The Annals of Statistics
1976, Vol. 4, No. 6, 1023–1037

THE 1975 WALD MEMORIAL LECTURES

ESTIMATION OF PARAMETERS IN A LINEAR MODEL[1]

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The first lecture in this series is devoted to a survey of contributions during the last five years to estimation of parameters by linear functions of observations in the Gauss–Markoff model. Some new results are also given.

The classes of BLE (Bayes linear estimators) and ALE (admissible linear estimators) are characterized when the loss function is quadratic. It is shown that ALE's are either BLE's or limits of BLE's. Biased estimators like ridge and shrunken estimators are shown to be special cases of BLE's.

Minimum variance unbiased estimation of parameters in a linear model is discussed with the help of a projection operator under very general conditions.

1. Introduction. Ever since Gauss introduced the theory of least squares there has been considerable interest in the estimation of parameters by linear functions of observations. Most of the contributions are devoted to unbiased estimators (Aitken, 1935; Zyskind and Martin, 1969; Rao, 1962, 1973a, to mention a few principal contributers). But with the advent of decision theory by Wald, attempts are being made to find estimators which may be biased but closer to the true values in some sense. Thus arose what are called ridge estimators (Hoel and Kennard, 1970a, b), shrunken estimators (Mayer and Wilke, 1973), a general class of homogeneous linear estimators which includes ridge and shrunken estimators (Rao, 1971), and other types of estimators (Marquardt, 1970). All these biased estimators are special cases of the class of Bayes linear estimators (BLE) which again are included in the class of admissible linear estimators (ALE) characterized by Cohen (1966) and Shinozaki (1975). Recently notable contributions have been made by Kuks and Olman (1972), Bunke (1975a, b, c) and Läuter (1975) on minimax linear estimation.

Some results in matrix algebra play an important role in the derivation of optimum linear estimators. These results are presented in a series of lemmas to

Received January 1976.

[1] First lecture.

AMS 1970 *subject classifications.* 62C10, 62C15, 62J05.

Key words and phrases. Gauss–Markoff model, ridge estimator, admissible estimator, minimax estimator, Bayes estimator, best homogeneous linear estimator.

make the paper self contained. The theorems of later sections on admissibility and minimax estimation may be deduced from certain general propositions in decision theory, but the object is to develop a separate theory for linear estimators which may be studied independently.

The literature on linear estimation is very vast and only some of the salient features in the development of the theory will be considered.

2. Some algebraic lemmas. The following notations and operations on a matrix $\mathbf{A}$ are used throughout.

$\mathscr{M}(\mathbf{A})$ = Linear manifold generated by the columns of $\mathbf{A}$.
$\mathbf{A}^-$ = Any g-inverse of $\mathbf{A}$ in the sense of Rao (1962), i.e., such that $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$.
$(\mathbf{A} : \mathbf{B})$ = Matrix obtained by adjoining the columns of $\mathbf{B}$ to those of $\mathbf{A}$.
$\mathbf{A}^\perp$ = Matrix of maximum rank such that $\mathbf{A}'\mathbf{A}^\perp = \mathbf{0}$.
$\mathbf{A} \geqq \mathbf{B}$ denotes $\mathbf{A} - \mathbf{B}$ is n.n.d. (nonnegative definite) matrix.

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be disjoint if $\mathscr{M}(\mathbf{A}) \cap \mathscr{M}(\mathbf{B})$ consists of the null vector only. The following lemma of Rao (1974) is useful in later applications.

LEMMA 2.1. *Let* $\mathbf{X}$ *be an* $n \times m$ *matrix,* $\mathbf{V}$ *be an* n.n.d. *matrix of order* n *and* $\mathbf{Z} = \mathbf{X}^\perp$. *Then*:

(i) $\mathbf{X}$ *and* $\mathbf{VZ}$ *are disjoint matrices.*
(ii) $\mathscr{M}(\mathbf{V} : \mathbf{X}) = \mathscr{M}(\mathbf{VZ} : \mathbf{X})$.

Projection operator. Let $\mathbf{A}$ and $\mathbf{B}$ be disjoint matrices each with the same number n of rows. Any vector $\mathbf{Y} \in \mathscr{M}(\mathbf{A} : \mathbf{B})$ has the unique decomposition.

