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MODELING HIGH-FREQUENCY FINANCIAL DATA BY PURE JUMP PROCESSES

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It is generally accepted that the asset price processes contain jumps. In fact, pure jump models have been widely used to model asset prices and/or stochastic volatilities. The question is: is there any statistical evidence from the high-frequency financial data to support using pure jump models alone? The purpose of this paper is to develop such a statistical test against the necessity of a diffusion component. The test is very simple to use and yet effective. Asymptotic properties of the proposed test statistic will be studied. Simulation studies and some real-life examples are included to illustrate our results.

1. Introduction. It is now widely accepted that the asset price processes contain jumps. This is partially based on many empirical evidences, such as heavy tails in the asset returns; see Cont and Tankov (2004) and Carr et al. (2002) and references therein. In the meantime, many statistical tests have been established to detect jumps from discretely observed prices [e.g., Jiang and Oomen (2005), Barndorff-Neilsen and Shepard (2006), Lee and Mykland (2008), Aït-Sahalia and Jacod (2010)], and these test results all seem to support the claim of the existence of jumps for the asset returns under their investigations.

In recent years, pure jump models have been widely used as an alternative model for price process to the classical model, which has a continuous martingale component; see Todorov and Tauchen (2010) and references within. The idea behind the pure-jump modeling is that small jumps can eliminate the need for a continuous martingale. The class of pure-jump models is extremely wide. It includes the normal inverse Gaussian [Rydberg (1997), Barndorff-Nielsen (1997, 1998)], the variance gamma [Madan, Carr and Chang (1998)], the CGMY model of Carr et al. (2002), the time-changed Levy models of Carr et al. (2003), the COGARCH model of Klüppelberg, Lindner and Maller (2004) for the financial prices, as well as the non-Gaussian Ornstein–Uhlenbeck-based models of Barndorff-Nielsen and Shephard (2001) and the Lévy-driven continuous-time moving average (CARMA)

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models of Brockwell (2001) for the stochastic volatility. Pure-jump models have been extensively considered and used for general options pricing [Huang and Wu (2004), Broadie and Detemple (2004), Levendorskii (2004), Schoutens (2006), Ivanov (2007)], and for foreign exchange options pricing [Huang and Hung (2005), Daal and Madan (2005), Carr and Wu (2007)]. Other applications of pure-jump models include reliability theory [Drosen (1986)], insurance valuation [Ballotta (2005)] and financial equilibrium analysis [Madan (2006)].

Given the wide usage of pure jump models, a natural question is: *is there any statistical evidence from the high-frequency financial data to support using the purely discontinuous models alone without any continuous diffusion components?* The question is of significance from both theoretical and practical viewpoints:

- Many empirical evidences indicate that pure jump models can fit the data well; see, for example, Cont and Tankov (2004), and Carr et al. (2002) and references therein. Therefore, it would be of theoretical interest to establish some statistical tests for this purpose.
- Given the existence of jumps, pure jump models are typically easier to handle than mixture models in practice, and a preferred choice to mixture models for users. However, before using a pure jump model, one must check its validity.
- Various jump models have been well studied in the literature, as mentioned earlier. Should we decide to use pure jump models, we would have an array of available tools at our disposal.
- Many results are strongly model dependent, and any model mis-specification could have a severe effect on the results. Therefore, it is imperative to choose the best possible model, and model selection is very critical.

To put our question into a mathematical context, suppose that the price process Y is a jump diffusion process of the form

$$Y_t = X_t + J_t, \tag{1.1}$$

for $t \in [0, T]$ with X_t and J_t being the continuous and discontinuous (or jump) components, defined as

$$\begin{aligned} X_t &= Y_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s, \\ J_t &= \int_0^t \int_{|x|\le 1} x(\mu - \nu)(ds, dx) + \int_0^t \int_{|x|>1} x\mu(ds, dx), \end{aligned} \tag{1.2}$$

where b and σ are some deterministic functions such that X has unique weak solution, μ is the jump measure, with ν its predictable compensator; for details on jump diffusion processes, see Jacod and Shiryaev (2003). Under this framework, the above question is tantamount to testing

$$H_0: \quad \int_0^T \sigma^2(X_s)\,ds > 0, \qquad \text{(i.e., diffusion effect is present)}, \tag{1.3}$$

$$H_1: \quad \int_0^T \sigma^2(X_s)\,ds = 0, \qquad \text{(i.e., diffusion effect is not present)}, \tag{1.4}$$

given the jump component J_t is present. Note that, under H_0, Y_t is a mixture model of diffusion and jumps while, under H_1, it is a pure jump model.

Cont and Mancini (2007) and Aït-Sahalia and Jacod (2010) considered the above test using threshold power variation. They assumed a general continuous semi-martingale form of X as opposed to a diffusion form in the present paper. However, to perform their test, one needs to impose the condition that J is of finite variation [e.g., Theorem 2 of Aït-Sahalia and Jacod (2010)]. This restriction rules out some interesting models used in finance, where the jumps are shown to be of infinite variation, as done in Aït-Sahalia and Jacod (2009), Zhao and Wu (2009) and some other references mentioned earlier.

In this paper, we propose a simple-to-use, general purpose and yet powerful goodness-of-fit test for differentiating a pure jump model from a mixture model. The CLTs are also derived for the test statistics under H_0, regardless whether the jump component is of finite or infinite variation. In that aspect, our proposed test works more generally than those proposed earlier by Cont and Mancini (2007) and Aït-Sahalia and Jacod (2010). Even for the situations where tests by Cont and Mancini (2007) and Aït-Sahalia and Jacod (2010) are applicable, our numerical results also show the superior performance of our proposed test.

The paper is organized as follows. In Section 2, we give some motivations via a simple example and then formally introduce our test statistics. Asymptotic results are derived in Section 3. Some review of alternative tests are given in Section 4. Numerical studies are given in Section 5. A real example is studied in Section 6. Some discussion on microstructure noise is given in Section 7. All technical proofs are postponed in the Appendix.

Throughout the paper, the available data set is denoted as $\{Y_{t_i}; 0 \le i \le n\}$ in the fixed interval $[0, T]$, which is discretely sampled from Y. For simplicity, we assume that $\{Y_{t_i}; 0 \le i \le n\}$ are equally spaced in $[0, T]$, that is, $t_i = i\Delta_n$ with $\Delta_n = T/n$ for $0 \le i \le n$. Denote the jth one-step increment by

$$\Delta_j^n Y = Y_{t_j} - Y_{t_{j-1}}, \qquad 1 \le i \le n.$$

2. Test statistics. We start with a simple motivating example first and then introduce our test statistics for testing (1.3) and (1.4).

2.1. *A simple motivating example.* We draw two respective samples $\{Y_{t_i}; 0 \le i \le n\}$ from the following two models:

$$H_0: \quad Y_t = \sigma W_t + S_t^\beta \qquad (\textit{a mixture model}),$$

$$H_1: \quad Y_t = S_t^\beta \qquad (\textit{a pure jump model}),$$

where W_t and S_t^β are a standard Brownian motion and a symmetric β-stable Lévy process, respectively. So the mixture model contains an extra continuous component σW_t, in comparison with the pure jump model. For illustration, we take $T = 1$, $\beta = 1.25$ and $\sigma = 0.25, 0.5$.

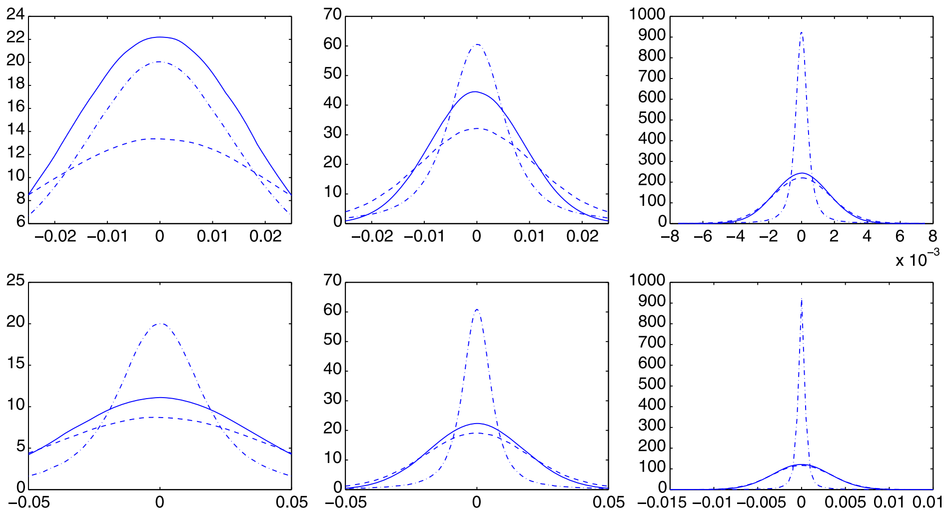

FIG. 1. *Smoothed histograms for the increment of the mixture model* (- -), *pure jump model* (-·), *and diffusion term alone* (-). *From left to right*, *the sample sizes are* 195, 780 *and* 23,400, *respectively*. *From top to bottom*, $\sigma = 0.25$ *and* 0.5, *respectively*.

The smoothed histograms (done by 10^6 replications) of the increments $\{\Delta_j^n Y, 1 \le j \le n\}$ under the two models are plotted in Figure 1 for sample sizes $n = 195$, 780, and 23,400, which corresponds to sampling every 2 minutes, 30 seconds, and every second in a 6.5 hour trading day. From Figure 1, we can see some very clear patterns:

(1) For small sample size n and small σ, it is difficult to distinguish the models under H_0 and H_1 (the dashed line and dash-dotted line). However, as n and/or σ increases, the difference is more significant under H_0 and H_1.
(2) The differences between the normal histogram and the mixture one (the solid line and the dashed line) are small in all cases and become even more negligible as n increases. Literally, the jump component has been "absorbed" by the diffusion component in the center.
(3) For fixed σ, as the sample size n increases, the differences between models under H_0 and H_1 are getting sharper. Take $n = 23{,}400$ and $\sigma = 0.5$, for example. The histogram under H_1 (dash-dotted line) shows a very narrow peak around the origin, while the histogram under H_0 (the dashed line) stays rather flat.

