

BAYESIAN NONPARAMETRIC ESTIMATION OF THE SPECTRAL DENSITY OF A LONG OR INTERMEDIATE MEMORY GAUSSIAN PROCESS¹

BY JUDITH ROUSSEAU, NICOLAS CHOPIN AND BRUNERO LISEO²

Université Paris Dauphine and CREST-ENSAE, Paris, CREST-ENSAE, Paris, and Sapienza Università di Roma

A stationary Gaussian process is said to be long-range dependent (resp., anti-persistent) if its spectral density $f(\lambda)$ can be written as $f(\lambda) = |\lambda|^{-2d}g(|\lambda|)$, where $0 < d < 1/2$ (resp., $-1/2 < d < 0$), and g is continuous and positive. We propose a novel Bayesian nonparametric approach for the estimation of the spectral density of such processes. We prove posterior consistency for both d and g , under appropriate conditions on the prior distribution. We establish the rate of convergence for a general class of priors and apply our results to the family of fractionally exponential priors. Our approach is based on the true likelihood and does not resort to Whittle's approximation.

1. Introduction. Let $\mathbf{X} = \{X_t, t = 1, 2, \dots\}$ be a real-valued stationary zero-mean Gaussian random process, with spectral density f , and covariance function $\gamma_f(\tau) = E(X_t X_{t+\tau})$, so that

$$(1) \quad \gamma_f(\tau) = \int_{-\pi}^{\pi} f(\lambda) e^{i\tau\lambda} d\lambda \quad (\tau = 0, \pm 1, \pm 2, \dots).$$

This process is long-range dependent (resp., anti-persistent) if there exist $C > 0$ and a value d , $0 < d < 1/2$ (resp., $-1/2 < d < 0$), such that $f(\lambda)|\lambda|^{2d} \rightarrow C$ when $\lambda \rightarrow 0$. This may be conveniently rewritten as $f(\lambda) = \lambda^{-2d}g(|\lambda|)$, where $g: [0, \pi] \rightarrow \mathbb{R}^+$ is a continuous positive function.

Interest in long-range dependent and anti-persistent time series has increased steadily in the last fifteen years; see [Beran \(1994\)](#) for a comprehensive introduction and [Doukhan, Oppenheim and Taqqu \(2003\)](#) for a review of theoretical aspects and fields of applications, including telecommunications, economics, finance, astrophysics, medicine and hydrology. Research in parametric inference for long and intermediate memory processes have been developed by [Mandelbrot and Van Ness \(1968\)](#), [Mandelbrot and Wallis \(1969\)](#), [Fox and Taqqu \(1986\)](#), [Dahlhaus \(1989\)](#),

Received July 2010; revised November 2011.

¹Supported by the 800-2007-2010 Grant ANR-07-BLAN-0237-01 "SP Bayes."

²Supported by the CEREMADE, Université Paris Dauphine.

MSC2010 subject classifications. Primary 62G20; secondary 62M15.

Key words and phrases. Bayesian nonparametric, consistency, FEXP priors, Gaussian long memory processes, rates of convergence.

Giraitis and Taqqu (1999), Geweke and Porter-Hudak (1983) and Beran (1993), among others. Unfortunately, parametric inference can be highly biased under misspecification of the true model. This limitation makes semiparametric approaches particularly appealing [Robinson (1995a)].

Under the representation $f(\lambda) = |\lambda|^{-2d}g(|\lambda|)$, one may like to estimate d as a measure of long-range dependence, without resorting to parametric assumptions on the nuisance parameter g . However, the existing procedures [see the review of Bardet et al. (2003)] either exploit the regression structure of the log-spectral density in a small neighborhood of the origin [Robinson (1995a)], or use an approximate likelihood function based on Whittle’s approximation [Whittle (1962)], where the original vector of observations $\mathbf{X}_n = (X_1, X_2, \dots, X_n)$ gets transformed into the periodogram $I(\lambda)$ computed at the Fourier frequencies $\lambda_j = 2\pi j/n, j = 1, 2, \dots, n$, and the artificial observations $I(\lambda_1), \dots, I(\lambda_n)$ are, under short range dependence, approximately independent. Whittle’s approximation is very convenient; the “observations” $I(\lambda_j)/f(\lambda_j)$ are approximately independent and identically distributed under short-range dependence. Unfortunately, this property does not hold under long-range dependence for the lowest frequencies [Robinson (1995b)].

We propose a Bayesian nonparametric approach to the estimation of the spectral density of the stationary Gaussian process based on the true likelihood, without resorting to Whittle’s approximation. We study the asymptotic properties of our procedure, including consistency and rates of convergence. Our study is based on standard tools for an asymptotic analysis of Bayesian approaches [e.g., Ghosal, Ghosh and van der Vaart (2000)]; that is, quantities of interest are the prior probability of a small neighborhood around the true spectral density, and some kind of entropy measure for the prior distribution. Most technical details differ, however, because of the long-range dependence.

The plan is as follows. In Section 2, we introduce the model and the notation. In Section 3, we provide a general theorem that states sufficient conditions to ensure consistency of the posterior distribution, and of several Bayes estimators. We also introduce the class of FEXP (Fractional Exponential) priors, based on the FEXP representation of Robinson (1991), and show that such prior distributions fulfill these sufficient conditions for posterior consistency. In Section 4, we study the rate of convergence of the posterior in the general case, and specialize our results for the FEXP class. Section 5 gives the proofs of the main theorems of the two previous sections. Section 6 discusses further research. The Appendix and the supplement contain technical lemmas.

2. Model and notation. The model consists of an observed vector $\mathbf{X}_n = (X_1, \dots, X_n)$ of n realizations from a zero-mean Gaussian stationary process, with spectral density f . The likelihood function is

$$(2) \quad \varphi(\mathbf{X}_n; f) = (2\pi)^{-n/2} |T_n(f)|^{-1/2} \exp\{-\frac{1}{2} \mathbf{X}_n^t T_n(f)^{-1} \mathbf{X}_n\},$$

where $T_n(f) = [\gamma_f(j - k)]_{1 \leq j, k \leq n}$ is the Toeplitz matrix associated to γ_f ; see (1). This model is parametrized by the pair (d, g) , which defines $f = F(d, g)$ through the factorization

$$F : (-1/2, 1/2) \times \mathcal{C}_+^0[0, \pi] \rightarrow \mathcal{F},$$

$$(d, g) \rightarrow f : f(\lambda) = |\lambda|^{-2d} g(|\lambda|),$$

where $\mathcal{C}_+^0[0, \pi]$ is the set of continuous, nonnegative functions over $[0, \pi]$, and \mathcal{F} denotes the set of spectral densities, that is, the set of even functions $f : [-\pi, \pi] \rightarrow \mathbb{R}^+$ such that $\int_{-\pi}^{\pi} f(\lambda) d\lambda < +\infty$.

The model is completed with a nonparametric prior distribution π for $(d, g) \in (-1/2, 1/2) \times \mathcal{C}_+^0[0, \pi]$. (There should be no confusion whether π refers to either the number or the prior distribution in the rest of the paper.) All our results will assume that the model is valid for some “true” parameter (d_0, g_0) , associated to some “true” spectral density $f_0 = F(d_0, g_0)$, where $d_0 \in (-1/2, 1/2)$; conditions on g_0 are detailed in the next section.

The Kullback–Leibler divergence for finite n is defined as

$$KL_n(f_0; f) = \frac{1}{n} \int_{\mathbb{R}^n} \varphi(\mathbf{X}_n; f_0) \{ \log \varphi(\mathbf{X}_n; f_0) - \log \varphi(\mathbf{X}_n; f) \} d\mathbf{X}_n$$

$$= \frac{1}{2n} \{ \text{tr}[T_n(f_0)T_n^{-1}(f) - \mathbf{I}_n] - \log \det[T_n(f_0)T_n^{-1}(f)] \},$$

where \mathbf{I}_n represents the identity matrix of order n . We also define a symmetrized version of KL_n , and its limit as $n \rightarrow \infty$,

$$h_n(f_0, f) = KL_n(f_0; f) + KL_n(f; f_0),$$

$$h(f_0, f) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[\frac{f_0(\lambda)}{f(\lambda)} + \frac{f(\lambda)}{f_0(\lambda)} - 2 \right] d\lambda = \frac{1}{2\pi} \int_0^{\pi} \left(\frac{f_0(\lambda)}{f(\lambda)} - 1 \right)^2 \frac{f(\lambda)}{f_0(\lambda)} d\lambda.$$

For technical reasons, we also define the pseudo-distance

$$b_n(f_0, f) = \frac{1}{n} \text{tr}[(T_n(f)^{-1}T_n(f_0 - f))^2]$$

and its limit as $n \rightarrow +\infty$,

$$b(f_0, f) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{f_0(\lambda)}{f(\lambda)} - 1 \right)^2 d\lambda.$$

Of course, asymptotic pseudo-distances are easier to interpret. In particular, our consistency results are expressed in terms of the standard distance h and posterior concentration results in the case of FEXP-type priors (see Theorem 4.2) are expressed in terms of the distance $l(\cdot, \cdot)$ defined in (3). The Kullback–Leibler divergence arises naturally in the study of asymptotic properties of the posterior distribution. The divergence measure $b_n(\cdot, \cdot)$ is the variance under f_0 of

$\log \varphi(\mathbf{X}_n; f_0) - \log \varphi(\mathbf{X}_n; f)$ and is also a common tool in such studies; see, for instance, Ghosal and van der Vaart (2007). The symmetrized Kullback–Leibler divergence, h_n is also encountered in Bayesian statistics and is sometimes referred to as the J divergence; see, for instance, Jeffreys (1946).

We also consider the L^2 distance between spectral log-densities, which is in particular used in Moulines and Soulier (2003),

$$(3) \quad \ell(f_0, f) = \int_{-\pi}^{\pi} \{\log f_0(\lambda) - \log f(\lambda)\}^2 d\lambda.$$

The advantage of l is that it always exists (for the models considered here) whereas the L^2 distance between spectral densities may not.

3. Consistency. We first state and prove the strong consistency of the posterior distribution under very general conditions on both π and $f_0 = F(d_0, g_0)$; that is, as $n \rightarrow \infty$, and for $\varepsilon > 0$ small enough,

$$P^\pi[\mathcal{A}_\varepsilon | \mathbf{X}_n] \rightarrow 1 \quad \text{a.s.},$$

where $P^\pi[\cdot | \mathbf{X}_n]$ denotes posterior probabilities associated with prior π , and

$$\mathcal{A}_\varepsilon = \{(d, g) \in (-1/2, 1/2) \times \mathcal{C}_+^0[0, \pi] : h(f_0, F(d, g)) \leq \varepsilon\}.$$

From this, we shall deduce the consistency of Bayes estimators of f and d . Finally, we shall introduce the class of FEXP priors, and show that they allow for posterior consistency.

3.1. *Main result.* Consider the following sets:

$$\mathcal{G}(m, M) = \{g \in \mathcal{C}^0[0, \pi] : m \leq g \leq M\};$$

$$\mathcal{G}(m, M, L, \rho) = \{g \in \mathcal{G}(m, M) : |g(\lambda) - g(\lambda')| \leq L|\lambda - \lambda'|^\rho\};$$

$$\mathcal{G}(t, m, M, L, \rho) = [-1/2 + t, 1/2 - t] \times \mathcal{G}(m, M, L, \rho)$$

for $\rho \in (0, 1]$, $L > 0$, $m \leq M$, $t \in (0, 1/2)$. Restricting the parameter space to such sets makes the model identifiable (boundedness of g , provided $m > 0$), and ensures that normalized traces of products of Toeplitz matrices that appear in the distances defined in the previous section converge (Hölder inequality). We now state our main consistency result.

THEOREM 3.1. *For $\varepsilon > 0$ small enough,*

$$P^\pi[\mathcal{A}_\varepsilon | \mathbf{X}_n] \rightarrow 1 \quad \text{a.s.}$$

as $n \rightarrow +\infty$, provided the following conditions are fulfilled:

- (1) *There exist $t, m, M, L > 0$, $\rho \in (0, 1]$, such that the set $\mathcal{G}(t, m, M, L, \rho)$ contains both the pair (d_0, g_0) that defines the true spectral density $f_0 = F(d_0, g_0)$ and the support of the prior distribution π .*

(2) For all $\varepsilon > 0$, $\pi(\mathcal{B}_\varepsilon) > 0$, where \mathcal{B}_ε is defined by

$$\mathcal{B}_\varepsilon = \{(d, g) \in \mathcal{G}(t, m, M, L, \rho) : h(f_0, F(d, g)) \leq \varepsilon, 16|d_0 - d| < \rho + 1 - t\}.$$

(3) For $\varepsilon > 0$ small enough, there exist a sequence \mathcal{F}_n such that $\pi(\mathcal{F}_n) \geq 1 - e^{-nr}$, $r > 0$, and a net (i.e., a finite collection)

$$\mathcal{H}_n \subset \{(d, g) \in [-1/2 + t, 1/2 - t] \times \mathcal{G}(m, M, L, \rho) : h(f_0; F(d, g)) > \varepsilon/2\}$$

such that, for n large enough, for all $(d, g) \in \mathcal{F}_n \cap A_\varepsilon^c$, $f = F(d, g)$, there exists $(d_i, g_i) \in \mathcal{H}_n$, $f_i = F(d_i, g_i)$, such that $8|d_i - d| \leq \rho + 1 - t$, $f \leq f_i$, and:

(a) if $8|d_i - d_0| \leq \rho + 1 - t$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f_i - f)(\lambda)}{f_0(\lambda)} d\lambda \leq h(f_0, f_i)/4;$$

(b) if $8(d_i - d_0) > \rho + 1 - t$,

$$b(f_i, f) \leq b(f_0, f_i)|\log \varepsilon|^{-1};$$

(c) otherwise, if $8(d_0 - d_i) > \rho + 1 - t$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f_i - f)(\lambda)}{f_i(\lambda)} d\lambda \leq b(f_i, f_0)|\log \varepsilon|^{-1}.$$

(4) The cardinality C_n of the net \mathcal{H}_n defined above is such that $\log C_n \leq n\varepsilon / \log(\varepsilon)$.