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2\,, \qquad \mathbf{Y}_1 \in \mathscr{M}(\mathbf{A})\,, \quad \mathbf{Y}_2 \in \mathscr{M}(\mathbf{B})\,. \tag{2.1}$$

Then $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ is said to be projection operator onto $\mathscr{M}(\mathbf{A})$ parallel to $\mathscr{M}(\mathbf{B})$ or along $\mathscr{M}(\mathbf{B})$ iff

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}}\mathbf{Y} = \mathbf{Y}_1 \quad \text{for every} \quad \mathbf{Y} \in \mathscr{M}(\mathbf{A} : \mathbf{B}) \quad \text{and the corresponding} \quad \mathbf{Y}_1\,. \tag{2.2}$$

LEMMA 2.2. *A necessary and sufficient condition that* $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ *is a projection operator in the sense of* (2.2) *is*

$$\mathbf{P}_{\mathbf{A}|\mathbf{B}}\mathbf{A} = \mathbf{A}\,, \qquad \mathbf{P}_{\mathbf{A}|\mathbf{B}}\mathbf{B} = \mathbf{0}\,. \tag{2.3}$$

For a proof see Rao (1974). Note that $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ may not be unique or idempotent as in the case when $\mathscr{M}(\mathbf{A} : \mathbf{B})$ coincides with the entire space.

Constrained g-inverse. Let $\mathbf{A}$ and $\mathbf{B}$ be disjoint matrices. $\mathbf{G}$ is said to be a g-inverse of $\mathbf{A}$ constrained by $\mathbf{B}$ iff (see Rao, 1974 and Rao and Mitra, 1973):

$$\mathbf{AGA} = \mathbf{A} \quad \text{and} \quad \mathbf{AGB} = \mathbf{0} \tag{2.4}$$

which can also be written as

$$\mathbf{AGA} = \mathbf{A} \quad \text{and} \quad \mathscr{M}(\mathbf{G}'\mathbf{A}') \subset \mathscr{M}(\mathbf{B}^\perp)\,. \tag{2.5}$$

Such an inverse may be explicitly denoted by $\mathbf{G}_{\mathbf{A}|\mathbf{B}}$ and the class of all such inverses by $\{\mathbf{G}_{\mathbf{A}|\mathbf{B}}\}$. One choice of $\mathbf{G}_{\mathbf{A}|\mathbf{B}}$ is

$$\mathbf{G}_{\mathbf{A}|\mathbf{B}} = (\mathbf{C}'\mathbf{A})^{-}\mathbf{C}', \qquad \mathbf{C} = \mathbf{B}^{\perp}, \tag{2.6}$$

and a general solution is obtained by adding to (2.6) a general solution of the equations $\mathbf{AXA} = \mathbf{0}$, $\mathbf{AXB} = \mathbf{0}$. The following lemma shows the relationship between $\mathbf{P}_{\mathbf{A}|\mathbf{B}}$ and $\mathbf{G}_{\mathbf{A}|\mathbf{B}}$, which follows from the definitions.

LEMMA 2.3.

$$\{\mathbf{P}_{\mathbf{A}|\mathbf{B}}\} = \{\mathbf{A}\mathbf{G}_{\mathbf{A}|\mathbf{B}}\}. \tag{2.7}$$

Using (2.7), we may write the projection of $\mathbf{Y}$ on $\mathbf{A}$ as $\mathbf{A}\mathbf{G}_{\mathbf{A}|\mathbf{B}}\mathbf{Y}$ which has a unique value whatever may be the choice of the g-inverse $\mathbf{G}_{\mathbf{A}|\mathbf{B}}$.

LEMMA 2.4. *Let* $\mathbf{D}$ *be a* $p \times p$ *diagonal matrix such that both* $\mathbf{D}$ *and* $\mathbf{I} - \mathbf{D}$ *are* n.n.d. *and* $\boldsymbol{\theta} \in R^p$. *Further let* $\mathbf{G}$ *be a diagonal and* $\mathbf{Q}$ *an orthogonal matrix of the same order. Then*

$$\operatorname{Tr} \mathbf{G}^2 + \boldsymbol{\theta}'\mathbf{Q}(\mathbf{I} - \mathbf{G})^2\mathbf{Q}'\boldsymbol{\theta} \leqq \operatorname{Tr} \mathbf{D}^2 + \boldsymbol{\theta}'(\mathbf{I} - \mathbf{D})^2\boldsymbol{\theta} \tag{2.8}$$

for all $\boldsymbol{\theta} \in R^p$ *and strict inequality for at least one value of* $\boldsymbol{\theta}$ *cannot hold.*

For (2.8) to be true

$$\operatorname{Tr} \mathbf{G}^2 \leqq \operatorname{Tr} \mathbf{D}^2, \tag{2.9}$$

and

$$(\mathbf{I} - \mathbf{D})^2 - \mathbf{Q}(\mathbf{I} - \mathbf{G})^2\mathbf{Q}' \quad \text{is n.n.d.} \tag{2.10}$$