The example shows that there is a huge difference around the origin between the models under H_0 and H_1. If we use the number of "small" increments as an indicator, $U_n = \sum_{i=1}^{n} \{|\Delta_i^n Y| \le u_n\}$ for some u_n, then it relies heavily on whether the diffusion is present or not, particularly when the sample size n gets large. To give a better idea, some values of U_n under the above two models are presented

TABLE 1
Numbers of increments $\leq \alpha\Delta_n^{\varpi}$ for Y, W and S^{β}, where $\alpha = 2$, $\varpi = 1$ and $\Delta_n = 1/23{,}400$. The numbers are averaged over 500 replications

Parameter	$\#\{\lvert\Delta_i^n Y\rvert \leq \alpha\Delta_n^{\varpi}\}$	$\#\{\lvert\Delta_i^n W\rvert \leq \alpha\Delta_n^{\varpi}\}$	$\#\{\lvert\Delta_i^n S\rvert \leq \alpha\Delta_n^{\varpi}\}$
$\beta = 1.50$	408	488	942
$\beta = 1.00$	485	489	16,491
$\beta = 0.50$	487	487	23,313

in Table 1 when $n = 23{,}400$. The drastic difference for the two models strongly suggests that we might be able to use U_n to test whether the diffusion is present or not.

2.2. *Test statistics*. Let us return to the testing problem given in (1.3) and (1.4). We observe from Section 2.1 that the increments from a pure jump model and a mixture model have fundamentally different behavior around the centers of their distributions. Namely, the distribution for the increments from a pure jump model shows a much higher peak in the center than that from a mixture model. In other words, the number of small increments from a pure jump model is far greater than than that from a mixture model. This suggests that we might use the number of small increments

$$U(\Delta_n) =: U(\alpha, \Delta_n, \varpi, T) = \sum_{i=1}^{[T/\Delta_n]} I(|\Delta_i^n Y| \leq \alpha\Delta_n^{\varpi}),$$

to define a test statistic. Note that $U(\Delta_n)$ simply counts the number of increments smaller than $\alpha\Delta_n^{\varpi}$, where $\alpha > 0$ and $\varpi > 1/2$. (Here, we suppress the dependence on α, ϖ and T for convenience.)

Under some mild conditions (given in Section 3), the behaviors of $U(\Delta_n)$ are different under H_0 and H_1. Here is a heuristic argument. Under H_0, we have $\Delta_i^n Y \approx \sigma(X_{t_{i-1}})\Delta_i^n W$, and hence

$$\begin{aligned} EU(\Delta_n) &\approx \sum_{i=1}^{[T/\Delta_n]} EP_{t_{i-1}}\big(|\Delta_i^n W| \leq \alpha\Delta_n^{\varpi}/\sigma(X_{t_{i-1}})\big) \\ &\approx 2\alpha\phi(0)\Delta_n^{\varpi-3/2}T\int_0^T E\sigma^{-1}(X_s)\,ds, \end{aligned}$$

where $\phi(x)$ is the density of the standard normal r.v., and $P_{t_{i-1}}$ is the probability conditioned at time t_{i-1}. Consequently, we have $U(\Delta_n)$ is of order $\Delta_n^{-3/2+\varpi}$ under H_0. Similarly, we can show that $U(\Delta_n)$ is of order $\Delta_n^{-(1+1/\beta)+\varpi}$ under H_1. Clearly, we have $\Delta_n^{-3/2+\varpi} \ll \Delta_n^{-(1+1/\beta)+\varpi}$. That is, there are far more small increments under the pure jump model (H_1) than those under the mixture model

(H_0), which agrees well with the above motivating example. Then, we will reject H_0 (a mixture model) in favor of H_1 (a pure jump model), if $U(\Delta_n)$ is large enough.

From both Proposition 1 and (A.24) in the Appendix, we see that the probability limit of $U(\Delta_n)$ depends on unknown population quantities, and hence can not be directly used for our testing purposes. To get around the problem, we adopt the same strategy as in Zhang, Mykland and Aït-Sahalia (2005) and Aït-Sahalia and Jacod (2010) by using a two-time scale test statistic,

$$V_n := \frac{U(\Delta_n)}{U(k\Delta_n)},$$

where $U(k\Delta_n) =: U(\alpha, k\Delta_n, \varpi, T) = \sum_{i=1}^{[T/(k\Delta_n)]} I(|\Delta_{(i-1)k+1}^n Y + \cdots + \Delta_{ik}^n Y| \leq \alpha(k\Delta_n)^{\varpi})$. As can be seen from (3.1) below, the distribution of V_n is model-free under H_0, and we can reject H_0 (a mixture model) in favor of H_1 (a pure jump model) if $V_n > C$ for some critical value $C > 0$.

We end the section by pointing out some differences between the above test and the one by Aït-Sahalia and Jacod (2010). The test statistic given in (16) and (19) of Aït-Sahalia and Jacod (2010) is based on the truncated pth power variations while our test statistic given by $U(\Delta_n)$ and V_n is simply based on the number of small increments. Further comparisons will be made later in the paper.

3. Main results. We first list some assumptions and then present the main results.

3.1. *Model assumptions.* Recall that $Y_t = X_t + J_t$. Assume that Y is defined on a filtered probability space $(\Omega, \mathcal{F}^Y, \mathcal{F}_t^Y)$, where $\mathcal{F}_t^Y$ is the history of Y up to time t.

ASSUMPTION 1. J_t has a jump measure $\mu(dx, dt)$ with compensator $\nu(\omega, dx, dt) = dt\, F_t(\omega, dx)$, such that, for all (ω, t), we have $F_t = F_t' + F_t''$, where:

(1) F_t' has the form

$$F_t'(dx) = \frac{1 + |x|^\gamma f(x)}{|x|^{1+\beta}}\big[a^{(+)} I(0 < x \leq \varepsilon^+) + a^{(-)} I(-\varepsilon^- \leq x < 0)\big]\, dx,$$

for some positive constants $a^{(+)}$, $a^{(-)}$, γ, ε^+ and ε^- and some bounded function $f(x)$, satisfying $1 + |x|^\gamma f(x) > 0$, $|f(x)| \leq L$.

(2) F_t'' is a singular measure with respect to F_t', satisfying $\int_R (|x|^{\beta'} \wedge 1) F_t''(\omega, dx) \leq L$.

ASSUMPTION 2. X and J are mutually independent.

ASSUMPTION 3. $b(\cdot)$ is a bounded continuous functions, $\sigma(\cdot)$ is bounded away from zero and infinity if it does not vanish and $\sigma'(\cdot)$ exists and is bounded.

Assumption 1 implies that the small jumps of J form a Lévy process with a β-stable-like Lévy density, while almost no condition is placed on the large jumps of J, and F_t'' could even be random. Assumption 1 includes a rich class of models, like the variance gamma model, CGMY model, tempered stable process, etc. Assumptions 2 and 3 are technical conditions.

3.2. *Asymptotic results*. Let $\mathcal{N}(0,1)$ denote a standard Gaussian random variable. We will use the stable convergence in law below, which is slightly stronger than weak convergence; see, for example, Jacod and Shiryaev (2003).

THEOREM 1. *Suppose that* $\varpi > \beta - 1/2$ *and that Assumptions* 1–3 *hold.*

(1) *We have*

$$V_n \to^P \begin{cases} k^{3/2-\varpi}, & \text{under } H_0, \\ k^{1+(1/\beta-\varpi)\wedge 0}, & \text{under } H_1 \text{ and Assumption 4 below}. \end{cases} \tag{3.1}$$

(2) *Let* $k=2$. *Under* H_0, *we have*

$$\Delta_n^{(\varpi-3/2)/2}(V_n - k^{3/2-\varpi}) \longrightarrow \sigma\mathcal{N}(0,1) \qquad \text{stably},$$

where $\mathcal{N}(0,1)$ *is independent of* Y *and*

$$\sigma^2 = \frac{(1+k^{3/2-\varpi})k^{3-2\varpi}}{2\alpha\phi(0)\int_0^T \sigma^{-1}(X_s)\,ds}.$$

To apply Theorem 1, one needs to estimate the unknown σ^2. However, in view of Proposition 1 in the Appendix and the stable convergence, we have the following.

COROLLARY 1. *Assuming the same assumptions as in Theorem* 1, *we have*

$$\Delta_n^{(\varpi-3/2)/2}(V_n - k^{3/2-\varpi})/\widehat{\sigma} \longrightarrow_d \mathcal{N}(0,1) \qquad \text{under } H_0,$$

where

$$\widehat{\sigma}^2 = \frac{(1+k^{3/2-\varpi})k^{3-2\varpi}}{\Delta_n^{3/2-\varpi}U(\Delta_n)}.$$

From Corollary 1, at significance level θ, we can reject H_0 if $V_n > k^{3/2-\varpi} + z_{1-\theta}\Delta_n^{3/4-\varpi/2}\widehat{\sigma}$ and $P(\mathcal{N}(0,1) > z_{1-\theta}) = \theta$. It follows from Corollary 1 that the size of the above test is asymptotically θ.

A slight variant of the test statistic V_n can be given below. Let

$$\tilde{V}_n = \frac{U(\Delta_n)}{U_L(2\Delta_n)},$$

where $U_L(2\Delta_n) = [U(2\Delta_n) + U'(2\Delta_n)]/2$ and

$$U'(2\Delta_n) = \sum_{i=1}^{[T/(2\Delta_n)]-1} I\big(|\Delta_{2i+1}^n Y + \Delta_{2i}^n Y| \le \alpha(2\Delta_n)^{\varpi}\big),$$

$$U(2\Delta_n) = \sum_{i=1}^{[T/(2\Delta_n)]} I\big(|\Delta_{2i}^n Y + \Delta_{2i-1}^n Y| \le \alpha(2\Delta_n)^{\varpi}\big).$$

In other words, we use linear combinations of $U(2\Delta_n)$ with different starting time points instead of a single $U(2\Delta_n)$ starting from time t_0 when nonoverlapping two-step increments of Y are sampled. Similarly to Corollary 1, we can easily derive the following result.