A proof is given in Section 5.1. Note that, in the above definition of the net \mathcal{H}_n , the $|\log \varepsilon|$ terms are here only to avoid writing inequalities in terms of awkward constants in the form m/M . If need be, we can replace the $|\log \varepsilon|$ by the correct constants as expressed in Appendix B. The definition of the above *entropy* is non-standard. The interest in expressing it in this general but nonstandard form lies in the difficulty in dealing with spectral densities which diverge at 0. In practice, the way one constructs the net \mathcal{H}_n should vary according to the form of the prior on the short memory part g .

The Bayes estimator associated to loss function l is

$$\hat{d} = E^\pi[d|\mathbf{X}_n], \quad \hat{g} : \lambda \rightarrow \exp\{E^\pi[\log g(\lambda)|\mathbf{X}_n]\}, \quad \hat{f} = F(\hat{d}, \hat{g}).$$

Consistency for these point estimates are easily deduced from Theorem 3.1, that is, $\hat{d} \rightarrow d_0$, $l(f_0, \hat{f}) \rightarrow 0$ a.s. as $n \rightarrow +\infty$; proof of these results are in the supplementary material [Rousseau, Chopin and Liseo (2012), Section 1], and follow Barron, Schervish and Wasserman (1999).

3.2. *The FEXP prior.* Following [Hurvich, Moulines and Soulier \(2002\)](#), we consider the FEXP parameterisation of spectral densities, that is, $f = \tilde{F}(d, k, \theta)$, where

$$(4) \quad \begin{aligned} &\tilde{F} : \mathcal{T} \rightarrow \mathcal{F}, \\ &(d, k, \theta) \rightarrow f : f(\lambda) = |1 - e^{i\lambda}|^{-2d} \exp \left\{ \sum_{j=0}^k \theta_j \cos(j\lambda) \right\} \end{aligned}$$

and $\mathcal{T} = (-1/2 + t, 1/2 - t) \times \{\bigcup_{k=0}^{+\infty} \{k\} \times \mathbb{R}^{k+1}\}$, for some fixed $t \in (0, 1/2)$. This FEXP representation is equivalent to our previous representation $f = F(d, g)$, provided $g = \psi^{-d} e^w$, $w(\lambda) = \{\sum_{j=0}^k \theta_j \cos(j\lambda)\}$ and $\psi(\lambda) = |1 - e^{i\lambda}|^2 / \lambda^2 = 2(1 - \cos \lambda) / \lambda^2$ for $\lambda \neq 0$, $\psi(0) = 1$. The function ψ is bounded, infinitely differentiable and positive for $\lambda \in [0, \pi]$. Thus g and w share the same regularity properties; that is, w is bounded and Hölder with exponent ρ implies that g is bounded and Hölder with exponent ρ , and vice versa. Under this parameterisation, the prior distribution π is expressed as a trans-dimensional prior distribution on the random vector (d, k, θ) , which, for convenience, factorizes as $\pi_d(d)\pi_k(k)\pi_\theta(\theta|k)$.

We assume that π puts mass one on the following Sobolev set:

$$(5) \quad \mathcal{S}(\beta, L) = \left\{ (d, k, \theta) \in \mathcal{T} : \sum_{j=0}^k \theta_j^2 (j+1)^{2\beta} \leq L \right\}$$

for some $\beta > 1/2$, $L > 0$. This ensures that the Fourier sum w , and thus the short-memory component g of the spectral density f , as explained above, belong to some set $\mathcal{G}(m, M, L', \rho)$, that is, both w and g are bounded and Hölder, for $\rho < \beta - 1/2$. To see this, note that, for $(d, k, \theta) \in \mathcal{S}(\beta, L)$,

$$(6) \quad \begin{aligned} \sum_{j=0}^k |\theta_j| j^r &\leq \sum_{j=0}^k \theta_j^2 (j+1)^{2\beta} + \sum_{j=0}^k |\theta_j| j^r \mathbb{1}(|\theta_j| j^r \geq \theta_j^2 (j+1)^{2\beta}) \\ &\leq L + \sum_{j=0}^{+\infty} (j+1)^{2r-2\beta} < +\infty, \end{aligned}$$

provided $2r - 2\beta < -1$. By taking $r = 0$, one sees that w is bounded, and by taking $r = \rho$, for any ρ , $0 < \rho < \beta - 1/2$, one sees that w is Hölder, with coefficient ρ , since, for $\lambda, \lambda' \in [-\pi, \pi]$,

$$\begin{aligned} |w(\lambda) - w(\lambda')| &\leq 2 \sum_{j=0}^k |\theta_j| \times |\{\cos(\lambda j) - \cos(\lambda' j)\} / 2|^\rho \\ &\leq 2^{1-\rho} \left(\sum_{j=0}^k |\theta_j| j^\rho \right) |\lambda - \lambda'|^\rho. \end{aligned}$$

Finally, we assume that π assigns positive prior probability to the intersection of $\mathcal{S}(\beta, L)$ with any rectangle set of the form $(a_d, b_d) \times \{k\} \times \prod_{j=1}^k (a_{\theta_j}, b_{\theta_j})$.

Alternatively, one could assume that the support of π is included in a set of the form $\{(d, k, \theta) \in \mathcal{T} : \sum_{j=0}^k |\theta_j| j^\rho \leq L\}$. However, Sobolev sets are more natural when dealing with rates of convergence (see Section 4.2), and are often considered in the nonparametric literature, so we restrict our attention to these sets.

In the same spirit, we assume that the true spectral density admits a FEXP representation associated to an infinite Fourier series,

$$f_0(\lambda) = |1 - e^{i\lambda}|^{-2d_0} \exp\left\{ \sum_{j=0}^{+\infty} \theta_{0j} \cos(j\lambda) \right\},$$

that is, $f_0 = F(d_0, g_0)$ with $g_0 = \psi^{-d_0} e^{w_0}$ and $w_0(\lambda) = \{\sum_{j=0}^{+\infty} \theta_{0j} \cos(j\lambda)\}$. In addition, we assume that w_0 satisfies the same type of Sobolev inequality, namely

$$(7) \quad L_0 = \sum_{j=0}^{+\infty} \theta_{0j}^2 (j + 1)^{2\beta} < L < +\infty,$$

which, as explained above, implies that $g_0 \in \mathcal{G}(m, M, L, \rho)$, for some well-chosen constants m, M, L, ρ . Note that it is essential to have a strict inequality in (7), that is, $L_0 < L$.

THEOREM 3.2. *Let π be a prior distribution $\pi_d(d)\pi_k(k)\pi_\theta(\theta|k)$ which fulfills the above conditions, and, in addition, such that $\pi_k(k) \leq \exp(-Ck \log k)$ for some $C > 0$ and k large enough. Then the conditions of Theorem 3.1 are fulfilled, and the posterior distribution is consistent.*

PROOF. Condition (1) of Theorem 3.1 is a simple consequence of (7) and (5), as explained above. For condition (2), we noted [see (6)] that $\sum_{j=0}^{+\infty} \theta_{0j}^2 (j + 1)^{2\beta} \leq L$ implies that $\sum_{j=0}^{+\infty} |\theta_{0j}| \leq L' < +\infty$. Let k such that $\sum_{j=k+1}^{\infty} |\theta_{0j}| \leq \varepsilon/14$, $\theta = (\theta_0, \dots, \theta_k)$ such that $\sum_{j=0}^k |\theta_{0j} - \theta_j| \leq \varepsilon/14$, d such that $|d - d_0| \leq \varepsilon/7$, and let $f = \tilde{F}(d, k, \theta)$. Using Lemma 14 (see Appendix D) one has $h(f, f_0) \leq \varepsilon$. Note that it is sufficient to prove that $\pi(\mathcal{B}_\varepsilon) > 0$ for ε small enough; hence we assume that $\varepsilon/7 < (\rho + 1 - t)/16$. Thus, condition (2) is verified as soon as the intersection of $\mathcal{S}(\beta, L)$ and the rectangle set

$$[d_0 - \varepsilon/7, d_0 + \varepsilon/7] \times \{k\} \times \prod_{j=1}^k [\theta_{0j} - \varepsilon/14k, \theta_{0j} + \varepsilon/14k]$$

is assigned positive prior probability. Now consider condition (3). Let $\varepsilon > 0$ and take

$$\mathcal{F}_n = \{(d, k, \theta) \in \mathcal{S}(\beta, L) : k \leq k_n\},$$

where $k_n = \lfloor \alpha n / \log n \rfloor$, for some $\alpha > 0$, so that, for some r depending on α , $\pi(\mathcal{F}_n^c) \leq \pi_k(k > k_n) \leq e^{-nr}$. Let $f = \tilde{F}(d, k, \theta)$, $f_i = (2e)^{c\varepsilon} \tilde{F}(d_i, k, \theta_i)$, such that $k \leq k_n$, $d_i - c\varepsilon \leq d \leq d_i$, and $\sum_{j=0}^k |\theta_j - \theta_{ij}| \leq c\varepsilon$, for some $c > 0$, then

$$\frac{f(\lambda)}{f_i(\lambda)} = (2e)^{-c\varepsilon} [2(1 - \cos \lambda)]^{d_i - d} \exp \left\{ \sum_{j=0}^k (\theta_j - \theta_{ij}) \cos(j\lambda) \right\} \leq 1,$$

$$\frac{f(\lambda)}{f_i(\lambda)} \geq (1 - \cos \lambda)^{c\varepsilon} 2^{-c\varepsilon} e^{-2c\varepsilon}.$$

If c is small enough, $f_i - f$ verifies the three inequalities considered in condition (3). The number \mathcal{C}_n of functions f_i necessary to ensure that, for any f in the support of π , at least one of them verify the above inequalities, can be bounded by, for n large enough, and some well-chosen constant C ,

$$\begin{aligned} \mathcal{C}_n &\leq k_n (Ck_n/\varepsilon)^{k_n+2} \leq k_n^{3k_n} \\ &\leq \exp\{3\alpha n[1 + (\log \alpha - \log \log n)/\log n]\} \\ &\leq \exp\{6\alpha n\}, \end{aligned}$$

so condition (4) is satisfied, provided one takes $\alpha = \varepsilon/6 \log \varepsilon$. □

A convenient default choice for π is as follows: π_d is uniform over $(-1/2 + t, 1/2 - t)$, π_k is Poisson and $\pi_{\theta|k}$ has the following structure: the sum $S = \sum_{j=0}^k \theta_j^2 (j + 1)^{2\beta}$ has a Gamma distribution truncated to interval $[0, L]$, independently of S , the vector $(\theta_0^2, \theta_1^2 2^{2\beta}, \dots, \theta_k^2 (k + 1)^{2\beta})/S$ is Dirichlet with some coefficients $\alpha_{1,k}, \dots, \alpha_{k,k}$ and the signs of $\theta_0, \dots, \theta_k$ have equal probabilities. In particular one may take $\alpha_{j,k} = 1$ for all $j \leq k$, or, if one needs to generate more regular spectral densities, $\alpha_{j,k} = j^{-\kappa}$, for some fixed or random $\kappa > 0$. Another interesting choice for the prior on θ is the following truncated Gaussian process: for each k , and each $j \leq k$, $\theta_j \sim \mathcal{N}(0, \tau_0^2 (1 + j)^{-2\beta})$ independently apart from the constraint, for some fixed, large $L > 0$,

$$\sum_{j=1}^k (1 + j)^{2\beta} \theta_j^2 \leq L.$$

Note that we can easily restrict ourselves to the important case $d \geq 0$, that is, processes having long or short memory but not intermediate memory.

4. Rates of convergence. In this section we first provide a general theorem relating rates of convergence of the posterior distribution to conditions on the prior. These conditions are, in essence, similar to the conditions obtained in the i.i.d. case [e.g., Ghosal, Ghosh and van der Vaart (2000)]: that is, a condition on the prior mass of Kullback–Leibler neighborhoods of the true spectral density, and an entropy condition on the support of the prior. We then present results specialized to the FEXP prior case.

