consistency of the above bootstrap procedure.

THEOREM 4.8. *Grant Assumptions 2.1, 4.3. Assume that (3.5) holds for some $0<\mathfrak{t}_0\le 1/2$, and that (S1) is valid. If in addition we have with*

$$\liminf_{d\to\infty}P\Big(\max_{h\in\mathcal{S}_d}B^{\sigma}_{n,h}(\mathfrak{t}_0)\ge x_d\Big)=1 \tag{4.14}$$

*for some monotone increasing sequence $x_d\to\infty$, then for any $0<\mathfrak{t}<\mathfrak{t}_0$*

$$\sup_{x\in\mathbb{R}}\Big|P_{|\mathcal{X}}\big(\widetilde{T}^{\widetilde{s}}_{d,L}(\mathfrak{t})\le x\big)-P\Big(\max_{h\in\mathcal{S}_d}B^{\widehat{\sigma}^*}_{n,h}(\mathfrak{t})\le x\Big)\Big|=\mathcal{o}_P(1).$$

REMARK 4.9. Assumption (4.14) is a mild nondegeneracy condition, and is only violated in the extreme case where $\lim_{d\to\infty}\max_{h\in\mathcal{S}_d}B^{\sigma}_{n,h}(\mathfrak{t}_0)=\mathcal{O}_P(1)$.

4.3. *Discussion of bootstrap procedures*. In the previous sections, we have seen that all three bootstrap Algorithms 4.1, 4.5 and 4.7 are consistent alternatives to Theorem 2.5 and Proposition 2.7. In particular, they do not require any assumptions on the spatial dependence structure. On the other hand, there are also some deficits that we will briefly outline.

**Computational cost:** Particularly if $d$ gets larger, the computational costs and time become a relevant issue.

**Homogeneity:** All bootstrap procedures require global blocks in order to reflect the underlying dependence structure. This in turn requires a certain homogeneity of temporal dependence of the data.

**Sensitivity:** As simulations reveal, the number and thus size of the blocks may have a huge impact, and in some cases the results appear to be rather sensitive in this respect, and there is also an interplay with the required homogeneity, mentioned above. This problem of blocklength selection is already well known in the literature in the univariate or multivariate case; see, for instance, [37]. A simple problematic example is given in Section 6.3.2, Table 8.

**Large $d$ small $n$:** If $d$ is rather large compared to $n$, one should take $L$ as large as possible to avoid or at least weaken some of the above problems. In particular, one should keep in mind that one multiplies and thus "models" the time series with only $L$ i.i.d. Gaussian random variables. However, a large $L$ results in a small $K$, and thus a possible failure in capturing the temporal dependence via the blocks.

From these considerations, a bootstrap procedure is only recommended if the dimension $d$ is not too large compared to the sample size $n$ ($d \leq n$ appears to still yield good results), and if the vast majority of the data can be expected to be homogenous (a few outliers do not hurt). Otherwise, the parametric bootstrap depicted in Proposition 2.7 is recommended.

**5. Examples.** In this section, we discuss some prominent and leading examples from the literature that fit into our framework. To keep the exposition at a reasonable length, our main focus lies on ARMA($\mathfrak{p}, \mathfrak{q}$) and GARCH($\mathfrak{p}, \mathfrak{q}$) models, but our setup also contains many more nonlinear time series, as will be briefly discussed. We mainly focus on examples that fulfill Assumptions 2.1 and 2.2. Of course, this implies that these are also valid examples for the bootstrap procedures. An important example are factor models in Example 5.6, which highlight the usefulness of bootstrap procedures.

In the setting of Theorem 2.5, the spatial decay condition (S3) plays a key role. The (multivariate) time series literature contains a huge variety of process that meet (T1). Especially for nonlinear time series such as GARCH-models, iterated random functions and the like, we refer to [25, 29, 44] and the many references there. We thus concentrate on giving examples for $\{\mathbf{X}_k\}_{k\in\mathbb{Z}}$, where (S3) holds.

More precisely, we give examples for two parameter processes $\{X_{k,h}\}_{k,h\in\mathbb{Z}}$ where the key conditions (T1) and (S3) are valid.

We recall the following convention. Throughout this section, $0 < C < \infty$ denotes an arbitrary, absolute constant that may vary from line to line.

EXAMPLE 5.1 (Linear processes). A common way to model multivariate linear models with finite dimension $d$ is by

$$\mathbf{X}_k = \sum_{l=0}^{\infty} \mathbf{R}_l \mathbf{Z}_{k-l}, \tag{5.1}$$

where $\{\mathbf{R}_i\}_{i\in\mathbb{N}}$ is a sequence of $d \times d$ matrices, and $\{\mathbf{Z}_k\}_{k\in\mathbb{Z}}$ is a sequence of $d$-dimensional vectors, usually i.i.d. However, describing (weak) spatial dependence in this model when $d$ is large is not at all straightforward, even if one assumes a simple spatial structure for $\mathbf{Z}_k$, for example, a linear process. In addition, using high-dimensional matrices for modelling purposes is only advisable if the matrices are sparse. The problem of transferring multivariate linear models, in particular the autoregressive multivariate setup to a high-dimensional setting is currently a very active field of research, particularly in connection with panel data or factor models. For example, in [16], various sparsity constraints are discussed to introduce the IVAR (infinite-dimensional vector-autoregression). Other approaches are offered in [13, 15]. Here, we will first follow the approach taken in the latter, before coming back to (5.1). Let $\{\varepsilon_{k,h}\}_{k,h\in\mathbb{Z}}$ be a sequence such that $\varepsilon_k = \{\varepsilon_{k,h}\}_{h\in\mathbb{Z}}$ is i.i.d. for $k \in \mathbb{Z}$. We then introduce the high-dimensional MA$(\infty, \varepsilon)$ process as

$$X_{k,h} = \sum_{i=0}^{\infty} \alpha_{i,h}\varepsilon_{k-i,h} \qquad \text{for } k, h \in \mathbb{Z} \text{ and } \alpha_{i,h} \in \mathbb{R}.$$

Naturally, we require some conditions on the numbers $\alpha_{i,h}$ to guarantee its existence. We do this in one sweep, by stating conditions such that Assumptions (T1) and (S3) are valid in addition.