Now let $e_i^2 = \mathbf{Q}_i(\mathbf{I} - \mathbf{G})^2\mathbf{Q}_i'$ where $\mathbf{Q}_i$ is the ith row of $\mathbf{Q}$, and d_i, g_i be the ith diagonal elements of $\mathbf{D}$ and $\mathbf{G}$ respectively. Then (2.10) implies

$$1 - d_i \geqq e_i \Rightarrow \sum d_i^2 \leqq \sum (1 - e_i)^2 .$$

$$\begin{aligned} \sum d_i^2 \leqq \sum (1 - e_i)^2 &= p + \sum e_i^2 - 2 \sum e_i \\ &= p + \sum (1 - g_i)^2 - 2 \sum e_i \\ &= \sum g_i^2 + 2 \sum (1 - g_i) - 2 \sum e_i \leqq \sum g_i^2 . \end{aligned} \tag{2.11}$$

Since

$$e_i^2 = \mathbf{Q}_i(\mathbf{I} - \mathbf{G})^2\mathbf{Q}_i' \geqq (\mathbf{Q}_i(\mathbf{I} - \mathbf{G})\mathbf{Q}'_i)^2$$

then

$$e_i \geqq \mathbf{Q}_i(\mathbf{I} - \mathbf{G})\mathbf{Q}_i' \qquad \text{or} \qquad \sum e_i \geqq \sum (1 - g_i) .$$

The relationship (2.11) contradicts (2.9) and the lemma is established.

LEMMA 2.5. *Let* $\mathbf{A}$ *be a fixed* $p \times q$ *and* $\mathbf{B}$ *be a* $r \times q$ *matrix such that* $\mathbf{B}'\mathbf{B} = \mathbf{C}$ *is fixed. Then*

$$\max_{\mathbf{B}'\mathbf{B}=\mathbf{C}} \operatorname{Tr} \mathbf{AB}' = \operatorname{Tr} (\mathbf{ACA}')^{\frac{1}{2}} \tag{2.12}$$

and the maximum is attained at $\mathbf{B}_*$ *satisfying the equation*

$$\mathbf{AB}_*' = (\mathbf{ACA}')^{\frac{1}{2}} . \tag{2.13}$$

We use the inequality $\operatorname{Tr} \mathbf{G} \leqq \operatorname{Tr} (\mathbf{GG}')^{\frac{1}{2}}$ with equality iff $\mathbf{G} = (\mathbf{GG}')^{\frac{1}{2}}$. Choosing $\mathbf{G} = \mathbf{AB}'$, the result (2.13) is established if we can show that there exists a $\mathbf{B}_*$ such that $\mathbf{AB}_*' = (\mathbf{ACA}')^{\frac{1}{2}}$. We need only choose $\mathbf{B}_*' = \mathbf{EDP}$ where $\mathbf{C} = \mathbf{ED}^2\mathbf{E}'$ is the spectral decomposition of $\mathbf{C}$ and $\mathbf{P}$ is an orthogonal matrix such that $\mathbf{KP} = (\mathbf{KK}')^{\frac{1}{2}}$ with $\mathbf{K} = \mathbf{AED}$. $\mathbf{P}$ exists for instance choosing $\mathbf{P} = \mathbf{HF}'$ where $\mathbf{K} = \mathbf{FMH}'$ is the singular value decomposition of $\mathbf{K}$.

3. Admissible linear estimator. In our study, we will be concerned with two kinds of loss functions associated with a vector estimator $\mathbf{t}$ of a vector parameter $\boldsymbol{\theta}$. One is the quadratic loss function

$$\text{(3.1)} \qquad \mathbf{L}(\mathbf{t}, \boldsymbol{\theta}) = \mathbf{E}(\mathbf{t} - \boldsymbol{\theta})'\mathbf{B}(\mathbf{t} - \boldsymbol{\theta})$$

where $\mathbf{B}$ is an n.n.d. matrix. Another is the matrix loss function

$$\text{(3.2)} \qquad \mathbf{M}(\mathbf{t}, \boldsymbol{\theta}) = \mathbf{E}(\mathbf{t} - \boldsymbol{\theta})(\mathbf{t} - \boldsymbol{\theta})' .$$

DEFINITION 3.1. An estimator $\mathbf{t}$ is said to be admissible for $\boldsymbol{\theta}$ under the loss function (3.1), if there exists no other estimator $\mathbf{t}_*$ such that

$$\text{(3.3)} \qquad \mathbf{L}(\mathbf{t}_*, \boldsymbol{\theta}) \leqq \mathbf{L}(\mathbf{t}, \boldsymbol{\theta})$$

with strict inequality for at least one value of $\boldsymbol{\theta}$.