4.1. *Main result.*

THEOREM 4.1. *Let (u_n) be a sequence of positive numbers such that $u_n \rightarrow 0$, $nu_n \rightarrow +\infty$ and \bar{B}_n a sequence of balls belonging to $\mathcal{G}(t, m, M, L, \rho)$, and defined as*

$$\bar{B}_n = \{(d, g) : KL_n(f_0; F(d, g)) \leq u_n/4, b_n(f_0, F(d, g)) \leq u_n, d_0 \leq d \leq d_0 + \delta\}$$

for some $\delta, L > 0, 0 < m \leq M, \rho \in (0, 1]$. Let π be a prior which satisfies all the conditions of Theorem 3.1, and, in addition, such that:

(1) *For n large enough, $\pi(\bar{B}_n) \geq \exp(-nu_n/2)$.*

(2) *There exists $\varepsilon > 0$ and a sequence of sets $\bar{F}_n \subset \{(d, g) : h(F(d, g), f_0) \leq \varepsilon\}$, such that, for n large enough,*

$$\pi(\bar{F}_n^c \cap \{(d, g) : h(F(d, g), f_0) \leq \varepsilon\}) \leq \exp(-2nu_n).$$

(3) *There exists a positive sequence (ε_n) , $\varepsilon_n^2 \geq u_n$, $\varepsilon_n^2 \rightarrow 0$, $n\varepsilon_n^2 \geq C \log n$, for some $C > 0$, satisfying the following conditions. Let*

$$\mathcal{V}_{n,l} = \{(d, g) \in \bar{F}_n; \varepsilon_n^2 l \leq h_n(f_0, F(d, g)) \leq \varepsilon_n^2(l + 1)\}$$

with $l_0 \leq l \leq l_n$, with fixed $l_0 \geq 2$ and $l_n = \lceil \varepsilon^2 / \varepsilon_n^2 \rceil - 1$. For each $l = l_0, \dots, l_n$, there exists a net (i.e., a finite collection) $\bar{\mathcal{H}}_{n,l} \subset \mathcal{V}_{n,l}$, with cardinality $\bar{C}_{n,l}$, such that for all $f = F(d, g)$, $(d, g) \in \mathcal{V}_{n,l}$, there exists $f_{i,l} = F(d_{i,l}, g_{i,l}) \in \bar{\mathcal{H}}_{n,l}$ such that $f_{i,l} \geq f$ and

$$0 \leq g_{i,l}(x) - g(x) \leq l\varepsilon_n^2 g_{i,l}/32, \quad 0 \leq d_{i,l} - d \leq l\varepsilon_n^2(\log n)^{-1},$$

where

$$\log \bar{C}_{n,l} \leq n\varepsilon_n^2 l^\alpha \quad \text{with } \alpha < 1.$$

Then, there exist $C, C' > 0$ such that, for n large enough,

$$(8) \quad E_0^n [P^\pi(h_n(f_0, F(d, g)) \geq l_0 \varepsilon_n^2 | \mathbf{X}_n)] \leq Cn^{-3} + 2e^{-C'\varepsilon_n^2} + e^{-nu_n/16}.$$

A proof is given in Section 5.2.

The conditions given in Theorem 4.1 are similar in spirit to those considered for rates of convergence of the posterior distribution in the i.i.d. case. The first condition is a condition on the prior mass of Kullback–Leibler neighborhoods of the true spectral density, the second one is necessary to allow for sets with infinite entropy (some kind of noncompactness) and the third one is an entropy condition. The inequality (8) obtained in Theorem 4.1 is nonasymptotic, in the sense that it is valid for all n . However, the distances considered in Theorem 4.1 heavily depend on n and, although they express the impact of the differences between f and f_0 on

the observations, they can be difficult to work with. Note that the metric h_n , which is a symmetrized version of the Kullback–Leibler divergence KL_n , leads to a strong convergence result since it implies in particular a similar posterior concentration rate for any metric smaller than h_n , which includes KL_n . For these reasons, the entropy condition is awkward and cannot be directly transformed into some more common entropy conditions. To state a result involving distances between spectral densities that might be more useful, we need to consider more specific class of priors. In the next section, we obtain rates of convergence in terms of the ℓ distance for the class of FEXP priors introduced in Section 3.2. The rates obtained are the optimal rates up to a $(\log n)$ term, at least on certain classes of spectral densities. It is to be noted that the calculations used when working on these classes of priors are actually more involved than those used to prove Theorem 4.1. This is quite usual when dealing with rates of convergence of posterior distributions; however, this is emphasized here by the fact that distances involved in Theorem 4.1 are strongly dependent on n . The method used in the case of the FEXP prior can be extended to other types of priors.

4.2. *Rates of convergence for the FEXP prior.* We apply Theorem 4.1 to the class of FEXP priors introduced in Section 3.2. Recall that under such a prior a spectral density f is parametrized as $f = \tilde{F}(d, k, \theta)$; see (4). We make the same assumptions as in Section 3.2. In particular, the prior $\pi(d, k, \theta)$ factorizes as $\pi_d(d)\pi_k(k)\pi_\theta(\theta|k)$; the right tail of π_k is such that

$$\exp\{-Ck \log k\} \leq \pi_k(k) \leq \exp\{-C'k \log k\}$$

for some $C, C' > 0$, and for k large enough; and there exists $\beta > 1/2$ such that the Sobolev set $S(\beta, L)$ contains the support of π . The last condition means that $S = \sum_{j=0}^k \theta_j^2 (j + 1)^{2\beta} \in [0, L]$ with prior probability one. In addition, we assume that the support of π_d is $[-1/2 + t, 1/2 - t]$, and, for $d \in [-1/2 + t, 1/2 - t]$, $\pi_d(d) \geq c_d > 0$. Similarly, we assume that $\pi_{\theta|k}$ is such that the random variable $S = \sum_{j=0}^k \theta_j^2 (j + 1)^{2\beta}$ is independent of k , and admits a probability density $\pi_S(s)$ with support $[0, L]$, and such that $\pi_S(s) \geq c_s > 0$ for $s \in [0, L]$.

THEOREM 4.2. *For the FEXP prior described above, there exist $C, C' > 0$ such that, for n large enough,*

$$(9) \quad E_0^n \left\{ P^\pi \left[\ell(f, f_0) > \frac{C \log n}{n^{2\beta/(2\beta+1)}} \mid \mathbf{X}_n \right] \right\} \leq \frac{C}{n^2},$$

where $f = \tilde{F}(d, k, \theta)$ and

$$(10) \quad E_0^n [\ell(\hat{f}, f_0)] \leq \frac{C'(\log n)}{n^{2\beta/(2\beta+1)}},$$

where $\log \hat{f}(\lambda) = E^\pi [\log f(\lambda) \mid \mathbf{X}_n]$.

A proof is given in Appendix C.

5. Proofs of Theorems 3.1 and 4.1.

5.1. *Proof of Theorem 3.1.* For the sake of conciseness, we introduce the following notation: for any pair (f, f_0) of spectral densities,

$$A(f_0, f) = T_n(f)^{-1}T_n(f_0),$$

$$B(f_0, f) = T_n(f_0)^{1/2}[T_n(f)^{-1} - T_n(f_0)^{-1}]T_n(f_0)^{1/2}.$$

The proof borrows ideas from Ghosal, Ghosh and van der Vaart (2000). The main difficulty is to formulate constraints on quantities such as $h_n(f, f_0)$ or $KL_n(f, f_0)$ in terms of distances between f, f_0 , independent on n , and uniformly over f . One has

$$P^\pi[A_\varepsilon^c|\mathbf{X}_n] = \frac{\int \mathbb{1}_{A_\varepsilon^c}(f)\varphi(\mathbf{X}_n; f)/\varphi(\mathbf{X}_n; f_0) d\pi(f)}{\int \varphi(\mathbf{X}_n; f)/\varphi(\mathbf{X}_n; f_0) d\pi(f)} \triangleq \frac{N_n}{D_n}.$$

Let $\delta \in (0, \varepsilon)$ and P_0^n be a generic notation for probabilities associated to the distribution of \mathbf{X}_n , under the true spectral density $f_0 = F(d_0, g_0)$. One has

$$(11) \quad P_0^n\{P^\pi[A_\varepsilon^c|\mathbf{X}_n] \geq e^{-n\delta}\} \leq P_0^n[D_n \leq e^{-n\delta}] + P_0^n[N_n \geq e^{-2n\delta}],$$

so that Theorem 3.1 follows from bounds on both terms of the right-hand side of the above inequality. The following lemma bounds the first term.

LEMMA 1. *There exists $C > 0$ such that*

$$(12) \quad P_0^n[D_n \leq e^{-n\delta}] \leq Cn^{-3}.$$

PROOF. Lemma 4 implies that, when n is large enough, $\tilde{\mathcal{B}}_n \supset \mathcal{B}_{\delta/8}$, where $\tilde{\mathcal{B}}_n = \{(d, g) \in [-1/2 + t, 1/2 - t] \times \mathcal{G}(m, M, L, \rho) : KL_n(f_0, F(d, g)) \leq \delta/4\}$, and condition (2) implies that, for n large enough, $\pi(\tilde{\mathcal{B}}_n) \geq \pi(\mathcal{B}_{\delta/8}) \geq 2e^{-n\delta/2}$. Consider the indicator function

$$\Omega_n = \mathbb{1}[-\mathbf{X}_n^t\{T_n(f)^{-1} - T_n(f_0)^{-1}\}\mathbf{X}_n + \log \det A(f_0, f) > -n\delta]$$

with implicit arguments (f, \mathbf{X}_n) , then, following Ghosal, Ghosh and van der Vaart (2000),

$$P_0^n[D_n \leq e^{-n\delta}] \leq P_0^n\left(\int \Omega_n \mathbb{1}_{\tilde{\mathcal{B}}_n}(f) \frac{\varphi(\mathbf{X}_n; f)}{\varphi(\mathbf{X}_n; f_0)} d\pi(f) \leq e^{-n\delta/2} \frac{\pi(\tilde{\mathcal{B}}_n)}{2}\right)$$

$$\leq P_0^n(E^\pi\{\Omega_n \mathbb{1}_{\tilde{\mathcal{B}}_n}(f)\} \leq \pi(\tilde{\mathcal{B}}_n)/2)$$

$$\leq P_0^n(E^\pi\{(1 - \Omega_n) \mathbb{1}_{\tilde{\mathcal{B}}_n}(f)\} \geq \pi(\tilde{\mathcal{B}}_n)/2)$$

$$\leq \frac{2}{\pi(\tilde{\mathcal{B}}_n)} \int_{\tilde{\mathcal{B}}_n} E_0^n\{1 - \Omega_n\} d\pi(f)$$

by Markov’s inequality. Besides,

$$\begin{aligned} E_0^n\{1 - \Omega_n\} &= P_0^n\{\mathbf{X}_n^t\{T_n(f)^{-1} - T_n(f_0)^{-1}\}\mathbf{X}_n - \log \det A(f_0, f) > n\delta\} \\ &= P_{\mathbf{Y}}\{\mathbf{Y}^t B(f_0, f)\mathbf{Y} - \text{tr}[B(f_0, f)] > D(f_0, f)\}, \end{aligned}$$

where $\mathbf{Y} \sim N_n(\mathbf{0}_n, \mathbf{I}_n)$, and, for $f \in \tilde{\mathcal{B}}_n$,

$$D(f_0, f) \triangleq n\delta + \log \det A(f_0, f) - \text{tr}[B(f_0, f)] > n\delta/2$$

thus

$$\begin{aligned} E_0^n[1 - \Omega_n] &\leq P_{\mathbf{Y}}\{\mathbf{Y}^t B(f_0, f)\mathbf{Y} - \text{tr}[B(f_0, f)] > n\delta/2\} \\ &\leq \frac{16}{n^4\delta^4} E_{\mathbf{Y}}[\{\mathbf{Y}^t B(f_0, f)\mathbf{Y} - \text{tr}[B(f_0, f)]\}^4] \\ &\leq \frac{C}{n^3\delta^4}, \end{aligned}$$

which concludes the proof. \square

A bound for the second term in (11) is obtained as follows:

$$\begin{aligned} (13) \quad P_0^n[N_n \geq e^{-2n\delta}] &\leq 2e^{2n\delta} \pi(\mathcal{F}_n^c) + p \\ &\leq 2e^{-n(r-2\delta)} + p \end{aligned}$$

using condition (3), where

$$p \triangleq P_0^n \left[\int \mathbb{1}_{(A_\varepsilon^c \cap \mathcal{F}_n)} \frac{\varphi(\mathbf{X}_n; f)}{\varphi(\mathbf{X}_n; f_0)} d\pi(f) \geq e^{-2n\delta} / 2 \right].$$

Assuming $2\delta < r$, we consider the following likelihood ratio tests for each $f_i \in \mathcal{H}_n$, and for some arbitrary values ρ_i ,

$$\phi_i = \mathbb{1}\{\mathbf{X}_n^t [T_n^{-1}(f_0) - T_n^{-1}(f_i)]\mathbf{X}_n \geq n\rho_i\}.$$

Lemmas 7, 8 and 9 given in Appendix B prove that, for each of the three cases in condition (3) of Theorem 3.1, and well-chosen values of ρ_i , one has

$$(14) \quad E_0^n[\phi_i] \leq e^{-nC_1\varepsilon}, \quad E_f^n[1 - \phi_i] \leq e^{-nC_1\varepsilon}$$

for all f_i , for f close to f_i [in the sense defined in cases (a), (b) and (c) in condition (3)], where $C_1 > 0$ is a constant that does not depend on f_i , and E_f^n stands for the expectation with respect to the likelihood $\varphi(\mathbf{X}_n; f)$. Then one concludes easily as follows. Let $\phi^{(n)} = \max_i \phi_i$; then, using Markov inequality, for n large enough,

$$\begin{aligned} (15) \quad p &\leq E_0^n[\phi^{(n)}] + 2e^{2n\delta} \int_{A_\varepsilon^c \cap \mathcal{F}_n} E_f[1 - \phi^{(n)}] d\pi(f) \\ &\leq C_n e^{-nC_1\varepsilon} + 2e^{2n\delta - nC_1\varepsilon} \leq e^{-nC_1\varepsilon/2}, \end{aligned}$$

provided $\delta < C_1\varepsilon/4$. Combining (12), (13) and (15), there exists $\delta > 0$ such that

$$P_0^n[P^\pi[A_\varepsilon^c|\mathbf{X}_n] > e^{-n\delta}] \leq Cn^{-3}$$

for n large enough, which implies that $P^\pi[A_\varepsilon^c|\mathbf{X}_n] \rightarrow 0$ a.s.