COROLLARY 2. *Assuming the same assumptions as in Theorem* 1, *we have*

$$\Delta_n^{(\varpi-3/2)/2}(\tilde{V}_n - 2^{3/2-\varpi})/\tilde{\sigma} \longrightarrow_d \mathcal{N}(0,1) \qquad \textit{under } H_0,$$

where

$$\tilde{\sigma}^2 = \frac{U(\Delta_n) + 2^{3/2-\varpi} U_L(2\Delta_n)/2}{\Delta_n^{3/2-\varpi} U_L(2\Delta_n)^2}.$$

Our final decision rule is: at significance level θ, we reject H_0 if

$$\tilde{V}_n > \tilde{C}, \tag{3.2}$$

where $\tilde{C} = 2^{3/2-\varpi} + z_{1-\theta}\Delta_n^{3/4-\varpi/2}\tilde{\sigma}$ and $P(\mathcal{N}(0,1) > z_{1-\theta}) = \theta$. It follows from Corollary 2 that the size of the above test in (3.2) is asymptotically θ.

REMARK 1. The requirement $\varpi > \beta - 1/2$ in Theorem 1 and Corollaries 1–2 can be easily satisfied by choosing $\varpi = 3/2$ as $\beta \in (0,2)$. Moreover, whatever the value of ϖ, H_0 and H_1 can be differentiated since $1 + 1/\beta > 3/2$ for all $\beta \in (0,2)$.

On the other hand, the behaviors of test statistics S_n under H_0 and H_1 in Aït-Sahalia and Jacod (2010) depend on the choice of p, that is, $2 > p > 1 \vee \beta$; see Theorem 1 in that paper. Aït-Sahalia and Jacod (2010). Since β is unknown, it is difficult to choose p. To be on the safe side, one might try to choose β close to 2. However, this will render the test with very low power since S_n converges in probability to roughly the same limit 1 under H_0 and H_1.

REMARK 2. In Theorem 1 and Corollaries 1–2, we have $\beta \in (0,2)$, and no further restriction on β is imposed, so that the jump component could be of finite variation or infinite variation. By contrast, The CLT under H_0 was developed by Aït-Sahalia and Jacod (2010) only when $\beta < 1$, namely when J is of finite variation.

3.3. *Asymptotic power.* Before discussing the power of our test statistic, we list one more condition, which basically assumes that the drift term is zero when $\beta \leq 1$. It is a standard assumption in the literature; see Jacod (2008) and Woerner (2003), and the references therein.

ASSUMPTION 4. If $\beta < 1$, we assume that $b(\cdot) \equiv 0$, and $\int_{|x|\leq 1} xF'(dx) \equiv 0$. If $\beta = 1$, we assume that $b(\cdot) \equiv 0$ and $F'(dx)$ is symmetric about 0.

The next theorem gives the asymptotic power of our proposed test (3.2).

THEOREM 2. *Under Assumptions* 1 *and* 4, *with prescribed level* θ *and for* $\varpi > 1$, *we have*

$$P(\tilde{V}_n > \tilde{C}|H_1) \longrightarrow 1,$$

that is, *the asymptotic power is* 1.

REMARK 3. We end this section with some remarks on finite sample performance of our test statistics. Intuitively, the closer β gets to 2, the more the pure jump process behaves like a diffusion process; thus, the more difficult it is to tell their difference apart, the less power our test will have. Similarly, the closer β gets to 0, the more power our test will have. In fact, simple algebra yields $\tilde{C} - 2^{3/2-\varpi} = O_p(\Delta_n^{(1+1/\beta-\varpi)/2})$, from which we can see that, as β becomes closer to 0, the power of our test increases soon. This is further confirmed in our simulation studies given later.

4. A review of other approaches. The testing problem considered in this paper has also been considered earlier by Cont and Mancini (2007) and Aït-Sahalia and Jacod (2010). Since the work in both papers is similar, we will only review the test by Aït-Sahalia and Jacod (2010) (hereafter AJ's test) below.

The building block of the AJ's test is based on the truncated p-power variation,

$$B(p, u_n, \Delta_n) = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n Y|^p I(|\Delta_i^n Y| \leq u_n), \tag{4.1}$$

where $p \in (1, 2)$, and u_n satisfies $u_n/\Delta_n^{\rho_-} \to 0$, $u_n/\Delta_n^{\rho_+} \to \infty$, for some $0 \leq \rho_- < \rho_+ < 1/2$. Similarly to Zhang, Mykland and Aït-Sahalia (2005), Aït-Sahalia and Jacod (2010) defined a two-time scale estimator

$$S_n = \frac{B(p, u_n, \Delta_n)}{B(p, u_n, k\Delta_n)} \qquad \text{for an integer } k \geq 2,$$

and showed that

$$S_n \to^P \begin{cases} k^{1-p/2}, & \text{under } H_0, \\ 1, & \text{under } H_1, \text{if } 2 > p > 1 \vee \beta \text{ and } \rho_+ \leq (p-1)/p \end{cases} \tag{4.2}$$

and that when $\beta < 1$,

$$(S_n - k^{1-p/2})/\sqrt{v_n} \longrightarrow_d \mathcal{N}(0,1) \qquad \text{under } H_0, \tag{4.3}$$

where $v_n^2 = CB(2p, u_n, \Delta_n)/B(p, u_n, \Delta_n)^2$ for some constant C. Noting $k^{1-p/2} > 1$, one would reject H_0 if $S_n \le C_0$, for some C_0 determined from the CLT. Aït-Sahalia and Jacod (2010) also showed that the asymptotic power of this test is 1.

We make several remarks regarding the AJ's test:

- From (4.2), the behaviors of test statistics S_n under H_0 and H_1 depend on the choice of p, that is, $2 > p > 1 \vee \beta$. Since β is unknown, it is difficult to choose p. To be on the safe side, one might try to choose β close to 2. However, this will render the test with very low power since S_n converges in probability to roughly the same limit 1 under H_0 and H_1.
- The CLT under H_0, (4.3), was established in Aït-Sahalia and Jacod (2010) only for the case $\beta < 1$, namely when J is of finite variation. However, when $\beta > 1$, that is, when J is of infinite variation, no CLT is available, and hence the size of the test cannot be controlled for that case. This rules out some interesting applications when $\beta > 1$.
- For $\beta \in (0, 1)$, where the CLT is available for AJ's test, we might expect that AJ's test should have very good power, particularly as β gets smaller toward 0. However, our simulation studies give some counterintuitive results; see Table 6.

5. Numerical studies. In this section, we conduct simulations to evaluate the performance of our proposed test statistics, and make some comparisons with that of Aït-Sahilia and Jocod (2010).

The test statistics V_n and $\widetilde{V}_n$ involve choosing the threshold level $u_n = \alpha\Delta_n^{\varpi}$. In view of the requirement $\varpi > \beta - 1/2$, a conservative choice of ϖ would be 1.5. To compensate for the conservative choice of ϖ, we choose a relatively large α by $\alpha_n = \delta(\log n)^{\kappa}$ for some positive constants δ and κ. This choice will not affect any of the asymptotic results in the paper.

Assume that the data generating process under the null and alternative hypotheses are, respectively,

$$H_0: \quad Y_t = X_t + \theta' S_{\beta,t}, \tag{5.1}$$

$$H_1: \quad Y_t = \exp(-\gamma t) + 0.5 S_{\beta,t}, \tag{5.2}$$

where X_t is an Ornstein–Urlenbeck process. $dX_t = -X_t\,dt + dW_t$, and W is a standard Brownian motion, and S_β is a symmetric β-stable process. Let $T = 1$, $\theta' = 0.5$. Also we take $n = 1560, 2340, 4680, 11{,}700$ and $23{,}400$, corresponding to an intra day data set recorded every 15, 10, 5, 2 and 1 seconds in a 6.5-hour trading day, respectively. We will simulate 10,000 samples from each model above.

TABLE 2
Sizes of the test (%) under different n's and β's, $(\delta = 2, \kappa = 2)$

Value of β	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**
$n = 1560$	2.97	4.18	4.71	4.08	4.39	3.99	4.13	4.27
$n = 2340$	3.94	4.42	4.41	4.27	4.55	4.42	4.25	4.37
$n = 4680$	3.98	4.36	4.46	4.82	5.31	4.81	4.93	4.56
$n = 11{,}700$	4.34	4.50	4.94	5.06	4.93	5.06	4.78	4.29

Asymptotic sizes. Fix the nominal level $\theta = 5\%$, so the critical value is $z_{0.95} = 1.645$. The size of the test is calculated by the percentage of samples such that (3.2) holds true over 10,000 samples.

Table 2 reports the asymptotic sizes for different sample sizes. From the table, we see that the type I error is well controlled by 5%; as the sample size n increases, the asymptotic sizes become closer to the true size 5%.

Table 3 reports the asymptotic sizes across different threshold levels which reflect the number of effective data. It shows that control of type I error is not affected much by changes of δ.

Asymptotic power. We also consider the power performance of $\widetilde{V}_n$. The power of the test is the percentage of samples with (3.2) violated over 10,000 samples. The results are listed in Table 4 for different values of β.

From Table 4, it is clear that, as the sample size n increases, the test becomes more powerful overall, as expected. The test is powerful especially when β is away from 2. When β approaches 2, the power gradually diminishes. This is easily understandable as in this case the behavior of the discontinuous process resembles that of a Brownian motion. This can also be seen from (3.1).

Finally, we examine the asymptotic sizes over different choices of θ'. We fix $n = 2340$, $\delta = 2$, $\kappa = 2$ and $\theta = 5\%$. In Figure 2, the asymptotic sizes for $\beta = 1.25$ and 1.5 are plotted against θ'. Clearly, the asymptotic sizes are not sensitive to choices of θ'.

TABLE 3
Sizes of the test (%) under different δ's and β's $(n = 23{,}400, \kappa = 2)$

Value of β	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**
$\delta = 1.0$	4.64	4.22	4.21	4.68	4.61	4.16	3.98	4.03
$\delta = 1.5$	4.23	4.16	4.65	4.54	4.62	4.97	4.35	4.65
$\delta = 2.0$	3.94	4.42	4.41	4.27	4.55	4.42	4.25	4.37
$\delta = 2.5$	4.30	4.62	4.16	4.38	4.47	4.49	4.65	4.08

TABLE 4
Powers of the test (%) under different n's and β's ($\delta = 2$, $\kappa = 2$)

Value of β	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**
$n = 1560$	100	100	100	94.10	48.79	19.86	9.85	5.38
$n = 2340$	100	100	100	91.36	46.46	20.53	10.43	5.82
$n = 4680$	100	100	100	89.55	48.52	21.85	17.55	6.14
$n = 11{,}700$	100	100	100	92.78	55.33	26.54	12.67	7.13
$n = 23{,}400$	100	100	100	96.01	63.80	31.05	14.59	7.34

Comparisons with AJ's test. Now we compare the performance of our estimator $\tilde{V}_n$ with that of AJ's estimator S_n, under the same settings as in (5.1) and (5.2). However, since AJ's test is only shown to be valid for the case $\beta \in (0, 1)$ (i.e., the jump process is of finite variation), our comparisons are also restricted to that case. Tables 5 and 6 report the sizes and powers of our test and AJ's test for various values of $\beta \in (0, 1)$, respectively.