DEFINITION 3.2. An estimator $\mathbf{t}$ is said to be admissible under (3.2), if there exists no other estimator $\mathbf{t}_*$ such that

$$\text{(3.4)} \qquad \mathbf{M}(\mathbf{t}, \boldsymbol{\theta}) - \mathbf{M}(\mathbf{t}_*, \boldsymbol{\theta})$$

is n.n.d. and is not zero for at least one value of $\boldsymbol{\theta}$.

If $\mathbf{t}$ is admissible for $\boldsymbol{\theta}$ in the sense of (3.4) then $\mathbf{p}'\mathbf{t}$ may not be admissible for $\mathbf{p}'\boldsymbol{\theta}$ under the loss function

$$\text{(3.5)} \qquad \mathbf{E}(\mathbf{p}'\mathbf{t} - \mathbf{p}'\boldsymbol{\theta})^2$$

for any given vector $\mathbf{p}$. Of course, if $\mathbf{p}'\mathbf{t}$ is admissible for $\mathbf{p}'\boldsymbol{\theta}$ in the sense of (3.5) for every $\mathbf{p}$, then $\mathbf{t}$ is admissible for $\boldsymbol{\theta}$ in the sense of (3.4).

We shall write $\mathbf{t} \sim \boldsymbol{\theta}$ to denote that $\mathbf{t}$ is admissible for $\boldsymbol{\theta}$. The following lemma due to Shinozaki (1975) plays an important role in discussing admissibility of estimators.

LEMMA 3.1. *Let* $\mathbf{t} \sim \boldsymbol{\theta}$ *under* (3.1) *for some* p.d. *matrix* $\mathbf{B} = \mathbf{B}_*$, *then* $\mathbf{t} \sim \boldsymbol{\theta}$ *under* (3.1) *for any* n.n.d. *matrix* $\mathbf{B}$.

If $\mathbf{t} \sim \boldsymbol{\theta}$ for $\mathbf{B} = \mathbf{B}_*$, then $\mathbf{B}_*^{\frac{1}{2}}\mathbf{t} \sim \mathbf{B}_*^{\frac{1}{2}}\boldsymbol{\theta}$ for $\mathbf{B} = \mathbf{I}$, so that we may take $\mathbf{B}_* = \mathbf{I}$ without loss of generality.

Now let $\mathbf{t} \sim \boldsymbol{\theta}$ for $\mathbf{B} = \mathbf{I}$ and not for $\mathbf{B} = \mathbf{C}$ where $\mathbf{C}$ is an n.n.d. matrix. Further let $\mathbf{t}_0 \sim \boldsymbol{\theta}$ for $\mathbf{B} = \mathbf{C}$. Choose

$$\text{(3.6)} \qquad \mathbf{t}_* = \mathbf{t} + \mathbf{F}(\mathbf{t}_0 - \mathbf{t})$$

where $\mathbf{F} = d^{-1}\mathbf{C}$ (d is the largest eigenvalue of $\mathbf{C}$) so that

$$\text{(3.7)} \qquad \mathbf{I} \geqq \mathbf{F} \quad \text{and} \quad E(\mathbf{t}_0 - \boldsymbol{\theta})'\mathbf{F}(\mathbf{t}_0 - \boldsymbol{\theta}) \leqq E(\mathbf{t} - \boldsymbol{\theta})'\mathbf{F}(\mathbf{t} - \boldsymbol{\theta}) .$$

Consider

$$(3.8)\qquad E(\mathbf{t}_* - \boldsymbol{\theta})'(\mathbf{t}_* - \boldsymbol{\theta}) = E[(\mathbf{t} - \boldsymbol{\theta})'(\mathbf{t} - \boldsymbol{\theta}) + (\mathbf{t}_0 - \mathbf{t})'F^2(\mathbf{t}_0 - \mathbf{t}) + (\mathbf{t} - \boldsymbol{\theta})'\mathbf{F}(\mathbf{t}_0 - \mathbf{t}) + (\mathbf{t}_0 - \mathbf{t})'\mathbf{F}(\mathbf{t} - \boldsymbol{\theta})]\,.$$

Replacing $\mathbf{F}^2$ by $\mathbf{F}$

$$(3.8) \leqq E(\mathbf{t} - \boldsymbol{\theta})'(\mathbf{t} - \boldsymbol{\theta}) + (\mathbf{t}_0 - \boldsymbol{\theta})'\mathbf{F}(\mathbf{t}_0 - \boldsymbol{\theta}) - (\mathbf{t} - \boldsymbol{\theta})'\mathbf{F}(\mathbf{t} - \boldsymbol{\theta}) \leqq E(\mathbf{t} - \boldsymbol{\theta})'(\mathbf{t} - \boldsymbol{\theta})\,,$$

which is a contradiction, unless equality holds. The lemma is proved.