5.2. *Proof of Theorem 4.1.* This proof uses the same notation as the previous section: C, C' denote generic constants, $f, d\pi(f)$ are short-hands for $f = F(d, g), d\pi(d, g)$, respectively, $A(f, f_0)$ and $B(f, f_0)$ have the same definition, and so on. In the proof of Theorem 3.1, we showed that $E_0^n[P^\pi(h(f, f_0) \geq \varepsilon | \mathbf{X}_n)] \leq Cn^{-3}$ for ε small enough, n large enough. Thanks to the uniform convergence in Lemmas 3 and 4 in Appendix A, one sees that the same inequality holds if h is replaced by h_n . Therefore, to obtain inequality (8), it is sufficient to bound the expectation of the sum of the following probabilities:

$$P^\pi((d, g) \in \mathcal{W}_{n,l} | \mathbf{X}_n) = \frac{\int \mathbb{1}_{\mathcal{W}_{n,l}}(d, g) (\varphi(\mathbf{X}_n; f) / \varphi(\mathbf{X}_n; f_0)) d\pi(f)}{\int (\varphi(\mathbf{X}_n; f) / \varphi(\mathbf{X}_n; f_0)) d\pi(f)} = \frac{N_{n,l}}{D_n}$$

for $l_0 \leq l \leq l_n$, where $\mathcal{V}_{n,l} = \mathcal{W}_{n,l} \cap \bar{\mathcal{F}}_n$ and

$$\mathcal{W}_{n,l} = \{(d, g) : h(f, f_0) \leq \varepsilon, \varepsilon_n^2 l \leq h_n(f_0, f) \leq \varepsilon_n^2(l + 1)\}.$$

To prove the theorem one can follow the same lines as in Section 5.1 to show that

$$\begin{aligned} E_0^n \left[\sum_{l=l_0}^{l_n} \frac{N_{n,l}}{D_n} \right] &\leq P_0^n(D_n \leq e^{-nu_n}/2) \\ (16) \qquad &+ E_0^n \left[\sum_{l=l_0}^{l_n} \frac{N_{n,l}}{D_n} \mathbb{1}(D_n \geq e^{-nu_n}/2) \right] \\ &:= A_n + B_n. \end{aligned}$$

Now we show that both A_n and B_n can be bounded.

5.2.1. *Boundedness of A_n .* A_n can be bounded as in Lemma 1; see Section 5.1: in fact,

$$\begin{aligned} P_0^n(D_n \leq e^{-nu_n}/2) &\leq P_0^n\left(D_n \leq \frac{e^{-nu_n/2} \pi(\bar{\mathcal{B}}_n)}{2}\right) \\ &\leq \frac{2 \int_{\bar{\mathcal{B}}_n} E_0^n[(1 - \Omega_n(f))] d\pi(f)}{\pi(\bar{\mathcal{B}}_n)}, \end{aligned}$$

where Ω_n is the indicator function of

$$\{(\mathbf{X}_n, f); \mathbf{X}_n^t(T_n^{-1}(f) - T_n^{-1}(f_0))\mathbf{X}_n - \log \det[A(f_0, f)] \leq nu_n\}.$$

Also note that, for $f \in \bar{\mathcal{B}}_n$, there exists $s_0 > 0$ such that for all $s \leq s_0$,

$$\mathbf{I}_n(1 + 2s) - 2sT_n(f_0)^{1/2}T_n(f)^{-1}T_n(f_0)^{1/2} \geq \mathbf{I}_n/2.$$

Using Chernoff-type inequalities as in Lemma 7, one can show that for $f = F(d, g)$, $d \geq d_0$, $g > 0$, and for all $0 < s \leq s_0$,

$$\begin{aligned} E_0^n[1 - \Omega_n] &\leq \exp\left\{-snu_n - s \log|T_n(f_0)T_n(f)^{-1}| \right. \\ &\quad \left. - \frac{1}{2} \log|\mathbf{I}_n(1 + 2s) - 2sT_n(f_0)^{1/2}T_n(f)^{-1}T_n(f_0)^{1/2}| \right\} \\ &\leq \exp\{-snu_n + 2snKL_n(f_0, f) + 4s^2nb_n(f_0, f)\} \\ &\leq \exp\left\{-\frac{snu_n}{2}(1 - 8s)\right\} \leq e^{-Cnu_n}. \end{aligned}$$

In the above derivation, the second inequality comes from a Taylor expansion in s of $\log|\mathbf{I}_n + 2s(\mathbf{I}_n - T_n(f_0)^{1/2}T_n(f)^{-1}T_n(f_0)^{1/2})|$, the third comes from the definition of \bar{B}_n and the last from choosing $s = \min(s_0, 1/16)$. Note that $s_0 \geq m/(M\pi)$ and that the constant C in the above inequality can be chosen as $C = m/(32M\pi)$.

5.2.2. *Boundedness of B_n .* B_n can be written as

$$\begin{aligned} (17) \quad B_n &= E_0^n \left[\sum_{l=l_0}^{l_n} \frac{N_{n,l}}{D_n} \mathbb{1}(D_n \geq e^{-nu_n}/2)(\bar{\phi}_l + 1 - \bar{\phi}_l) \right] \\ &\leq \sum_{l=l_0}^{l_n} E_0^n[\bar{\phi}_l] + 2e^{nu_n} \sum_{l=l_0}^{l_n} E_0^n[N_{n,l}(1 - \bar{\phi}_l)], \end{aligned}$$

where $\bar{\phi}_l = \max_{i: f_{i,l} \in \tilde{\mathcal{T}}_{n,l}} \phi_{i,l}$, $\phi_{i,l}$ is a test function defined as in Section 5.1,

$$\begin{aligned} \phi_{i,l} &= \mathbb{1}\{\mathbf{X}'_n(T_n^{-1}(f_0) - T_n^{-1}(f_{i,l}))\mathbf{X}_n \geq \text{tr}[\mathbf{I}_n - T_n(f_0)T_n^{-1}(f_{i,l})] \\ &\quad + nh_n(f_0, f_{i,l})/4\}. \end{aligned}$$

We now show that both terms in the right-hand side of (17) are bounded. For the first term, we first derive a bound for the logarithm of $E_0^n[\phi_{i,l}]$: using inequality (23) in Lemma 7, one has

$$(18) \quad \log E_0^n[\phi_{i,l}] \leq -Cnh_n(f_0, f_i) \min\left(\frac{h_n(f_0, f_i)}{b_n(f_0, f_i)}, 1\right)$$

for some universal constant C , and n large enough. In addition, one has

$$\frac{b_n(f_0, f_i)}{h_n(f_0, f_i)} \leq \|T_n(f_0)^{1/2}T_n(f_i)^{-1/2}\|^2 \leq C'n^{2\max(d_0-d_i, 0)}.$$

The first inequality comes from Lemma 2 of Appendix A.1, and the second inequality comes from Lemma 3 in Lieberman, Rosemarin and Rousseau (2012). Hence for all $C > 0$, if $2|d_0 - d_i| \leq C/\log n$, $b_n(f_0, f_i) \leq C'e^C h_n(f_0, f_i)$. Moreover for all $\delta > 0$, there exists $C_\delta > 0$ such that if $2|d_0 - d_i| > C_\delta(\log n)^{-1}$, then

$h_n(f_0, f_i) \geq n^{-\delta}$. Indeed, equation (21) of Lemma 6 implies that if $h_n(f_0, f_i) \geq \varepsilon_n^2$, then

$$h_n(f_0, f_i) \geq \frac{C}{n} \text{tr}[T_n(f_0^{-1})T_n(f_i - f_0)T_n(f_i^{-1})T_n(f_i - f_0)]$$

and Lemma 5 (see Appendix A.3) implies that, for all $a > 0$,

$$\left| \frac{1}{n} \text{tr}[T_n(f_0^{-1})T_n(f_i - f_0)T_n(f_i^{-1})T_n(f_i - f_0)] - (2\pi)^3 \int_{-\pi}^{\pi} \frac{(f_i - f_0)^2}{f_i f_0} d\lambda \right| \leq n^{-\rho+a}.$$

Lemma 11 in Appendix D implies that there exists $a > 0$ such that, if $2|d_0 - d_i| > C_\delta(\log n)^{-1}$,

$$\int_{-\pi}^{\pi} \frac{(f_i - f_0)^2}{f_i f_0} dx \geq C e^{-a \log n / C_\delta} \geq n^{-\delta}$$

as soon as C_δ is large enough. Choosing $\delta < \rho$ we finally obtain that $h_n(f_0, f_i) \geq C'n^{-\delta}$. This and the definition of $\bar{\mathcal{H}}_{n,l}$ implies that $l \geq C'n^{-\delta}\varepsilon_n^{-2}$, and therefore $ln^{-\max(d_0-d_i, 0)} \geq 2l^\alpha / C'$, for all $\alpha < 1$ as soon as $|d_0 - d_i|$ is small enough. This implies that (18) becomes

$$\log E_0^n[\phi_{i,l}] \leq -cl\varepsilon_n^2 n^{1-\max(d_0-d_i, 0)} \leq -2n\varepsilon_n^{2l^\alpha}.$$

Also, condition (3) implies that

$$E_0^n[\bar{\phi}_l] \leq \sum_i E_0^n[\phi_{i,l}] \leq \bar{C}_{n,l} \exp\{-2n\varepsilon_n^{2l^\alpha}\} \leq \exp\{-n\varepsilon_n^{2l^\alpha}\}$$

so that $\sum_l E_0^n[\bar{\phi}_l] \leq 2 \exp\{-n\varepsilon_n^{2l^\alpha}\}$ for n large enough.

The second term of the right-hand side of (17) is bounded by considering that, from condition (3) on f and $f_{i,l}$, one has

$$0 \leq f_{i,l} - f \leq h_n(f_0, f_{i,l}) f_{i,l} \left(\frac{\pi^{2(d_i-d)}}{32} + \frac{2|\log|\lambda||}{\log n} \right)$$

for n is large enough; hence $\text{tr} A(f_{i,l} - f, f_0) \leq nh_n(f_0, f_{i,l})/4$, and we obtain the first part of equation (24),

$$\log E_f^n[1 - \phi_{i,l}] \leq -\frac{n}{64} \min\left(\frac{h_n(f_0, f_{i,l})^2}{b_n(f, f_0)}, 4h_n(f_0, f_{i,l})\right).$$

We also have

$$b_n(f, f_0) \leq b_n(f_{i,l}, f_0) + \frac{h_n^2(f_{i,l}, f_0)}{32} + 2\sqrt{b_n(f_0, f_{i,l})h_n(f_{i,l}, f_0)},$$

hence $\log E_f^n[1 - \phi_{i,l}] \leq -cnl^\alpha \varepsilon_n^2$, using the same arguments as before, and

$$\begin{aligned} \sum_{l=l_0}^{l_n} E_0^n[(1 - \bar{\phi}_l)N_{n,l}] &= \int \left\{ \sum_{l=l_0}^{l_n} \mathbb{1}_{\mathcal{V}_{n,l}}(f) E_f(1 - \bar{\phi}_l) \right\} d\pi(f) \\ &\leq P^\pi(f \in \mathcal{F}_n^c \cap \{h(f, f_0) \leq \varepsilon\}) \\ &\quad + \sum_{l=l_0}^{l_n} \int \mathbb{1}_{\mathcal{V}_{n,l}}(f) E_f^n(1 - \bar{\phi}_l) d\pi(f) \\ &\leq e^{-n\varepsilon_n^2} + \sum_{l=l_0}^{l_n} e^{-Cn\varepsilon_n^2 l^\alpha} \leq 2e^{-n\varepsilon_n^2}. \end{aligned}$$

6. Discussion. In this paper we have considered the theoretical properties of Bayesian nonparametric estimates of the spectral density for Gaussian long memory processes. Some general conditions on the prior and on the true spectral density are provided to ensure consistency and to determine concentration rates of the posterior distributions in terms of the pseudo-metric $h_n(f_0, f)$. To derive a posterior concentration rate in terms of a more common metric such as $l(\cdot, \cdot)$, we have considered a specific family of priors based of the FEXP models that are also used in the frequentist literature. Gaussian long memory processes lead to complex behaviors, which makes the derivation of concentration rates a difficult task. This paper is thus a step in the direction of better understanding the asymptotic behavior of the posterior distribution in such models and could be applied to various types of priors on the short memory part—other than the FEXP priors.

The rates we have derived are optimal (up to a $\log n$ term) in Sobolev balls but not adaptive since the estimation procedure depends on the smoothness β . Another limitation is that the prior is restricted to Sobolev balls with fixed though large radius. But, even in the parametric framework, current asymptotic results on likelihood-based approaches all assume the parameter space to be compact. The technical reason is that all these results rely on the short memory part of the spectral density being uniformly bounded.

A related and fundamental problem is the practical implementation of the model described in the paper. [Liseo and Rousseau \(2006\)](#) adopted a Population MC algorithm which easily deals with the trans-dimensional parameter space issue. We are currently working on alternative computational approaches.

APPENDIX A: TECHNICAL LEMMAS ON CONVERGENCE RATES OF PRODUCTS OF TOEPLITZ MATRICES

We first give a set of inequalities on norms of matrices that are useful throughout the proofs. We then give three technical lemmas on the uniform convergence of traces of products of Toeplitz matrices, in the spirit of [Lieberman, Rousseau and](#)

Zucker (2003) and Lieberman, Rosemarin and Rousseau (2012), but extending those previous results to functional classes instead of parametric classes.