- For both tests, all sizes are close to to the nominal level, 5%, with AJ's test being slightly closer overall.
- Our test outperforms AJ's in terms of power throughout. In fact, our test has full power for all $\beta \in (0, 1]$, even for sample size $n = 1560$. On the other hand, AJ's test has very low power in detecting the alternatives for $\beta \leq 0.7$, even when $n = 23{,}400$.

The very low powers of AJ's test for small β came as a surprise to us. Some more detailed analysis suggests that the reason might be due to the large variation of S_n for finite sample size n under H_1. More precisely, from (69) in Aït-Sahalia and Jacod (2010), we have $S_n = O_P(u_n^{\beta/2})$ under H_1. So for finite sample n, S_n may not be close to 0 for small β, which often places the test statistic S_n wrongly within the acceptance region, resulting in low power. It also explains why the problem is mostly pronounced if β is closer to 0.

Figure 3 displays the histograms of the studentized S_n as given in (4.3) when $n = 4680$. We see that values of studentized S_n's are seldom less than $z_{0.05} = -1.645$, except in the case $\beta = 1$.

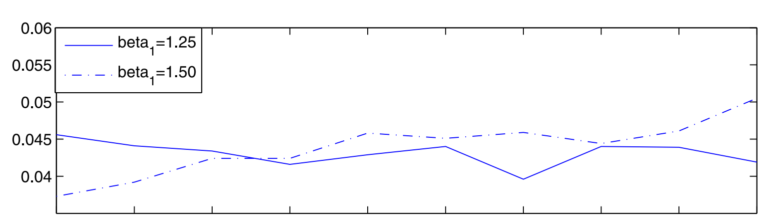

FIG. 2. *Sensitivity plot of asymptotic sizes to choices of θ'.*

TABLE 5
Size of our test v.s. that of AJ's test, (%), $\delta = 2$, $\kappa = 2$

	β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$n = 1560$	$\tilde{V}_n$	4.19	3.81	4.18	3.90	4.08	3.84	3.62	3.59	4.06	4.11
	AJ	4.08	4.23	4.22	3.94	3.99	4.29	4.24	4.10	4.37	4.41
$n = 2340$	$\tilde{V}_n$	4.31	4.37	3.97	3.98	4.16	4.10	4.18	3.93	4.11	4.32
	AJ	4.36	4.33	4.23	3.89	4.67	4.47	4.61	4.29	4.54	4.47
$n = 4680$	$\tilde{V}_n$	4.07	4.38	4.01	4.57	3.87	4.24	4.34	4.18	4.52	4.43
	AJ	4.52	4.53	4.33	4.64	4.56	4.43	4.41	4.71	4.89	4.76

Sensitivity to model misspecification of our test. In the model assumptions, we assumed a local volatility function. Now we conduct a simulation study to check the sensitivity of our test to model misspecification; see Figure 4. Instead of using an Ornstein–Urlenbeck process as the continuous part of the full model, we use a stochastic volatility process here, that is, $dX_t = \sigma_t \, dW_t$ with $\sigma_t = v_t^{1/2}$, $dv_t = \kappa(\eta - v_t)\, dt + \gamma v_t^{1/2} \, dB_t$, $E[dW_t \, dB_t] = \rho \, dt$. We take $\eta = 1/16$, $\gamma = 0.5$, $\kappa = 5$, $\rho = -0.5$. We use $\theta' S_{\beta,t}$ as the jump process as in last two simulations. Now we fix $n = 23{,}400$, $\delta = 1$, $\kappa = 2$, $\theta' = 0.25$ and $\theta = 5\%$. All simulations are run 10,000 times. From Figure 4, the asymptotic sizes are not much affected by using a stochastic volatility model as the continuous part.

6. A real data set analysis. In this section, we implement our test to some real data sets. We use the stock price records of Microsoft (MFST) in there trading days, Nov. 1, Dec. 1 and Dec. 11 in the year 2000. All data sets are from the TAQ database. For prices recorded simultaneously, we use their averages. To weaken the possible effect from microstructure noise, we sparsely sample observations every 10 seconds and the sample sizes for the aforementioned three trading days are

TABLE 6
Power comparisons of our test v.s. that of AJ's test when $\beta \in (0, 1]$, (%), $\delta = 2$, $\kappa = 2$

	β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$n = 1560$	$\tilde{V}_n$	100	100	100	100	100	100	100	100	100	100
	AJ	0	0	0	0	0	0.01	0.25	2.03	7.61	24.03
$n = 2340$	$\tilde{V}_n$	100	100	100	100	100	100	100	100	100	100
	AJ	0	0	0	0.01	0	0.07	0.43	3.32	14.82	48.36
$n = 4680$	$\tilde{V}_n$	100	100	100	100	100	100	100	100	100	100
	AJ	0	0	0	0	0	0.08	0.97	8.66	48.26	97.02
$n = 23{,}400$	$\tilde{V}_n$	100	100	100	100	100	100	100	100	100	100
	AJ	0	0	0	0	0	0.30	6.40	85	100	100

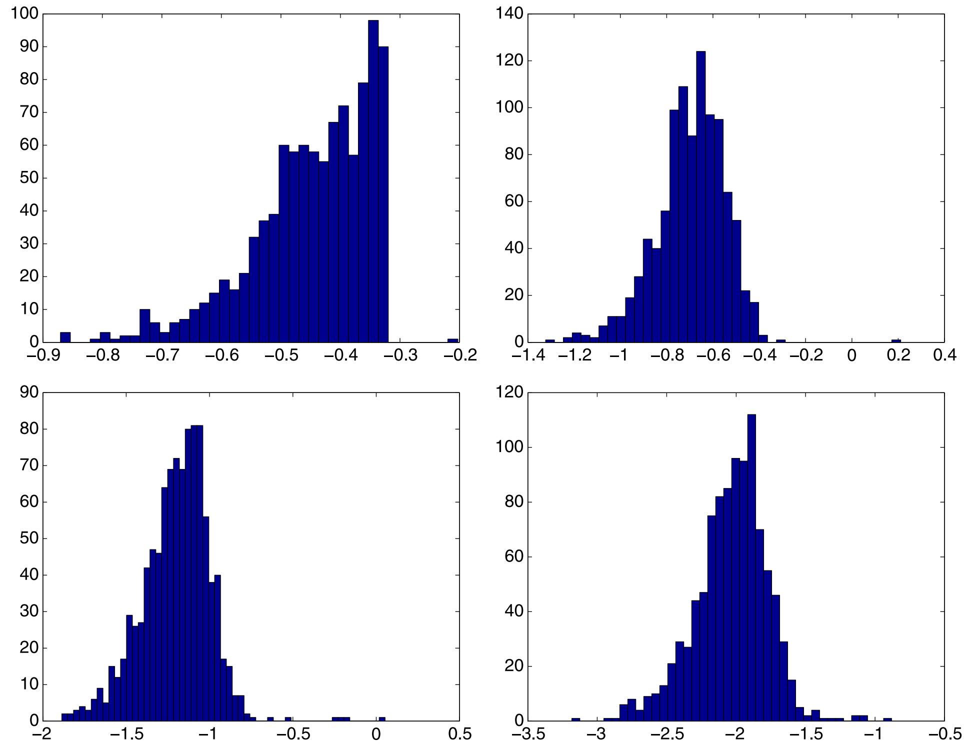

FIG. 3. *Upper left panel*: $\beta = 0.25$; *upper right panel*: $\beta = 0.50$; *lower left panel*: $\beta = 0.75$; *lower right panel*: $\beta = 1.00$.

1343, 1701 and 1253, respectively. Finally, we take the logarithm of the sparsely sampled prices and use the log prices to calculate the test statistics. We set $T = 1$ (day) consisting of 6.5 hours of trading time.

We now discuss how to choose the parameters δ, κ and ϖ. As argued theoretically at the beginning of Section 5, we fix $\varpi = 1.5$. Since κ and δ are dependent parameters, we fix $\kappa = 2$ and consider a grid of points of δ such that

$$\delta(\log n)^{\kappa} \Delta_n^{\varpi} \leq \hat{\sigma}^* \Delta_n^{1/2}, \tag{6.1}$$

where $\hat{\sigma}^*$ is approximately the averaged standard deviation of the diffusion component of one 10-second log return in case the diffusion term exists in the underlying

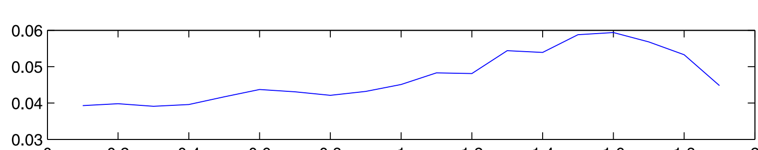

FIG. 4. *The data generating process is the combination of a stochastic volatility process and a standard symmetric stable process.*

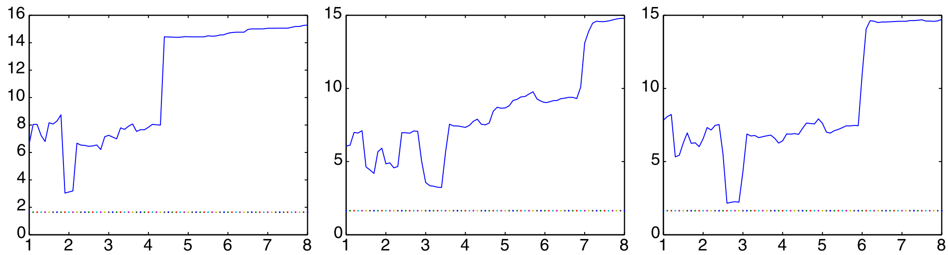

FIG. 5. *The statistics evaluated over different values of* δ. *From left to right*: *test statistics for* 01, *Nov*., 01, *Dec. and* 11, *Dec*., *respectively. The horizontal axis stands for the value of* δ *while the vertical axis stands for the value of the test statistics*.

dynamics. Mathematically, $\hat{\sigma}^*$ is defined as

$$\hat{\sigma}^{*2} =: \frac{1}{T}\sum(\Delta_i^n X)^2 I(|\Delta_i^n X| \le \Delta_n^{1/4}) \to^P \frac{1}{T}\int_0^T \sigma^2(X_s)\,ds;$$

see Jacod (2008) for example. In virtue of (6.1), we can choose δ conservatively as the grid points from 1 to 8 with equal step length 0.1 for all three data sets. The plots are displayed in Figure 5.