We shall prove some general theorems on ALE's (admissible linear estimators). Theorem 3.1 is essentially due to Cohen (1966) as extended by Shinozaki (1975). The statements of the theorem are, however, more general and the proof is mainly algebraic in character.

THEOREM 3.1. *Let* $\mathbf{Y}$ *be a k-vector random variable such that* $E(\mathbf{Y}) = \boldsymbol{\theta}$ *and* $D(\mathbf{Y}) = \sigma^2\mathbf{I}$. *Then under the quadratic loss function* (3.1), *with* $\mathbf{B} = \mathbf{I}$, *the following hold*:

(a) $\mathbf{LY} \sim \boldsymbol{\theta}$ iff $\mathbf{L}$ *is a symmetric matrix and has all its eigenvalues in the closed interval* [0, 1].

(b) $\mathbf{SLY} \sim \mathbf{S}\boldsymbol{\theta}$ *for any* $\mathbf{S}$ *if* $\mathbf{LY} \sim \boldsymbol{\theta}$.

(c) *If* $\mathbf{SLY} \sim \mathbf{S}\boldsymbol{\theta}$ *and* $\mathbf{S}$ *is nonsingular then* $\mathbf{LY} \sim \boldsymbol{\theta}$.

To prove (a) we proceed as follows. Consider the singular value decomposition $\mathbf{I} - \mathbf{L} = \mathbf{PGQ}'$ where $\mathbf{G}$ is diagonal. Then

$$(3.9)\qquad E(\mathbf{LY} - \boldsymbol{\theta})'(\mathbf{LY} - \boldsymbol{\theta}) = \sigma^2 \operatorname{Tr}(\mathbf{I} - \mathbf{QGP}')(\mathbf{I} - \mathbf{PGQ}') + \boldsymbol{\theta}'\mathbf{QG}^2\mathbf{Q}'\boldsymbol{\theta} = \sigma^2 \sum (g_i^2 - 2r_i g_i + 1) + \sum g_i^2\phi_i^2$$

where g_i and r_i are the ith diagonal elements of $\mathbf{G}$ and $\mathbf{P}'\mathbf{Q}$, and ϕ_i is the ith component of $\mathbf{Q}'\boldsymbol{\theta}$. Since $g_i \geqq 0$, the expression (3.9) can be made smaller in value by choosing $r_i = 1$ or $\mathbf{P} = \mathbf{Q}$, and also choosing $g_i = 1$ if it exceed 1. Thus the necessity of (a) is proved.

With $\mathbf{P} = \mathbf{Q}$ and $0 \leqq g_i \leqq 1$, the loss (3.9) can be written as

$$(3.10)\qquad \sigma^2 \operatorname{Tr}(\mathbf{1} - \mathbf{G})^2 + \boldsymbol{\phi}'\mathbf{G}^2\boldsymbol{\phi}\,.$$

If $\mathbf{MY}$ is an alternative estimator with $\mathbf{M} = \mathbf{S}(\mathbf{I} - \mathbf{D})\mathbf{S}'$ where $\mathbf{D}$ is diagonal, then the expected loss is

$$(3.11)\qquad \sigma^2 \operatorname{Tr}(\mathbf{I} - \mathbf{D})^2 + \boldsymbol{\phi}'\mathbf{RDR}'\boldsymbol{\phi}$$

where $\mathbf{R}$ is an orthogonal matrix. By Lemma 2.4, (3.11) cannot be uniformly (i.e., for all $\boldsymbol{\phi}$) less than (3.10). Thus sufficiency is established.

The results (b) and (c) are direct consequences of Lemma 3.1.

COROLLARY 3.1. *Let* $E(\mathbf{Y}) = \boldsymbol{\theta}$ *and* $D(\mathbf{Y}) = \sigma^2\mathbf{V}$ *where* $\mathbf{V}$ *is nonsingular, and* $\mathbf{LY} \sim \boldsymbol{\theta}$ *under* (3.1) *for any* p.d. *matrix* $\mathbf{B}$. *Then it is necessary and sufficient that*

LV *or* $\mathbf{V}^{-1}\mathbf{L}$ *is symmetric and the eigenvalues of* **L** *are in the closed interval* [0, 1]. *Further* $\mathbf{LY} \sim \boldsymbol{\theta}$ *under* (3.1) *for any* n.n.d. *matrix* **B**.