PROPOSITION 5.2. *Suppose that* $|\gamma^{\varepsilon}_{i,j}| = |\mathbb{E}[\varepsilon_{k,i}\varepsilon_{k,j}]| \le C\log(|i-j|)^{-2-\delta}$ *for* $|i-j| \ge 2$ *and* $\delta > 0$. *If* (T2) *holds and also*

$$\sup_h{}^{*} |\alpha_{i,h}| \lesssim i^{-\mathfrak{u}} \qquad \textit{with } \mathfrak{u} > 5/2,$$

*then* $\{X_{k,h}\}_{k,h\in\mathbb{Z}}$ *meets Assumptions* (T1) *and* (S3).

As a special case, we may now consider ARMA$(\mathfrak{p}, \mathfrak{q}, \varepsilon)$ processes, which we introduce as

$$X_{k,h} = \alpha^*_{1,h}\varepsilon_{k,h} + \cdots + \alpha^*_{\mathfrak{p},h}\varepsilon_{k-\mathfrak{p},h} + \beta^*_{1,h}X_{k-1,h} + \cdots + \beta^*_{\mathfrak{q},h}X_{k-\mathfrak{q},h},$$

$\alpha^*_{1,h}, \ldots, \alpha^*_{\mathfrak{p},h}, \beta^*_{1,h}, \ldots, \beta^*_{\mathfrak{q},h} \in \mathbb{R}$. As in the univariate case, we consider the polynomials

$$\mathbf{A}_h(z) = \sum_{j=0}^{\mathfrak{p}} \alpha^*_{j,h} z^j, \qquad \mathbf{B}_h(z) = \sum_{j=0}^{\mathfrak{q}} \beta^*_{j,h} z^j, \tag{5.2}$$

where $\mathbf{A}_h(z)$ and $\mathbf{B}_h(z)$ are assumed to be relative prime. Then following, for instance, [11], one readily verifies the following result.

PROPOSITION 5.3. *If the associated polynomials* $\mathbf{C}_h(z) = \mathbf{A}_h(z)\mathbf{B}_h^{-1}(z)$ *satisfy* $\inf_h^* |\mathbf{C}_h(z)| > 0$ *for* $|z| \le 1$, *then* $X_{k,h}$ *admits a causal representation*

$$X_{k,h} = \sum_{i=0}^{\infty} \alpha_{i,h} \varepsilon_{k-i,h} \qquad \text{where } \sup_h^* |\alpha_{k,h}| \lesssim q^k \text{ for } 0 < q < 1.$$

It is now easy to see that we may choose $\mathfrak{a} > 5/2$ arbitrarily large, hence Assumption (T1) holds. The validity of (S3) can be obtained as in Proposition 5.2. Next, we demonstrate how model (5.1) fits into our framework. Recall that $\mathbf{Z}_k = \{Z_{k,h}\}_{h \in \mathbb{Z}}$. We impose the following conditions.

ASSUMPTION 5.4. The sequence $\{\mathbf{Z}_k\}_{k \in \mathbb{Z}}$ is IID, and for $p > 4$:

(i) $\gamma^Z_{i,j} = \mathbb{E}[Z_{k,i} Z_{k,j}] \le C \log(|i-j|+2)^{-2-\delta}$, $\delta > 0$,

(ii) $\mathbf{R}_l = (r^{(l)}_{i,j})_{1 \le i,j \le d}$ with $|r^{(l)}_{i,j}| \le C(l+1)^{-\mathfrak{q}}(|i-j|+1)^{-\mathfrak{p}}$, $\mathfrak{q}, \mathfrak{p} > 2$.

Condition (ii) is mild, allowing for a large variety of matrix sequences $\mathbf{R}_l$. We have the following result.

PROPOSITION 5.5. *Assume that Assumptions* 5.4 *and* (T2) *hold. Then* $\{X_{k,h}\}_{k,h \in \mathbb{Z}}$ *meets Assumptions* (T1) *and* (S3).

Based on the above proposition, one can derive a related result for multivariate ARMA processes, we omit the details.

EXAMPLE 5.6 (Factor models). In econometric theory, it is often believed that the dynamics of a multivariate or high-dimensional time series $\mathbf{X}_k$ can be described via so-called (normally unobserved) common factors $\mathbf{Z}_k \in \mathbb{R}^{d'}$, where it is usually assumed in the literature that $d'$ is much smaller than $d$. This amounts to the model

$$\mathbf{X}_k = \mathbf{R}\mathbf{Z}_k + \boldsymbol{\xi}_k, \qquad k \in \mathbb{Z}, \tag{5.3}$$

where $\mathbf{R} = (r_{i,j})_{1 \le i \le d, 1 \le j \le d'}$ is a $d \times d'$ matrix of factor loadings, and $\boldsymbol{\xi}_k = \{\xi_{k,h}\}_{h \in \mathbb{Z}}$ denotes the noise sequence. We also denote with $s^2_{h,\xi}$ the coordinate-wise standard deviation of $\{\xi_{k,h}\}_{k \in \mathbb{Z}}$, and with $\phi^Z_{k,i,j} = \mathbb{E}[Z_{k,i} Z_{0,j}]$, $\phi^{\xi}_{k,i,j} = \mathbb{E}[\xi_{k,i} \xi_{0,j}]$. We then make the following assumptions.

ASSUMPTION 5.7. For $\delta > 0$ and $p > 4$, we have:

(i) $\{\mathbf{Z}_k\}_{k\in\mathbb{Z}}$ and $\{\boldsymbol{\xi}_k\}_{k\in\mathbb{Z}}$ are independent and both satisfy Assumption (T1),
(ii) $\phi^{Z}_{k,i,j}, \phi^{\xi}_{k,i,j} \le C(k+1)^{-\mathfrak{q}}(|i-j|+1)^{-\mathfrak{p}}$, $\mathfrak{q}, \mathfrak{p} > 2$ and $\inf_h^* s^2_{h,\xi} > 0$,
(iii) $\sup_i^* \sum_{j=1}^{d'} |r_{i,j}| < \infty$ and $|\sum_{j=1}^{d'} r_{i_1,j} r_{i_2,j}| \le C(\log|i_1 - i_2|)^{-2-\delta}$ for $|i_1 - i_2| \ge 2$.

The above assumptions are related to those of Assumption 5.4. This comes as no surprise, since both process are very similar. As we shall see, rather straightforward computations show that the corresponding spatial correlation matrix $\Sigma_d = (\rho_{i,j})_{1\le i,j\le d}$ only needs to satisfy

$$|\rho_{i,j}| \le C\big(\log|i-j|\big)^{-2-\delta} \quad \delta > 0 \qquad \text{if } |i-j| \ge 2. \tag{5.4}$$

We now have the following result.

