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TESTING FOR PURE-JUMP PROCESSES FOR HIGH-FREQUENCY DATA

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Pure-jump processes have been increasingly popular in modeling high-frequency financial data, partially due to their versatility and flexibility. In the meantime, several statistical tests have been proposed in the literature to check the validity of using pure-jump models. However, these tests suffer from several drawbacks, such as requiring rather stringent conditions and having slow rates of convergence. In this paper, we propose a different test to check whether the underlying process of high-frequency data can be modeled by a pure-jump process. The new test is based on the realized characteristic function, and enjoys a much faster convergence rate of order $O(n^{1/2})$ (where n is the sample size) versus the usual $o(n^{1/4})$ available for existing tests; it is applicable much more generally than previous tests; for example, it is robust to jumps of infinite variation and flexible modeling of the diffusion component. Simulation studies justify our findings and the test is also applied to some real high-frequency financial data.

1. Introduction. Itô's semimartingales are widely used in modeling the log prices of an asset since they fit many stylized features of asset returns, and in option pricing due to absence of arbitrage in efficient market. Mathematically, they consist of two parts: a continuous local martingale term and a pure-jump process with both big and small jumps. Itô's semimartingale with a continuous local martingale is in common use in the literature, for example, the Black and Scholes (1973) model (geometric Brownan motion), the Merton (1976) model and Kou (2002) model (geometric Brownan motion plus finitely many jumps).

On the other hand, in recent years pure-jump processes have also been accepted as an alternative model for log price processes or even the latent spot volatility process to the classic models mentioned earlier; see, for example, Todorov and Tauchen (2010, 2014) and references therein. The idea behind the pure-jump

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modeling is that small jumps can eliminate the need for a continuous martingale. Pure-jump models are also very flexible. They include the normal inverse Gaussian [Rydberg (1997); Barndorff-Nielsen (1997, 1998)], the variance gamma [Madan, Carr and Chang (1998)], the CGMY model of Carr et al. (2003b), the time-changed Lévy mdoels of Carr et al. (2003a), the non-Gaussian Ornstein–Uhlenbeck-based models of Barndorff-Nielsen and Shephard (2001) and the Lévy-driven continuous-time moving average (CARMA) models of Brockwell (2001) for the stochastic volatility. Pure-jump models have been extensively used for general option pricing [Huang and Wu (2004); Broadie and Detemple (2004); Levendorskiĭ (2004); Schoutens (2006); Ivanov (2007)] and for foreign exchange option pricing [Huang and Hung (2005); Daal and Madan (2005); Carr and Wu (2007)]. Other applications of pure-jump models include reliability theory [Drosen (1986)], insurance valuation [Ballotta (2005)] and fianancial equilibrium analysis [Madan (2006)].

Statistically, this forces us to reconsider the necessity of including the local martingale part driven by Brownian motion in modeling high-frequency data. This begs the following question: "Is it sufficient to model high frequency data by pure-jump process alone," or equivalently, "is it necessary to add a Brownian force underlying the high frequency data?" The answer to this question serves as a model selection purpose. For more motivation and explanation, we refer to Aït-Sahalia and Jacod (2010) and Jing, Kong and Liu (2012).

For ease of presentation, let X_t be a semimartingale defined on some filtered probability space $(\Omega, \mathcal{F}, P)$,

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s + X_t^d,$$

where X_0 is the initial value, $\int_0^t b_s\,ds$ is the drift term with b_s being the time-varying drift coefficient which is an optional and càdlàg process, $\int_0^t \sigma_s\,dW_s$ is a continuous local martingale with σ_s being an adapted process and W_s a standard Brownian motion and the last term is a pure-jump component with the jump activity index β defined by

$$\beta = \inf\left\{r; \sum_{0\le s\le T} |\Delta_s X|^r \le \infty\right\}, \tag{1.1}$$

where $\Delta_s X = X_s - X_{s-}$; see Aït-Sahalia and Jacod (2009) and Jing et al. (2012). Then the above question can be formulated as a hypothesis testing problem as

$$H_0: \int_0^T \sigma_s^2\,ds > 0 \quad \text{v.s.} \quad H_1: \int_0^T \sigma_s^2\,ds = 0, \tag{1.2}$$

where T is the time span of the high-frequency data.

The testing problem (1.2) was studied by several authors. For instance, Cont and Mancini (2007), Aït-Shalia and Jacod (2010) used threshold power variation to construct their test statistics. However, there are two main drawbacks with the threshold power variation method:

- First, their tests require that X^d be of finite variation, which rules out many interesting models in finance since empirical evidences in some real data analysis show that the jumps are of infinite variation; see, for example, Aït-Sahalia and Jacod (2009) and Zhao and Wu (2009).
- Second, their tests are not very powerful, even when β ($0 \leq \beta < 2$) is close to 0. This is rather counterintuitive since probabilistically the smaller the value of β is, the farther X^d is from a continuous semimartingale.

Interestingly, Todorov and Tauchen (2011) invented a test based on the point estimator of the JAI cleverly constructed as the smallest power for which the realized power variation (without thresholding) does not explode. Surprisingly, a test based on this estimator for the presence of Brownian motion has the property that it has more power for lower level of activity. However, since it is from realized power variation, once more, one has to assume that X^d is of finite variation when $C_T = \int_0^T \sigma_s^2\, ds$ does not vanish in order to have available central limit theorem. It is also worth noticing that Todorov and Tauchen (2014) test for presence of Brownian motion by checking whether "devolatilized" truncated returns are i.i.d. normal assuming finite activity jumps present in the underlying log price processes.

Testing the existence of a nonvanishing continuous local martingale is challenging when the jumps are of infinite variation. Jing, Kong and Liu (2012) used the number of small increments to propose a test, which mitigates the abovementioned difficulties, and can handle jumps of infinite variation. However, it still has the following deficiencies:

- First, the local volatility model is too restrictive. For example, it does not even cover the Heston model under H_0.
- Second, the spot volatility of the continuous component is assumed to be positive almost everywhere in time t. So if H_0 is rejected, it is quite possible that the continuous component vanishes only in certain subintervals, but is still present in other subintervals; see the simulation in Section 4 for more illustration.

In this paper, we develop a novel test to (1.2) to overcome the difficulties encountered in previous approaches. The convergence rate of our new test under H_0 is of order $n^{-1/2}$ when the jump component is of infinite variation, which is faster than that of all existing tests. The idea of the test is based on the realized characteristic function, which was introduced in Todorov and Tauchen (2012) to investigate the distributional property of volatilities at different time points; see also Todorov, Tauchen and Grynkiv (2011) and Jacod and Todorov (2014). With observable i.i.d. increments of a class of Lévy process with either finite activity or infinite activity jumps, Chen, Delaigle and Hall (2010) proposed a regression method based on the empirical characteristic function to estimate the parameters of the drift, scale, stable index and the distribution of the jump size of a compound Poisson process, while in our paper, we assume a flexible Itô semimartingale with stochastic volatility and stochastic coefficient of jump measures, and assume that the time lag of successive observations shrinks to 0 (high-frequency data) rather than fixed, as implicitly assumed in Chen, Delaigle and Hall (2010). However, we remark that direct application of the realized characteristic function does not work in testing (1.2), and some other novel statistical techniques are needed.

The paper is organized as follows. In Section 2, we give some assumptions and introduce our test statistics. Main results are presented in Section 3. Section 4 gives some simulation studies and real data analysis. The main proofs are postponed to the Appendix, and the proofs of some lemmas are provided in the supplementary material [Kong, Liu and Jing (2015)] to this paper.

Throughout the paper, we assume that the available data set is $\{X_{t_i}; 0 \le i \le n\}$ which is discretely sampled from X, and is equally spaced in the fixed interval $[0, T]$, that is, $t_i = i\Delta_n$ with $\Delta_n = T/n$ for $0 \le i \le n$. Denote the jth one-step increment by

$$\Delta_j^n X = X_{t_j} - X_{t_{j-1}}, \qquad 1 \le j \le n.$$

2. Methodology. The key idea behind our test statistic is that the characteristic function of the increments of the Itô's semimartingale is dominated by the continuous local martingale part.

For illustration, let us take the following simple example:

$$X_t = \sigma W_t + \gamma Y_t,$$

where $\sigma \ge 0$ is a constant spot volatility, γ is some constant and Y_t is a symmetric β-stable process. Then the logarithm of the characteristic function is

$$\log \psi_n(u) \equiv \log E\left[e^{\sqrt{-1}u\Delta_i^n X/\sqrt{\Delta_n}}\right] = -\tfrac{1}{2}\sigma^2 u^2 - |\gamma|^\beta u^\beta \Delta_n^{1-\beta/2}. \tag{2.1}$$

As $\Delta_n \to 0$, the last term in (2.1) induced by the jump part decreases at a rate of $\Delta_n^{1-\beta/2}$. Note that when $\beta < 1$ (i.e., Y_t is of finite variation), in the context of estimating σ (or its functionals), the bias caused by the jump part is of negligible size $o(\Delta_n^{1/2})$. This implies that an estimator of σ_t (or its functionals) for a general semimartingale based on the characteristic function would very likely be robust to jumps of finite variation, which is confirmed in Todorov and Tauchen (2012) and Jacod and Todorov (2014). On the other hand, the problem becomes more challenging when $\beta > 1$ since the last term in (2.1) is no longer a negligible bias term. In testing (1.2), under H_0, the right-hand side of (2.1) is a nonvanishing constant while under H_1 it is almost zero. This is a major feature we will explore later to differentiate the null and the alternative hypotheses.

We shall now introduce our test statistic. To start with, we split the data into m_n nonoverlapping blocks with each block length equal to $2v_n$ consisting of $2k_n$ intervals of length Δ_n, where k_n is some integer depending on n. Motivated by (2.1), and in view of $X_{t+s} - X_t \approx \sigma_t(W_{t+s} - W_t) + \gamma_{t-}^+(Y_{t+s}^+ - Y_t^+) + \gamma_{t-}^-(Y_{t+s}^- - Y_t^-)$ where $Y^\pm$ are two independent "stable like" Lévy processes and $\gamma^\pm$ are two càdlàg processes that will be specified later in Assumption 3.1. When s is close to 0, we can estimate $\sigma^2_{2jv_n}$ $(0 \le j \le m_n - 1)$ locally by

$$c^0_j(u) = -\frac{1}{u^2}\log\left(L^0_j(u) \vee \frac{1}{\sqrt{k_n}}\right), \tag{2.2}$$

where

$$L^0_j(u) = \frac{1}{k_n - 1}\sum_{l=1}^{k_n-1} \cos\bigl(u\bigl(\Delta^n_{2jk_n+2l+1}X - \Delta^n_{2jk_n+2l}X\bigr)/\Delta_n^{1/2}\bigr). \tag{2.3}$$

Summing over $c^0_j(u)$ for all $j \le m_n$ and properly normalizing it, one easily gets an estimator of the integrated volatility process,

$$C_t \equiv \int_0^t \sigma^2_s\, ds.$$

Jacod and Todorov (2014) introduced a bias-corrected estimator of C_t as

$$\hat{C}_0(u_n) = 2v_n \sum_{j=0}^{[t/(2v_n)]-1}\left(c^0_j(u_n) - \frac{1}{u_n^2(k_n-1)}\bigl(\sinh\bigl(u_n^2 c^0_j(u_n)\bigr)\bigr)^2\right), \tag{2.4}$$

and further showed that

$$\hat{C}_0(u_n) = C_t + A_0(u_n)^n_t + O_p\bigl(\Delta_n^{1/2}\bigr), \tag{2.5}$$

where

$$A_0(u)^n_t = 2u^{\beta-2}\Delta_n^{1-\beta/2}\int_0^t a_s\, ds$$

with $a_s = \chi(\beta)(|\gamma_s^+|^\beta + |\gamma_s^-|^\beta)$ and $\chi(\beta) = \int_0^\infty y^{-\beta}\sin y\, dy$. Then a natural test statistic which can differentiate the null and alternative hypotheses is

$$T'_n \equiv \frac{\hat{C}_0(2u_n) - \hat{C}_0(u_n)}{\hat{C}_0(u_n)} \longrightarrow_p \begin{cases} 0, & \text{on } \{C_T > 0\}, \\ 2^{\beta-2} - 1 < 0, & \text{on } \{C_T = 0\}. \end{cases}$$

The problem with T'_n is that no central limit theorem is available as $\beta > 1$, so that one cannot find the rejection region when jumps are of infinite variation. We will fix this problem with some manipulations to T'_n below.