In view of Lemma 3.1, the choice of **B** is immaterial. Consider $\mathbf{Y} = \mathbf{V}^{\frac{1}{2}}\mathbf{Z}$, so that $D(\mathbf{Z}) = \sigma^2\mathbf{I}$, and let $E(\mathbf{Z}) = \boldsymbol{\phi}$

$$\mathbf{LY} \sim \boldsymbol{\theta} \Leftrightarrow \mathbf{LV}^{\frac{1}{2}}\mathbf{Z} \sim \mathbf{V}^{\frac{1}{2}}\boldsymbol{\phi} \Leftrightarrow \mathbf{V}^{-\frac{1}{2}}\mathbf{LV}^{\frac{1}{2}}\mathbf{Z} \sim \boldsymbol{\phi}\,. \tag{3.12}$$

Then from (a) of Theorem (3.1), $\mathbf{V}^{-\frac{1}{2}}\mathbf{LV}^{\frac{1}{2}}$ is symmetric with eigenvalues in [0, 1] and hence the result of the corollary.

COROLLARY 3.2. *If under conditions of Corollary* 3.1, $\mathbf{a} + \mathbf{LY} \sim \boldsymbol{\theta}$, *then the additional* n.s. *condition is* $\mathbf{a} \in \mathscr{M}(\mathbf{L} - \mathbf{I})$.

THEOREM 3.2. *Let* **Y** *be a k-vector such that* $E(\mathbf{Y}) = \boldsymbol{\theta}$ *and* $D(\mathbf{Y}) = \sigma^2\mathbf{V}$ *where* **V** *may be singular. Then* $\mathbf{q}'\mathbf{Y} \sim \mathbf{p}'\boldsymbol{\theta}$ *where* **q** *and* **p** *are k-vectors* iff $\mathbf{q}'\mathbf{Vq} \leqq \mathbf{p}'\mathbf{Vq}$.

$$E(\mathbf{q}'\mathbf{Y} - \mathbf{p}'\boldsymbol{\theta})^2 = \sigma^2\mathbf{q}'\mathbf{Vq} + [\boldsymbol{\theta}'(\mathbf{q} - \mathbf{p})]^2\,. \tag{3.13}$$

Let us compare $\mathbf{q}'\mathbf{Y}$ with $\mathbf{m}'\mathbf{Y}$ where $\mathbf{m}' = \mathbf{p}' + c(\mathbf{q}' - \mathbf{p}')$ with $0 < c < 1$.

$$E(\mathbf{m}'\mathbf{Y} - \mathbf{p}'\boldsymbol{\theta})^2 = \sigma^2[(1 - c^2)\mathbf{p}'\mathbf{Vp} + c^2\mathbf{q}'\mathbf{Vq} + 2c(1 - c)\mathbf{p}'\mathbf{Vq}] + c^2[\boldsymbol{\theta}'(\mathbf{q} - \mathbf{p})]^2\,. \tag{3.14}$$

If $\mathbf{q}'\mathbf{Y}$ is admissible, we must have

$$\mathbf{q}'\mathbf{Vq} \leqq (1 - c^2)\mathbf{p}'\mathbf{Vp} + c^2\mathbf{q}'\mathbf{Vq} + 2c(1 - c)\mathbf{p}'\mathbf{Vq} \tag{3.15}$$

or

$$(1 + c)\mathbf{q}'\mathbf{Vq} \leqq (1 - c)\mathbf{p}'\mathbf{Vp} + 2c\mathbf{p}'\mathbf{Vq}\,. \tag{3.16}$$

Taking limits as $c \to 1$, $\mathbf{q}'\mathbf{Vq} \leqq \mathbf{p}'\mathbf{Vq}$ which proves necessity.

To prove sufficiency, we have only to show that if $\mathbf{q}'\mathbf{Vq} \leqq \mathbf{p}'\mathbf{Vq}$, then (3.15) holds for all $0 \leqq c \leqq 1$. This can be easily verified.

We shall now prove a theorem which is more general than Theorems 3.1 and 3.2.