PROPOSITION 5.8. *Assume that $d' = d'_n$ with $d'_n \to \infty$ and $d' \le d$. Suppose that Assumptions 5.7, (S1) and (S2) hold. Then Theorem 2.5 is valid.*

The above result shows that under reasonable assumptions, high-dimensional factor models fit into our framework. Note that the limit distribution is pivotal, no additional information on $\mathbf{R}$ is required. This is important from a statistical point of view, since the factor loadings $\mathbf{R}$ are usually unobservable in practice. In this context, the question arises whether a structural condition like

$$\Big|\sum_{j=1}^{d'} r_{i_1,j} r_{i_2,j}\Big| \lesssim \big(\log|i_1 - i_2|\big)^{-2-\delta} \qquad \text{if } |i_1 - i_2| \ge 2 \tag{5.5}$$

is necessary to obtain a pivotal limit distribution. If (5.5) does not hold, one can still show via Theorems 2.5, 2.8 and the triangle inequality that with probability one

$$\liminf_n \frac{\max_{1\le h\le d} B^{\widehat{\sigma}}_{n,h}}{\sqrt{\log d}} > 0 \quad \text{and} \quad \limsup_n \frac{\max_{1\le h\le d} B^{\widehat{\sigma}}_{n,h}}{\sqrt{\log d}} \le \frac{1}{\sqrt{2}}, \tag{5.6}$$

hence $\sqrt{\log d}$ is the right scaling, even without (5.5). However, determining the exact limit distribution in the absence of (5.5) seems to be very difficult, and is likely to depend on $\mathbf{R}$, questioning its usefulness for applications. In fact, if we drop condition (5.5) in Assumption 5.7(iii), a pivotal result like Theorem 2.5 cannot hold as the next result shows.

PROPOSITION 5.9. *Assume that the conditions of Proposition 5.8 hold, with the exception that we do not have (5.5). Then universal sequences $a_d$, $b_d$, only depending on $d$ such that*

$$a_d\Big(\max_{1\le h\le d} B^{\widehat{\sigma}}_{n,h} - b_d\Big)$$

*converges in distribution to a nondegenerate limit do not exist.*

Proposition 5.9 tells us that an exact fluctuation control without any intrinsic knowledge on $\mathbf{R}$ is not possible. In this sense, Assumption 5.7 seems to be near the minimum requirements to obtain a pivotal, nonparametric result like Theorem 2.5. In any case, relation (5.6) tells us that we always remain in control of the Type I error, and the possible loss in power is only marginal.

EXAMPLE 5.10 (GARCH process). In this example, we discuss one possible way to extend the constant conditional GARCH model (CCG) of Bollerslev [9]. If the dimension $d$ is fixed, related multivariate extensions can be found in the literature; see, for instance, [3]. Here, we define the GARCH$(\mathfrak{p}, \mathfrak{q}, \varepsilon)$ sequence as

$$X_{k,h} = \varepsilon_{k,h} s_{k,h} \qquad \text{where } \{s_{k,h}\}_{k,h\in\mathbb{Z}} \text{ meets,}$$

$$s_{k,h}^2 = \eta_h + \alpha_{1,h} s_{k-1,h}^2 + \cdots + \alpha_{\mathfrak{p},h} s_{k-\mathfrak{p},h}^2 + \beta_{1,h} X_{k-1,h}^2 + \cdots + \beta_{\mathfrak{q},h} X_{k-\mathfrak{q},h}^2,$$

with $\eta_h, \alpha_{1,h}, \ldots, \alpha_{\mathfrak{p},h}, \beta_{1,h}, \ldots, \beta_{\mathfrak{q},h} \in \mathbb{R}$. Note that $\mathfrak{p}$ and $\mathfrak{q}$ denote the maximal degree of $\alpha_{i,h}, \beta_{i,h}$. Possible undefined $\alpha_{i,h}$ and $\beta_{i,h}$ are replaced with zeros. As in the univariate case, a crucial quantity in this context is

$$\gamma_C = \max_{1\le h\le d} \sum_{i=1}^{r} \|\alpha_{i,h} + \beta_{i,h}\varepsilon_{i,h}^2\|_{p/2} \qquad \text{with } r = \max\{\mathfrak{p}, \mathfrak{q}\}. \tag{5.7}$$

If $\gamma_C < 1$, then it can be shown that $\{X_{k,h}\}_{k,h\in\mathbb{Z}}$ is stationary (cf. [10]). We have the following result, establishing a link between the underlying parameters and Assumption 2.1.

PROPOSITION 5.11. *Suppose that* $\gamma_C < 1$ *and* $\gamma_{i,j}^{\varepsilon} = \mathbb{E}[\varepsilon_{k,i}\varepsilon_{k,j}]$ *satisfies*

$$|\gamma_{i,j}^{\varepsilon}| \le C\big(\log|i-j|\big)^{-2-\delta}\delta > 0 \qquad \text{if } |i-j| \ge 2.$$

*Then* $\{X_{k,h}\}_{k,\in\mathbb{Z},h\in\mathbb{N}}$ *meets Assumptions* (T1) *and* (S3).

In [33], we additionally discuss time series that arise as iterated random functions. Moreover, as in the univariate case, many more examples can be constructed based on the vast time series literature (cf. [25, 29, 44]). Also note that any combination of the previous examples fulfills Assumption 2.1. This means that in one coordinate we may have a GARCH model, but in another coordinate, the process has a linear dynamic, and so on.

**6. Empirical results and applications.** In the empirical part of the paper, we first discuss the implications and relevance of our assumptions for real data sets. We then move on to the computation of critical values. In the third part, we asses the accuracy and behavior of $\widehat{\mathcal{S}}_d^c$ in a small simulation study. In [33], the S&P 500 companies over a period of one year, with a particular emphasis of detecting companies with an unusual behavior.

6.1. *Assumptions and real data.* The necessary assumptions of Theorem 2.5 can be divided into temporal and spatial conditions. Assumption 2.1 concerns temporal dependence, and is standard in the literature (cf. [50]). We therefore focus on the spatial conditions, in particular (S3). This condition implicitly assumes that the coordinates of the data-vector $\mathbf{X}_k$ can be ordered such that two coordinates $X_{k,i}$ and $X_{k,j}$ become less dependent as the difference $|i-j|$ increases. In many cases, the data at hand already has a natural ordering with corresponding weak spatial dependence. Such examples can be found in the ever growing literature on high-dimensional covariance estimation (cf. [12]), where spatial dependence is modelled (or expressed) by a banding or block-wise structure of the matrix. Note that in this case, the order of the coordinates is essential for the covariance estimator and needs to be specified in advance. In our setup, however, the advantageous order need not be known explicitly to the practitioner due to the maximum statistic.