To do this, we replace $\hat{C}_0(u)$ by a similarly defined quantity. Let the c^1_j's and $\hat{C}_1(u)$ be similarly defined as the c^0_j's and $\hat{C}_0(u)$ with $\Delta^n_{2jk_n+2l+1}X - \Delta^n_{2jk_n+2l}X$ replaced by $\Delta^n_{2jk_n+2l}X - \Delta^n_{2jk_n+2l-1}X$, for $l = 1, \dots, k_n - 1$. A seemingly better test statistic is then

$$T^*_n = \frac{(\hat{C}_0(2u_n) - \hat{C}_1(u_n)) - (\hat{C}_0(2u_n) - \hat{C}_0(u_n))}{\hat{C}_1(u_n)} = \frac{\hat{C}_0(u_n) - \hat{C}_1(u_n)}{\hat{C}_1(u_n)}, \tag{2.6}$$

which works under H_0 because the numerator is equal to

$$\begin{aligned}
&\big[\big(\hat{C}_0(2u_n) - C_t - A_0(2u_n)_t^n\big) - \big(\hat{C}_1(u_n) - C_t - A_0(u_n)_t^n\big)\big] \\
&\quad -\big[\big(\hat{C}_0(2u_n) - C_t - A_0(2u_n)_t^n\big) - \big(\hat{C}_0(u_n) - C_t - A_0(u_n)_t^n\big)\big] \\
&\quad = O_p(\Delta_n^{1/2}) - o_p(\Delta_n^{1/2}).
\end{aligned} \tag{2.7}$$

The second term in (2.7) is $o_p(\Delta_n^{1/2})$ since $\hat{C}_0(2u)$ and $\hat{C}_0(u)$ are calculated in the same way, except for using different arguments, and are asymptotically perfectly correlated as $u = u_n \to 0$; see also (a) in Theorem 1 of Jacod and Todorov (2014). However, the first term in (2.7) is $O_p(\Delta_n^{1/2})$ since $\hat{C}_1(u_n)$ uses the data points one grid after those in $\hat{C}_0(2u_n)$, which decreases the overlap of the data and hence has lower dependency between the terms with argument $2u_n$ and u_n; see Theorem 3.2 below.

Although $T_n^*/\Delta_n^{1/2}$ is tight under H_0, it can be close to zero with a large probability under H_1 since the signal in the numerator is swept away in the bias correction. This causes difficulty in successfully detecting pure-jump processes under H_1 and hence results in a low power. This difficulty can be remedied by adding a bias of order $o(\Delta_n^{1/2})$ onto the numerator of T_n^*.

Our final test statistic is

$$T_n = \frac{\hat{C}_0(u_n) - \hat{C}_1(u_n) - \gamma_n \Delta_n^{1/2}}{\hat{C}_1(u_n)}, \tag{2.8}$$

where γ_n is some chosen constant satisfying $\gamma_n \to 0$ of which the explicit form will be given in Section 3.3. It can be shown that

$$T_n/\Delta_n^{1/2} \begin{cases} = O_p(1), & \text{on } \{C_T > 0\}, \\ \to^P -\infty & \text{on } \{C_T = 0\}. \end{cases} \tag{2.9}$$

This means that $T_n/\Delta_n^{1/2}$ can be used to differentiate H_0 and H_1.

3. Main results.

3.1. *Model assumptions.*

We need the following assumptions.

ASSUMPTION 3.1.

$$X_t^d = \int_0^t \gamma_{s-}^+ \, dY_s^+ + \int_0^t \gamma_{s-}^- \, dY_s^- + \int_0^t \int_R \delta(s,z) p(ds,dz),$$

where Y^+ and Y^- are two independent Lévy processes with positive jumps and Lévy triplet equal to $(0, 0, F^{\pm})$, $\gamma^{\pm}$ are two càdlàg adapted processes and p is a Poisson random measure on $R_+ \times R$ with intensity $q(dt, dx) = dt \otimes dx$. We assume further that, for some $\beta > 1 > r$, the Lévy measure satisfies

$$\left|\overline{F}^{\pm}(x) - \frac{1}{x^\beta}\right| = \left|F^{\pm}\big((x,\infty)\big) - \frac{1}{x^\beta}\right| \le g(x), \qquad x \in (0,1],$$

with $g(x)$ a decreasing function s.t. $\int_0^1 x^{r-1} g(x)\,dx < \infty$, and $|\delta(t,x)|^r \wedge 1 \le J(x)$ with $J(x)$ Lebesgue integrable on R.

ASSUMPTION 3.2. σ_t is an Itô semimartingale of the form

$$\begin{aligned}\sigma_t = \sigma_0 &+ \int_0^t b_s^\sigma\,ds + \int_0^t H_s^\sigma\,dW_s + \int_0^t H_s'^\sigma\,dW_s' \\ &+ \int_0^t \int_{\{|\delta^\sigma(s,x)|\le 1\}} \delta^\sigma(s,x)(p-q)(ds,dx) \\ &+ \int_0^t \int_{\{|\delta^\sigma(s,x)|> 1\}} \delta^\sigma(s,x)p(ds,dx),\end{aligned}$$

where all the integrands are optional processes satisfying the integrable condition in Itô's sense, and q is the compensator of p. Assume that W and W' are two independent Brownian motions that are further independent of (p, Y^+, Y^-).

ASSUMPTION 3.3. We have a sequence τ_n of stopping times increasing to infinity, a sequence a_n of numbers and a nonnegative Lebsgue-integrable function J on R, such that the processes b, H^σ, $\gamma^\pm$ are càdlàg adapted, the coefficients δ, δ^σ are predictable, the processes b^σ, H'^σ are progressively measurable and

$$\begin{aligned}&t < \tau_n \Rightarrow |\delta(t,z)|^r \wedge 1 \le a_n J(z),\ |\delta^\sigma(t,z)|^2 \wedge 1 \le a_n J(z),\\ &t < \tau_n,\ V = b, b^\sigma, H^\sigma, H'^\sigma, \gamma^\pm \Rightarrow |V_t| \le a_n,\\ &V = b, H^\sigma, \gamma^\pm \Rightarrow |E(V_{(t+s)\wedge\tau_n} - V_{t\wedge\tau_n}|\mathcal{F}_t)| + E(|V_{(t+s)\wedge\tau_n} - V_{t\wedge\tau_n}|^2|\mathcal{F}_t) \le a_n s.\end{aligned}$$

Assumption 3.1 is the same as the Assumption (A) given in Jacod and Todorov (2014). It essentially states that X^d can be decomposed into two components: active and less active jumps. Here, the first two components are the stable-like jumps assumed to have the jump activity index $\beta > 1$. (This can be extended to cover the case for $r < \beta \le 1$ with extra efforts and possibly more stringent conditions. However, if we have a priori that $\beta < 1$, more straightforward tests will be possible.) Another reason we restrict attention to $\beta > 1$ is because this is more interesting and challenging statistically. The last term consists of jumps with finite variation (but possibly of infinite activity) which is expected to disappear in a limiting sense as inspired by the finding following (2.1). In Aït-Sahalia and Jacod (2010), it is assumed that $\beta < 1$ since otherwise no asymptotic distribution theory could be used under H_0 to calculate the rejection region.

Assumption 3.2 is a standard assumption in the literature which allows for the "leverage" effect due to the common driving forces in X and σ. In Assumption 3.2, the jumps of σ_t are assumed, without restriction, to be driven by the same Poisson measure as X.

Assumption 3.3 is the same as the Assumption (B) in Jacod and Todorov (2014) and a rather general assumption which is satisfied by the multifactor stochastic volatility models that are widely used in financial econometrics, for example, the popular affine jump diffusion models in Duffie, Pan and Singleton (2000). Assumptions 3.2 and 3.3 admit a rather general Itô semimartingale as the continuous part under H_0. As a comparison, Jing, Kong and Liu (2012) require that the volatility be of form $\sigma(X_t)$, a smooth function of X_t bounded away from 0. Hence our assumptions on the continuous component is far less restrictive than that in Jing, Kong and Liu (2012).

3.2. *Main theorems.* We first state a central limit theorem for the joint distribution of $(\hat{C}_0(u_n), \hat{C}_1(u_n))$.

THEOREM 3.1. *Suppose k_n, u_n, γ_n and Δ_n satisfy*

$$k_n\Delta_n^{1/2} \to 0, \qquad k_n\Delta_n^{1/2-\varepsilon} \to \infty, \qquad u_n \to 0, \qquad \sup_n \frac{k_n\Delta_n^{1/2}}{u_n^4} < \infty, \tag{3.1}$$
$$\gamma_n \to 0,$$

for any $\varepsilon > 0$. Let $c_s = \sigma_s^2$. Then on the set $\{C_t > 0\}$ we have

$$\begin{aligned} &\frac{1}{\Delta_n^{1/2}}\begin{pmatrix} \hat{C}_0(u_n) - A_0(u_n)_t^n - C_t \\ \hat{C}_1(u_n) - A_0(u_n)_t^n - C_t \end{pmatrix} \\ &\quad \to^{\mathcal{L}_s} 2\begin{pmatrix} \int_0^t c_s\, d\tilde{W}_s \\ \int_0^t c_s d\Big(\frac{1}{2}\tilde{W}_s + \sqrt{3}/2\tilde{W}^\perp\Big) \end{pmatrix}, \end{aligned} \tag{3.2}$$

where $\tilde{W}$ and $\tilde{W}^\perp$ are two mutually independent standard Brownian motions defined on an extension of the original probability space and are further independent of $\mathcal{F}$, and $\mathcal{L}_s$ stands for stable convergence.

In Theorem 1 of Jacod and Todorov (2014), a similar multivariate central limit theorem related to the bias corrected estimator of C_t in (2.4) with distinct arguments was obtained. While in (3.10) and (3.11) of Theorem 1 of their paper, the vector of component estimators with distinct multiples of u_n are formed by using the same way of aggregating the high-frequency data, Theorem 3.1 in our paper considers a bivariate central limit theorem for $(\hat{C}_0(u_n), \hat{C}_1(u_n))$, with $\hat{C}_0(u_n)$ collecting the high-frequency data one lag after $\hat{C}_1(u_n)$. By simple application of Theorem 3.1 and the continuous mapping theorem, we soon have the following null distribution of T_n.

THEOREM 3.2. *Under the conditions in Theorem 3.1, we have in restriction to $\{C_t > 0\}$,*

$$\Delta_n^{-1/2}T_n \to^{\mathcal{L}_s} G_t,$$

where G_t is a centered Gaussian process with conditional variance $\kappa_t = \frac{4\int_0^t c_t^2\,dt}{C_t^2}$.

It follows from Theorem 3.2 that the convergence rate of T_n is of order $\Delta_n^{1/2}$, in contrast to $\Delta_n^{3/4-\varpi/2}$ in Jing, Kong and Liu (2012), where $\varpi > \beta - 1/2$ is some constant (practically ϖ is taken as $3/2$ since β is usually unknown) or $v_n^{\beta'/2}$ in Aït-Sahalia and Jacod (2010), where $\beta' < 1$ and v_n satisfies

$$v_n/\Delta_n^{\rho-} \to 0, \qquad v_n/\Delta_n^{\rho+} \to \infty, \qquad 0 < \rho- < \rho+ < 1/2.$$

Theorem 3.2 is not directly applicable in determining the rejection region since the conditional variance is unknown. The denominator of the conditional variance can be consistently estimated by $(\hat{C}_1(u_n))^2$, thanks to (2.5). Inspired by the construction of $\hat{C}_k(u)$ $(k = 0, 1)$, we use the following linear combination of sample variances to estimate the integral in the numerator of κ_T. Define

$$\hat{I}_n \equiv \tfrac{1}{2}(\hat{I}_{n,0} + \hat{I}_{n,1}), \tag{3.3}$$

where

$$\hat{I}_{n,k} = 2v_n \sum_{j=0}^{[t/(2v_n)]-1} \left(c_j^k(u_n) - \frac{(\sinh(u_n^2 c_j^k(u_n)))}{u_n^2(k_n - 1)} \right)^2, \qquad k = 0, 1. \tag{3.4}$$

Now we have the following studentized central limit theorem.