A.1. Some matrix inequalities. Let A and B be n -dimensional matrices. We consider the following two norms:

$$|A|^2 = \text{tr}[AA^t], \quad \|A\|^2 = \sup_{|x|=1} (x^t AA^t x).$$

We recall that: $|\text{tr}[AB]| \leq |A||B|$, $|AB| \leq \|A\||B|$, $|A| \leq \|A\|$, $\|AB\| \leq \|A\|\|B\|$. Using these inequalities we prove the following basic lemma:

LEMMA 2. *Let f_1, f_2 be two spectral densities, then*

$$2nb_n(f_1, f_2) \leq n\|T_n(f_2)^{-1/2}T_n(f_1)^{1/2}\|^2 h_n(f_1, f_2).$$

PROOF. One has

$$\begin{aligned} & 2nb_n(f_1, f_2) \\ &= \text{tr}[T_n(f_1)^{1/2}T_n(f_2)^{-1}T_n(f_1)^{1/2}(T_n(f_1)^{-1/2}T_n(f_1 - f_2)T_n(f_2)^{-1/2})^2] \\ &= |T_n(f_2)^{-1/2}T_n(f_1)^{1/2}(T_n(f_1)^{-1/2}T_n(f_1 - f_2)T_n(f_2)^{-1/2})|^2 \\ &\leq \|T_n(f_2)^{-1/2}T_n(f_1)^{1/2}\|^2 |T_n(f_2)^{-1/2}T_n(f_1 - f_2)T_n(f_2)^{-1/2}|^2 \\ &= n\|T_n(f_2)^{-1/2}T_n(f_1)^{1/2}\|^2 h_n(f_1, f_2). \quad \square \end{aligned}$$

A.2. Uniform convergence: Lemmas 3 and 4. We state two technical lemmas, which are extensions of Lieberman, Rousseau and Zucker (2003) on uniform convergence of traces of Toeplitz matrices, and which are repeatedly used in the paper.

LEMMA 3. *Let $t > 0$, $M, L > 0$ and $\rho \in (0, 1]$, let p be a positive integer, we have, as $n \rightarrow +\infty$,*

$$\sup_{\substack{f_i=F(d_1, g_i), f'_i=F(d_2, g'_i) \\ 2p(d_1+d_2)\leq 1-t \\ g_i \in \mathcal{G}(-M, M, L, \rho) \\ g'_i \in \mathcal{G}(-M, M, L, \rho)}} \left| \frac{1}{n} \text{tr} \left[\prod_{i=1}^p T_n(f_i)T_n(f'_i) \right] - \frac{\int_{-\pi}^{\pi} \prod_{i=1}^p f_i(\lambda) f'_i(\lambda) d\lambda}{(2\pi)^{1-2p}} \right| \rightarrow 0.$$

This lemma is a direct adaptation from Lieberman, Rousseau and Zucker (2003); the only nonobvious part is the change from the condition of continuous differentiability in that paper to the Lipschitz condition of order ρ . This different assumption affects only equation (30) of Lieberman, Rousseau and Zucker (2003), with η_n replaced by η_n^ρ , which does not change the convergence results.

LEMMA 4. *Let $t > 0$, $M, L, m > 0$ and $\rho_1, \rho_2 \in (0, 1]$, let p be a positive integer, we have, as $n \rightarrow +\infty$,*

$$\sup_{\substack{f_i=F(d_1, g_i) f'_i=F(d_2, g'_i) \\ 4p(d_1-d_2) \leq \rho_2+1-t \\ g_i \in \mathcal{G}(-M, M, L, \rho_1) \\ g'_i \in \mathcal{G}(m, M, L, \rho_2)}} \left| \frac{1}{n} \operatorname{tr} \left[\prod_{i=1}^p T_n(f_i) T_n(f'_i)^{-1} \right] - \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{i=1}^p \frac{f_i(\lambda)}{f'_i(\lambda)} d\lambda \right| \rightarrow 0.$$

PROOF. This result is a direct consequence of Lemma 3, as in Lieberman, Rousseau and Zucker (2003). The only difference is in the proof of Lemma 5.2. of Dahlhaus (1989), that is, in the study of terms in the form

$$|\mathbf{I}_n - T_n(f)^{1/2} T_n((4\pi^2 f)^{-1}) T_n(f)^{1/2}|$$

with $f = F(d_2, g'_i)$ for any $i \leq p$. For simplicity's sake we write $f = F(d, g)$ in the following calculations. Following Dahlhaus's (1989) proof, we obtain an upper bound of $|f(\lambda_1)/f(\lambda_2) - 1|$ which is different from Dahlhaus (1989). If $g \in \mathcal{G}(m, M, L, \rho_2)$, the Lipschitz condition in ρ_2 implies that

$$\left| \frac{f(\lambda_1)}{f(\lambda_2)} - 1 \right| \leq K \left(|\lambda_1 - \lambda_2|^{\rho_2} + \frac{|\lambda_1 - \lambda_2|^{1-\delta}}{|\lambda_1|^{1-\delta}} \right).$$

Calculations as in Lemma 5.2 of Dahlhaus (1989) imply that

$$|I - T_n(f)^{1/2} T_n((4\pi^2 f)^{-1}) T_n(f)^{1/2}|^2 = O(n^{1-\rho_2} \log n^2) + O(n^\delta) \quad \forall \delta > 0.$$

From this we prove the lemma following Lieberman, Rosemarin and Rousseau [(2012), Lemma 7], the bounds being uniform over the considered class of functions. \square

A.3. Approximations: Lemmas 5 and 6. We now propose a generalization of Lieberman and Phillips (2004), whose proof is given in the supplementary material; see Lemma 1, Section 3, in Rousseau, Chopin and Liseo (2012).

LEMMA 5. *Let $1/2 > a > 0$, $L > 0$, $M > 0$ and $0 < \rho \leq 1$. Then for all $\delta > 0$, there exists $C > 0$ such that for all $n \in \mathbb{N}^*$,*

$$(19) \quad \sup_{\substack{p(d_1+d_2) \leq a \\ g_j, g'_j \in \mathcal{G}(-M, M, L, \rho)}} \left| \frac{1}{n} \operatorname{tr} \left[\prod_{j=1}^p T_n(F(d_1, g_j)) T_n(F(d_2, g'_j)) \right] - (2\pi)^{2p-1} \int_{-\pi}^{\pi} \prod_{j=1}^p F(d_1, g_j) F(d_2, g'_j)(x) dx \right| \leq C n^{-\rho+\delta+2a_+},$$

where $d_1, d_2 > -1/2$ and $a_+ = \max(a, 0)$.

LEMMA 6. *Let $f_j, j \in \{1, 2\}$ be such that $f_j(\lambda) = F(d_j, g_j)$, where $d_j \in (-1/2, 1/2)$, $0 < m \leq g_j \leq M < +\infty$ for some positive constant m, M and consider b a bounded function on $[-\pi, \pi]$. Assume that $|d_1 - d_2| < \delta$, with $\delta \in (0, 1/4)$; then, provided $d_1 > d_2, \forall a > 2\delta$,*

$$(20) \quad \begin{aligned} & \frac{1}{n} \text{tr}[T_n(f_1)^{-1} T_n(f_1 b) T_n(f_2)^{-1} T_n(f_1 b)] \\ & \leq C(\log n)[|b|_2^2 + (\delta + n^{-1+6a})|b|_\infty^2] \end{aligned}$$

and, without assuming $d_1 > d_2$,

$$(21) \quad \begin{aligned} & \frac{1}{n} \text{tr}[T_n(f_1^{-1}) T_n(f_1 - f_2) T_n(f_2^{-1}) T_n(f_1 - f_2)] \\ & \leq C[h_n(f_1, f_2) + n^{\delta-1/2} \sqrt{h_n(f_1, f_2)}]. \end{aligned}$$

APPENDIX B: CONSTRUCTION OF TESTS: LEMMAS 7, 8 AND 9

LEMMA 7. *If $8|d_0 - d_i| \leq \rho + 1 - t$ [case (a) of condition (1)], the inequalities in (14) are verified provided $\rho_i = \text{tr}[\mathbf{I}_n - T_n(f_0) T_n^{-1}(f_i)]/n + h_n(f_0, f_i), f \leq f_i$ and*

$$(22) \quad \frac{1}{2\pi} \int_0^\pi \frac{f_i(\lambda) - f(\lambda)}{f_0(\lambda)} d\lambda \leq h(f_0, f_i)/4.$$

PROOF. For all $s \in (0, 1/4)$, using Markov inequality,

$$\begin{aligned} E_0^n[\phi_i] & \leq \exp\{-sn\rho_i\} E_0^n[\exp\{-s\mathbf{X}_n^t\{T_n^{-1}(f_i) - T_n^{-1}(f_0)\}\mathbf{X}_n\}] \\ & = \exp\{-sn\rho_i - \frac{1}{2} \log \det[\mathbf{I}_n + 2sB(f_0, f_i)]\} \\ & \leq \exp\{-sn\rho_i - s \text{tr}[B(f_0, f_i)] \\ & \quad + s^2 \text{tr}[(\mathbf{I}_n + 2s\tau B(f_0, f_i))^{-2} B(f_0, f_i)^2]\} \\ & \leq \exp\{-sn\rho_i - s \text{tr}[B(f_0, f_i)] + 4s^2 \text{tr}[B(f_0, f_i)^2]\}, \end{aligned}$$

where $\tau \in (0, 1)$, using a Taylor expansion of the log-determinant around $s = 0$, and the following inequality:

$$\begin{aligned} & \mathbf{I}_n + 2s\tau B(f_0, f_i) \\ & = (1 - 2s\tau)\mathbf{I}_n + 2s\tau T_n(f_0)^{1/2} T_n(f)^{-1} T_n(f_0) \\ & \geq \frac{1}{2}\mathbf{I}_n, \end{aligned}$$

since $s\tau < 1/4$. Substituting ρ_i with its expression, the polynomial above is minimal for $s_{\min} = h_n(f_0, f_i)/8b_n(f_0, f_i)$. According to $s_{\min} \in (0, 1/4)$ or not, that is,

whether $h_n(f_0, f_i) < 2b_n(f_0, f_i)$ or not, one has

$$\begin{aligned}
 \frac{1}{n} \log E_0^n[\phi_i] &\leq -\frac{h_n(f_0, f_i)^2}{16b_n(f_0, f_i)} \mathbb{1}\{h_n(f_0, f_i) < 2b_n(f_0, f_i)\} \\
 (23) \quad &\quad -\frac{h_n(f_0, f_i) - b_n(f_0, f_i)}{4} \mathbb{1}\{h_n(f_0, f_i) \geq 2b_n(f_0, f_i)\} \\
 &\leq -\frac{h_n(f_0, f_i)}{16} \min\left\{\frac{h_n(f_0, f_i)}{b_n(f_0, f_i)}, 2\right\}.
 \end{aligned}$$

Since $8|d_0 - d_i| \leq \rho + 1 - t$, the convergences $b_n(f_0, f_i) \rightarrow b(f_0, f_i)$ and $h_n(f_0, f_i) \rightarrow h(f_0, f_i)$ are uniform on the support of the prior π ; see Lemma 2. One deduces that, for any $a > 0$ and n large enough,

$$\frac{1}{n} \log E_0^n[\phi_i] \leq -\frac{n}{16} \min\left\{\frac{h(f_0, f_i)^2 - a}{b(f_0, f_i) + a}, 2h(f_0, f_i) - a\right\}.$$

Since $f_i \in \mathcal{A}_\varepsilon^c$, $h(f_0, f_i) > \varepsilon$, and one may take $a = \varepsilon^2/2$ to obtain

$$\frac{1}{n} \log E_0^n[\phi_i] \leq -\frac{nh(f_0, f_i)}{32} \min\left\{\frac{h(f_0, f_i)}{b(f_0, f_i) + \varepsilon^2/2}, 2\right\}.$$

Since $|d_0 - d_i| \leq (\rho + 1 - t)/8 \leq 1/4$, Lemma 12 (see Appendix D) implies that there exists $C_1 > 0$ such that $E_0^n[\phi_i] \leq \exp(-nC_1\varepsilon)$ for ε small enough.

If f is in the support of π and satisfies $f \leq f_i$, and $8(d_i - d) \leq \rho + 1 - t$, using the same kind of calculations and the fact that

$$\mathbf{I}_n - 2sT_n^{1/2}(f)\{T_n^{-1}(f_i) - T_n^{-1}(f_0)\}T_n^{1/2}(f) \geq \mathbf{I}_n + 2sB(f, f_0)$$

as $T_n(f) \leq T_n(f_i)$, we obtain for $s \in (0, 1/4)$,

$$\begin{aligned}
 E_f^n[1 - \phi_i] &\leq \exp\{ns\rho_i - s \operatorname{tr}[B(f, f_0)] + 4s^2 \operatorname{tr}[B(f, f_0)^2]\} \\
 &\leq \exp\{-nsh_n(f_0, f_i) + s \operatorname{tr}[A(f_i - f, f_0)] \\
 &\quad + 4s^2 \operatorname{tr}[B(f, f_0)^2]\} \\
 &\leq \exp\{-nsh_n(f_0, f_i)/2 + 4s^2 \operatorname{tr}[B(f, f_0)^2]\},
 \end{aligned}$$

where the last inequality comes from (22), which implies $\operatorname{tr}[A(f_i - f, f_0)]/n \leq h_n(f_0, f_i)/2$ for n large enough, uniformly in f , using Lemma 2. Doing the same calculations as above, for n large enough,

$$\begin{aligned}
 \frac{1}{n} \log E_f^n[1 - \phi_i] &\leq -\frac{1}{64} \min\left\{\frac{h_n(f_0, f_i)^2}{b_n(f, f_0)}, 4h_n(f_0, f_i)\right\} \\
 (24) \quad &\leq -\frac{1}{64} \min\left\{\frac{h(f_0, f_i)^2/2}{b(f, f_0) + \varepsilon^2/2}, 2h(f_0, f_i)\right\}.
 \end{aligned}$$

To conclude, note that $f \leq f_i$, and (22) implies that

$$\begin{aligned}
 b(f, f_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{f^2}{f_0^2} + 1 - 2\frac{f}{f_0} \right\} d\lambda \\
 &\leq b(f_i, f_0) + h(f_0, f_i)/2 \leq (C + 1/2)h(f_0, f_i),
 \end{aligned}$$

according to Lemma 12. One concludes that there exists $C_1 > 0$ such that $E_f^n[1 - \phi_i] \leq e^{-nC_1\varepsilon}$. \square

LEMMA 8. *If $8(d_i - d_0) > \rho + 1 - t$ [case (b) of condition (3)], the inequalities (14) are verified provided $\rho_i = \text{tr}[\mathbf{I}_n - T_n(f_0)T_n^{-1}(f_i)]/n + 2KL_n(f_0; f_i)$, for any f_i such that $f \leq f_i$ and*

$$(25) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{f_i}{f} - 1 \right) d\lambda \leq \left(\frac{M}{\pi^2 m} \right)^4 \frac{b(f_0, f_i)}{64}, \quad b(f_i, f) \leq b(f_0, f_i).$$

For ε small enough, if $b(f_i, f) \leq b(f_0, f_i)|\log \varepsilon|^{-1}$, (25) is satisfied.