If a spatial condition like (S3) cannot be assumed to hold (see Example 5.6), we can use the bootstrap procedures from Section 4. However, at least some preliminary considerations should be made; see the short discussion in Section 4.3. One way to check whether the permutation bootstrap is necessary is by means of a PCA. The literature on factor models provides a simple heuristic test (cf. [17]) in this direction. Compute the largest empirical eigenvalue $\widehat{\lambda}_1$ of the empirical correlation matrix $\widehat{\Sigma}_d$. In the presence of a common factor, $\widehat{\lambda}_1$ will explode with rate $d$, that is,

$$\liminf_{d\to\infty} \widehat{\lambda}_1/d \geq C > 0 \qquad \text{a.s.} \tag{6.1}$$

Hence, if $\widehat{\lambda}_1/d$ is small, a common factor is rather unlikely or its overall influence very weak, and a bootstrap is not necessary. As a final remark, let us mention that if controlling the Type I error is essential, the parametric bootstrap is highly recommended as a first tool for inference. The empirical results regarding the bootstrap in Section 6.3.2 reveal that the behavior may be significantly influenced by the choice of the number of blocks $L$ and the connected size $K$, which makes controlling the Type I error not so easy.

6.2. *Critical values.* Deriving reasonable critical values for extreme value statistics is a delicate issue. The root of the problem typically lies in the slow convergence rate of extreme value statistics. In our case, the domain of attraction is the Gumbel distribution, and the rate of convergence (for Gaussian random variables) is no better than $\mathcal{O}(\log n^{-1})$; see [28]. Hence, using the normalizing sequences $e_d$, $f_d$ given in Theorem 2.5 may not be the best thing to do. On the other hand, as is demonstrated by Proposition 2.7, approximative critical values can either be obtained by a parametric bootstrap or numerical computations. In principle, there are two methods for obtaining critical values in case of the parametric bootstrap.

(a) Simulate $\max_{1\leq h\leq d} B^Z_{n,h}$ directly.

TABLE 1
*Parametric bootstrap. Sample size* $n \in \{100, 250, 500\}$, *dimension* $d \in \{100, 250, 500\}$

| | $n = 100$ | | | $n = 250$ | | | $n = 500$ | | | **Numerical** | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | **100** | **250** | **500** | **100** | **250** | **500** | **100** | **250** | **500** | **100** | **250** | **500** |
| $\mathbf{q}_{0.9}$ | 1.83 | 1.93 | 2.00 | 1.88 | 1.99 | 2.07 | 1.9 | 2.02 | 2.10 | 1.95 | 2.05 | 2.14 |
| $\mathbf{q}_{0.95}$ | 1.91 | 2.00 | 2.10 | 1.97 | 2.07 | 2.15 | 1.99 | 2.10 | 2.19 | 2.03 | 2.15 | 2.22 |
| $\mathbf{q}_{0.975}$ | 1.98 | 2.07 | 2.15 | 2.05 | 2.15 | 2.22 | 2.08 | 2.19 | 2.28 | 2.12 | 2.21 | 2.30 |
| $\mathbf{q}_{0.99}$ | 2.07 | 2.17 | 2.24 | 2.15 | 2.25 | 2.31 | 2.19 | 2.30 | 2.36 | 2.22 | 2.32 | 2.40 |

(b) Estimate $F_n(x) = P(B_{n,h}^Z \le x)$, and obtain the critical values via $1 - \alpha = F_n(z_\alpha)^d$.

Method (b) is more flexible and was used to obtain the results. The corresponding critical values are tabulated in Table 1. A total of $10^6$ MC-runs was used to compute each critical value. Generally speaking, the quantiles obtained by numerical computations (Table 1, column "Numerical") are larger. This can be explained by the fact that in the "discrete" version $B_{n,h}^Z$, the maximum is taken over the set $\{1, \dots, n\}$, whereas in the limit $\mathcal{B}_h$, the supremum is taken over the whole interval $[0, 1]$, which is a larger set, and hence leads to this relation. In case of the permutation bootstrap, very similar results are obtained in the same Gaussian setup. Empirical evidence for the validity of the permutation bootstrap in the presence of dependence and change points is given in Section 6.3.2, where critical values are tabulated in Table 7.

6.3. *Simulation study.* In this subsection, we investigate the Type I error and power of the estimator $\widehat{\mathcal{S}}_d^c$ in a small simulation study. We consider estimates originating from the parametric as well as the permutation bootstrap. To assess the power, several alternatives are considered: we insert artificial changes in certain coordinates $h$ at $\tau_h \in \{(2i + 1)/10\}$, $0 \le i \le 4$ with size $\delta/10$ where $\delta \in \{0, 0.25, 0.5, 0.75, 1\}$. We then study the behavior and estimation accuracy on the sets

$$\mathcal{S}_d^c = \mathcal{S}_{d,1}^c \uplus \mathcal{S}_{d,2}^c \uplus \mathcal{S}_{d,3}^c \uplus \mathcal{S}_{d,4}^c \uplus \mathcal{S}_{d,5}^c,$$

where

$$\mathcal{S}_{d,i}^c = \{h \in \mathcal{S}_d^c : \tau_h \in [(i-1)/5, i/5)\}, \qquad 1 \le i \le 5.$$

Note that this means that we check whether the test detects a change and also classifies the time of change correctly. As a measure of comparison, we evaluate the relative estimation accuracy (in %) as

$$r_{d,i}^c = \frac{\mathbb{E}[|\widehat{\mathcal{S}}_{d,i}^c \cap \mathcal{S}_{d,i}^c|]}{|\mathcal{S}_{d,i}^c|} \times 100, \qquad 1 \le i \le 5,$$

TABLE 2
*Sample size* $n = 100$, *dimension* $d = 100$, $\mathrm{TI}_h^* = \mathrm{TI}_h \times 100$, $\alpha = 0.05$, $\widehat{\sigma}_h^*$

| | **Bootstrap II $d = 100$** | | | | | | **Parametric $d = 100$** | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ |
| 0 | – | – | – | – | – | 3.12 | – | – | – | – | – | 1.86 |
| 0.025 | 0.3 | 3.3 | 8.7 | 3.2 | 0.2 | 3.2 | 0.3 | 2.6 | 6.6 | 1.8 | 0.1 | 1.86 |
| 0.05 | 1.3 | 18.6 | 34.6 | 18.1 | 1.0 | 3.52 | 0.8 | 12.8 | 26.4 | 11.6 | 0.6 | 1.86 |
| 0.075 | 3.6 | 48.2 | 68.4 | 46.5 | 3.3 | 3.76 | 2.7 | 39.3 | 59.3 | 37.2 | 1.6 | 1.86 |
| 0.1 | 8.7 | 72.0 | 85.9 | 70.1 | 6.2 | 2.9 | 6.2 | 65.8 | 82.6 | 64.3 | 5.0 | 1.86 |

where the mean $\mathbb{E}[|\widehat{\mathcal{S}}_{d,i}^c \cap \mathcal{S}_d^c|]$ is estimated from the overall simulated sample. This gives an accurate measure of the performance of the test procedure. We also consider the coordinatewise Type I error, described by the probability

$$\mathrm{TI}_h = P\big(h \in \widehat{\mathcal{S}}_d^c \cap \mathcal{S}_d\big), \qquad h \in \mathcal{S}_d.$$