THEOREM 3.3. *Let $\hat{\kappa}_T = 4\hat{I}_n/(\hat{C}_1(u_n))^2$. Then we have under the conditions in Theorem 3.1, in restriction to $\{C_T > 0\}$,*

$$\mathcal{T}_n \equiv \frac{1}{\Delta_n^{1/2}} \frac{T_n}{\hat{\kappa}_T^{1/2}} \equiv \frac{\hat{C}_0(u_n) - \hat{C}_1(u_n) - \gamma_n \Delta_n^{1/2}}{2\hat{I}_n^{1/2}\Delta_n^{1/2}} \to^{\mathcal{L}_s} \mathcal{N}(0, 1), \tag{3.5}$$

where $\mathcal{N}(0, 1)$ is a standard normal random variable independent of $\mathcal{F}$.

From Theorem 3.3, we can reject H_0 if $\mathcal{T}_n < -z_\alpha$ where $P(\mathcal{N}(0, 1) > z_\alpha) = \alpha$ for $\alpha \in (0, 1)$. Now we state a result on the convergence rate of $\mathcal{T}_n$ under H_1.

THEOREM 3.4. *Suppose Assumptions 3.1 and 3.3 hold, $k_n\Delta_n^{1/2} \to 0$, $k_n\Delta_n^{1/2-\varepsilon} \to \infty$ for any $\varepsilon > 0$, $\sup_n k_n\Delta_n^{1/2}/u_n^4 < \infty$ and u_n is bounded. Then on the set $\{C_t = 0, \int_0^t a_s\,ds \neq 0\}$, we have*

$$\hat{C}_0(u_n) - \hat{C}_1(u_n) = O_p\big(u_n^{-2}\Delta_n^{1-\beta/(2(\beta+1-r))} + u_n^{\beta/2-2}\Delta_n^{1-\beta/4}\big) \tag{3.6}$$

and

$$\hat{I}_n = 4u_n^{2\beta-4}\Delta_n^{2(1-\beta/2)} \int_0^t a_s^2\,ds + o_p\big(u_n^{2\beta-4}\Delta_n^{2(1-\beta/2)}\big). \tag{3.7}$$

The following result concerning the size and power performance of the test is a straightforward consequence of Theorems 3.3 and 3.4.

COROLLARY 3.1. (1) *Under the conditions in Theorem* 3.1, *we have* $P(\mathcal{T}_n < -z_\alpha|\{C_T \neq 0\}) \to \alpha$;

(2) *under the conditions in Theorem* 3.4, *if*

$$\gamma_n(u_n^2 \Delta_n^{\beta/(2(\beta+1-r))-1/2} + u_n^{2-\beta/2} \Delta_n^{\beta/4-1/2}) \to \infty,$$

we have $P(\mathcal{T}_n < -z_\alpha|C_T = 0, \int_0^T a_s\, ds \neq 0) \to 1$.

REMARK 3.1. Corollary 3.1 shows that our new test achieves asymptotic nominal level α and the asymptotic power 1. It follows from the proof of Corollary 3.1 that $\mathcal{T}_n$ goes to $-\infty$ with rate $O_p(\gamma_n(\frac{u_n^2}{\Delta_n})^{(2-\beta)/2})$ under H_1 and conditions in 2. Thus the test becomes more powerful as β gets closer to 0, which will be further confirmed by our simulation studies. This overcomes the drawbacks of the test by Aït-Sahalia and Jacod (2010).

3.3. *Choice of tuning parameters*. We now study how to choose tuning parameters k_n, u_n and γ_n. The major role of k_n is to balance the bias and variance of $\hat{C}_0(u_n) - C_t$ and $\hat{C}_1(u_n) - C_t$. The larger the k_n, the smaller the bias and the larger the variance. Hence we could choose $k_n = -c'\Delta_n^{1/2} \log \Delta_n$ for some constant $c' > 0$.

Now we turn to u_n. The rationale for letting $u_n \to 0$ under H_0 is to guarantee the convergence in probability in (A.13). As in Jacod and Todorov (2014), we choose u_n so that $u_n^2 \int_0^T c_s\, ds \to 0$ by setting $u_n = c(\log(1/\Delta_n))^{-1/30} {BV_T}^{-1/2}$, where $BV_T = (\pi/2) \sum_{i=1}^{n-1} |\Delta_i^n X\| \Delta_{i+1}^n X|$ is the bipower variation, which is a consistent estimator of $\int_0^T c_s\, ds$. Another advantage of such choice of u_n is that it would be enlarged under H_1, which in turn increases the power as is seen from Corollary 3.1 and Remark 3.1. Choosing an optimal c is quite hard. In order not to incur much approximation error in (A.13), we suggest to choose small c when n is moderate, say $c = 0.18$. Simulation studies where the data is generated from a fitted model (no guarantee of good fitting accuracy) assuming H_0 given in Jacod and Todorov (2014) show that choosing c around 0.18 would work well.

Finally, we look at γ_n. On the one hand, γ_n should be close to 0 under H_0 in order not to produce a big bias for $\mathcal{T}_n$; on the other hand, γ_n should converge to 0 with a rate of $u_n^{-2}\Delta_n^{1/2-\beta/2(\beta+1-r)} + u_n^{\beta/2-2}\Delta_n^{1/2-\beta/4}$ so that the test has good power. This is easily achieved by setting $\gamma_n = c^*/\log(u_n^2/\Delta_n)$ when u_n is determined by the aforementioned method. To be conservative, one can choose small c^* when n is moderate, say $c^* = 0.2$.

TABLE 1
Empirical sizes and the empirical powers of the new test; the nominal level is 5%; $(n = 1170, k_n = 50)$; $(n = 2340, k_n = 78)$; $(n = 4680, k_n = 100)$

	Empirical sizes			**Empirical power**		
β	$n = 1170$	$n = 2340$	$n = 4680$	$n = 1170$	$n = 2340$	$n = 4680$
1.0	0.0610	0.0586	0.0574	0.9988	0.9998	1.0000
1.1	0.0616	0.0624	0.0610	0.9984	0.9990	1.0000
1.2	0.0640	0.0635	0.0634	0.9936	0.9986	0.9996
1.3	0.0604	0.0601	0.0608	0.9596	0.9948	0.9986
1.4	0.0522	0.0616	0.0616	0.6508	0.8414	0.9650
1.5	0.0566	0.0624	0.0610	0.2902	0.3810	0.5290
1.6	0.0612	0.0514	0.0524	0.1328	0.1698	0.2138
1.7	0.0594	0.0624	0.0554	0.0942	0.1068	0.1208
1.8	0.0578	0.0550	0.0594	0.0776	0.0804	0.0804
1.9	0.0572	0.0568	0.0558	0.0748	0.0790	0.0728

4. Numerical experiments.

4.1. *Simulation studies.* In this section, we conduct simulation studies to check the performance of the new test and make comparisons with the test given in Jing, Kong and Liu (2012). We first consider the performance on control of type I error probability. As in Jacod and Todorov (2014), we generate simulation data for 5000 times from the following stochastic volatility model:

$$X_t = X_0 + \int_0^t \sqrt{c_s}\, dW_s + 0.5 Y_t, \qquad 0 \le t \le T, \tag{4.1}$$

$$c_t = c_0 + \int_0^t 0.03(1.0 - c_s)\, ds + 0.15 \int_0^t \sqrt{c_s}\, dW'_s, \tag{4.2}$$

for $0 \le t \le 3T/4$ and $c_t \equiv 0$ if $3T/4 \le t \le T$. In order to incorporate the leverage effect, we set $\mathrm{corr}(dW, dW') = -0.5$. The parameters in the volatility dynamic are specified by fitting actual financial data in the same reference paper. The volatility c_t is a square root diffusion process which is widely used in financial applications. We tuned k_n, u_n and γ_n as in Section 3.3 with $c = 0.18$ and $c^* = 0.2$. We consider $n = 1170, 2340, 4680$ which corresponds to sample the data per 20, 10, 5 seconds, respectively. In the simulation, we let T be one day consisting of 6.5 trading hours.

Table 1 displays the empirical sizes of the new test. Clearly, they are slightly higher than the nominal level but acceptable across the board due to the small bias added artificially. Figure 1 gives the QQ-plot of the test statistics for $n = 2340$ and $\beta = 1.2, 1.5$, showing that the normal approximation works well.

For comparison, we choose $\beta = 1.2$ and $n = 2340$ and carry out the test given in Jing, Kong and Liu (2012), referred to as JKL's test below. No comparisons will be made with the test given in Aït-Sahalia and Jacod (2010) (AJ's test), since it was

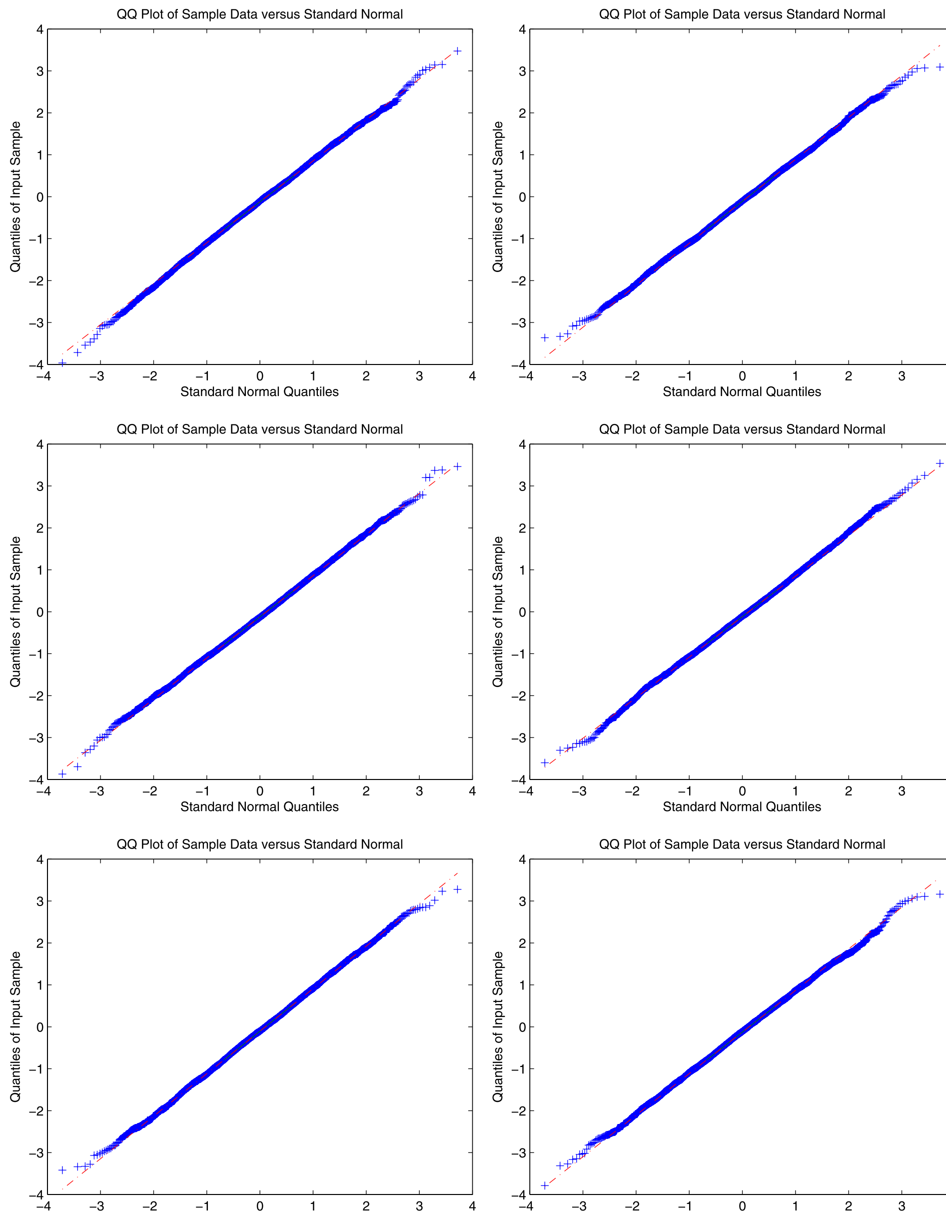

FIG. 1. *QQ-plot of the test statistics under* H_0 *for* $\beta = 1.2$ (*left panels*), 1.5 (*right panels*); *from top to bottom*, $c = 0.15, 0.18, 0.2$; $n = 2340$.

outperformed by the JKL's test in extensive simulation studies given in Jing, Kong and Liu (2012). Table 2 lists the empirical sizes of JKL's test where δ^* is a tuning parameter determining how many small increments are used to compute the test

TABLE 2

Empirical sizes of JKL's test; $\beta = 1.2$, $n = 2340$; *the nominal level is* 5%; *Empirical sizes** *stand for the empirical sizes when* c_s *follows the same square root process for* $3/4T \le t \le T$

δ*	0.50	0.75	1.00	1.25	1.50	1.75
Empirical sizes	0.3032	0.4442	0.5916	0.7312	0.8532	0.9402
Empirical sizes*	0.0298	0.0400	0.0358	0.0406	0.0402	0.0436

statistics. Clearly, the JKL's test is too liberal since the type I error probabilities are out of control, showing that the JKL's test fails when the continuous process vanishes in some subintervals. The reason for the failure is that the JKL's test statistic has a nonnegligible bias, even for large enough n.