THEOREM 3.3. *Let* **Y** *be a k-vector random variable such that* $E(\mathbf{Y}) = \boldsymbol{\theta}$ *and* $D(\mathbf{Y}) = \sigma^2\mathbf{I}$, *and* **S** *be a* $r \times k$ *matrix. Then for* $\mathbf{LY} \sim \mathbf{S}\boldsymbol{\theta}$ *it is* n.s. *that*

(i) $\mathbf{LS}'$ *is symmetric, and*
(ii) $\mathbf{LL}' \leqq \mathbf{LS}'$.

Let $\mathbf{LY} \sim \mathbf{S}\boldsymbol{\theta}$. Consider

$$E(\mathbf{LY} - \mathbf{S}\boldsymbol{\theta})'(\mathbf{LY} - \mathbf{S}\boldsymbol{\theta}) = \sigma^2\mathbf{LL}' + \boldsymbol{\theta}'(\mathbf{S} - \mathbf{L})\boldsymbol{\theta}\,. \tag{3.17}$$

By Lemma 2.5 there exists an $\mathbf{L}_*$ such that $(\mathbf{S} - \mathbf{L}_*)'(\mathbf{S} - \mathbf{L}_*) = (\mathbf{S} - \mathbf{L})'(\mathbf{S} - \mathbf{L})$ and $\mathrm{Tr}\,\mathbf{L}_*'\mathbf{L}_* \leqq \mathrm{Tr}\,\mathbf{L}'\mathbf{L}$ with equality iff $\mathbf{LS}'$ is symmetrical. Thus the necessity of (i) is established.

By the corollary to Lemma 3.1, $\mathbf{p}'\mathbf{LY} \sim \mathbf{p}'\mathbf{S}\boldsymbol{\theta}$ for any vector $\mathbf{p}$ and therefore using the result of Theorem 3.2.

$$\mathbf{p}'\mathbf{LS}'\mathbf{p} \geqq \mathbf{p}'\mathbf{LL}'\mathbf{p}\,, \quad \forall\, \mathbf{p} \Leftrightarrow \mathbf{LS}' \geqq \mathbf{LL}' \tag{3.18}$$

which establishes the necessity of (ii).

To prove sufficiency we show that under (i) and (ii), $\mathbf{L}$ can be written as $\mathbf{SM}$ where $\mathbf{M}$ is symmetrical and has eigenvalues in the range [0, 1]. This would prove that $\mathbf{MY} \sim \boldsymbol{\theta}$ and hence by Theorem 3.1, $\mathbf{SMY} = \mathbf{LY} \sim \mathbf{S}\boldsymbol{\theta}$.

Let $r \leqq k$ and $\mathbf{L} = \mathbf{PGQ}'$ be the singular value decomposition of $\mathbf{L}$ where $\mathbf{P}$ is $r \times r$ orthogonal, $\mathbf{G}$ is $r \times r$ diagonal and $\mathbf{Q}'$ is $r \times k$ semiorthogonal matrices. Then

$$\mathbf{LL}' = \mathbf{PG}^2\mathbf{P}' \leqq \mathbf{PGQ}'\mathbf{S}' \Rightarrow \mathbf{G}^2 \leqq \mathbf{GQ}'\mathbf{S}'\mathbf{P} = \mathbf{GT}' \quad \text{(say)}\,. \tag{3.19}$$

Let

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \mathbf{T}' = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix}.$$

Then from (3.19) and the fact that $\mathbf{GT}'$ is symmetrical it follows that $\mathbf{T}_{11}$ is nonsingular and $\mathbf{T}_{12} = 0$. Now let $\mathbf{A}$ be such that $\mathbf{G}_1 = \mathbf{T}_{11}'\mathbf{A}$. Then $\mathbf{A}$ is symmetrical since $\mathbf{G}_1\mathbf{T}_{11}$ is symmetrical and $\mathbf{T}_{11}$ is nonsingular. Further

$$\mathbf{G} = \mathbf{T}\left(\begin{smallmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right) = \mathbf{P}'\mathbf{SQ}\left(\begin{smallmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right), \tag{3.20}$$

$$\mathbf{L} = \mathbf{PGQ}' = \mathbf{SQ}\left(\begin{smallmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right)\mathbf{Q}' = \mathbf{SM}\,, \tag{3.21}$$

where $\mathbf{M}$ is symmetrical as is required to be shown. Now

$$\mathbf{LL}' = \mathbf{SMM}'\mathbf{S}' \leqq \mathbf{SMS}'\,, \tag{3.22}$$

where $\mathbf{M}$ is as defined in (3.21). Then (3.22) implies that the eigenvalues of $\mathbf{M}$ chosen as in (3.21) are in the interval [0, 1].