Note that if $\{X_{k,h}\}_{k\in\mathbb{Z},h\in\mathbb{N}}$ is a stationary random field (which is the case in all of our simulations), then $\mathrm{TI}_h$ is the same for all $h \in \mathcal{S}_d$ and can be written as

$$\mathrm{TI}_h = \frac{\mathbb{E}[|\widehat{\mathcal{S}}_d^c \setminus \mathcal{S}_d^c|]}{|\mathcal{S}_d|}, \qquad h \in \mathcal{S}_d.$$

To allow for reproducibility and transparency, all simulations use exactly the same random seed, and also the sets $\mathcal{S}_{d,i}^c$ remain the same. This implies that for $n, d$ fixed, the Type I error $\mathrm{TI}_h$ remains *the same* for all $\delta$. Natural exceptions are only when $\delta = 0$ or the long run simulations in Tables 2 and 3 concerning the bootstrap results. The number of change points for each $\mathcal{S}_{d,i}^c$ is set to 10 for $d = 100$ and 15 for $d = 250$. This gives a total amount of changes $|\mathcal{S}_{100}^c| = 50$ and $|\mathcal{S}_{250}^c| = 75$. As sample size, we consider $n \in \{100, 250\}$ and 1000 MC runs for each setting, unless stated otherwise. We use two different models for our simulations, namely

TABLE 3
*Sample size* $n = 100$, *dimension* $d = 100$, $\mathrm{TI}_h^* = \mathrm{TI}_h \times 100$, $\alpha = 0.05$, $\widehat{\sigma}_h^*$

| | **Bootstrap III $d = 100$** | | | | | | **Parametric $d = 100$** | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ |
| 0 | – | – | – | – | – | 2.3 | – | – | – | – | – | 1.86 |
| 0.025 | 0.3 | 2.9 | 7.3 | 2.5 | 0.1 | 2.4 | 0.3 | 2.6 | 6.6 | 1.8 | 0.1 | 1.86 |
| 0.05 | 0.8 | 15.2 | 30.0 | 14.0 | 0.8 | 2.52 | 0.8 | 12.8 | 26.4 | 11.6 | 0.6 | 1.86 |
| 0.075 | 3.0 | 43.5 | 63.0 | 40.7 | 2.2 | 2.66 | 2.7 | 39.3 | 59.3 | 37.2 | 1.6 | 1.86 |
| 0.1 | 9.3 | 70.5 | 85.0 | 68.8 | 5.9 | 2.92 | 6.2 | 65.8 | 82.6 | 64.3 | 5.0 | 1.86 |

TABLE 4
*Sample size* $n = 100$, *dimension* $d \in \{100, 250\}$, $\mathrm{TI}_h^* = \mathrm{TI}_h \times 100$, $\alpha = 0.05$, $\widehat{\sigma}_h^*$

| | **Parametric $d = 100$** | | | | | | **Parametric $d = 250$** | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ |
| 0 | – | – | – | – | – | 2.01 | – | – | – | – | – | 1.23 |
| 0.025 | 0.19 | 2.46 | 6.67 | 2.24 | 0.11 | 2.01 | 0.06 | 1.43 | 5.05 | 1.76 | 0.06 | 1.23 |
| 0.05 | 0.74 | 13.1 | 28.1 | 12.5 | 0.55 | 2.01 | 0.37 | 9.97 | 22.8 | 9.87 | 0.27 | 1.23 |
| 0.075 | 2.54 | 38.8 | 58.4 | 38.0 | 1.85 | 2.01 | 1.52 | 32.3 | 51.6 | 30.7 | 0.95 | 1.23 |
| 0.1 | 7.21 | 67.2 | 82.5 | 65.8 | 4.61 | 2.01 | 5.03 | 61.2 | 77.3 | 59.8 | 2.90 | 1.23 |

autoregressive and factor models. In case of the factor model, we also investigate the behavior of the bootstrap Algorithms 4.5 and 4.7.

6.3.1. *Autoregressive models*. We use the following model. We take $Y_{k,h}$ as an MA(100) process

$$Y_{k,h} = \sum_{i=0}^{99} \alpha_i \varepsilon_{k,h-i}, \qquad \alpha_i = 0.1|i|^{-3} \text{ and } \varepsilon_{k,h} \sim \mathcal{N}(0, s^2), s = 0.1. \tag{6.2}$$

We then consider the ARMA(2, 2) model

$$X_{k,h} = 0.2X_{k-1,h} - 0.3X_{k-2,h} - 0.1Y_{k,h} + 0.2Y_{k-1,h}, \qquad 1 \le h \le d. \tag{6.3}$$

Note that we stick to the same model in each coordinate, which makes the comparison and analysis easier and more transparent. Throughout this section, the nominal level of all tests is $\alpha = 0.05$, that is, we always use the corresponding quantiles $\mathbf{q}_{0.95}$. We first analyze the parametric bootstrap. The corresponding results are given in Tables 4 and 5. Note that in both tables, the row with $\delta = 0$ corresponds to the empirical levels of the test. The Type I error is slightly different from the cases where $\delta > 0$ (not visible due to rounded values), which is due to the fact

TABLE 5
*Sample size* $n = 250$, *dimension* $d \in \{100, 250\}$, $\mathrm{TI}_h^* = \mathrm{TI}_h \times 100$, $\alpha = 0.05$, $\widehat{\sigma}_h^*$

| | **Parametric $d = 100$** | | | | | | **Parametric $d = 250$** | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ |
| 0 | – | – | – | – | – | 0.94 | – | – | – | – | – | 0.50 |
| 0.025 | 0.04 | 4.96 | 13.9 | 5.02 | 0.04 | 0.94 | 0.02 | 3.22 | 10.2 | 3.52 | 0.03 | 0.50 |
| 0.05 | 0.76 | 42.5 | 65.8 | 41.5 | 0.56 | 0.94 | 0.25 | 34.1 | 58.7 | 33.4 | 0.27 | 0.50 |
| 0.075 | 5.35 | 84.7 | 95.5 | 83.9 | 4.08 | 0.94 | 1.98 | 79.6 | 93.6 | 79.9 | 2.28 | 0.50 |
| 0.1 | 19.7 | 96.3 | 99.6 | 95.6 | 16.9 | 0.94 | 10.7 | 96.2 | 99.6 | 95.6 | 11.4 | 0.50 |