It seems that choosing δ^* small would have satisfactory control of type I error. However, when δ^* is small, the normal approximation is actually no longer reliable. For $\delta^* = 0.05$, there are roughly 5 small increments (effective data) used in calculation of the test statistics, which affects the accuracy of the normal approximation. Figure 2 gives the QQ-plot for the test statistics given in Jing, Kong and Liu (2012) for $\delta^* = 0.05$ (left panel), 0.5 (right panel) when $\beta = 1.2$ and $n = 2340$. From the left panel, we see a clear concavity pattern, which implies that the distribution of the test statistic is left-skewed, yet the empirical size is 0.07. Apparent improvement in skewness could be seen in the right panel for $\delta^* = 0.5$ since more effective data (roughly 40) were added in calculation of the test statistics. However, we see a clear bias in the QQ-plot.

Next we investigate the power of the new test. We generate the data for 5000 times from the above model, except that $c_s \equiv 0$. The empirical powers for various β values are given in Table 1. We make the following observations:

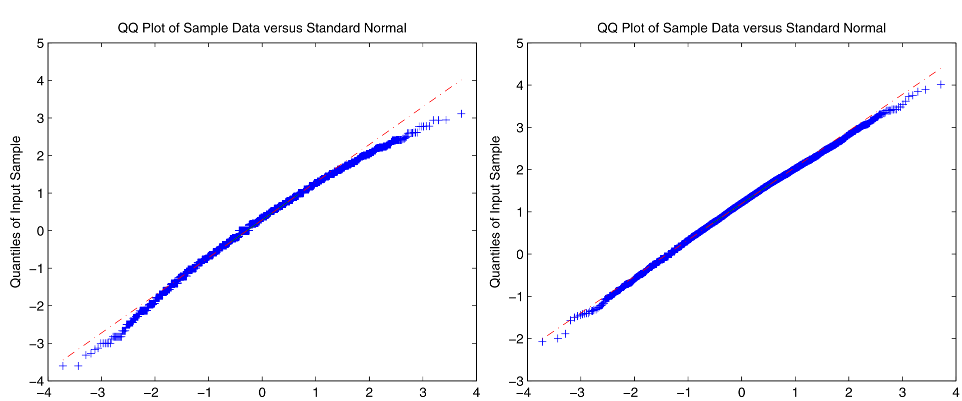

FIG. 2. *QQ-plot of the test statistics given in Jing et al.* (2012) *for* $\delta^* = 0.05$ (*left panel*), 0.5 (*right panel*) *under* H_0 *when* $\beta = 1.2$; $n = 2340$.

TABLE 3
Empirical sizes and the empirical powers of the new test for different pairs of (c, k_n); *the nominal level is* 5%; n=2340

	Empirical sizes			**Empirical power**		
β	**(0.15, 50)**	**(0.15, 78)**	**(0.2, 78)**	**(0.15, 50)**	**(0.15, 78)**	**(0.2, 78)**
1.0	0.0630	0.0604	0.0634	0.9986	0.9994	0.9992
1.1	0.0604	0.0604	0.0608	0.9984	0.9986	0.9992
1.2	0.0634	0.0618	0.0624	0.9970	0.9982	0.9982
1.3	0.0558	0.0592	0.0638	0.9842	0.9896	0.9968
1.4	0.0580	0.0562	0.0618	0.7432	0.7708	0.8786
1.5	0.0584	0.0560	0.0614	0.3148	0.3242	0.4146
1.6	0.0576	0.0613	0.0608	0.1670	0.1498	0.1814
1.7	0.0558	0.0496	0.0568	0.0908	0.0906	0.1102
1.8	0.0558	0.0540	0.0582	0.0780	0.0778	0.0788
1.9	0.0542	0.0544	0.0566	0.0702	0.0702	0.0744

- the power of the new test decreases as β increases since, as β increases to 2, the pure-jump process fluctuates more like a Brownian motion;
- as the sample size increases, the empirical power increases overall, as can be expected.

We also did a sensitivity study to k_n when it is chosen in the proposed range. In the sensitivity study we take $c^* = 0.2$ and $k_n = 50, 78$, $c = 0.15$ or 0.2 when $n = 2340$. The results on both the size and power performance are reported in Table 3, where we can see that the empirical sizes and power do not change much. We also conducted other sensitivity studies for $c \approx 0.18$ and $n = 1170$ with k_n in the corresponding range and reached similar conclusions (hence not presented here).

4.2. *Real data analysis*. In this section, we implement our test on some real data sets. We first investigate the stock price records of Microsoft (MFST) in two trading days, December 1, and 12, 2000, which were also included in Jing, Kong and Liu (2012). All data sets are from the TAQ database. As in Jing, Kong and Liu (2012), to weaken the possible effect from microstructure noise, we sample observations every 1/3 minutes. Finally, we use logarithms of the sampled prices to calculate the test statistics.

We set $T = 1$ (day) consisting of 6.5 hours of trading time. As in the simulation studies, we set $k_n = 50$ and $\gamma_n = 0.2/\log(u_n^2/\Delta_n)$. To be on the safe side, let u_n take values in the grid points in (0, 1] with step length equal to 0.01. Figure 3 plots the test statistics against u_n for two data sets. We see from the figure that for all configurations of u_n, the test statistics are far lower than -1.645, hence providing significant evidence against the existence of a Brownian force. This confirms the

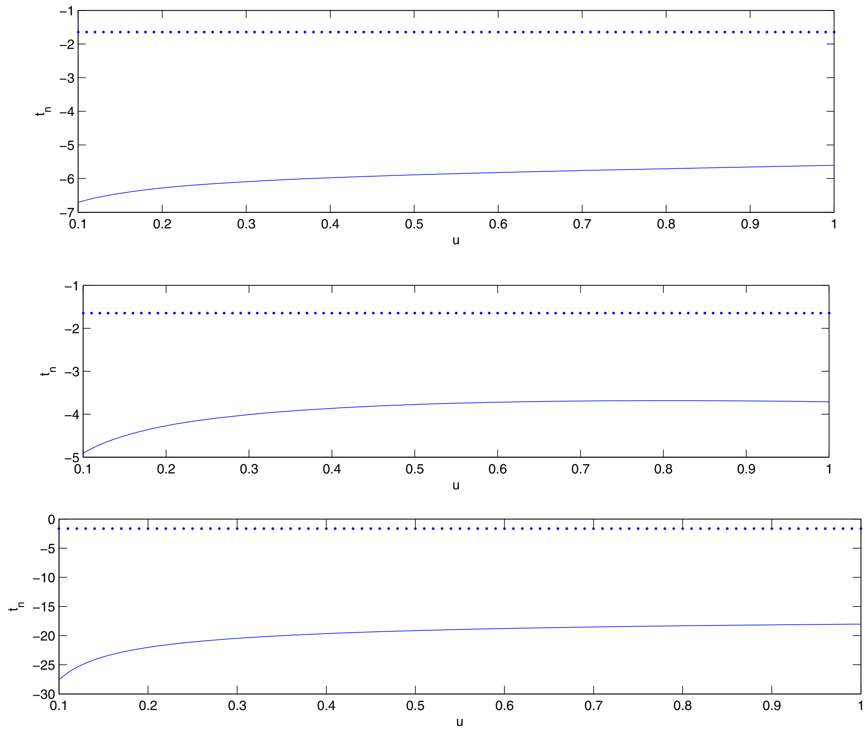

FIG. 3. *Observed test statistics for the trading date*, *December* 1 (*middle panel*) *and December* 12 (*upper panel*), *in* 2000, *and the* 5-*mins S&P* 500 *index* (*lower panel*) *data during January* 4–29, 2010. *The horizontal line has level* -1.645.

empirical results in Jing, Kong and Liu (2012) and in the meantime rules out the possibility that Brownian force exists in some subintervals.

Next we implement our test the S&P 500 index data which are sampled every 5 minutes during January 4–29, 2010. The tuning parameters are used as given above for those two stock data. The observed test statistics are plotted against u in the lower panel of Figure 3. We obtain the same conclusion that during the specified time period, the underlying log price should be modeled by a pure-jump process.

5. Conclusion and discussion. In this paper, we have developed a new test based on the realized characteristic function to check whether the underlying process of a high frequency data set can be modeled as a pure-jump process, and shown its advantages over existing tests. Here are some future problems worth pursing in future research work:

- The effect of the microstructure noise, in the testing problem (1.2) or even in estimating the functionals of the volatility, is unclear and worthy of investigation in both theory and practice. Here we could explore the two-time-scale technique or multi-time-scale technique [Aït-Sahalia, Mykland and Zhang (2005), Zhang (2006)] or the pre-averaging approach [Jacod et al. (2009)].

- In the present paper, our inference is with the price process. It is of interest to make inference on the volatility process which, as recommended in Todorov and Tauchen (2014), could be modeled by a pure-jump process. The challenge of this problem is that the volatility process is unobservable. Studies on this topic is still undergoing.

APPENDIX: PROOFS OF MAIN THEOREMS

This appendix contains the proofs of main theorems. The proofs of Lemmas A.4–A.6 as well as some interesting supplemental lemmas are given in Kong, Liu and Jing (2015), a supplementary material [Kong, Liu and Jing (2015)] to this paper that is not for purpose of publication. By the standard localization procedure, it is enough to prove the main results under the following strengthened assumption.

ASSUMPTION A.1. $b, \sigma, \gamma^+, \gamma^-, b^\sigma, H^\sigma$ and H'^σ are bounded.

Before we prove the theorems, we introduce some notation and give an outline of our proof. Let $U_t(u) = \exp(-u^2 c_t - 2\Delta_n^{1-\beta/2} u^\beta a_t)$ where $a_t = \chi(\beta)(|\gamma_t^+|^\beta + |\gamma_t^-|^\beta)$ with $\chi(\beta) = \int_0^\infty y^{-\beta}\sin(y)\,dy$. For ease of notation, $U_j(u) \equiv U_{2jv_n}(u)$ and sometimes we write $E_{\mathcal{F}_t} V_s = E(V_s|\mathcal{F}_t)$ for a stochastic process V_t. Let $\xi_{k,j}(u) = L_j^k(u)/U_j(u) - 1$, $k = 0, 1$. Let $\Omega(k, n, t) = \{\omega, \max_{k,j} |\xi_{k,j}(u, \omega)| \le 1/2\}$. By Lemma 7 of Jacod and Todorov (2014),

$$P(\Omega^c(k, n, t)) \to 0, \tag{A.1}$$

irrespective of whether the continuous component exists or not.