PROOF. The upper bound of $E_0^n[\phi_i]$ is computed similarly to (23) so that

$$\frac{1}{n} \log E_0^n[\phi_i] \leq -\frac{1}{4} \min \left\{ \frac{KL_n(f_0, f_i)^2}{b_n(f_0, f_i)}, KL_n(f_0, f_i) \right\}.$$

According to Lemma 11 and since $8(d_i - d_0) \geq \rho + 1 - t$, there exists $C > 0$, such that $b(f_0, f_i) \geq C$. Using the uniform convergence results of Appendix A, this means that $b_n(f_0, f_i) \geq C/2$, for n large enough, independently of f_i . Using Lemma 13, there exists a constant $C_1 \leq 1$ such that $KL_n(f_0, f_i) \geq C_1 b_n(f_0, f_i)$. Thus, there exists $C_2 > 0$ such that

$$\frac{1}{n} \log E_0^n[\phi_i] \leq -nC_2 b(f_0, f_i)$$

and, for ε small enough, and some $C_3 > 0$, $E_0^n[\phi_i] \leq \exp\{-nC_3\varepsilon\}$.

As in the previous lemma, let $h \in (0, 1)$,

$$\begin{aligned}
 &\log E_f^n[1 - \phi_i] \\
 &\leq (1 - h)n\rho_i/2 \\
 &\quad - \frac{1}{2} \log \det[\mathbf{I}_n - (1 - h)T_n(f)^{1/2}\{T_n^{-1}(f_i) - T_n^{-1}(f_0)\}T_n(f)^{1/2}] \\
 &\leq (1 - h)n\rho_i/2 - \frac{1}{2} \log \det[\mathbf{I}_n + (1 - h)B(f, f_0)] \\
 &= (1 - h)n\rho_i/2 - \log \det[A(f, f_0)]/2 \\
 &\quad - \frac{1}{2} \log \det[\mathbf{I}_n(1 - h) + hT_n^{-1/2}(f)T_n(f_0)T_n^{-1/2}(f)].
 \end{aligned}$$

Substituting ρ_i with its expression, that is, $n\rho_i - \log \det A(f, f_0) = \log \det A(f_i, f)$ and using the same kind of expansions as in the previous lemma, one obtains

$$\begin{aligned} & \frac{1}{n} \log E_f^n[1 - \phi_i] \\ & \leq \frac{1}{n} \log \det[A(f_i, f)] + (h/2) \operatorname{tr}[T_n(f_0)\{T_n^{-1}(f_i) - T_n^{-1}(f)\}] \\ & \quad - hnKL_n(f_0; f_i) + h^2 \operatorname{tr}[\{\mathbf{I}_n - T_n^{-1}(f)T_n(f_0)\}^2] \\ & \leq \frac{1}{n} \log \det[A(f_i, f)] - hnKL_n(f_0; f_i) + h^2 \operatorname{tr}[\{\mathbf{I}_n - T_n^{-1}(f)T_n(f_0)\}^2] \\ & \leq +\frac{1}{n} \log \det[A(f_i, f)] - n \min\left(\frac{KL_n(f_0, f_i)^2}{4 \operatorname{tr} B(f_0, f)^2/n}, \frac{KL_n(f_0, f_i)}{4}\right). \end{aligned}$$

Note that we use the fact $f \leq f_i$ in the second line.

Since $\log \det A(f_i, f) = \log \det\{\mathbf{I}_n + T_n(f_i - f)T_n(f)^{-1}\}$, using a Taylor expansion of $\log \det$ around \mathbf{I}_n , we obtain that for n large enough,

$$\frac{1}{n} \log \det A(f_i, f) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_i - f}{f} d\lambda + a,$$

where a can be chosen as small as necessary. In addition, we use Lemma 13 and the uniform convergence results of Lemmas 3, 4 to obtain that

$$\frac{(nKL_n(f_0, f_i))^2}{\operatorname{tr}[B(f_0, f)^2]} \geq \frac{nm^4(b(f_0, f_i)^2 - a)^2}{16\pi^8 M^4(b(f_0, f) + a)}$$

and, since $d \geq d_0$ and (25),

$$\begin{aligned} b(f_0, f) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{f_0}{f} - 1\right)^2 d\lambda \leq 2\left(b(f_0, f_i) + \frac{M^2\pi^4}{m^2}b(f_i, f)\right) \\ &\leq 2b(f_0, f_i)\left(1 + \frac{M^2\pi^4}{m^2}\right); \end{aligned}$$

hence, under the constraint (25), there exists $C_1 > 0$ such that, for n large enough, ε small enough, $E_f^n[1 - \phi_i] \leq \exp\{-nC_1b(f_0, f_i)\} \leq e^{-n\varepsilon}$. \square

LEMMA 9. *If $8(d_0 - d_i) > \rho + 1 - t$ [case (c) of condition (3)], the inequalities (14) are verified provided $\rho_i = \log \det[T_n(f_i)T_n(f_0)^{-1}]/n$ if*

$$(26) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_i - f}{f_0}(\lambda) d\lambda \leq \frac{m^2}{4M^2\pi^4}b(f_i, f_0), \quad b(f, f_i) \leq b(f_i, f_0).$$

For $\varepsilon > 0$ small if $\int(f_i - f)f_i^{-1} d\lambda \leq b(f_i, f_0)|\log \varepsilon|^{-1}$, (26) is satisfied.

PROOF. For $0 < h < 1$, following the same lines as above, one has

$$\begin{aligned} \frac{1}{n} \log E_0^n[\phi_i] &\leq -(1-h)n\rho_i/2 + \log \det[A(f_0, f_i)]/2 \\ &\quad - \frac{1}{2} \log \det[\mathbf{I}_n(1-h) + hT_n^{-1/2}(f_0)T_n(f_i)T_n^{-1/2}(f_0)] \\ &\leq -nhKL_n(f_i, f_0) + h^2 \operatorname{tr}[B(f_i, f_0)^2] \leq -\varepsilon. \end{aligned}$$

Moreover, for all $f \leq f_i$, satisfying $8(d_i - d) \leq \rho + 1 - t$, using the same calculations as in the proof of Lemma 7, we bound $\log E_f^n[1 - \phi_i]$ by the maximum of the two following quantities:

$$\begin{aligned} &-\frac{\{nKL_n(f_i, f_0) - \operatorname{tr}[A(f_i - f, f_0)]/2\}^2}{4n\{b(f, f_0) + a\}}, \\ &-\frac{n}{4}KL_n(f_i, f_0) + \frac{1}{8} \operatorname{tr}[A(f_i - f, f_0)], \end{aligned}$$

where a is any positive constant and n is large enough. Using Lemma 13, one has

$$nKL_n(f_i, f_0) \geq \frac{nm^2}{2\pi^4 M^2} b(f_i, f_0),$$

and the constraints (26) we finally obtain that there exists constant $c_1, C_1 > 0$ such that, for ε small enough,

$$\begin{aligned} E_f^n[1 - \phi_i] &\leq \exp\{-2n(KL_n(f_i, f_0) - \operatorname{tr}[A(f_i - f, f)]/2n) + 4s^2nb_n(f, f_0)\} \\ &\leq e^{-nc_1b(f_i, f_0)} \leq e^{-nC_1\varepsilon}. \end{aligned} \quad \square$$

APPENDIX C: PROOF OF THEOREM 4.2

We re-use some of the notation of Section 5.1; in particular, C, C' denote generic constants.

The proof of the theorem is divided in two parts. First, we show that

$$(27) \quad E_0^n \left[P^\pi \left\{ f : h_n(f, f_0) \geq \frac{\log n}{n^{2\beta/(2\beta+1)}} \mid \mathbf{X}_n \right\} \right] \leq \frac{C}{n^2}.$$

Second, we show that, for $f \in \tilde{\mathcal{F}}_n$, and n large enough,

$$(28) \quad h_n(f, f_0) \leq Cn^{-2\beta/(2\beta+1)} \log n \quad \Rightarrow \quad h(f, f_0) \leq C'n^{-2\beta/(2\beta+1)} \log n.$$

Since $\ell(f, f_0) \leq h(f, f_0)$ (see the proof of Corollary 2 in the supplementary material [Rousseau, Chopin and Liseo (2012)]), the right-hand side inequality of

(28) implies that

$$\begin{aligned} E_0^n \{ E^\pi [\ell(f, f_0) | \mathbf{X}_n] \} &\leq C \frac{\log n}{n^{2\beta/(2\beta+1)}} \\ &\quad + \bar{\ell} E_0^n \left\{ P^\pi \left(h_n(f, f_0) > \frac{\log n}{n^{2\beta/(2\beta+1)}} \mid \mathbf{X}_n \right) \right\} \\ &\leq C n^{-2\beta/(2\beta+1)} \log n + C' n^{-2} \end{aligned}$$

for large n , where $\bar{\ell} < +\infty$ is an upper bound for $\ell(f, f_0)$ which is easily deduced from the fact that f, f_0 belongs to some Sobolev class of functions. This implies Theorem 4.2.

To prove (27), we show that conditions (1) and (2) of Theorem 4.1 are fulfilled for $u_n = n^{-2\beta/(2\beta+1)}(\log n)$. In order to establish condition (1), we show that, for n large enough, $\bar{\mathcal{B}}_n \supset \hat{\mathcal{B}}_n$, the set containing all the $f = \tilde{F}(d, k, \theta)$ such that $k \geq \bar{k}_n$, for $\bar{k}_n = k_0 n^{1/(2\beta+1)}$, $d - u_n n^{-a} \leq d_0 \leq d$ and, for $j = 0, \dots, k$,

$$(29) \quad |\theta_j - \theta_{0j}| \leq (j + 1)^{-2\beta} u_n n^{-a},$$

where $a > 0$ is some small constant. Then it is easy to see that $\pi(\bar{\mathcal{B}}_n) \geq \pi(\hat{\mathcal{B}}_n) \geq \exp\{-nu_n/2\}$, provided k_0 is small enough, since $\pi_k(k \geq \bar{k}_n) \geq \exp\{-C\bar{k}_n \log \bar{k}_n\}$, and (29) for all j implies that

$$\begin{aligned} \sum_{j=0}^k \theta_j^2 (j + 1)^{2\beta} &= \sum_{j=0}^k (\theta_{0j} - \theta_j + \theta_j)^2 (j + 1)^{2\beta} \\ &\leq L_0 + u_n^2 n^{-2a} \sum_{j=0}^k (1 + j)^{-2\beta} + 2u_n n^{-a} \left(\sum_{j=1}^k |\theta_{0j}| \right) \\ &< L \end{aligned}$$

for n large enough, since $L_0 = \sum_j \theta_{0j} (j + 1)^{2\beta} < L$, and $\sum_{j=1}^k |\theta_{0j}|$ is bounded according to (6).

Let $f = \tilde{F}(d, k, \theta)$, with $(d, k, \theta) \in \hat{\mathcal{B}}_n$. To prove that $(d, k, \theta) \in \bar{\mathcal{B}}_n$, it is sufficient to prove that $h_n(f, f_0) \leq u_n/4$, since $h_n(f, f_0) = KL_n(f_0; f) + KL_n(f; f_0)$, and $KL_n(f; f_0) \geq Cb_n(f_0, f)$, using the same calculation as in Dahlhaus [(1989), page 1755] and the fact that $d \leq d_0$.