TABLE 6
*Sample size* $n = 250$, *dimension* $d \in \{100, 250\}$, $\mathrm{TI}_h^* = \mathrm{TI}_h \times 100$, $\alpha = 0.05$, $\widehat{\sigma}_h^{\diamond}$

| | **Parametric $d = 100$** | | | | | | **Parametric $d = 250$** | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ | $r_{d,1}^c$ | $r_{d,2}^c$ | $r_{d,3}^c$ | $r_{d,4}^c$ | $r_{d,5}^c$ | $\mathrm{TI}_h^*$ |
| 0 | – | – | – | – | – | 1.51 | – | – | – | – | – | 0.99 |
| 0.025 | 0.12 | 8.12 | 16.6 | 8.64 | 0.06 | 1.42 | 0.04 | 6.25 | 12.6 | 6.47 | 0.05 | 1.01 |
| 0.05 | 1.58 | 53.7 | 64.1 | 54.0 | 1.26 | 1.42 | 0.80 | 48.0 | 57.5 | 48.4 | 0.68 | 1.01 |
| 0.075 | 9.12 | 88.6 | 90.2 | 88.6 | 8.56 | 1.42 | 5.95 | 87.5 | 85.6 | 86.8 | 5.93 | 1.01 |
| 0.1 | 29.5 | 96.8 | 97.9 | 96.4 | 29.4 | 1.42 | 23.5 | 96.2 | 95.6 | 96.1 | 23.2 | 1.01 |

that $\mathcal{S}_d = \{1, \dots, d\}$ if $\delta = 0$, and $\mathcal{S}_d \subset \{1, \dots, d\}$ otherwise. Observe that small changes are found with difficulty if the sample size is small, and this effect naturally gets amplified in higher dimensions. The power for bigger samples/changes is however very reasonable. As expected, the test loses power if the time of change $\tau_h$ moves away from the center $1/2$. Unreported simulations exhibit a similar behavior in case of GARCH-sequences, or tests for changes in the second moment or variance.

Next, we briefly discuss a possible effect in the choice of variance estimator. In the previous results, estimator $\widehat{\sigma}_h^*$ was used; see Section 3 to recall the definition. As one comparison, we now use $\widehat{\sigma}_h^{\diamond}$; see Table 6 for corresponding results. An interesting phenomena appears. We note that $\widehat{\sigma}_h^*$ yields the better results if $\tau_h = 1/2$, and $\widehat{\sigma}_h^{\diamond}$ if the change is more away from $1/2$. This is a little surprising, since one can show that for large enough $n$, $\widehat{\sigma}_h^*$ has the smaller MSE. A possible explanation could be the quality of estimation of $\widehat{\tau}_h$, and the actual choice of $B_\tau$.

We now turn to the behavior of the bootstrap procedures, more precisely, we consider Algorithms 4.5 and 4.7, where we "illegally" set $\mathfrak{t} = 0$. We use the same model (6.3). In order to obtain an overall feasible computational time, we restrict ourselves to the setup where $n = 100$, $d = 100$ and we only used 100 overall simulations for comparison (note: comparing the parametric results indicates that there actually is not much difference between 100 or 1000 simulations). Moreover, we only use $M = 100$ Monte Carlo runs for the bootstrap procedures. Arguably, this might be too low to obtain a necessary accuracy for a 95% quantile, but it turns out that this is not the case. We choose $L = 25$ as the number of blocks, and thus $K = 4$ for the block length. The results of Algorithm 4.5 are given in Table 2. Even though we only set $M = 100$, we get slightly better results than the parametric procedure. Observe that the results are also conservative. The behavior of Algorithm 4.7 in Table 3 is slightly worse, but overall very similar.

6.3.2. *Factor models and number of block length effect.* We consider a factor model that shows that block and parametric bootstrap may behave very differently. As explained in Example 5.6, this is the case if overall dependence on certain

factors is present. To allow for a comparison to the autoregressive model, we use the same model for the general dynamics. We take $Y_{k,h}$ as an MA(100) process

$$Y_{k,h} = \sum_{i=0}^{99} \alpha_i \varepsilon_{k,h-i}, \qquad \alpha_i = 0.1|i|^{-3} \text{ and } \varepsilon_{k,h} \sim \mathcal{N}(0, s^2), s = 0.1. \tag{6.4}$$

We then consider the ARMA(2, 2)-factor model

$$X_{k,h} = \alpha_F F_k + 0.2X_{k-1,h} - 0.3X_{k-2,h} - 0.1Y_{k,h} + 0.2Y_{k-1,h}, \qquad 1 \le h \le d,$$

where $\alpha_F \ge 0$ is a constant, and the factors $\{F_k\}_{k\in\mathbb{Z}}$ are IID standard Gaussian random variables. The primary focus in this section is to demonstrate the effect of high spatial dependence and the size $L$ on the quantiles. The pronounced effect of the factors is visible in Table 7, where the critical values of Algorithms 4.5 and 4.7 are tabulated for $\alpha_F \in \{0.1, 0.3\}$. A value of $\delta = 0$ and $M = 1000$ were used in the simulations. We see that the factor has an expected significant reciprocal impact on the quantiles, that is, larger factors result in lower quantiles. We also observe that in this setup, the number of blocks $L \in \{25, 50\}$ does not have a notable impact. A slight outlier seems to be the results of Algorithm 4.5(II) in case of $L = 25$. Also note that particularly the results about the more extremal quantiles $\mathbf{q}_{0.975}$ and $\mathbf{q}_{0.99}$ have to be considered with care, since "only" $M = 1000$ was used.

Unreported simulations show that the power and size are different from the autoregressive model. Particularly if $\alpha_F$ is large, (e.g., $\alpha_F = 0.3$), one has to consider a larger size of change $\delta > 0.1$ in order to obtain visible effects. The reason for this loss in power is the (considerably) larger long-run variance $\sigma_h$ in this model, induced by the factor loading $\alpha_F$. More precisely, since we scale by a (larger) consistent estimate of $\sigma_h$, changes become harder to detect [see also (3.2)].