A.1. Proof of results under H_0. Assuming the continuous local martingale exists, our proof depends heavily on the following decomposition:

$$c_j^k(u) = c_{2jv_n} + 2u^{\beta-2}\Delta_n^{1-\beta/2} a_{2jv_n} - \frac{1}{u^2}\xi_{k,j}(u) + \frac{1}{2u^2}\xi_{k,j}^2(u) + r_{k,j}(u), \tag{A.2}$$

where $r_{k,j}(u)$ represents the remaining term which will be shown to be negligible. By summing up the terms in (A.2) over j, one soon has

$$\begin{aligned} \hat{C}_k(u) = & \sum_{j=0}^{[t/(2v_n)]-1} 2v_n c_{2jv_n} + \sum_{j=0}^{[t/(2v_n)]-1} 2v_n\left(2u^{\beta-2}\Delta_n^{1-\beta/2} a_{2jv_n}\right) \\ & - \sum_{j=0}^{[t/(2v_n)]-1} 2v_n \xi_{k,j}(u)/u^2 \\ & + \sum_{j=0}^{[t/(2v_n)]-1} 2v_n\left(\frac{\xi_{k,j}^2(u)}{2u^2} - \frac{1}{(k_n-1)u^2}\left(\sinh\left(u^2 c_j^k(u)\right)\right)^2\right) + R_k(u). \end{aligned} \tag{A.3}$$

We will first show that the first and second term converge to some limits, and the fourth and last term in (A.3) are $o_p(\Delta_n^{1/2})$, while the third term is $O_p(\Delta_n^{1/2})$ and converges to a conditionally centered Gaussian random variable stably. This proves the univariate central limit theorem in Theorem 3.1. After that we proceed with the proof of the bivariate central limit theorem by investigation into the covariation of those two marginal sequences, which ends up with Theorem 3.1. Theorem 3.2 is a consequence of Theorem 3.1 and the continuous mapping theorem. Theorem 3.3 can be proved by showing that $\hat{\kappa}_T$ is consistent to κ_T. In the sequel, K will be a constant that has different values at different appearances.

We now cite three lemmas from Jacod and Todorov (2014), whose proof can be found in the same reference paper. Lemma A.1 is concerned with the first and second term in (A.3), that is, the discretization error terms. Lemma A.2 gives the stochastic order of $\xi_{k,j}(u_n)$, $k = 0, 1$, while Lemma A.3 shows that the fourth term and the remainder term in (A.3) are asymptotically negligible.

LEMMA A.1 [Lemma 8 in Jacod and Todorov (2014)]. *Under Assumptions 3.1–A.1 and assuming (3.1), we have*

$$\sum_{j=0}^{[t/(2v_n)]-1} 2v_n c_{2jv_n} - \int_0^t c_s\,ds = o_p\big(u_n^2 \Delta_n^{1/2}\big), \tag{A.4}$$

$$\sum_{j=0}^{[t/(2v_n)]-1} 2v_n\big(2u_n^{\beta-2}\Delta_n^{1-\beta/2} a_{2jv_n}\big) - A_0(u_n)_t^n = o_p\big(u_n^2 \Delta_n^{1/2}\big). \tag{A.5}$$

LEMMA A.2 [Lemma 14 in Jacod and Todorov (2014)]. *Under Assumptions 3.1–A.1 and assuming (3.1), we have, for $k = 0, 1$,*

$$\big|E_{\mathcal{F}_{2jv_n}} \xi_{k,j}(u_n)\big| \le K u_n^4 \Delta_n^{1/2} \phi_n, \tag{A.6}$$

$$\left| E_{\mathcal{F}_{2jv_n}} \xi_{k,j}^2(u_n) - \frac{U_j(2u_n) + U_j(0) - 2U_j^2(u_n)}{2(k_n - 1)U_j^2(u_n)} \right| \le K u_n^4 \Delta_n^{1/2} \phi_n, \tag{A.7}$$

and for $q \ge 2$,

$$E_{\mathcal{F}_{2jv_n}} \big|\xi_{k,j}(u_n)\big|^q \le K\big(u_n^{2q}/k_n^{q/2} + u_n^4 v_n\big), \tag{A.8}$$

where ϕ_n is some sequence of numbers converging to 0.

LEMMA A.3 [Lemma 9 in Jacod and Todorov (2014)]. *Under Assumptions 3.1–A.1 and assuming (3.1), we have $R_k(u) = o_p(u_n^2 \Delta_n^{1/2})$ $k = 0, 1$ and*

$$\sum_{j=0}^{[t/(2v_n)]-1} 2v_n \left(\frac{\xi_{k,j}^2(u_n)}{2u_n^2} - \frac{1}{(k_n - 1)u_n^2} \big(\sinh\big(u_n^2 c_j^k(u_n)\big)\big)^2 \right) = o_p\big(u_n^2 \Delta_n^{1/2}\big). \tag{A.9}$$

The following lemma provides a formula for the limit of the conditional real part of the characteristic function of a linear combination of three successive increments. The proof can be found in the supplementary material [Kong, Liu and Jing (2015)] to this paper.

LEMMA A.4. *Let* $u_n^* = |a_{n,0}| \vee |a_{n,1}| \vee |a_{n,2}|$, *under Assumptions* 3.1–A.1, *and assume* (3.1) *with* u_n^* *replacing* u_n, *so we have*

$$\begin{aligned}
\Bigg| & E_{\mathcal{F}_{(i-1)\Delta_n}} \cos\left(\sum_{l=0}^{2} a_{n,l} \frac{\Delta_{i+l}^n X}{\Delta_n^{1/2}}\right) \\
& - \exp\left(-\frac{1}{2}\sigma_{(i-1)\Delta_n}^2 \sum_{l=0}^{2} a_{n,l}^2 \right. \\
& \left. + \Delta_n^{1-\beta/2} \chi(\beta) \sum_{l=0}^{2} \left(|a_{n,l}\gamma_{(i-1)\Delta_n}^{+}|^{\beta} + |a_{n,l}\gamma_{(i-1)\Delta_n}^{-}|^{\beta}\right)\right) \qquad \text{(A.10)} \\
& \times \cos\left(\Delta_n^{1-\beta/2} \chi'(\beta) \sum_{l=0}^{2} \left(\{a_{n,l}\gamma_{(i-1)\Delta_n}^{+}\}^{\beta} + \{a_{n,l}\gamma_{(i-1)\Delta_n}^{-}\}^{\beta}\right)\right)\Bigg| \\
& \leq K u_n^{*4} \Delta_n^{1/2} \phi_n,
\end{aligned}$$

where $\{x\}^{\beta} = \operatorname{sign}(x)|x|^{\beta}$ *and* $\chi'(\beta) = \int_0^{\infty} \frac{1-\cos(y)}{y^{\beta}}\, dy$.

PROOF OF THEOREM 3.1. By Lemmas A.1, A.3 and (A.3), it suffices to prove that

$$\frac{1}{\Delta_n^{1/2}}\left(\sum_{j=0}^{[t/(2v_n)]-1} 2v_n \xi_{0,j}(u_n)/u_n^2, \sum_{j=0}^{[t/(2v_n)]-1} 2v_n \xi_{1,j}(u_n)/u_n^2\right)$$

converges to the right-hand side of (3.2) stably. By Lemma A.2, we have

$$\sum_{j=0}^{[t/(2v_n)]-1} 2v_n E\left(\xi_{k,j}(u_n)/u_n^2 | \mathcal{F}_{2jv_n}\right) = o_p\left(u_n^2 \Delta_n^{1/2}\right), \qquad k = 0, 1.$$

Hence it is enough to prove the bivariate central limit theorem with stable convergence for the following centered discrete bivariate martingale with respect to $(\mathcal{F}_{2jv_n})_{j=0}^{[t/(2v_n)]-1}$:

$$\begin{aligned}
& \frac{2v_n}{\Delta_n^{1/2}}\left(\sum_{j=0}^{[t/(2v_n)]-1} \left(\xi_{0,j}(u_n)/u_n^2 - E\left(\xi_{0,j}(u_n)/u_n^2 | \mathcal{F}_{2jv_n}\right)\right),\right. \qquad \text{(A.11)} \\
& \left.\sum_{j=0}^{[t/(2v_n)]-1} \left(\xi_{1,j}(u_n)/u_n^2 - E\left(\xi_{1,j}(u_n)/u_n^2 | \mathcal{F}_{2jv_n}\right)\right)\right).
\end{aligned}$$

Let

$$\chi_j^{n,0} = \frac{2v_n}{\Delta_n^{1/2}}\big(\xi_{0,j}(u_n)/u_n^2 - E\big(\xi_{0,j}(u_n)/u_n^2|\mathcal{F}_{2jv_n}\big)\big),$$

$$\chi_j^{n,1} = \frac{2v_n}{\Delta_n^{1/2}}\big(\xi_{1,j}(u_n)/u_n^2 - E\big(\xi_{1,j}(u_n)/u_n^2|\mathcal{F}_{2jv_n}\big)\big).$$

By Theorem 7.28 in Chapter IX of Jacod and Shiyayev (2003), we only need to prove that

$$\text{(A.12)}\quad \begin{cases} \sup_t\left|\sum_{j=0}^{[t/(2v_n)]-1} E\big(\chi_j^{n,k}|\mathcal{F}_{2jv_n}\big)\right| \to^P 0; & k=0,1,\\ \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}}\big(\chi_j^{n,k}\big)^2 \to^P 4\int_0^t c_s^2\,ds; & k=0,1,\\ \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}}\big(\chi_j^{n,0}\chi_j^{n,1}\big) \to^P 2\int_0^t c_s^2\,ds; & \\ \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}}\big(\chi_j^{n,k}\big)^2 I_{\{|\chi_j^{n,k}|>\varepsilon\}} \to^P 0; & k=0,1,\\ \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}}\big(\chi_j^{n,k}(M_{2(j+1)v_n} - M_{2jv_n})\big) \to^P 0; & k=0,1,\end{cases}$$

for any square-integrable martingale M. The first equation holds automatically since $(\chi_j^{n,k})_{j=0}^{[t/(2v_n)]-1}$ form a sequence of $\mathcal{F}_{2(j+1)v_n}$-martingale differences.

Now we calculate the conditional variances of the marginal sequences. By (3.1), Lemma A.2 and the fact that $|U_j(u_n) - e^{-u_n^2\sigma^2_{(i-1)\Delta_n}}| \le K\Delta_n^{1-\beta/2}u_n^\beta$, we have

$$\begin{aligned} &\frac{4v_n^2}{\Delta_n u_n^4}\sum_{j=0}^{[t/(2v_n)]-1} E\big(\big(\xi_{k,j}(u_n) - E\big(\xi_{k,j}(u_n)|\mathcal{F}_{2jv_n}\big)\big)^2|\mathcal{F}_{2jv_n}\big)\\ &\quad= \frac{4v_n^2}{\Delta_n u_n^4}\sum_{j=0}^{[t/(2v_n)]-1}\big(E\big[\xi_{k,j}^2(u_n)|\mathcal{F}_{2jv_n}\big] - \big(E\big[\xi_{k,j}(u_n)|\mathcal{F}_{2jv_n}\big]\big)^2\big)\\ &\quad= \frac{4v_n^2}{\Delta_n u_n^4}\sum_{j=0}^{[t/(2v_n)]-1} E\big(\xi_{k,j}^2(u_n)|\mathcal{F}_{2jv_n}\big) + o_p(1)\end{aligned}$$

$$\text{(A.13)}\quad = \frac{4v_n^2}{2(k_n-1)\Delta_n u_n^4}\sum_{j=0}^{[t/(2v_n)]-1}\frac{U_j(2u_n) + 1 - 2U_j^2(u_n)}{U_j^2(u_n)} + o_p(1)$$

$$= \frac{(1+o_p(1))}{u_n^4} \sum_{j=0}^{[t/(2v_n)]-1} \big(U_j(2u_n) + 1 - 2U_j^2(u_n)\big)2v_n + o_p(1)$$

$$= \frac{(1+o_p(1))}{u_n^4} \sum_{j=0}^{[t/(2v_n)]-1} \big(e^{-4u_n^2 c_{2jv_n}} + 1 - 2e^{-2u_n^2 c_{2jv_n}}\big)2v_n + o_p(1)$$

$$= \frac{\int_0^t (e^{-4u_n^2 c_s} + 1 - 2e^{-2u_n^2 c_s})\, ds}{u_n^4} + o_p(1) \to^P 4\int_0^t c_s^2\, ds,$$

where in obtaining the convergence in probability, we used the Taylor expansion of e^x when x is near 0. This proves the second equation in (A.12).