Since $f_0 \in \mathcal{S}(\beta, L)$, and for the particular choice of \bar{k}_n above,

$$(30) \quad \sum_{j=\bar{k}_n}^{+\infty} \theta_{0j}^2 \leq L(\bar{k}_n + 1)^{-2\beta}$$

and

$$\sum_{j=\bar{k}_n}^{+\infty} |\theta_{0j}| \leq \left(\sum_{j=\bar{k}_n}^{+\infty} \theta_{0j}^2 (j + 1)^{2\beta} \right)^{1/2} \left(\sum_{j=\bar{k}_n}^{+\infty} (j + 1)^{-2\beta} \right)^{1/2} \leq C\bar{k}_n^{1/2-\beta}.$$

Let

$$(31) \quad \begin{aligned} f_{0n}(\lambda) &= |1 - e^{i\lambda}|^{-2d_0} \exp\left(\sum_{j=0}^{\bar{k}_n} \theta_{0j} \cos(j\lambda)\right), \\ b_n(\lambda) &= \exp\left(-\sum_{j \geq \bar{k}_n+1} \theta_{0j} \cos(j\lambda)\right) - 1 \end{aligned}$$

and $g_n = 1 - f_{0n}/f$. Then $f - f_0 = f_0 b_n + f g_n$, where b_n and g_n are bounded as follows. From (31), one gets that, for n large enough, $|b_n|_\infty \leq C \bar{k}_n^{1/2-\beta}$, and

$$|b_n|_2^2 = \int_{-\pi}^{\pi} b_n(\lambda)^2 d\lambda \leq 2 \sum_{j=\bar{k}_n+1}^{\infty} \theta_{0j}^2 \leq 2L \bar{k}_n^{-2\beta} \leq 2Lk_0^{-2\beta} \frac{u_n}{\log n}$$

according to (30). In addition since $1 - x \leq -\log x$, for $x > 0$,

$$\begin{aligned} g_n(\lambda) &\leq (d_0 - d) \log(1 - \cos \lambda) + \sum_{j \leq \bar{k}_n} |\theta_{0j} - \theta_j| \\ &\leq C u_n n^{-a} (|\log|\lambda|| + 1). \end{aligned}$$

Moreover, since $\text{tr}\{(A + B)^2\} \leq 2 \text{tr} A^2 + 2 \text{tr} B^2$ for square matrices A and B , one has

$$\begin{aligned} h_n(f_0, f) &\leq \frac{1}{n} \text{tr}[T_n(f_0 b_n) T_n^{-1}(f) T_n(f_0 b_n) T_n^{-1}(f_0)] \\ &\quad + \frac{1}{n} \text{tr}[T_n(f g_n) T_n^{-1}(f) T_n(f g_n) T_n^{-1}(f_0)] \\ &\leq C \log n \{|b_n|_2^2 + u_n n^{-a} |b_n|_\infty^2\} \\ &\quad + C u_n^2 n^{-1-2a} \text{tr}[(T_n(f(|\log|\lambda|| + 1)) T_n^{-1}(f))^2] \\ &\leq c u_n, \end{aligned}$$

where c may be chosen as small as necessary, since k_0 is arbitrarily large. Note that the first two terms above come from (20) in Lemma 6, and the third term comes from Lemma 4.

To establish condition (2) is straightforward, since the prior has the same form as in Section 3.2, and we can use the same reasoning as in the proof of Theorem 3.2; that is, we take, for some suitably chosen δ ,

$$\tilde{\mathcal{F}}_n = \{(d, k, \theta) \in \mathcal{S}(\beta, L) : |d - d_0| \leq \delta, k \leq \tilde{k}_n\},$$

where $\tilde{k}_n = k_1 n^{1/(2\beta+1)}$ so that, using Lemma 10,

$$\pi(\tilde{\mathcal{F}}_n^c \cap \{f, h(f, f_0) < \varepsilon\}) \leq \pi_k(k \geq \tilde{k}_n) \leq e^{-C \tilde{k}_n \log \tilde{k}_n}$$

for n large enough. Choosing k_1 large enough leads to condition (2).

We now verify condition (3) of Theorem 4.2. Let $\varepsilon_n^2 \geq u_n$ and $l_0 \leq l \leq l_n$, and consider $f = \tilde{F}(d, k, \theta)$, $(d, k, \theta) \in \mathcal{V}_{n,l}$, as defined in Theorem 4.1, and $f_{i,l} = (2e)^{l\varepsilon_n^2} \tilde{F}(d_i, k, \theta_i)$, where dependencies on l in d_i and θ_i are dropped for convenience. If for some positive $c > 0$ to be chosen accordingly $|\theta_j - \theta_{ij}| \leq cl\varepsilon_n^2/(k + 1)$, for $j = 0, \dots, k$, one obtains

$$\frac{g_{i,l}(\lambda)}{g(\lambda)} = (2e)^{l\varepsilon_n^2} \exp \left\{ \sum_{j=0}^k (\theta_j - \theta_{ij}) \cos(j\lambda) \right\} \leq (2e^2)^{cl\varepsilon_n^2}$$

and $f_{i,l}/f \geq 1$ so that the constraints of condition (3) of Theorem 4.2 are verified by choosing c small enough. The cardinal of the smallest possible net under these constraints needed to cover $\mathcal{V}_{n,l}$ is bounded by

$$\bar{C}_{n,l} \leq k_n \left(\frac{1}{cl\varepsilon_n^2} \right) \left(\frac{L'k_n}{cl\varepsilon_n^2} \right)^{k_n+1}$$

since for all l $|\theta_l| \leq L$. This implies that $\log \bar{C}_{n,l} \leq Cnu_n$, and condition (3) is verified with $\varepsilon_n^2 = \varepsilon_0^2 u_n$. This achieves the proof of (27), which provides a rate of convergence in terms of the distance $h_n(\cdot, \cdot)$.

Finally, we prove (28) to obtain a rate of convergence in terms of the distance $h(\cdot, \cdot)$. Consider f such that

$$h_n(f_0, f) = \frac{1}{2n} \text{tr}[T_n^{-1}(f_0)T_n(f - f_0)T_n^{-1}(f)T_n(f - f_0)] \leq \varepsilon_n^2.$$

Equation (21) of Lemma 6 implies that

$$\begin{aligned} & \frac{1}{2n} \text{tr}[T_n(f_0^{-1})T_n(f - f_0)T_n(f^{-1})T_n(f - f_0)] \\ (32) \quad & \leq C\varepsilon_n[\varepsilon_n + n^{-1/2+\delta}] \\ & \leq C\varepsilon_n^2. \end{aligned}$$

We now prove that

$$\begin{aligned} & \text{tr}[T_n(f_0^{-1})T_n(f - f_0)T_n(f^{-1})T_n(f - f_0)] \\ & - \text{tr}[T_n(f_0^{-1}(f - f_0))T_n(f^{-1}(f - f_0))] \\ & \leq \frac{C(\log n)^2}{n^{1-2a}} \end{aligned}$$

for some small $a > 0$. By symmetry we consider only the case $d \geq d_0$. Let $h_0 = (1 - \cos \lambda)^{d_0}$, $h = (1 - \cos \lambda)^d$, then $fh \leq C$, $f_0h_0 \leq C$ and $|f - f_0|h \leq C$ for some $C \geq 0$, and it is sufficient to study the difference below. Note that the

calculations below follow the same lines and the same notation as the treatment of $\gamma(b)$ in Lemma 6; see Appendix A; in particular, $\Delta_n(\lambda) = \sum_{j=1}^n \exp(-i\lambda j)$, and $L_n(\lambda) = n$ for $|\lambda| \leq 1/n$, $L_n(\lambda) = |\lambda|^{-1}$ otherwise.

$$\begin{aligned} & \frac{1}{n} \operatorname{tr}[T_n(h_0(f - f_0))T_n(h(f - f_0))] \\ & - \frac{1}{n} \operatorname{tr}[T_n(h_0)T_n(f - f_0)T_n(h)T_n(f - f_0)] \\ & = -\frac{1}{n} \int_{[-\pi, \pi]^3} (f - f_0)(\lambda_2)h_0(\lambda_2)(f - f_0)(\lambda_4)h(\lambda_4) \left(\frac{h_0(\lambda_1)}{h_0(\lambda_2)} - 1 \right) \\ & \quad \times \Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_4)\Delta_n(\lambda_4 - \lambda_1) d\lambda \\ & \quad - \frac{1}{n} \int_{[-\pi, \pi]^3} (f - f_0)(\lambda_2)h_0(\lambda_1)(f - f_0)(\lambda_4)h(\lambda_4) \left(\frac{h(\lambda_3)}{h(\lambda_4)} - 1 \right) \\ & \quad \times \Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_3)\Delta_n(\lambda_3 - \lambda_4)\Delta_n(\lambda_4 - \lambda_1) d\lambda \\ & \leq \frac{C(\log n)}{n} \int_{[-\pi, \pi]^2} |\lambda_2|^{-2(d-d_0)} |\lambda_1|^{-1+a} L_n(\lambda_1 - \lambda_2)^{1+a} d\lambda \\ & \quad + \frac{C}{n} \int_{[-\pi, \pi]^4} \frac{|\lambda_1|^{2d}}{|\lambda_2|^{2d} |\lambda_3|^{1-a}} L_n(\lambda_1 - \lambda_2)L_n(\lambda_2 - \lambda_3) \\ & \quad \quad \times L_n(\lambda_3 - \lambda_4)^a L_n(\lambda_4 - \lambda_1) d\lambda \\ & \leq \frac{C(\log n)^2}{n^{1-a}} \int_{[-\pi, \pi]^2} |\lambda_2|^{-2(d-d_0)} |\lambda_1|^{-1+a} L_n(\lambda_2 - \lambda_1) d\lambda \\ & \quad + \frac{C(\log n)}{n^{1-a}} \int_{[-\pi, \pi]^3} \frac{|\lambda_1|^{2d}}{|\lambda_2|^{2d} |\lambda_3|^{1-a}} L_n(\lambda_1 - \lambda_2)L_n(\lambda_2 - \lambda_3) d\lambda \\ & \leq \frac{C(\log n)^2}{n^{1-2a}}, \end{aligned}$$

provided $d - d_0 \leq a/4$, using standard calculations. Combined with (32), this result implies that

$$\frac{1}{n} \operatorname{tr}[T_n(h_0(f - f_0))T_n(h(f - f_0))] \leq C\epsilon_n^2.$$

Finally, to obtain (28), we bound

$$\begin{aligned} & |\operatorname{tr}[T_n(h_0(f - f_0))T_n(h(f - f_0))] - \operatorname{tr}[T_n(h_0h(f - f_0)^2)]| \\ & = C \left| \int_{[-\pi, \pi]^2} \{h_0(f - f_0)\}(\lambda_1)[\{h(f - f_0)\}(\lambda_2) - \{h(f - f_0)\}(\lambda_1)] \right. \\ & \quad \left. \times \Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_1) d\lambda \right| \end{aligned}$$

$$\begin{aligned} &\leq C \left| \int_{[-\pi, \pi]^2} \{h(f - f_0)\}(\lambda_1)(f - f_0)(\lambda_2)[h(\lambda_2) - h(\lambda_1)] \right. \\ &\qquad \qquad \qquad \left. \times \Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_1) d\lambda \right| \\ &+ C \left| \int_{[-\pi, \pi]^2} \{hh_0(f - f_0)\}(\lambda_1)[f_0(\lambda_2) - f_0(\lambda_1)] \right. \\ &\qquad \qquad \qquad \left. \times \Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_1) d\lambda \right| \\ &+ C \left| \int_{[-\pi, \pi]^2} \{hh_0(f - f_0)\}(\lambda_1)[f(\lambda_2) - f(\lambda_1)] \right. \\ &\qquad \qquad \qquad \left. \times \Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_1) d\lambda \right|. \end{aligned}$$

The first term is of order $O(n^{2a} \log n)$, from the same calculations as above. We consider the last term, but the calculations for the second term follow exactly the same lines. Recall that $f = he^w$, where $w(\lambda) = \sum_{j=0}^k \theta_j \cos(j\lambda)$ is not necessarily continuously differentiable, for example, when $\beta < 1$. Thus

$$\begin{aligned} f(\lambda_2) - f(\lambda_1) &= [h(\lambda_2)^{-1} - h(\lambda_1)^{-1}]e^{w(\lambda_2)} \\ &\quad + h(\lambda_1)^{-1}[e^{w(\lambda_2)} - e^{w(\lambda_1)}]. \end{aligned}$$

The first term is dealt with using (5) and (6) in the supplementary material [Rousseau, Chopin and Liseo (2012)], leading to a bound of order $(\log n)^2 n^{2a}$. For the second term, and $k \leq k_n$,

$$\begin{aligned} &\left| \int_{[-\pi, \pi]^2} h_0(f - f_0)(\lambda_1)[g(\lambda_2) - g(\lambda_1)]\Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_1) d\lambda \right| \\ &\leq C \int_{[-\pi, \pi]^2} h_0|f - f_0|(\lambda_1) \left| \sum_{j=0}^k \theta_j (\cos(j\lambda_2) - \cos(j\lambda_1)) \right| \\ &\qquad \qquad \qquad \times L_n(\lambda_1 - \lambda_2)L_n(\lambda_2 - \lambda_1) d\lambda \\ &\leq C(\log n) \left(\sum_{j=0}^k |\theta_j|j \right) \int_{-\pi}^{\pi} \{h_0|f - f_0|\}(\lambda_1) d\lambda_1 \\ &\leq C(\log n) \left(\sum_{j=0}^k |\theta_j|j \right) \left(\int_{-\pi}^{\pi} \{hh_0(f - f_0)^2\}(\lambda) d\lambda \right)^{1/2}, \end{aligned}$$

where the latter inequality holds because $\int_{-\pi}^{\pi} \{h_0/h\}(\lambda) d\lambda$ is bounded when $|d - d_0|$ is small enough. The same computations can be made on f_0 so that for all

$a > 4|d - d_0|$, we finally obtain that

$$\begin{aligned} & |\text{tr}[T_n(h_0(f - f_0))T_n(h(f - f_0))] - \text{tr}[T_n(h_0h(f - f_0)^2)]| \\ & \leq C(\log n)n^{2a} + (\log n) \sum_{j=0}^k j(|\theta_j| + |\theta_{0j}|) \left(\int_{[-\pi, \pi]} g_0g(f - f_0)^2(\lambda) d\lambda \right)^{1/2}. \end{aligned}$$

Splitting the indices of the sum above into into $\{j : j|\theta_j| \leq j^{2\beta+r}\theta_j^2\}$ and its complementary, for some r , we get that

$$\sum_{j=0}^k j|\theta_j| \leq \sum_{j=0}^k j^{2\beta+r}\theta_j^2 + \sum_{j=0}^k j^{1-2\beta-r} \leq C(k^r + k^{2-2\beta-r}) \leq Ck_n,$$

provided we take $r = 3/2 - \beta$. One concludes by doing the same computation for f_0 , so as to obtain that, for $\beta \geq 1/2$, $\int_{-\pi}^{\pi} h_0h(f_0 - f)^2 d\lambda \leq C\varepsilon_n^2$.