From the plots, the observed test statistics are all larger than 1.645. Therefore we can reject the existence of the diffusion component and simply use a pure jump model to characterize the underlying dynamics of the prices for those three days.

7. Discussions on microstructure noise. It is widely accepted nowadays that microstructure noise is present. Various methods have been studied to handle the issue of the microstructure noise in the context of the integrated volatility estimation for high-frequency data. See, for example, Aït-Sahalia, Mykland and Zhang (2005), Zhang, Mykland and Aït-Sahalia (2005), Zhang (2006), Fan and Wang (2007), Podolskij and Vetter (2009) and Jacod et al. (2009), among others. A very effective technique in handling microstructure noise is the so-called "pre-averaging method"; see Jacod et al. (2009) and Podolskij and Vetter (2009).

Suppose that the observation at time t_i is

$$Z_{t_i} = Y_{t_i} + \varepsilon_{t_i}, \qquad i = 1, \dots, n,$$

where Y_t is an unobserved semi-martingale of the form (1.1) and (1.2), and ε_{t_i} with mean 0 and variance σ^2 is the microstructure noise at time t_i. We wish to test (1.3) and (1.4), that is, whether Y_t can be modeled as a pure jump process, or not.

So far, we have not seen any work in the testing framework in the presence of microstructure noise. We now apply the simplest pre-averaging technique as follows. We first separate the full data set Z_{t_i}, $1 \le i \le n$ into n/M nonoverlapped blocks,

$$\{Z_{t_1}, \dots, Z_{t_M}\}, \dots, \{Z_{t_{kM+1}}, \dots, Z_{(k+1)M}\}.$$

Then within each block, we take the average of all K-step increments, that is,

$$\overline{Z}_j = \frac{1}{n/M - K + 1} \sum_{i=kM+K+1}^{(k+1)M} (Z_{t_i} - Z_{t_{i-K}}) := \overline{X}_j + \overline{J}_j + \overline{\varepsilon}_j, \qquad j = 1, \dots, n/M.$$

Simple calculation yields $\overline{X}_j = O_p(M^{-1/2}), \overline{J}_j = O_p(M^{-1/\beta}), \overline{\varepsilon} = O_p((M/n)^{1/2})$. By properly tuning M, for example, $M = o(n^{1/2})$, one could make $\overline{\varepsilon}_j$ asymptotically negligible. Based on the modified data set $\overline{Z}_1, \dots, \overline{Z}_M$, the test statistics can be defined (similarly to V_n) as

$$\overline{V}_n = \frac{\overline{U}(\Delta_M)}{\overline{U}(k\Delta_M)},$$

where $\Delta_M = T/M$, $\overline{U}(\Delta_M)$ and $\overline{U}(k\Delta_M)$ are defined as $U(\Delta_M)$ and $U(k\Delta_M)$ by replacing Y_i with $\overline{Z}_i$ and by replacing Δ_n by Δ_M, for example, $U(\Delta_M) = \sum_{i=1}^{M} I(|\overline{Z}_i| \le \alpha(\Delta_M)^{\varpi})$.

Under appropriate conditions, the results obtained in the paper should be expected to hold here as well, for instance,

$$\overline{V}_n \to^P \begin{cases} k^{1.5-\varpi}, & \text{under } H_0, \\ k^{1+(1/\beta-\varpi)\wedge 0}, & \text{under } H_1. \end{cases} \tag{7.1}$$

Let us conduct a simple simulation study to investigate the feasibility of the test statistic $\overline{V}_n$. Take $Y_t = W_t + S_t$ under H_0 and $Y_t = S_t$ under H_1, where W_t and S_t are a standard Brownian motion and a symmetric Cauchy process (i.e., $\beta = 1$), respectively. Also take $\sigma^2 \sim N(0, \sigma^2)$ with $\sigma = 0.01$. We let $T = 1$, $n = 23{,}400$ and $k = 2$. We further take $M = 234$, $K = 50, \alpha = 9, \varpi = 1.5$. Note that the choice of $M = 234$ corresponds to taking averages about every 4 minutes. The simulation is repeated 5000 times. Each time, we calculate $\overline{V}_n$ both under H_0 and H_1. Their histograms under H_0 and H_1 are plotted in Figure 6.

From Figure 6, we see that the means of $\overline{V}_n$ under H_0 and H_1 (marked by $*$ in the horizontal axis) are 1.0578 and 1.4781, respectively. These are rather close to the asymptotic values 1 and 1.414, given by (7.1). Note that the effective sample size after pre-averaging is 23,400/234 = 100, a rather small sample size for this testing purpose. This explains partly why the variances of histograms plots are rather large, and there are substantial overlaps between the plots under H_0 and H_1. If we choose $M = 120$, or even 60, then the histograms under H_0 and H_1 will become thinner and more easily separable.

The above simple simulation study suggests that the pre-averaging method would work well in handling microstructure noise in the testing problems. Of course, there remain many theoretical and practical issues to be resolved. For example, we need to establish a CLT under H_0; to study its asymptotic power; to find a data-driven method to determine parameter M, etc. We will pursue these and other related issues in our future work.

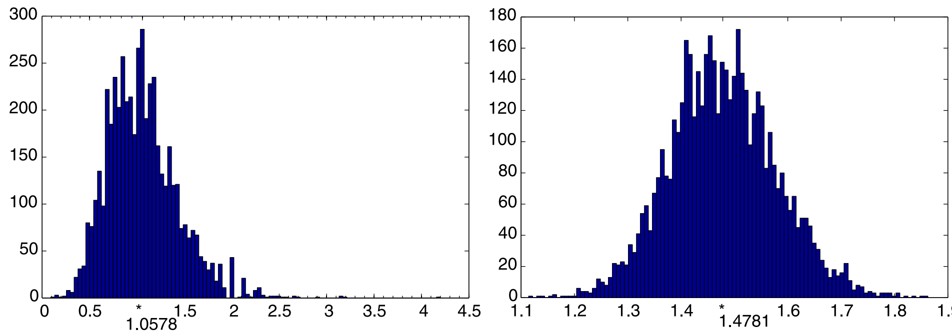

FIG. 6. *Histograms of* $\overline{V}_n$.

APPENDIX

In the sequel, C will denote a constant which may take different values in different places, and χ is an arbitrarily small positive number. Also, $P_{t_{i-1}}$ and $E_{t_{i-1}}$ denote probability and expectation given time t_{i-1}, respectively.

A.1. Proof of Theorem 1. Let $\sigma_0^2 = 2\alpha\phi(0)\int_0^T \sigma^{-1}(X_s)\,ds$. Now

$$\begin{aligned}\Delta_n^{(\varpi-3/2)/2}&(V_n - k^{3/2-\varpi})\\ &= \frac{\Delta_n^{(\varpi-3/2)/2}(\tilde{U}(\Delta_n)-\sigma_0^2) - k^{(3/2-\varpi)/2}(k\Delta_n)^{(\varpi-3/2)/2}(\tilde{U}(k\Delta_n)-\sigma_0^2)}{k^{\varpi-3/2}\tilde{U}(k\Delta_n)}\\ &:= \frac{A}{B}.\end{aligned}$$

By Proposition 1 (below), $A \to^S \sigma_0 z_1 - k^{(3/2-\varpi)/2}\sigma_0 z_2$ with z_1 and z_2 independent Gaussian random variables independent of $\mathcal{F}^Y$, while $B \to^P k^{\varpi-3/2}\sigma_0^2$, which is random but depending only on $\mathcal{F}^Y$. Then Theorem 1 is proved.

Now, we prove Proposition 1, in which we need the following two lemmas. Lemma 2 implies that the proportion of paths of a jump diffusion process having "small" increments is the same as that of the diffusion component. This has its own interest.

LEMMA 1. *Let* $A_i = \{\omega : |\Delta_i^n X + x| \le \alpha\Delta_n^{\varpi}\}$.

(1) *For* $|x| < \Delta_n^{1/2}$, $|P_{t_{i-1}}(A_i) - \frac{2\alpha\phi(0)\Delta_n^{\varpi-1/2}}{\sigma(-x+X_{t_{i-1}})}| \le C\Delta_n^{\varpi}(x^2\Delta_n^{-3/2} + \Delta_n^{-\chi})$.

(2) *For any* $x \in R/\{0\}$, *we have* $P_{t_{i-1}}(A_i) \le C\Delta_n^{\varpi-1/2}$.

PROOF. Define $f(x) = \int_0^x \sigma^{-1}(y)\,dy$, then $f'(x) = \sigma^{-1}(x)$ and $f''(x) = -\sigma'(x)/\sigma^2(x)$. So $f(x)$ is strictly increasing. Let $\Xi_t = f(X_t)$, or equivalently,

$X_t = f^{-1}(\Xi_t)$. By Itô's formula,

$$
\begin{aligned}
d\Xi_t &= \left(\frac{b \circ f^{-1}(\Xi_t)}{\sigma \circ f^{-1}(\Xi_t)} - \frac{1}{2}\sigma' \circ f^{-1}(\Xi_t)\right) dt + dW_t \\
&:= \overline{b} \circ f^{-1}(\Xi_t)\, dt + dW_t.
\end{aligned} \tag{A.1}
$$

Let $\mathcal{F}'_t = \mathcal{F}_{t+t_{i-1}}$ where t_{i-1} is the $(i-1)$th observation time defined at the end of the Introduction, and $\tilde{W}_t = W_{t+t_{i-1}}$, $t \ge 0$. It is easy to see that $\tilde{W}$ is a martingale under $(\Omega, \mathcal{F}, \mathcal{F}'_t, P_{t_{i-1}})$ with quadratic variation t. Thus by Lévy's characterization theorem, $\tilde{W}_t$ is a Brownian motion under $(\Omega, \mathcal{F}, \mathcal{F}'_t, P_{t_{i-1}})$. By the Girsanov theorem, there exists a probability measure $Q_{t_{i-1}}$, locally equivalent to $P_{t_{i-1}}$, satisfying

$$
\left.\frac{dQ_{t_{i-1}}}{dP_{t_{i-1}}}\right|_{\mathcal{F}'_t} = \exp\left(-\int_0^t \overline{b}(X_{s+t_{i-1}})\, d\tilde{W}_s - \frac{1}{2}\int_0^t \overline{b}^2(X_{s+t_{i-1}})\, ds\right), \tag{A.2}
$$

such that $\Xi_{t+t_{i-1}}$, $t \ge 0$, is a Brownian motion under $Q_{t_{i-1}}$.