Next, we are going to check the third equation in (A.12). By Lemma A.2, we have

$$\begin{aligned}
&\frac{4v_n^2}{\Delta_n u_n^4} \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}}\big(\xi_{0,j}(u_n) - E_{\mathcal{F}_{2jv_n}}\xi_{0,j}(u_n)\big) \\
&\qquad\qquad \times\big(\xi_{1,j}(u_n) - E_{\mathcal{F}_{2jv_n}}\xi_{1,j}(u_n)\big) \\
&\quad = \frac{4v_n^2}{\Delta_n u_n^4} \sum_{j=0}^{[t/(2v_n)]-1} \big(E_{\mathcal{F}_{2jv_n}}\xi_{0,j}(u_n)\xi_{1,j}(u_n) \\
&\qquad\qquad - E_{\mathcal{F}_{2jv_n}}\xi_{0,j}(u_n)E_{\mathcal{F}_{2jv_n}}\xi_{1,j}(u_n)\big) \\
&\quad = \frac{4v_n^2}{\Delta_n u_n^4} \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}}\xi_{0,j}(u_n)\xi_{1,j}(u_n) + o_p(1).
\end{aligned} \tag{A.14}$$

Now we investigate the summand in (A.14). Let

$$\begin{aligned}
\zeta_k(j,l) &= \cos\Big(u_n \frac{\Delta^n_{2jk_n+2l-k+1}X - \Delta^n_{2jk_n+2l-k}X}{\Delta_n^{1/2}}\Big) \\
&\quad - E_{\mathcal{F}_{(2jk_n+2l-k-1)\Delta_n}} \cos\Big(u_n \frac{\Delta^n_{2jk_n+2l-k+1}X - \Delta^n_{2jk_n+2l-k}X}{\Delta_n^{1/2}}\Big)
\end{aligned}$$

and

$$\zeta'_k(j,l) = \cos\Big(u_n \frac{\Delta^n_{2jk_n+2l-k+1}X - \Delta^n_{2jk_n+2l-k}X}{\Delta_n^{1/2}}\Big) - U_j(u_n),$$

$k = 0, 1$. By (6.22) and (6.29) in Jacod and Todorov (2014), we have

$$\begin{aligned}
&\big|\zeta'_k(j,l) - \zeta_k(j,l)\big| \\
&\quad \le K u_n^4 \Delta_n^{1/2}\phi_n + \big|U_{2jv_n+(2l-k-1)\Delta_n}(u_n) - U_j(u_n)\big|,
\end{aligned} \tag{A.15}$$

which, together with Lemma A.2 and the property of $U_t(u_n)$, shows that

$$\begin{aligned}
&\left|E_{\mathcal{F}_{2jv_n}}\xi_{1,j}(u_n)\left(\frac{(1/(k_n-1))\sum_{l=1}^{k_n-1}(\zeta_0'(j,l)-\zeta_0(j,l))}{U_j(u_n)}\right)\right| \\
&\quad\le \sqrt{E_{\mathcal{F}_{2jv_n}}\xi_{1,j}^2(u_n)}\sqrt{E_{\mathcal{F}_{2jv_n}}\left(\frac{(1/(k_n-1))\sum_{l=1}^{k_n-1}|\zeta_0'(j,l)-\zeta_0(j,l)|}{U_j(u_n)}\right)^2} \\
&\quad\le K\frac{u_n^2}{\sqrt{k_n}}\left(u_n^4\Delta_n^{1/2}\phi_n+\sqrt{\max_l E_{\mathcal{F}_{2jv_n}}\left(U_{2jv_n+(2l-1)\Delta_n}-U_j(u_n)\right)^2}\right) \\
&\quad\le K\frac{u_n^4\sqrt{v_n}}{\sqrt{k_n}}.
\end{aligned} \tag{A.16}$$

Similarly, by the property of $U_t(u_n)$, (6.22) and (6.29) in Jacod and Todorov (2014), and Hölder's inequality, we have

$$\begin{aligned}
&\left|E_{\mathcal{F}_{2jv_n}}\frac{\sum_{l=1}^{k_n-1}\zeta_0(j,l)}{(k_n-1)U_j(u_n)}\left(\frac{(1/(k_n-1))\sum_{l=1}^{k_n-1}(\zeta_1'(j,l)-\zeta_1(j,l))}{U_j(u_n)}\right)\right| \\
&\quad\le \sqrt{\frac{\sum_{l=1}^{k_n-1}E_{\mathcal{F}_{2jv_n}}\zeta_0^2(j,l)}{(k_n-1)^2U_j^2(u_n)}} \\
&\qquad\times\sqrt{E_{\mathcal{F}_{2jv_n}}\left(\frac{(1/(k_n-1))\sum_{l=1}^{k_n-1}|\zeta_0'(j,l)-\zeta_0(j,l)|}{U_j(u_n)}\right)^2} \\
&\quad\le K\frac{u_n^4\sqrt{v_n}}{\sqrt{k_n}}.
\end{aligned} \tag{A.17}$$

Equations (A.16) and (A.17) yield

$$\begin{aligned}
&E_{\mathcal{F}_{2jv_n}}\xi_{0,j}(u_n)\xi_{1,j}(u_n) \\
&\quad=E_{\mathcal{F}_{2jv_n}}\frac{(1/(k_n-1))\sum_{l=1}^{k_n-1}\zeta_0(j,l)(1/(k_n-1))\sum_{l=1}^{k_n-1}\zeta_1(j,l)}{U_j(u_n)U_j(u_n)}+r_j,
\end{aligned} \tag{A.18}$$

where r_j satisfies $|r_j|\le K\sqrt{v_n}u_n^4/\sqrt{k_n}$. By the definition of $\zeta_k(j,l)$, we have

$$\begin{aligned}
&E_{\mathcal{F}_{2jv_n}}\frac{(1/(k_n-1))\sum_{l=1}^{k_n-1}\zeta_0(j,l)(1/(k_n-1))\sum_{l=1}^{k_n-1}\zeta_1(j,l)}{U_j(u_n)U_j(u_n)} \\
&\quad=\frac{1}{(k_n-1)^2U_j(u_n)U_j(u_n)}\sum_{l=1}^{k_n-1}E_{\mathcal{F}_{2jv_n}}\zeta_0(j,l)\zeta_1(j,l) \\
&\qquad+\sum_{l=1}^{k_n-2}E_{\mathcal{F}_{2jv_n}}\zeta_0(j,l)\zeta_1(j,l+1).
\end{aligned} \tag{A.19}$$

By Lemmas 11–12 in Jacod and Todorov (2014), we have

$$
\begin{aligned}
&E_{\mathcal{F}_{2jv_n}}\zeta_0(j,l)\zeta_1(j,l)\\
&\quad= E_{\mathcal{F}_{2jv_n}}\zeta_0(j,l)\cos\left(u_n\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\Delta_n^{1/2}}\right)\\
&\quad= E_{\mathcal{F}_{2jv_n}}\cos\left(u_n\frac{\Delta^n_{2jk_n+2l+1}X-\Delta^n_{2jk_n+2l}X}{\Delta_n^{1/2}}\right)\\
&\qquad\times\cos\left(u_n\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\Delta_n^{1/2}}\right)\\
&\qquad-U_j(u_n)U_j(u_n)+r_{2j},
\end{aligned}
\tag{A.20}
$$

where r_{2j} satisfies $|r_{2j}|\le Ku_n^2\sqrt{v_n}$. Since $\cos(x)\cos(y)=\frac{1}{2}(\cos(x+y)+\cos(x-y))$, we have by Lemma A.4,

$$
\begin{aligned}
&E_{\mathcal{F}_{2jv_n}}\cos\left(u_n\frac{\Delta^n_{2jk_n+2l+1}X-\Delta^n_{2jk_n+2l}X}{\Delta_n^{1/2}}\right)\cos\left(u_n\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\Delta_n^{1/2}}\right)\\
&\quad=\frac{1}{2}E_{\mathcal{F}_{2jv_n}}\left(\cos\left(u_n\frac{\Delta^n_{2jk_n+2l+1}X}{\Delta_n^{1/2}}-u_n\frac{\Delta^n_{2jk_n+2l-1}X}{\Delta_n^{1/2}}\right)\right.\\
&\qquad\left.+\cos\left(u_n\frac{\Delta^n_{2jk_n+2l+1}X}{\Delta_n^{1/2}}-2u_n\frac{\Delta^n_{2jk_n+2l}X}{\Delta_n^{1/2}}+u_n\frac{\Delta^n_{2jk_n+2l-1}X}{\Delta_n^{1/2}}\right)\right)\\
&\quad=\frac{1}{2}E_{\mathcal{F}_{2jv_n}}\left(\exp\left(-u_n^2c_{2jv_n+(2l-2)\Delta_n}\right)+\exp\left(-3u_n^2c_{2jv_n+(2l-2)\Delta_n}\right)\right)+r_{3,j}\\
&\quad=\frac{1}{2}\left(\exp\left(-u_n^2c_{2jv_n}\right)+\exp\left(-3u_n^2c_{2jv_n}\right)\right)+r_{3,j}+r_{4,j},
\end{aligned}
\tag{A.21}
$$

where $|r_{3,j}|\le K\Delta_n^{1-\beta/2}$ and $|r_{4,j}|\le Ku_n^2v_n$ by second-order Taylor expansion on e^x for x around the origin and (S.1.3) with $V=c$. Now substituting (A.21) back into (A.20), we have

$$
\begin{aligned}
E_{\mathcal{F}_{2jv_n}}\zeta_0(j,l)\zeta_1(j,l)&=\tfrac{1}{2}\left(\exp\left(-u_n^2c_{2jv_n}\right)+\exp\left(-3u_n^2c_{2jv_n}\right)\right)\\
&\quad-\exp\left(-2u_n^2c_{2jv_n}\right)+r_{5,j},
\end{aligned}
\tag{A.22}
$$

where $|r_{5j}|\le K(\sqrt{v_n}+\Delta_n^{1-\beta/2})$. Similarly, we have

$$
\begin{aligned}
&E_{\mathcal{F}_{2jv_n}}\zeta_0(j,l)\zeta_1(j,l+1)\\
&\quad=\tfrac{1}{2}\left(\exp\left(-u_n^2c_{2jv_n}\right)+\exp\left(-3u_n^2c_{2jv_n}\right)\right)\\
&\qquad-\exp\left(-2u_n^2c_{2jv_n}\right)+r_{6,j},
\end{aligned}
\tag{A.23}
$$

where $|r_{6,j}| \le K(\sqrt{v_n} + \Delta_n^{1-\beta/2})$. Substitute (A.22) and (A.23) into (A.19), and then substitute the latter into (A.18), and we have

$$
\begin{aligned}
&E_{\mathcal{F}_{2jv_n}} \xi_{0,j}(u_n)\xi_{1,j}(u_n) \\
&\quad = \frac{\exp(-u_n^2 c_{2jv_n}) + \exp(-3u_n^2 c_{2jv_n}) - 2\exp(-2u_n^2 c_{2jv_n})}{(k_n-1)U_j^2(u_n)} + r_j^* \\
&\quad = \frac{u_n^4 c_{2jv_n}^2 + r_{7j}}{(k_n-1)U_j^2(u_n)} + r_j^* = \frac{u_n^4 c_{2jv_n}^2 + r_{8j}}{k_n-1} + r_j^*,
\end{aligned} \tag{A.24}
$$

where $|r_{7j}| \vee |r_{8j}| \le K u_n^6$, $|r_j^*| \le |r_j| + \frac{|r_{2j}|+|r_{3j}|+|r_{4j}|+|r_{5j}|+|r_{6j}|}{k_n-1} + \frac{1}{(k_n-1)^2}$. Now a combination of (A.24) and (A.14) yields

$$
\begin{aligned}
&\frac{4v_n^2}{\Delta_n u_n^4} \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}} \bigl(\xi_{0,j}(u_n) - E_{\mathcal{F}_{2jv_n}} \xi_{0,j}(u_n)\bigr) \\
&\qquad \times \bigl(\xi_{1,j}(u_n) - E_{\mathcal{F}_{2jv_n}} \xi_{1,j}(u_n)\bigr) \to^p 2\int_0^t c_s^2\,ds.
\end{aligned} \tag{A.25}
$$

This proves the third equation in (A.12).