Finally, we take a look at Table 8, which reveals that the choice of $L$ may have a serious impact. Here, we set $\alpha_F = 0$ to allow for a comparison to the results in Sections 6.2 and 6.3.1. We observe that raising $K$ only by one from $K = 4$ to $K = 5$

TABLE 7
*Bootstrap Alg.* II, III. *Sample size* $n = 250$, *dimension* $d = 100$, $\delta = 0$, $\alpha_F \in \{0.1, 0.3\}$

| | **Bootstrap $\alpha_F = 0.1$** | | | | **Bootstrap $\alpha_F = 0.3$** | | | |
|---|---|---|---|---|---|---|---|---|
| $K \times L$ | $n = 250$ $5 \times 50$ | | $n = 250$ $10 \times 25$ | | $n = 250$ $5 \times 50$ | | $n = 250$ $10 \times 25$ | |
| $d$ | 100 | | 100 | | 100 | | 100 | |
| **Algorithm** | **II** | **III** | **II** | **III** | **II** | **III** | **II** | **III** |
| $\mathbf{q}_{0.9}$ | 1.69 | 1.73 | 1.69 | 1.77 | 1.47 | 1.49 | 1.44 | 1.51 |
| $\mathbf{q}_{0.95}$ | 1.75 | 1.83 | 1.78 | 1.87 | 1.56 | 1.59 | 1.56 | 1.61 |
| $\mathbf{q}_{0.975}$ | 1.81 | 1.89 | 1.91 | 1.94 | 1.69 | 1.68 | 1.63 | 1.75 |
| $\mathbf{q}_{0.99}$ | 1.89 | 1.95 | 2.05 | 2.14 | 1.82 | 1.80 | 1.73 | 1.85 |

TABLE 8
*Bootstrap Alg.* II, III. *Sample size* $n = 100$, *dimension* $d = 100$, $\delta = 0$, $\alpha_F = 0$

| $K \times L$ | $n = 100$ $5 \times 20$ | | $n = 100$ $4 \times 25$ | |
|---|---|---|---|---|
| $d$ | 100 | | 100 | |
| **Algorithm** | **II** | **III** | **II** | **III** |
| $\mathbf{q}_{0.9}$ | 1.91 | 2.04 | 1.75 | 1.81 |
| $\mathbf{q}_{0.95}$ | 2.08 | 2.23 | 1.89 | 1.92 |
| $\mathbf{q}_{0.975}$ | 2.25 | 2.44 | 1.95 | 2.02 |
| $\mathbf{q}_{0.99}$ | 2.38 | 2.68 | 2.05 | 2.11 |

leads to much larger quantiles. In view of the results presented in Section 6.3.1, these would lead to a high loss in power. The setup itself appears rather harmless; we note, however, that $d/n = 1$ have the same size, unlike to the situation in Table 7 where $d/n = 2/5$. It appears that at least if $d/n \geq 1$, block bootstrap procedures based on multipliers can require careful tuning. Particularly if $d \gg n$, the parametric bootstrap seems to be the more stable option.

**7. Proofs.** All proofs together with additional results are given in detail in [33].

**Acknowledgements.** I am indebted to the Associate Editor and the reviewers for a careful reading of the manuscript, and the many thoughtful remarks and comments that considerable helped to improve the results and presentation. I warmly thank Wei Biao Wu for his comments on the proofs, and Lajos Horváth for his advice. I also thank Holger Drees for his expertise on extremal quantile estimation.

## SUPPLEMENTARY MATERIAL

**Supplement to "Uniform change point tests in high dimension."** (DOI: 10.1214/15-AOS1347SUPP; .pdf). The supplement contains all proofs. In addition, a detailed analysis of a data set (S&P 500) is presented.

## REFERENCES

[1] ANDREWS, D. W. K. (1984). Nonstrong mixing autoregressive processes. *J. Appl. Probab.* **21** 930–934. MR0766830

[2] ANTOCH, J. and HUŠKOVÁ, M. (2001). Permutation tests in change point analysis. *Statist. Probab. Lett.* **53** 37–46. MR1843339

[3] AUE, A., HÖRMANN, S., HORVÁTH, L. and REIMHERR, M. (2009). Break detection in the covariance structure of multivariate time series models. *Ann. Statist.* **37** 4046–4087. MR2572452

[4] AUE, A. and HORVÁTH, L. (2013). Structural breaks in time series. *J. Time Series Anal.* **34** 1–16. MR3008012

[5] BANERJEE, A. and URGA, G. (2005). Modelling structural breaks, long memory and stock market volatility: An overview. *J. Econometrics* **129** 1–34. MR2209656

[6] BERKES, I., GABRYS, R., HORVÁTH, L. and KOKOSZKA, P. (2009). Detecting changes in the mean of functional observations. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **71** 927–946. MR2750251

[7] BERKES, I., GOMBAY, E. and HORVÁTH, L. (2009). Testing for changes in the covariance structure of linear processes. *J. Statist. Plann. Inference* **139** 2044–2063. MR2497559

[8] BERKES, I., HÖRMANN, S. and SCHAUER, J. (2009). Asymptotic results for the empirical process of stationary sequences. *Stochastic Process. Appl.* **119** 1298–1324. MR2508575

[9] BOLLERSLEV, T. (1986). Generalized autoregressive conditional heteroskedasticity. *J. Econometrics* **31** 307–327. MR0853051

[10] BOUGEROL, P. and PICARD, N. (1992). Strict stationarity of generalized autoregressive processes. *Ann. Probab.* **20** 1714–1730. MR1188039

[11] BROCKWELL, P. J. and DAVIS, R. A. (1991). *Time Series*: *Theory and Methods*, 2nd ed. Springer, New York. MR1093459

[12] CAI, T. T. and JIANG, T. (2011). Limiting laws of coherence of random matrices with applications to testing covariance structure and construction of compressed sensing matrices. *Ann. Statist.* **39** 1496–1525. MR2850210

[13] CHEN, S. X. and QIN, Y.-L. (2010). A two-sample test for high-dimensional data with applications to gene-set testing. *Ann. Statist.* **38** 808–835. MR2604697

[14] CHEN, X., XU, M. and WU, W. B. (2013). Covariance and precision matrix estimation for high-dimensional time series. *Ann. Statist.* **41** 2994–3021. MR3161455

[15] CHO, H. and FRYZLEWICZ, P. (2015). Multiple change-point detection for high dimensional time series via sparsified binary segmentation. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **77** 475–507. MR3310536

[16] CHUDIK, A. and PESARAN, M. H. (2011). Infinite-dimensional VARs and factor models. *J. Econometrics* **163** 4–22. MR2803662

[17] CONNOR, G. and KORAJCZYK, R. A. (1993). A test for the number of factors in an approximate factor model. *J. Finance* **48** 1263–1291.