APPENDIX D: TECHNICAL LEMMAS

The three following lemmas provide inequalities involving

$$b(f, f_0) = \frac{1}{2\pi} \int_0^\pi (f/f_0 - 1)^2 d\lambda, \quad h(f, f_0) = \frac{1}{2\pi} \int_0^\pi (f/f_0 - 1)^2 \frac{f_0}{f} d\lambda$$

for $f = F(d, g)$, $f_0 = F(d_0, g_0)$, $d, d_0 \in (-1/2, 1/2)$, $g, g_0 \in \mathcal{G}(m, M)$, $0 < m < M$.

LEMMA 10. *For any $\varepsilon > 0$, $|d - d_0| \geq \varepsilon \Rightarrow h(f, f_0) \geq \frac{1}{\pi} \left(\frac{4M}{m}\right)^{-1/2\varepsilon}$.*

PROOF. Without loss of generality, take $d \geq d_0$, then, since $(x - 1)^2/x \geq x/2$ for $x \geq 4$,

$$h(f, f_0) \geq \frac{m}{4\pi M} \int_0^\pi \mathbb{1}\{\lambda^{-2(d-d_0)} \geq 4M/m\} \lambda^{-2(d-d_0)} d\lambda \geq \frac{1}{\pi} \left(\frac{4M}{m}\right)^{-1/2\varepsilon}. \quad \square$$

LEMMA 11. *There exists $C > 0$ such that, for any $\varepsilon > 0$,*

$$|d - d_0| \geq \varepsilon \quad \Rightarrow \quad b(f, f_0) \geq C^{-1/2\varepsilon}.$$

PROOF. If $d \geq d_0$, then, since $(x - 1)^2 \geq x^2/2$ for $x \geq 4$,

$$b(f, f_0) \geq \frac{m^2}{4\pi M^2} \int_0^\pi \mathbb{1}\{\lambda^{-2(d-d_0)} \geq 4M/m\} \lambda^{-4(d-d_0)} d\lambda \geq \frac{4}{\pi} \left(\frac{4M}{m}\right)^{-1/2\varepsilon}.$$

Otherwise, if $d < d_0$, one has $(x - 1)^2 \geq 1/4$ for $0 \leq x \leq 1/2$, so

$$b(f, f_0) \geq \frac{1}{8\pi} \int_0^\pi \mathbb{1}\{\lambda^{2(d_0-d)} \leq m/2M\} d\lambda \geq \frac{1}{8\pi} \left(\frac{2M}{m}\right)^{-1/2\varepsilon}. \quad \square$$

LEMMA 12. For any $\tau \in (0, 1/4)$, there exists $C > 0$ such that

$$d - d_0 < \frac{1}{4} - \tau \implies b(f, f_0) \leq Ch(f, f_0).$$

PROOF. If $d \leq d_0$, the bound is trivial, since $f/f_0 \leq M/m\pi^{2(d_0-d)}$. Assume $d > d_0$, and let $A \geq 1/2$ some arbitrary large constant. Since $(x - 1)^2 \leq x^2$ for $x \geq 1/2$, one has

$$\begin{aligned} b(f, f_0) &\leq Ah(f, f_0) + \frac{M^2}{2\pi m^2} \int_0^\pi \mathbb{1}\{f(\lambda)/f_0(\lambda) \geq A\} \lambda^{-4(d-d_0)} d\lambda \\ (33) \quad &\leq Ah(f, f_0) + \frac{M^2}{2\pi m^2} \int_0^\pi \mathbb{1}\{\lambda^{-2(d-d_0)} \geq Am/M\} \lambda^{-4(d-d_0)} d\lambda \\ &\leq Ah(f, f_0) + \frac{C'(Am/M)^{2-1/2(d-d_0)}}{1-4t}, \end{aligned}$$

provided $A \geq M/m$ and $C' = M^2/2\pi m^2$. In turn, since $(x - 1)^2 \geq x^2/2$ for $x \geq 4$, and assuming $A \geq 4M^2/m^2$, then $\lambda^{-2(d-d_0)} \geq Am/M$ implies that $f/f_0 \geq Am^2/M^2 \geq 4$, and $(f/f_0 - 1)^2 f_0/f \geq f/2 f_0 \geq Am^2/2M^2$. Therefore

$$(34) \quad h(f, f_0) \geq \frac{1}{2\pi} \int_0^\pi \mathbb{1}\{\lambda^{-2(d-d_0)} \geq Am/M\} (f/f_0 - 1)^2 \frac{f_0}{f} d\lambda$$

$$(35) \quad \geq (Am/M)^{2-1/2(d-d_0)} / 4\pi A.$$

One concludes by combining (33) with (35) and taking $A = 4M^2/m^2$. \square

The lemma below makes the same assumptions with respect to f and f_0 .

LEMMA 13. $d > d_0 \implies KL_n(f_0; f) \geq \frac{m^2}{M^2\pi^2} b_n(f_0, f)$.

PROOF. Dahlhaus [(1989), page 1755] proves that $KL_n(f_0; f) \geq C^{-2} b_n(f_0, f)$ where C is the largest eigenvalue of $T_n(f_0)T_n^{-1}(f)$. In our case, $f_0/f \leq M\pi^{2(d-d_0)}/m$, hence $C^{-2} = m^2/M^2\pi^{2(d-d_0)}$. \square

The last lemma applies to the FEXP formulation of Section 3.2.

LEMMA 14. For $\varepsilon \in (0, 1/4)$, $f_0(\lambda) = (2 - 2\cos \lambda)^{-d_0} \exp\{w_0(\lambda)\}$, $f(\lambda) = (2 - 2\cos \lambda)^{-d} \exp\{w(\lambda)\}$, one has

$$|d - d_0| \leq \varepsilon, \quad |w - w_0| \leq \varepsilon \implies h(f, f_0) \leq 7\varepsilon.$$

PROOF. Without loss of generality, take $d - d_0 \geq 0$. Then $f_0/f - 1 \leq 2^\varepsilon e^\varepsilon - 1 \leq (1 + \log 2)\varepsilon$, since $e^x \leq 1 + 2x$ for $x \in [0, 1]$. Moreover, since $2(1 - \cos \lambda) \geq \lambda^2/3$ for $\lambda \in (0, \pi)$, one has

$$\int_0^\pi \frac{f(\lambda)}{f_0(\lambda)} d\lambda = e^\varepsilon 3^{(d-d_0)} \int_0^\pi \lambda^{-2(d-d_0)} d\lambda \leq \frac{\pi e^\varepsilon 3^\varepsilon}{1-2\varepsilon}$$

and, to conclude, as again $e^x \leq 1 + 2x$ for $x \in [0, 1]$, and $e^{\varepsilon(1+\log 3)}(1 - 2\varepsilon)^{-1} - 1 \leq 10\varepsilon$, for $\varepsilon \leq 1/4$,

$$h(f, f_0) = \frac{1}{2\pi} \int_0^\pi \left(\frac{f(\lambda)}{f_0(\lambda)} + \frac{f_0(\lambda)}{f(\lambda)} - 2 \right) d\lambda \leq (6 + \log 2)\varepsilon. \quad \square$$

Acknowledgments. Part of this work was done while the third author was visiting the Université Paris Dauphine, CEREMADE, whom he thanks for warm hospitality and financial support.

SUPPLEMENTARY MATERIAL

Bayesian nonparametric estimation of the spectral density of a long or intermediate memory Gaussian process: Supplementary material (DOI: [10.1214/11-AOS955SUPP](https://doi.org/10.1214/11-AOS955SUPP); .pdf). Proof of technical lemmas and theorems stated in the paper.

REFERENCES

- BARDET, J.-M., LANG, G., OPPENHEIM, G., PHILIPPE, A., TAQQU, M. S. and STOEVE, S. (2003). Semi-parametric estimation of the long-range dependence parameter: A survey. In *Theory and Applications of Long-range Dependence* 557–577. Birkhäuser, Boston, MA. [MR1957508](#)
- BARRON, A., SCHERVISH, M. J. and WASSERMAN, L. (1999). The consistency of posterior distributions in nonparametric problems. *Ann. Statist.* **27** 536–561. [MR1714718](#)
- BERAN, J. (1993). Fitting long-memory models by generalized linear regression. *Biometrika* **80** 817–822. [MR1282790](#)
- BERAN, J. (1994). *Statistics for Long-memory Processes. Monographs on Statistics and Applied Probability* **61**. Chapman & Hall, New York. [MR1304490](#)
- DAHLHAUS, R. (1989). Efficient parameter estimation for self-similar processes. *Ann. Statist.* **17** 1749–1766. [MR1026311](#)
- DOUKHAN, P., OPPENHEIM, G. and TAQQU, M. S., eds. (2003). *Theory and Applications of Long-range Dependence*. Birkhäuser, Boston, MA. [MR1956041](#)
- FOX, R. and TAQQU, M. S. (1986). Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series. *Ann. Statist.* **14** 517–532. [MR0840512](#)
- GEWEKE, J. and PORTER-HUDAK, S. (1983). The estimation and application of long memory time series models. *J. Time Series Anal.* **4** 221–238. [MR0738585](#)
- GHOSAL, S., GHOSH, J. K. and VAN DER VAART, A. W. (2000). Convergence rates of posterior distributions. *Ann. Statist.* **28** 500–531. [MR1790007](#)
- GHOSAL, S. and VAN DER VAART, A. (2007). Convergence rates of posterior distributions for non-i.i.d. observations. *Ann. Statist.* **35** 192–223. [MR2332274](#)
- GIRAITIS, L. and TAQQU, M. S. (1999). Whittle estimator for finite-variance non-Gaussian time series with long memory. *Ann. Statist.* **27** 178–203. [MR1701107](#)
- HURVICH, C. M., MOULINES, E. and SOULIER, P. (2002). The FEXP estimator for potentially non-stationary linear time series. *Stochastic Process. Appl.* **97** 307–340. [MR1875337](#)
- JEFFREYS, H. (1946). An invariant form for the prior probability in estimation problems. *Proc. R. Soc. London. Ser. A* **186** 453–461. [MR0017504](#)
- LIEBERMAN, O. and PHILLIPS, P. C. B. (2004). Error bounds and asymptotic expansions for Toeplitz product functionals of unbounded spectra. *J. Time Series Anal.* **25** 733–753. [MR2089192](#)

- LIEBERMAN, O., ROSEMARIN, R. and ROUSSEAU, J. (2012). Asymptotic theory for maximum likelihood estimation in stationary fractional Gaussian processes, under short, long and intermediate memory. *Econometric Theory* **28** 457–470.
- LIEBERMAN, O., ROUSSEAU, J. and ZUCKER, D. M. (2003). Valid asymptotic expansions for the maximum likelihood estimator of the parameter of a stationary, Gaussian, strongly dependent process. *Ann. Statist.* **31** 586–612. [MR1983543](#)
- LISEO, B. and ROUSSEAU, J. (2006). Sequential importance sampling algorithm for Bayesian nonparametric long range inference. *Atti della XLIII Riunione Scientifica della Societa Italiana di Statistica* **2** 43–46.
- MANDELBROT, B. B. and VAN NESS, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** 422–437. [MR0242239](#)
- MANDELBROT, B. and WALLIS, J. (1969). Some long run properties of geophysical records. *Water Resources Research* **5** 321–340.
- MOULINES, E. and SOULIER, P. (2003). Semiparametric spectral estimation for fractional processes. In *Theory and Applications of Long-range Dependence* 251–301. Birkhäuser, Boston, MA. [MR1956053](#)
- ROBINSON, P. M. (1991). Nonparametric function estimation for long memory time series. In *Nonparametric and Semiparametric Methods in Econometrics and Statistics (Durham, NC, 1988)* (W. A. Barnett, J. Powell and G. E. Tauchen, eds.) 437–457. Cambridge Univ. Press, Cambridge. [MR1174983](#)
- ROBINSON, P. M. (1995a). Gaussian semiparametric estimation of long range dependence. *Ann. Statist.* **23** 1630–1661. [MR1370301](#)
- ROBINSON, P. M. (1995b). Log-periodogram regression of time series with long range dependence. *Ann. Statist.* **23** 1048–1072. [MR1345214](#)
- ROUSSEAU, J., CHOPIN, N. and LISEO, B. (2012). Supplement to “Bayesian nonparametric estimation of the spectral density of a long or intermediate memory Gaussian process.” DOI:[10.1214/11-AOS955SUPP](#).
- WHITTLE, P. (1962). Gaussian estimation in stationary time series. *Bull. Inst. Internat. Statist.* **39** 105–129. [MR0162345](#)

J. ROUSSEAU
 N. CHOPIN
 CREST-ENSAE
 3, AVENUE PIERRE LAROUSSE
 92245 MALAKOFF
 FRANCE
 E-MAIL: rousseau@ceremade.dauphine.fr
nicolas.chopin@ensae.fr

B. LISEO
 SAPIENZA UNIVERSITÀ DI ROMA
 VIA DEL CASTRO LAURENZIANO
 9 00161 ROMA
 ITALY
 E-MAIL: brunero.liseo@uniroma1.it