Now

$$
\begin{aligned}
P_{t_{i-1}}(A_i) &= P_{t_{i-1}}(-x - \alpha\Delta_n^{\varpi} + X_{t_{i-1}} \le X_{t_i} \le -x + \alpha\Delta_n^{\varpi} + X_{t_{i-1}}) \\
&= P_{t_{i-1}}\big(f(-x + X_{t_{i-1}} - \alpha\Delta_n^{\varpi}) \\
&\qquad \le f(X_{t_i}) \le f(-x + X_{t_{i-1}} + \alpha\Delta_n^{\varpi})\big) \\
&= P_{t_{i-1}}(l_i \le \Xi_{t_i} - \Xi_{t_{i-1}} \le u_i),
\end{aligned} \tag{A.3}
$$

where $l_i = f(-x + X_{t_{i-1}} - \alpha\Delta_n^{\varpi}) - f(X_{t_{i-1}})$ and $u_i = f(-x + X_{t_{i-1}} + \alpha\Delta_n^{\varpi}) - f(X_{t_{i-1}})$. Taking $t = t_i - t_{i-1}$ in (A.2), we then have

$$
\begin{aligned}
P_{t_{i-1}}(A_i) &= \int_{A_i} \frac{dP_{t_{i-1}}}{dQ_{t_{i-1}}}\, dQ_{t_{i-1}} \\
&= \int_{A_i} \exp\left(\int_{t_{i-1}}^{t_i} \overline{b} \circ f^{-1}(\Xi_s)\, dW_s \right. \\
&\qquad\qquad \left. + \frac{1}{2}\int_{t_{i-1}}^{t_i} (\overline{b} \circ f^{-1})^2(\Xi_s)\, ds\right) dQ_{t_{i-1}}.
\end{aligned} \tag{A.4}
$$

Since $|\exp(x) - 1| \le 2|x|$ for $|x| \le \log 2$, by boundedness of the diffusion coefficient, Lévy's theorem of continuity modulus and change of time,

$$
\begin{aligned}
&\big|P_{t_{i-1}}(A_i) - \big(\Phi(u_i/\sqrt{\Delta_n}) - \Phi(l_i/\sqrt{\Delta_n})\big)\big| \\
&\qquad \le C\Delta_n^{1/2-\chi}\big(\Phi(u_i/\sqrt{\Delta_n}) - \Phi(l_i/\sqrt{\Delta_n})\big),
\end{aligned}
$$

for any arbitrarily small $\chi > 0$. On the other hand, by the mean value theorem,

$$
\Phi(u_i/\sqrt{\Delta_n}) - \Phi(l_i/\sqrt{\Delta_n}) = \phi(\xi)\frac{(u_i - l_i)}{\sqrt{\Delta_n}} = 2\alpha\Delta_n^{\varpi-1/2}\frac{\phi(\xi)}{\sigma(\eta)}, \tag{A.5}
$$

where $\xi \in \frac{1}{\sqrt{\Delta_n}}(l_i, u_i)$ and $\eta \in (-x + X_{t_{i-1}} - \alpha\Delta_n^{\varpi}, -x + X_{t_{i-1}} + \alpha\Delta_n^{\varpi})$. Then as $\Delta_n \to 0$, by Assumption 3, we have

$$|\sigma(\eta) - \sigma(-x + X_{t_{i-1}})| \le C\Delta_n^{\varpi}. \tag{A.6}$$

Since as n large enough $|u_i| \le C|x|$ and $|l_i| \le C|x|$ which yields that $\xi \in \frac{1}{\sqrt{\Delta_n}}(-C|x|, C|x|)$. Then, since $\phi'(0) = 0$ and $\phi''(\cdot)$ is bounded,

$$|\phi(\xi) - \phi(0)| \le C(x)^2\Delta_n^{-1}. \tag{A.7}$$

The combination of (A.3)–(A.7) completes the proof. □

LEMMA 2. *Let $B_i = \{\omega : |\Delta_i^n Y| \le \alpha\Delta_n^{\varpi}\}$. Then,*

$$\left| P_{t_{i-1}}(B_i) - \frac{2\alpha\phi(0)}{\sigma(X_{t_{i-1}})}\Delta_n^{\varpi-1/2} \right| \le C(\Delta_n^{\varpi-1/2+1-\beta/2} + \Delta_n^{\varpi-\chi}).$$

PROOF. We write

$$\begin{aligned} &P_{t_{i-1}}(B_i) \\ &\quad = \left(\int_{|x|<\sqrt{\Delta_n}} + \int_{|x|\ge\sqrt{\Delta_n}}\right) P_{t_{i-1}}(|\Delta_i^n X + x| \le \alpha\Delta_n^{\varpi})\, dP_{t_{i-1}}(\Delta_i^n J \le x) \\ &\quad =: P_{i,1} + P_{i,2}. \end{aligned} \tag{A.8}$$

Since J is purely discontinuous, we can take the exponent in (64) of Aït-Sahalia and Jacod (2009) as $1/2$, and then by Lemma 1,

$$P_{i,2} \le C\Delta_n^{\varpi-1/2} P_{t_{i-1}}\big(|\Delta_i^n J| \ge \sqrt{\Delta_n}\big) \le C\Delta_n^{\varpi-1/2+1-\beta/2}. \tag{A.9}$$

By Lemma 1,

$$P_{i,1} = \int_{|x|<\sqrt{\Delta_n}} \frac{2\alpha\phi(0)\Delta_n^{\varpi-1/2}}{\sigma(-x + X_{t_{i-1}})}\, dP_{t_{i-1}}(\Delta_i^n J \le x) + R_{n,i}. \tag{A.10}$$

Similarly, we can obtain

$$\begin{aligned} |R_{n,i}| &\le \int_{|x|<\sqrt{\Delta_n}} C[(x)^2\Delta_n^{\varpi-3/2} + \Delta_n^{\varpi-\chi}]\, dP_{t_{i-1}}(\Delta_i^n J \le x) \\ &= C\Delta_n^{\varpi-3/2} E_{t_{i-1}}(\Delta_i^n J)^2 I\big(|\Delta_i^n J| < \sqrt{\Delta_n}\big) + C\Delta_n^{\varpi-\chi} \\ &\le C(\Delta_n^{\varpi-1/2+1-\beta/2} + \Delta_n^{\varpi-\chi}). \end{aligned} \tag{A.11}$$

Since $|x| < \sqrt{\Delta_n}$, $|\sigma(-x + X_{t_{i-1}}) - \sigma(X_{t_{i-1}})| \le C\sqrt{\Delta_n}$, by boundedness of the diffusion coefficient,

$$\begin{aligned} &\int_{|x|<\sqrt{\Delta_n}} 2\alpha\phi(0)\Delta_n^{\varpi-1/2}\left(\frac{1}{\sigma(-x + X_{t_{i-1}})} - \frac{1}{\sigma(X_{t_{i-1}})}\right) dP_{t_{i-1}}(\Delta_i^n J \le x) \\ &\quad \le C\Delta_n^{\varpi}. \end{aligned} \tag{A.12}$$

On the other hand, as in (A.9), we have

$$\left|\frac{2\alpha\phi(0)\Delta_n^{\varpi-1/2}}{\sigma(X_{t_{i-1}})}\left(\int_{|x|\le\sqrt{\Delta_n}} dP_{t_{i-1}}(\Delta_i^n D\le x)-1\right)\right| \le C\Delta_n^{\varpi-1/2+1-\beta/2}. \tag{A.13}$$

Combining (A.10), (A.12) and (A.13) gives

$$\left|P_{i,1}-\frac{2\alpha\phi(0)\Delta_n^{\varpi-1/2}}{\sigma(X_{t_{i-1}})}\right|\le C(\Delta_n^{\varpi-1/2+1-\beta/2}+\Delta_n^{\varpi-\chi}),$$

which together with (A.9) completes the proof. $\square$

Define $\tilde U(\Delta_n)=\Delta_n^{3/2-\varpi}U(\Delta_n)$, and so $\tilde U(k\Delta_n)=(k\Delta_n)^{3/2-\varpi}U(k\Delta_n)$. Then we have

PROPOSITION 1. *We have*

$$\Delta_n^{(\varpi-3/2)/2}\big(\tilde U(\Delta_n)-\sigma_0^2, k^{(\varpi-3/2)/2}[\tilde U(k\Delta_n)-\sigma_0^2]\big) \to \sigma_0(z_1,z_2), \qquad \mathcal{F}^Y\text{-stably},$$

where z_1 and z_2 are two independent Gaussian variables independent of $\mathcal{F}^Y$.