By Lemma A.2, we also have

$$
\begin{aligned}
&\sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}} (\chi_j^{n,k})^2 I(|\chi_j^{n,k}| > \varepsilon) \\
&\quad \le \frac{1}{\varepsilon} \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}} |\chi_j^{n,k}|^3 \\
&\quad \le \frac{K}{\varepsilon} \sum_{j=0}^{[t/(2v_n)]-1} \left(\frac{2v_n}{\Delta_n^{1/2}}\right)^3 \frac{1}{u_n^6} E_{\mathcal{F}_{2jv_n}} |\xi_{k,j}(u_n)|^3 \to 0.
\end{aligned}
$$

This proves the Linderberg condition [equation four in (A.12)].

Taking $\kappa = 2$ and $\zeta_j^n = 1$ in Lemma 15 of Jacod and Todorov (2014), we have

$$
\begin{aligned}
&\sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}} \chi_j^{n,k} (M_{2(j+1)v_n} - M_{2jv_n}) \\
&\quad = \sum_{j=0}^{[t/(2v_n)]-1} \frac{2v_n}{u_n^2 \Delta_n^{1/2}} E_{\mathcal{F}_{2jv_n}} \xi_{k,j}(u_j)(M_{2(j+1)v_n} - M_{2jv_n}) \to^P 0.
\end{aligned}
$$

This proves the final equation in (A.12) and completes the proof of the bivariate central limit theorem with stable convergence. □

PROOF OF THEOREM 3.2. Let $T_{n1} = \hat{C}_0(u_n) - A_0(u_n)_t^n - C_t - (\hat{C}_1(u_n) - A_0(u_n)_t^n - C_t)$. By (2.5),

$$T_n \equiv \frac{T_{n1} - \gamma_n \Delta_n^{1/2}}{C_t + A_0(u_n)_t^n + O_p(\Delta_n^{1/2})} = \frac{T_{n1}}{C_t + o_p(1)} + o_p(\Delta_n^{1/2}). \tag{A.26}$$

Then Theorem 3.2 is a straightforward consequence of Theorem 3.1, (A.26), the stable convergence mode and the continuous mapping theorem. □

PROOF OF THEOREM 3.3. By Theorem 3.1, $\hat{C}_1(u_n) = C_t + A_0(u_n)_t^n + O_p(\Delta_n^{1/2}) = C_t + o_p(1)$. This shows that the denominator of $\hat{\kappa}_T$ converges to C_T^2 in probability. By (A.2), we have

$$\begin{aligned}&\left(c_j^k(u_n) - \frac{(\sinh(u_n^2 c_j^k(u_n)))^2}{(k_n-1)u_n^2}\right)^2 \\ &\quad = c_{2jv_n}^2 + \tilde{c}_{j,1}^k(u_n) + \tilde{c}_{j,2}^k(u_n) + \tilde{c}_{j,3}^k(u_n),\end{aligned} \tag{A.27}$$

where

$$\tilde{c}_{j,1}^k = \left(\frac{\xi_{k,j}(u_n)}{u_n^2}\right)^2 + \left(\frac{\xi_{k,j}^2(u_n)}{2u_n^2} - \frac{(\sinh(u_n^2 c_j^k(u_n)))^2}{(k_n-1)u_n^2}\right)^2 + \left(r_{k,j}(u_n)\right)^2,$$

$$\tilde{c}_{j,2}^k = 2c_{2jv_n}\left(-\frac{\xi_{k,j}(u_n)}{u_n^2} + \frac{\xi_{k,j}^2(u_n)}{2u_n^2} - \frac{(\sinh(u_n^2 c_j^k(u_n)))^2}{(k_n-1)u_n^2} + r_{k,j}(u_n)\right),$$

$$\tilde{c}_{j,3}^k(u_n) = 4c_{2jv_n} u_n^{\beta-2}\Delta_n^{1-\beta/2} a_{2jv_n} + \left(2u_n^{\beta-2}\Delta_n^{1-\beta/2} a_{2jv_n}\right)^2.$$

By (A.1),

$$\sum_{j=0}^{[t/(2v_n)]-1} 2v_n \tilde{c}_{j,1}^k I_{\Omega^c(k,n,t)} = o_p(1), \qquad \sum_{j=0}^{[t/(2v_n)]-1} 2v_n \tilde{c}_{j,2}^k I_{\Omega^c(k,n,t)} = o_p(1).$$

By Lemma A.2, $\sum_{j=0}^{[t/(2v_n)]-1} 2v_n (\frac{\xi_{k,j}(u_n)}{u_n^2})^2 = o_p(1)$. On $\Omega(k,n,t)$, $|\frac{\xi_{k,j}^2(u_n)}{2u_n^2} - \frac{(\sinh(u_n^2 c_j^k(u_n)))^2}{(k_n-1)u_n^2}|$ is bounded by K/u_n^2, hence

$$\begin{aligned}&\left(\frac{\xi_{k,j}^2(u_n)}{2u_n^2} - \frac{(\sinh(u_n^2 c_j^k(u_n)))^2}{(k_n-1)u_n^2}\right)^2 I_{\Omega(k,n,t)} \\ &\qquad \le \frac{K}{u_n^2}\left|\frac{\xi_{k,j}^2(u_n)}{2u_n^2} - \frac{(\sinh(u_n^2 c_j^k(u_n)))^2}{(k_n-1)u_n^2}\right| I_{\Omega(k,n,t)} \\ &\qquad \le \frac{K}{u_n^2}\left(\left|\frac{\xi_{k,j}^2(u_n)}{2u_n^2} - E_{\mathcal{F}_{2jv_n}}\frac{\xi_{k,j}^2(u_n)}{2u_n^2}\right|\right.\end{aligned} \tag{A.28}$$

$$+\left|E_{\mathcal{F}_{2jv_n}}\frac{\xi_{k,j}^2(u_n)}{2u_n^2}-\frac{U_j(2u_n)+1-2U_j^2(u_n^2)}{4u_n^2(k_n-1)U_j^2(u_n)}\right|$$
$$+\left|\frac{U_j(2u_n)+1-2U_j^2(u_n^2)}{4u_n^2(k_n-1)U_j^2(u_n)}-\frac{(\sinh(u_n^2c_j^k(u_n)))^2}{(k_n-1)u_n^2}\right|\Bigg)I_{\Omega(k,n,t)}.$$

By the property of $U_j(u_n)$ and the definition of $c_j^k(u_n)$, the expectation of the third absolute value conditional on $\mathcal{F}_{2jv_n}$ is smaller than $K(u_n^{\beta-2}\Delta_n^{1-\beta/2}/k_n + u_n^{-2}/k_n^{3/2}+\Delta_n^{1/2}\phi_n/k_n)$. By Lemma A.2, the second absolute value is smaller than $Ku_n^4\Delta_n^{1/2}\phi_n$. By Hölder's inequality and Lemma A.2 with $q=4$, the expectation of the first absolute value conditional on $\mathcal{F}_{2jv_n}$ is smaller than $K(u_n^2/k_n+\sqrt{v_n})$. In summary, we conclude that

$$\sum_{j=0}^{[t/(2v_n)]-1}2v_n\left(\frac{\xi_{k,j}^2(u_n)}{2u_n^2}-\frac{(\sinh(u_n^2c_j^k(u_n)))^2}{k_nu_n^2}\right)^2 I_{\Omega(k,n,t)}=o_p(1). \tag{A.29}$$

By (A.1), $\sum_{j=0}^{[t/(2v_n)]-1}2v_n(r_{k,j}(u_n))^2I_{\Omega^c(k,n,t)}=o_p(1)$. On $\Omega(k,n,t)$, $|r_{k,j}|\le K\frac{|\xi_{k,j}(u_n)|^3}{u_n^2}$. By Lemma A.2 with $q=6$, we have $\sum_{j=0}^{[t/(2v_n)]-1}2v_n(r_{k,j}(u_n))^2\times I_{\Omega(k,n,t)}=o_p(1)$. Combining all the results of the terms on the right-hand side of the decomposition of $\tilde{c}_{j,1}^k$, we have $\sum_{j=0}^{[t/(2v_n)]-1}2v_n\tilde{c}_{j,1}^k=o_p(1)$. Similarly, one easily proves that $\sum_{j=0}^{[t/(2v_n)]-1}2v_n\tilde{c}_{j,2}^k=o_p(1)$. By boundedness of c and a, $\sum_{j=0}^{[t/(2v_n)]-1}2v_n\tilde{c}_{j,3}^k=o_p(1)$. This shows that

$$\hat{I}_{nk}=\sum_{j=0}^{[T/(2v_n)]-1}2v_nc_{2jv_n}^2+o_p(1)=\int_0^T c_s^2\,ds+o_p(1),\qquad k=0,1. \tag{A.30}$$

This shows that the numerator of $\hat{\kappa}_T$ converges to $4\int_0^T c_s^2\,ds$ in probability, and hence $\hat{\kappa}_T$ itself converges to κ_T in probability. On the other hand, by Theorem 3.2, $T_n/\Delta_n^{1/2}$ converges to G_T stably. By the stable convergence mode, $\mathcal{T}_n$ converges to standard normal distribution stably. □

A.2. Proof of results under H_1. In the sequel we assume that X is a pure-jump process. We rewrite

$$\begin{aligned} c_j^k(u_n)&=-\frac{\log U_j(u_n)}{u_n^2}-\frac{\log(1+\xi_{k,j}(u_n))}{u_n^2}\\ &=2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{2jv_n}-\frac{\xi_{k,j}(u_n)}{u_n^2}+\tilde{r}_{k,j},\qquad k=0,1, \end{aligned} \tag{A.31}$$

where $|\tilde{r}_{k,j}| \le K\xi^2_{k,j}(u_n)/u_n^2$ on $\Omega(k,n,t)$. Recall the definition of T_{n1} in (A.26), and we have

$$T_{n1} = 2v_n \sum_{j=0}^{[t/(2v_n)]-1} -\frac{\xi_{0,j}(u_n) - \xi_{1,j}(u_n)}{u_n^2} + \tilde{R}_{n,t}, \tag{A.32}$$

where

$$\begin{aligned}\tilde{R}_{n,t} = 2v_n \sum_{j=0}^{[t/(2v_n)]-1} \Bigg[&(\tilde{r}_{0,j} - \tilde{r}_{1,j}) \\ &+ \left(\frac{(\sinh(u_n^2 c_j^1(u_n)))^2 - (\sinh(u_n^2 c_j^0(u_n)))^2}{u_n^2(k_n-1)}\right)\Bigg].\end{aligned} \tag{A.33}$$

Similar to Lemma A.2, we have the following. The proof is provided in the supplementary material [Kong, Liu and Jing (2015)].

LEMMA A.5. *Assume Assumptions* 3.1, 3.3 *and* A.1, *and suppose* u_n *is bounded*, *so we have on the set* $\{C_t = 0\}$,

$$\begin{aligned}&\left|E_{\mathcal{F}_{2jv_n}}\xi_{k,j}(u_n)\right| \\ &\quad\le K\big(\Delta_n^{(1-r/2)\wedge((3-\beta)/2-\varepsilon')\wedge(1-\beta/(2(\beta+1-r)))} \\ &\qquad + u_n^\beta \Delta_n^{1-\beta/2} v_n^{\beta/2} + u_n^{2\beta}\Delta_n^{2-\beta} v_n\big),\end{aligned} \tag{A.34}$$

and if further $k_n\Delta_n^{1/2-\varepsilon} \to \infty$ *for any* $\varepsilon > 0$, *and* $\sup_n \frac{k_n\Delta_n^{1/2}}{u_n^4} < \infty$ *is satisfied*,

$$E_{\mathcal{F}_{2jv_n}}\xi^2_{k,j}(u_n) \le K\frac{u_n^\beta \Delta_n^{1-\beta/2}}{k_n}. \tag{A.35}$$

The following lemma gives the convergence rate of the terms on the right-hand side of (A.32). The proof can be found in the supplementary material [Kong, Liu and Jing (2015)] to this paper.