[18] CRAINICEANU, C. M. and VOGELSANG, T. J. (2007). Nonmonotonic power for tests of a mean shift in a time series. *J. Stat. Comput. Simul.* **77** 457–476. MR2405424

[19] CSÖRGŐ, M. and HORVÁTH, L. (1993). *Weighted Approximations in Probability and Statistics*. Wiley, Chichester. MR1215046

[20] CSÖRGŐ, M. and HORVÁTH, L. (1997). *Limit Theorems in Change-Point Analysis*. Wiley, Chichester. MR2743035

[21] CSÖRGŐ, M., SZYSZKOWICZ, B. and WANG, Q. (2003). Darling–Erdős theorem for self-normalized sums. *Ann. Probab.* **31** 676–692. MR1964945

[22] CSÖRGŐ, M., SZYSZKOWICZ, B. and WANG, Q. (2003). Donsker's theorem for self-normalized partial sums processes. *Ann. Probab.* **31** 1228–1240. MR1988470

[23] DARLING, D. A. and ERDÖS, P. (1956). A limit theorem for the maximum of normalized sums of independent random variables. *Duke Math. J.* **23** 143–155. MR0074712

[24] FAN, J., LV, J. and QI, L. (2011). Sparse high-dimensional models in economics. *Annual Review of Economics* **3** 291–317.

[25] FAN, J. and YAO, Q. (2003). *Nonlinear Time Series*: *Nonparametric and Parametric Methods*. Springer, New York. MR1964455

[26] FREMDT, S., STEINEBACH, J. G., HORVÁTH, L. and KOKOSZKA, P. (2013). Testing the equality of covariance operators in functional samples. *Scand. J. Stat.* **40** 138–152. MR3024036

[27] GROEN, J. J. J., KAPETANIOS, G. and PRICE, S. (2010). Multivariate methods for monitoring structural change. Working papers 658, Queen Mary, Univ. of London, School of Economics and Finance.

[28] HALL, P. (1979). On the rate of convergence of normal extremes. *J. Appl. Probab.* **16** 433–439. MR0531778

[29] HÖRMANN, S. (2009). Berry–Esseen bounds for econometric time series. *ALEA Lat. Am. J. Probab. Math. Stat.* **6** 377–397. MR2557877

[30] HÖRMANN, S. and KOKOSZKA, P. (2010). Weakly dependent functional data. *Ann. Statist.* **38** 1845–1884. MR2662361

[31] HORVÁTH, L. and HUŠKOVÁ, M. (2012). Change-point detection in panel data. *J. Time Series Anal.* **33** 631–648. MR2944843

[32] JIANG, T. (2004). The asymptotic distributions of the largest entries of sample correlation matrices. *Ann. Appl. Probab.* **14** 865–880. MR2052906

[33] JIRAK, M. (2015). Supplement to "Uniform change point tests in high dimension." DOI:10.1214/15-AOS1347SUPP.

[34] JIRAK, M. (2012). Change-point analysis in increasing dimension. *J. Multivariate Anal.* **111** 136–159. MR2944411

[35] JIRAK, M. (2012). Simultaneous confidence bands for Yule–Walker estimators and order selection. *Ann. Statist.* **40** 494–528. MR3014315

[36] KIRCH, C. (2007). Block permutation principles for the change analysis of dependent data. *J. Statist. Plann. Inference* **137** 2453–2474. MR2325449

[37] LAHIRI, S. N. (2003). *Resampling Methods for Dependent Data*. Springer, New York. MR2001447

[38] LEADBETTER, M. R. (1973/74). On extreme values in stationary sequences. *Z. Wahrsch. Verw. Gebiete* **28** 289–303. MR0362465

[39] LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York. MR0691492

[40] LÉVY-LEDUC, C. and ROUEFF, F. (2009). Detection and localization of change-points in high-dimensional network traffic data. *Ann. Appl. Stat.* **3** 637–662. MR2750676

[41] OMBAO, H., VON SACHS, R. and GUO, W. (2005). SLEX analysis of multivariate nonstationary time series. *J. Amer. Statist. Assoc.* **100** 519–531. MR2160556

[42] EL MACHKOURI, M., VOLNÝ, D. and WU, W. B. (2013). A central limit theorem for stationary random fields. *Stochastic Process. Appl.* **123** 1–14. MR2988107

[43] PERRON, P. (2005). Dealing with structural breaks. Working papers Series WP2005-017, Dept. of Economics, Boston Univ.

[44] PÖTSCHER, B. M. and PRUCHA, I. R. (1997). *Dynamic Nonlinear Econometric Models: Asymptotic Theory*. Springer, Berlin. MR1468737

[45] SHAO, X. and WU, W. B. (2007). Asymptotic spectral theory for nonlinear time series. *Ann. Statist.* **35** 1773–1801. MR2351105

[46] SHAO, X. and ZHANG, X. (2010). Testing for change points in time series. *J. Amer. Statist. Assoc.* **105** 1228–1240. MR2752617

[47] SHORACK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR0838963

[48] SPOKOINY, V. (2009). Multiscale local change point detection with applications to value-at-risk. *Ann. Statist.* **37** 1405–1436. MR2509078

[49] TARTAKOVSKY, A. G. and VEERAVALLI, V. V. (2004). Change-point detection in multichannel and distributed systems. In *Applied Sequential Methodologies*. *Statist. Textbooks Monogr.* **173** 339–370. Dekker, New York. MR2159163

[50] WU, W. B. (2005). Nonlinear system theory: Another look at dependence. *Proc. Natl. Acad. Sci. USA* **102** 14150–14154 (electronic). MR2172215

[51] XIAO, H. and WU, W. B. (2013). Asymptotic theory for maximum deviations of sample covariance matrix estimates. *Stochastic Process. Appl.* **123** 2899–2920. MR3054550

[52] XIE, Y. and SIEGMUND, D. (2013). Sequential multi-sensor change-point detection. *Ann. Statist.* **41** 670–692. MR3099117

[53] ESMA, In European Securities and Markets Authority. Basel III: Level I, Level II Reports.

INSTITUT FÜR MATHEMATIK
UNTER DEN LINDEN 6
D-10099 BERLIN
GERMANY
E-MAIL: jirak@math.hu-berlin.de