PROOF. Without loss of generality, assume $k=2$. Denote $I_i=I(|\Delta_i^nY|\le\alpha\Delta_n^{\varpi})$. In view of Lemma 2,

$$\left|\Delta_n^{3/2-\varpi}\sum_{i=1}^{[T/\Delta_n]}E_{t_{i-1}}I_i-\sigma_0^2\right|\le C(\Delta_n^{1-\beta/2}+\Delta_n^{1/2-\chi}). \tag{A.14}$$

Since χ could be made arbitrarily small, and $\varpi>\beta-1/2$, or equivalently, $1-\beta/2>\frac{3/2-\varpi}{2}$,

$$\Delta_n^{(\varpi-3/2)/2}\big(\tilde U(\Delta_n)-\sigma_0^2\big)=\Delta_n^{3/2-\varpi/2}\sum_{i=1}^{[T/\Delta_n]}(I_i-E_{t_{i-1}}I_i)+o(1). \tag{A.15}$$

Now the summands in (A.15) are centered martingale difference sequences w.r.t. $\mathcal{F}_{t_{i-1}}$, $1\le i\le[T/\Delta_n]$. In view of Lemma 2, and making use of (A.14) again,

$$\Delta_n^{3/2-\varpi}\sum_{i=1}^{[T/\Delta_n]}E_{t_{i-1}}(I_i-E_{t_{i-1}}I_i)^2=\Delta_n^{3/2-\varpi}\sum_{i=1}^{[T/\Delta_n]}E_{t_{i-1}}I_i+o_P(1) = \sigma_0^2+o_P(1).$$

Since the indicator function is bounded, the Linderberg condition for the martingale central limit theorem holds automatically. Then by (A.14) and (A.15),

$$\Delta_n^{(\varpi-3/2)/2}\big(\tilde U(\Delta_n)-\sigma_0^2\big)\to\sigma_0 z_1 \tag{A.16}$$

$\mathcal{F}^Y$-stably if the following holds [c.f. Theorem IX 7.28 of Jacod and Shiryaev (2003)]: for any bounded martingale $N \in \mathcal{F}^Y$

$$\Delta_n^{(3/2-\varpi)/2} \sum_{i=1}^{[T/\Delta_n]} E_{t_{i-1}}(\Delta_i^n N) I_i \to^P 0. \tag{A.17}$$

Since $\mathcal{F}^Y = \mathcal{F}^X \vee \mathcal{F}^J$, it suffices to show (A.17) with N replaced by X and $N_1 \in \mathcal{F}^J$, respectively, where N_1 is a bounded martingale. By Lévy's theorem of continuity modulus, (A.14) and $\varpi > 1/2$,

$$\begin{aligned} &\Delta_n^{(3/2-\varpi)/2} \sum_{i=1}^{[T/\Delta_n]} E_{t_{i-1}}(\Delta_i^n X) I_i \\ &\quad \le C \Delta_n^{(3/2-\varpi)/2+1/2-\chi} \sum_{i=1}^{[T/n]} E_{t_{i-1}} I_i \to^P 0. \end{aligned} \tag{A.18}$$

Next, by independence of X from $\mathcal{F}^{\mathcal{J}}$ and Lemma 1,

$$\Delta_n^{(3/2-\varpi)/2} \sum_{i=1}^{[T/\Delta_n]} E_{t_{i-1}}(\Delta_i^n N_1) I_i \le C \Delta_n^{\varpi/2+1/4} \sum_{i=1}^{[T/\Delta_n]} E_{t_{i-1}} |\Delta_i^n N_1|. \tag{A.19}$$

By Cauchy–Schwarz and Jensen's inequalities, the orthogonality of the martingale increments, the expectation of the right-hand side in (A.19) is

$$\begin{aligned} &\le C \Delta_n^{\varpi/2+1/4} E\left(\sum_{i=1}^{[T/\Delta_n]} \sqrt{E_{t_{i-1}}(\Delta_i^n N_1)^2} \right) \\ &\le C \Delta_n^{\varpi/2+1/4} \frac{T}{\Delta_n} \sqrt{\frac{\Delta_n}{T} E\left(\sum_{i=1}^{[T/\Delta_n]} E_{t_{i-1}}(\Delta_i^n N_1)^2 \right)} \\ &\le C \Delta_n^{(\varpi-1/2)/2} \sqrt{E(N_{1,T} - N_{1,0})^2}. \end{aligned} \tag{A.20}$$

Since $\varpi > 1/2$, (A.17) holds. Similarly, we can deduce that

$$(k\Delta_n)^{(\varpi-3/2)/2} (\tilde{U}(k\Delta_n) - \sigma_0^2) \to \sigma_0 z_2 \tag{A.21}$$

$\mathcal{F}^Y$-stably. Finally in view of (A.16) and (A.21), and by virtue of Lemma 2 and (A.14), to complete the proof, it suffices to show that

$$\begin{aligned} &\Delta_n^{3/2-\varpi} \sum_{i=1}^{[T/k\Delta_n]} E_{t_{i-1}}\left(I(|\Delta_{i,k}^n Y| \le \alpha (k\Delta_n)^{\varpi}) \sum_{j=i}^{i+k-1} I(|\Delta_j^n Y| \le \alpha \Delta_n^{\varpi}) \right) \\ &\quad \to^P 0, \end{aligned} \tag{A.22}$$

where $\Delta_{i,k}^n Y = \sum_{j=i}^{i+k-1} \Delta_j^n Y$. To this end, we give an estimate of the summands in (A.22). Let $\Delta_{i,k}^{n,-j} Y = \Delta_{i,k}^n Y - \Delta_j^n Y$, $\Delta_{i,k}^{n,j-} = \sum_{l=i}^{j-1} \Delta_l^n Y$ and $\Delta_{i,k}^{n,j+} = \sum_{l=j+1}^{i+k-1} \Delta_l^n Y$, for $i \le j \le i+k-1$. We make the convention that $\Delta_{i,k}^{n,i-} = \Delta_{i,k}^{n,(i+k-1)+} = 0$. Then there exists a constant C such that

$$\begin{aligned}&\{|\Delta_{i,k}^n Y| \le \alpha(k\Delta_n)^{\varpi}\} \cap \{|\Delta_j^n Y| \le \alpha\Delta_n^{\varpi}\}\\ &\qquad \subset \{|\Delta_{i,k}^{n,-j} Y| \le C\Delta_n^{\varpi}\} \cap \{|\Delta_j^n Y| \le \alpha\Delta_n^{\varpi}\},\end{aligned}$$

and consequently, in view of $k = 2$, we have

$$\begin{aligned}&E_{t_{i-1}} I\big(|\Delta_{i,2}^n Y| \le \alpha(2\Delta_n)^{\varpi}\big) I(|\Delta_j^n Y| \le \alpha\Delta_n^{\varpi})\\ &\quad \le E_{t_{i-1}}[I(|\Delta_{i,2}^{n,j-} Y| \le C\Delta_n^{\varpi}) E_{t_{j-1}} I(|\Delta_j^n Y| \le \alpha\Delta_n^{\varpi})]\\ &\qquad + E_{t_{i-1}}[I(|\Delta_j^n Y| \le \alpha\Delta_n^{\varpi}) E_{t_j} I(|\Delta_{i,2}^{n,j+} Y| \le C\Delta_n^{\varpi})]\\ &\quad \le C\Delta_n^{2\varpi-1} \qquad \text{(by Lemma 2)}.\end{aligned} \tag{A.23}$$

Substituting (A.23) into the left-hand side of (A.22), we deduce that the left-hand side of (A.22) is less than $C\Delta_n^{\varpi-1/2}$. Since $\varpi > 1/2$, (A.22) is proved. $\square$

A.2. Proof of Theorem 2. We start with the proof of the following equation which is implied by Lemmas 3 and 4:

$$\Delta_n^{1+(1/\beta-\varpi)\wedge 0} U(\Delta_n) \longrightarrow^P 2\alpha C_\beta \qquad \text{under } H_1, \tag{A.24}$$

where C_β is some constant. Then Theorem 2 is a straight consequence of (A.24), since now $\tilde{V}_n \to^P 2^{1+1/\beta-\varpi} > 2^{3/2-\varpi}$ and $\tilde{C} \to^P 2^{3/2-\varpi}$.

Now X vanishes to a deterministic drift satisfying $dX_t = b(X_t)\,dt$. For simplicity, we assume that $\varepsilon^- = \varepsilon^+ =: \varepsilon$. Then Y admits the following decomposition:

$$\begin{aligned}Y_t &= X_t + \int_0^t \int_{|x|\le\varepsilon} x(\mu-\nu)(dx,ds) + \int_0^t \int_{|x|>\varepsilon} x\mu(dx,ds)\\ &\quad - \int_0^t \int_{\varepsilon<|x|\le 1} x F_s''(dx)\,ds\\ &:= X_t + J_{1,t} + J_{2,t} + J_{3,t}.\end{aligned}$$

The next lemma reveals that the count of small increments has almost nothing to do with the large jumps.

LEMMA 3. *Under the conditions in Theorem 2,*

$$\big|P_{t_{i-1}}(|\Delta_i^n Y| \le \alpha\Delta_n^{\varpi}) - P_{t_{i-1}}\big(|\Delta_i^n (Y - J_2)| \le \alpha\Delta_n^{\varpi}\big)\big| \le C\Delta_n.$$

PROOF. Let $M_t = \sum_{0\le s\le t} I(|\Delta_s Y| > \varepsilon)$. Then M is a Poisson counting process with ω wise time dependent intensity function $\int_{|x|>\varepsilon} F_s''(dx)$ and

$$P_{t_{i-1}}(\Delta_i^n M \ge 1) \le 1 - \exp\left(-\int_{t_{i-1}}^{t_i}\int_{|x|>\varepsilon} F_s''(dx)\,ds\right) \le C\Delta_n. \tag{A.25}$$

Notice that on $\Delta_i^n M = 0$, $\Delta_i^n Y = \Delta_i^n (Y - J_2)$, so the difference within the absolute value sign is

$$\begin{aligned} &E_{t_{i-1}}\big[I(|\Delta_i^n Y| \le \alpha\Delta_n^{\varpi}) - I\big(|\Delta_i^n (Y-J_2)| \le \alpha\Delta_n^{\varpi}\big); \Delta_i^n M = 0 \text{ or } \ge 1\big] \\ &\quad = E_{t_{i-1}}\big[I(|\Delta_i^n Y| \le \alpha\Delta_n^{\varpi}) - I\big(|\Delta_i^n (Y-J_2)| \le \alpha\Delta_n^{\varpi}\big); \Delta_i^n M \ge 1\big]. \end{aligned} \tag{A.26}$$

Lemma 3 is a consequence of (A.25) and (A.26). □

LEMMA 4. *Under Assumption 4*,

$$P_{t_{i-1}}\big(|\Delta_i^n (X + J_1 + J_3)| \le \alpha\Delta_n^{\varpi}\big) = C_\beta \Delta_n^{(\varpi - 1/\beta)\wedge 0} + o_P(\Delta_n^{\varpi - 1/\beta}).$$

PROOF. Let $\tilde{l}_i = -\alpha\Delta_n^{\varpi-1/\beta} - \Delta_n^{-1/\beta}(\Delta_i^n X + \Delta_i^n J_3)$ and $\tilde{u}_i = \alpha\Delta_n^{\varpi-1/\beta} - \Delta_n^{-1/\beta}(\Delta_i^n X + \Delta_i^n J_3)$. The required probability is equal to

$$P_{t_{i-1}}(\tilde{l}_i \le \Delta_n^{-1/\beta}\Delta_i^n J_1 \le \tilde{u}_i). \tag{A.27}$$

Now we prove the lemma in two cases: (i) $\beta \ge 1$ and (ii) $\beta < 1$.