LEMMA A.6. *Assume Assumption* 3.1, 3.3 *and* A.1, *and suppose* u_n *is bounded and* $k_n\Delta_n^{1/2} \to 0$, *so we have on the set* $\{C_t = 0\}$,

(1)

$$\begin{aligned}&\left|2v_n \sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}}\left(\frac{\xi_{0,j}(u_n) - \xi_{1,j}(u_n)}{u_n^2}\right)\right| \\ &\quad\le K u_n^{-2}\big(\Delta_n^{(1-r/2)\wedge((3-\beta)/2-\varepsilon')\wedge(1-\beta/(2(\beta+1-r)))} + u_n^\beta\Delta_n^{3/2-\beta/2}\big);\end{aligned} \tag{A.36}$$

(2)

$$\left|\sum_{j=0}^{[t/(2v_n)]-1} E_{\mathcal{F}_{2jv_n}}\left(\frac{\xi_{0,j}(u_n)-\xi_{1,j}(u_n)-E_{\mathcal{F}_{2jv_n}}(\xi_{0,j}(u_n)-\xi_{1,j}(u_n))}{u_n^2/(2v_n)}\right)^2\right| \tag{A.37}$$
$$\le K u_n^{-4}\big(u_n^\beta \Delta_n^{2-\beta/2} + \Delta_n^{(2-r/2)\wedge((5-\beta)/2-\varepsilon')}\big);$$

for any $\varepsilon' > 0$;

(3)

$$\tilde{R}_{nt} = O_p\left(\frac{u_n^{\beta-2}\Delta_n^{1-\beta/2}}{k_n}\right). \tag{A.38}$$

PROOF OF THEOREM 3.4. We first prove the first equation. By (A.31), we have

$$\hat{C}_0(u_n) - \hat{C}_1(u_n) = T_{n,1} \tag{A.39}$$
$$= 2v_n \sum_{j=0}^{[t/(2v_n)]-1} -\frac{\xi_{0,j}(u_n)-\xi_{1,j}(u_n)}{u_n^2} + \tilde{R}_{n,t}.$$

Now by Lemma A.6, we have

$$\hat{C}_0(u_n) - \hat{C}_1(u_n) = O_p(\delta_{n,1} + \sqrt{\delta_{n,2}} + \delta_{n,3}), \tag{A.40}$$

where

$$\delta_{n,1} = u_n^{-2}\big(\Delta_n^{(1-r/2)\wedge((3-\beta)/2-\varepsilon')\wedge(1-\beta/(2(\beta+1-r)))} + u_n^\beta \Delta_n^{3/2-\beta/2}\big),$$
$$\sqrt{\delta_{n,2}} = u_n^{-2}\big(u_n^{\beta/2}\Delta_n^{1-\beta/4} + \Delta_n^{(1-r/4)\wedge((5-\beta)/4-\varepsilon'/2)}\big),$$
$$\delta_{n,3} = u_n^{\beta-2}\Delta_n^{1-\beta/2}/k_n.$$

Now, notice that: (1) $1 - r/4 > 1 - r/2 > 1 - \beta/2(\beta+1-r)$; (2) $\delta_{n,3} > u_n^{\beta-2}\Delta_n^{3/2-\beta/2}$; (3) $\frac{3-\beta}{2} < \frac{5-\beta}{4}$; (4) $\frac{\delta_{n,3}}{u_n^{-2}\Delta_n^{3/2-\beta/2-\varepsilon'}} \le u_n^\beta/(k_n\Delta_n^{1/2-\varepsilon'}) \le K$; (5) $\frac{u_n^{-2}\Delta_n^{3/2-\beta/2-\varepsilon'}}{u_n^{\beta/2-2}\Delta_n^{1-\beta/4}} = u_n^{4-\beta/2}\frac{k_n\Delta_n^{1/2}}{u_n^4}\frac{1}{k_n\Delta_n^{\beta/4+\varepsilon'}} \le K$. By choosing $\varepsilon' > 0$ small enough and the conditions on u_n and k_n, we have

$$\hat{C}_0(u_n) - \hat{C}_1(u_n) = O_p\big(u_n^{-2}\Delta_n^{1-\beta/(2(\beta+1-r))} + u_n^{\beta/2-2}\Delta_n^{1-\beta/4}\big).$$

Next, we prove the second equation. By (A.31), we have

$$\big(c_j^k(u_n)\big)^2 - \big(2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{2jv_n}\big)^2 - \left(\frac{\xi_{k,j}(u_n)}{u_n^2}\right)^2 - (\tilde{r}_{k,j})^2 \tag{A.41}$$
$$= 2\big(2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{2jv_n}\big)\left(-\frac{\xi_{k,j}(u_n)}{u_n^2} + \tilde{r}_{k,j}\right) - 2\left(\frac{\xi_{k,j}(u_n)}{u_n^2}\right)\tilde{r}_{k,j}.$$

Now we use several steps to show that under H_1 the principal term of $(c_j^k(u_n))^2$ is $(2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{2jv_n})^2$ and $E_{\mathcal{F}_{2jv_n}}(c_j^k(u_n))^2 \le K u_n^{2\beta-4}\Delta_n^{2-\beta}$. By Lemma A.5, we have

$$E_{\mathcal{F}_{2jv_n}}\left(\frac{\xi_{k,j}(u_n)}{u_n^2}\right)^2 \le K\big(u_n^{\beta-2}\Delta_n^{1-\beta/2}\big)^2 \frac{k_n\Delta_n^{1/2}}{u_n^4}\frac{u_n^{4-\beta}}{k_n^2\Delta_n^{3/2-\beta/2}}, \tag{A.42}$$

which is $o_p((u_n^{\beta-2}\Delta_n^{1-\beta/2})^2)$ by the conditions on k_n and u_n given in Theorem 3.4. By Lemma A.5 and (A.42), we have on $\Omega(k,n,t)$ (on which $|\tilde{r}_{k,j}| \le K\xi_{k,j}^2/u_n^2$ and $|\xi_{k,j}|$ is bounded),

$$\begin{aligned} E_{\mathcal{F}_{2jv_n}}(\tilde{r}_{k,j})^2 I_{\Omega(k,n,t)} &\le K E_{\mathcal{F}_{2jv_n}}|\tilde{r}_{k,j}| I_{\Omega(k,n,t)} \\ &\le K E_{\mathcal{F}_{2jv_n}}\left(\frac{\xi_{k,j}(u_n)}{u_n^2}\right)^2 \\ &\le K\big(u_n^{\beta-2}\Delta_n^{1-\beta/2}\big)^2 \frac{k_n\Delta_n^{1/2}}{u_n^4}\frac{u_n^{4-\beta}}{k_n^2\Delta_n^{3/2-\beta/2}}. \end{aligned} \tag{A.43}$$

By (A.42) and (A.43), we have by Hölder's inequality,

$$\begin{aligned} &E_{\mathcal{F}_{2jv_n}}\left|\big(2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{2jv_n}\big)\left(-\frac{\xi_{k,j}(u_n)}{u_n^2}+\tilde{r}_{k,j}\right)\right| I_{\Omega(k,n,t)} \\ &\quad \le K\big(u_n^{\beta-2}\Delta_n^{1-\beta/2}\big)^2\left(\frac{u_n^{4-\beta/2}}{k_n^{3/2}\Delta_n^{1-\beta/4}}+\frac{u_n^2}{k_n^2\Delta_n^{1/2}}\right) \end{aligned} \tag{A.44}$$

and

$$\begin{aligned} E_{\mathcal{F}_{2jv_n}}\left|\frac{\xi_{k,j}(u_n)}{u_n^2}\right||\tilde{r}_{k,j}| I_{\Omega(k,n,t)} &\le K E_{\mathcal{F}_{2jv_n}}\left(\frac{\xi_{k,j}(u_n)}{u_n^2}\right)^2 \\ &\le K\big(u_n^{\beta-2}\Delta_n^{1-\beta/2}\big)^2 \frac{k_n\Delta_n^{1/2}}{u_n^4}\frac{u_n^{4-\beta}}{k_n^2\Delta_n^{3/2-\beta/2}}. \end{aligned} \tag{A.45}$$

Combining (A.42)–(A.45) yields that

$$E_{\mathcal{F}_{2jv_n}}\big|\big(c_j^k(u_n)\big)^2 - \big(2u_n^{\beta-2}\Delta_n^{1-\beta/2}\big)^2\big| I_{\Omega(k,n,t)} = o(1), \tag{A.46}$$

where $o(1)$ holds uniformly in j.

By the form of $c_j^k(u_n)$, we have $u_n^2|c_j^k(u_n)|I_{\Omega(k,n,t)} \le K$, and hence by Taylor expansion on the exponential function, we have

$$\big(\sinh\big(u_n^2 c_j^k(u_n)\big)\big)^2 I_{\Omega(k,n,t)} \le K u_n^4\big(c_j^k(u_n)\big)^2 I_{\Omega(k,n,t)} \le K. \tag{A.47}$$

By virtue of (A.47), we have

$$
\begin{aligned}
&E_{\mathcal{F}_{2jv_n}}\left(\frac{(\sinh(u_n^2 c_j^k(u_n)))^2 I_{\Omega(k,n,t)}}{u_n^2(k_n-1)}\right)^2 \\
&\quad\le \frac{K}{u_n^4 k_n^2} E_{\mathcal{F}_{2jv_n}}\bigl(\sinh\bigl(u_n^2 c_j^k(u_n)\bigr)\bigr)^2 I_{\Omega(k,n,t)} \\
&\quad\le \frac{K}{k_n^2} E_{\mathcal{F}_{2jv_n}}\bigl(c_j^k(u_n)\bigr)^2 I_{\Omega(k,n,t)} \le \frac{K}{k_n^2}\bigl(u_n^{\beta-2}\Delta_n^{1-\beta/2}\bigr)^2,
\end{aligned}
\tag{A.48}
$$

and further by the Cauchy inequality,

$$
E_{\mathcal{F}_{2jv_n}}\left|c_j^k(u_n)\frac{(\sinh(u_n^2 c_j^k(u_n)))^2}{u_n^2(k_n-1)}\right| I_{\Omega(k,n,t)} \le \frac{K}{k_n}\bigl(u_n^{\beta-2}\Delta_n^{1-\beta/2}\bigr)^2. \tag{A.49}
$$

Now combining (A.46), (A.48), (A.49) and (A.1), we have

$$
\begin{aligned}
\hat{I}_{n,k} &= \bigl(2u_n^{\beta-2}\Delta_n^{1-\beta/2}\bigr)^2\left(\sum_{j=0}^{[t/(2v_n)]-1} 2v_n a_{2jv_n}^2 + o_p(1)\right) \\
&= \bigl(2u_n^{\beta-2}\Delta_n^{1-\beta/2}\bigr)^2\left(\int_0^t a_s^2\,ds + o_p(1)\right),
\end{aligned}
$$

for $k = 0, 1$. This proves the second equation of Theorem 3.4. □

PROOF OF COROLLARY 3.1. Part 1 is a straight consequence of Theorem 3.3. Now we prove part 2. By Theorem 3.4, we have by the condition on γ_n,

$$
\begin{aligned}
\mathcal{T}_n &= \frac{-\gamma_n + O_p(u_n^{-2}\Delta_n^{1/2-\beta/(2(\beta+1-r))} + u_n^{\beta/2-2}\Delta_n^{1/2-\beta/4})}{4u_n^{\beta-2}\Delta_n^{1-\beta/2}\sqrt{\int_0^t a_s^2\,ds + o_p(1)}} \\
&= \frac{-\gamma_n(1+o_p(1))}{4u_n^{\beta-2}\Delta_n^{1-\beta/2}\sqrt{\int_0^t a_s^2\,ds + o_p(1)}}.
\end{aligned}
\tag{A.50}
$$

Since $\gamma_n u_n^{2-\beta/2}\Delta_n^{\beta/4-1/2} \to \infty$ and $\frac{u_n^{2-\beta/2}\Delta_n^{\beta/4-1/2}}{u_n^{2-\beta}\Delta_n^{\beta/2-1}} \le u_n^{\beta/2}\Delta_n^{1/2-\beta/4} \to 0$,

$$
\frac{-\gamma_n(1+o_p(1))}{4u_n^{\beta-2}\Delta_n^{1-\beta/2}\sqrt{\int_0^t a_s^2\,ds + o_p(1)}} \to^P -\infty.
$$

This proves part 2 on the performance of the power of the test. □

SUPPLEMENTARY MATERIAL

Supplement to "Testing for pure-jump processes for high-frequency data" (DOI: 10.1214/14-AOS1298SUPP; .pdf). This supplement contains technical proofs of the Lemmas A.4–A.6 as well as some interesting supplemental lemmas.

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