Case (i): $\beta > 1$. By the Lévy–Khintchine formula,

$$E_{t_{i-1}} \exp(i\theta\Delta_n^{-1/\beta}\Delta_i^n J_1) = \exp(\Delta_n \psi(\Delta_n^{-1/\beta}\theta)),$$

where $\psi(u) = \int_R \{\exp(iuy) - 1 - iuyI(|y| \le 1)\}F'(dy)$. By a change of variable, we have

$$\begin{aligned} \psi(\Delta_n^{-1/\beta}\theta) &= \int_R \big(\exp(i\theta z) - 1 - i\theta z I(|z| \le 1)\big) F'(\Delta_n^{1/\beta} z)\Delta_n^{1/\beta}\,dz \\ &\quad - \int_R i\theta z I(1 \le |z| \le \Delta_n^{-1/\beta}) F'(\Delta_n^{1/\beta} z)\Delta_n^{1/\beta}\,dz. \end{aligned} \tag{A.28}$$

Hence,

$$\Delta_n F'(\Delta_n^{1/\beta} z)\Delta_n^{1/\beta} \to \frac{1}{|z|^{1+\beta}}\big(a^{(+)} I(z>0) + a^{(-)} I(z<0)\big) := \tilde{\nu}(z).$$

By the dominant convergence theorem, $E\exp(i\theta\Delta_n^{-1/\beta}\Delta_i^n J_1)$ converges to

$$\int_R \big(\exp(i\theta z) - 1 - i\theta z I(|z| \le 1)\big)\tilde{\nu}(z)\,dz + i\theta/(\beta - 1)\big(a^{(+)} + a^{(-)}\big). \tag{A.29}$$

Therefore we have $\Delta_n^{-1/\beta}\Delta_i^n J_1$ converges in distribution to a stable random variable. Since $(\Delta_i^n X + \Delta_i^n J_3)\Delta_n^{-1/\beta} = o(1)$, by (A.27) and Assumption 4, the lemma is proved in this case.

Case (ii): $\beta < 1$. In this case, we can further decompose J_1 as follows:

$$J_1 = -\int_0^t \int_{|x|\le\varepsilon} x\nu(dx, ds) + \int_0^t \int_{|x|\le\varepsilon} x\mu(dx, ds) := J_{11} + J_{12}.$$

Then the required probability in (A.27) could be rewritten as

$$P_{t_{i-1}}(\tilde{l}_i - \Delta_n^{-1/\beta}\Delta_i^n J_{11} \le \Delta_n^{-1/\beta}\Delta_i^n J_{12} \le \tilde{u}_i - \Delta_n^{-1/\beta}\Delta_i^n J_{12}). \tag{A.30}$$

By similar calculation to (A.29), one gets $\Delta_n^{-1/\beta}\Delta_i^n J_{12}$ converges to a stable random variable. First, consider the case where $\varpi > 1/\beta$. Now by (A.30) and Assumption 4, the lemma is obtained straightforwardly. Second, if $\varpi \le 1/\beta$, by (A.30), the required probability is asymptotically a constant. $\square$

By Lemmas 3 and 4,

$$\Delta_n^{1+(1/\beta-\varpi)\wedge 0}\sum_{i=1}^n E_{t_{i-1}} I(|\Delta_i^n Y| \le \alpha\Delta_n^{\varpi}) \to^P C_\beta 2\alpha,$$

which implies that the conditional variance goes to zero in probability, since

$$\sum_{i=1}^n \Delta_n^{2(1+(1/\beta-\varpi)\wedge 0)} E_{t_{i-1}} I^2(|\Delta_i^n Y| \le \alpha\Delta_n^{\varpi}) \to^P 0.$$

Therefore, a direct use of Lenglart's inequality yields (A.24).

REFERENCES

AÏT-SAHALIA, Y. and JACOD, J. (2009). Estimating the degree of activity of jumps in high frequency data. *Ann. Statist.* **37** 2202–2244. MR2543690

AÏT-SAHALIA, Y. and JACOD, J. (2009). Testing for jumps in a discretely observed process. *Ann. Statist.* **37** 184–222. MR2488349

AÏT-SAHALIA, Y. and JACOD, J. (2010). Is Brownian motion necessary to model high-frequency data? *Ann. Statist.* **38** 3093–3128. MR2722465

AÏT-SAHALIA, Y., MYKLAND, P. A. and ZHANG, L. (2005). How often to sample a continuous-time process in the presence of market microstructure noise? *Review of Financial Studies* **18** 351–416.

BALLOTTA, L. (2005). A Lévy process-based framework for the fair valuation of participating life insurance contracts. *Insurance Math. Econom.* **37** 173–196. MR2172097

BARNDORFF-NIELSEN, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scand. J. Stat.* **24** 1–13. MR1436619

BARNDORFF-NIELSEN, O. E. (1998). Processes of normal inverse Gaussian type. *Finance Stoch.* **2** 41–68. MR1804664

BARNDORFF-NIELSEN, O. E. and SHEPHARD, N. (2001). Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **63** 167–241. MR1841412

BARNDORFF-NIELSEN, O. E. and SHEPHARD, N. (2006). Econometrics of testing for jumps in financial economics using bipower variation. *Journal of Financial Econometrics* **2** 1–48.

BROADIE, M. and DETEMPLE, J. B. (2004). Option pricing: Valuation models and applications. *Management Science* **50** 1145–1177.

BROCKWELL, P. J. (2001). Continuous-Time ARMA Processes. In *Stochastic Processes*: *Theory and Methods*. *Handbook of Statist*. **19** 249–276. North-Holland, Amsterdam. MR1861726

CARR, P., GEMAN, H., MADAN, D. B. and YOR, M. (2002). The fine structure of asset returns: An empirical investigation. *Journal of Business* **75** 305–332.

CARR, P., GEMAN, H., MADAN, D. B. and YOR, M. (2003). Stochastic volatility for Lévy processes. *Math*. *Finance* **13** 345–382. MR1995283

CARR, P. and WU, L. (2007). Stochastic skew for currency options. *Journal of Financial Economics* **86** 213–247.

CONT, R. and MANCINI, C. (2007). Nonparametric tests for probing the nature of asset price processes. Technical report, Univ. Firenze.

CONT, R. and TANKOV, P. (2004). *Financial Modelling with Jump Processes*. Chapman & Hall/CRC, Boca Raton, FL. MR2042661

DAAL, E. and MADAN, D. B. (2005). An empirical examination of the variance-gamma model for foreign currency options. *Journal of Business* **78** 2121–2152.

DROSEN, J. W. (1986). Pure jump shock models in reliability. *Adv. in Appl*. *Probab.* **18** 423–440. MR0840102

FAN, J. and WANG, Y. (2007). Multi-scale jump and volatility analysis for high-frequency financial data. *J*. *Amer. Statist*. *Assoc*. **102** 1349–1362. MR2372538

HUANG, J. Z. and WU, L. (2004). Specification analysis of option pricing models based on time-changed Levy processes. *J*. *Finance* **59** 1405–1440.

HUANG, S. and HUNG, M. W. (2005). Pricing foreign equity options under Levy processes. *Journal of Futures Markets* **25** 917–944.

IVANOV, R. V. (2007). On the pricing of American options in exponential Lévy markets. *J*. *Appl*. *Probab.* **44** 409–419. MR2340207

JACOD, J. (2008). Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Process*. *Appl*. **118** 517–559. MR2394762

JACOD, J., LI, Y., MYKLAND, P. A., PODOLSKIJ, M. and VETTER, M. (2009). Microstructure noise in the continuous case: The pre-averaging approach. *Stochastic Process*. *Appl*. **119** 2249–2276. MR2531091

JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

JIANG, G. J. and OOMEN, R. C. (2005). A new test for jumps in asset prices. Technical report, Warwick Business School, Univ. Warwick.

KLÜPPELBERG, C., LINDNER, A. and MALLER, R. (2004). A continuous-time GARCH process driven by a Lévy process: Stationarity and second-order behaviour. *J*. *Appl*. *Probab.* **41** 601–622. MR2074811

LEE, S. and MYKLAND, P. A. (2008). Jumps in financial markets: A new nonparametric test and jump dynamics. *Review of Financial Studies* **21** 2535–2563.

LEVENDORSKIĬ, S. Z. (2004). Early exercise boundary and option prices in Lévy driven models. *Quant*. *Finance* **4** 525–547. MR2241294

MADAN, D. B. (2006). Equilibrium asset pricing: With non-Gaussian factors and exponential utilities. *Quantitative Finance* **6** 455–463.

MADAN, D., CARR, P. and CHANG, E. (1998). The variance gamma process and option pricing. *European Finance Review* **2** 79–105.

PODOLSKIJ, M. and VETTER, M. (2009). Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps. *Bernoulli* **15** 634–658. MR2555193

RYDBERG, T. H. (1997). The normal inverse Gaussian Lévy process: Simulation and approximation. *Comm*. *Statist*. *Stochastic Models* **13** 887–910. MR1482297

SCHOUTENS, W. (2006). Exotic options under Lévy models: An overview. *J*. *Comput*. *Appl*. *Math*. **189** 526–538. MR2202995

TODOROV, V. and TAUCHEN, G. (2010). Activity signature functions for high-frequency data analysis. *J. Econometrics* **154** 125–138. MR2558956

WOERNER, J. H. C. (2003). Variational sums and power variation: A unifying approach to model selection and estimation in semimartingale models. *Statist. Decisions* **21** 47–68. MR1985651

ZHANG, L. (2006). Efficient estimation of stochastic volatility using noisy observations: A multi-scale approach. *Bernoulli* **12** 1019–1043. MR2274854

ZHANG, L., MYKLAND, P. A. and AÏT-SAHALIA, Y. (2005). A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *J. Amer. Statist. Assoc.* **100** 1394–1411. MR2236450

ZHAO, Z. and WU, W. B. (2009). Nonparametric inference of discretely sampled stable Lévy processes. *J. Econometrics* **153** 83–92. MR2558496

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