If $r > k$, the argument is similar by choosing $\mathbf{P}$ to be semiorthogonal. The theorem is proved.

Note that no restriction has been placed on the order or rank of the matrix $\mathbf{S}$ in Theorem 3.3, and it is thus more general than Theorems 3.1 and 3.2.

COROLLARY 3.3. *If in Theorem* 3.3, $D(\mathbf{Y}) = \sigma^2\mathbf{V}$ *where* $\mathbf{V}$ *is nonsingular then* n.s. *conditions are*

(i) $\mathbf{LVS}'$ *is symmetrical, and*
(ii) $\mathbf{LVL}' \leqq \mathbf{LVS}'$.

THEOREM 3.4. *Let* $\mathbf{Y}$ *be a k-vector such that* $E(\mathbf{Y}) = \boldsymbol{\theta}$ *and* $D(\mathbf{Y}) = \sigma^2\mathbf{V}$ *where* $\mathbf{V}$ *is nonsingular. Then the two statements are equivalent*:

(i) $\mathbf{LV}$ *is symmetric and* $\mathbf{p}'\mathbf{LY} \sim \mathbf{p}'\boldsymbol{\theta}$ *for every* $\mathbf{p}$ *under a quadratic loss function.*
(ii) $\mathbf{LY}$ *is admissible for* $\boldsymbol{\theta}$ *under a quadratic loss function* (3.1) *with a* p.d. *matrix* $\mathbf{B}$.

That (ii) $\Rightarrow$ (i) follows from the corollary to Lemma 3.1. To prove that

(i) $\Rightarrow$ (ii) we observe that $\mathbf{p}'\mathbf{LY} \sim \mathbf{p}'\boldsymbol{\theta}$ iff

$$\mathbf{p}'\mathbf{LVL}'\mathbf{p} \leqq \mathbf{p}'\mathbf{LVp}\,, \tag{3.23}$$

using the result of Theorem 3.2. Since (3.23) is true for all $\mathbf{p}$ and $\mathbf{LV}$ is symmetrical

$$\mathbf{LVL}' \leqq \mathbf{LV} \Leftrightarrow \mathbf{S}'\mathbf{S} \leqq \mathbf{S} \tag{3.24}$$

where $\mathbf{S} = \mathbf{V}^{-\frac{1}{2}}\mathbf{LV}^{\frac{1}{2}}$ is symmetric by hypothesis. If $\mathbf{S} = \mathbf{PDP}'$ where $\mathbf{D}$ is diagonal and $\mathbf{P}$ is orthogonal then $\mathbf{S}'\mathbf{S} \leqq \mathbf{S} \Rightarrow (\mathbf{D}^2 - \mathbf{D}) \leqq \mathbf{0}$, i.e., each diagonal element of $\mathbf{D}$ is in the closed interval [0, 1], which is a n.s. condition for $\mathbf{LY} \sim \boldsymbol{\theta}$.

Note that (i) of Theorem 3.4 indicates the possibility of obtaining an estimator $\mathbf{LY}$ inadmissible for $\boldsymbol{\theta}$ under (3.1) and yet $\mathbf{p}'\mathbf{LY}$ is admissible for $\mathbf{p}'\boldsymbol{\theta}$ for every $\mathbf{p}$. For this we need only choose a matrix $\mathbf{L}$ such that

$$2\mathbf{LVL}' \leqq \mathbf{LV} + \mathbf{VL}' \tag{3.25}$$

and $\mathbf{LV}$ is not symmetrical.

4. Bayes linear estimator. Let $\mathbf{Y}$ be a k-vector such that $E(\mathbf{Y}) = \boldsymbol{\theta}$ and $D(\mathbf{Y} \mid \boldsymbol{\theta}) = \sigma^2\mathbf{V}$ where $\mathbf{V}$ is nonsingular. We shall say that $\mathbf{L}_*\mathbf{Y}$ is a Bayes homogeneous linear estimator (BHLE) if

$$E_{\boldsymbol{\theta}} E_{\mathbf{Y}}(\mathbf{L}_*\mathbf{Y} - \boldsymbol{\theta})'(\mathbf{L}_*\mathbf{Y} - \boldsymbol{\theta}) \leqq E_{\boldsymbol{\theta}} E_{\mathbf{Y}}(\mathbf{LY} - \boldsymbol{\theta})'(\mathbf{LY} - \boldsymbol{\theta}) \tag{4.1}$$

for any $\mathbf{L}$, where the first expectation is taken with respect to a prior distribution for $\boldsymbol{\theta}$. Now

$$E_{\boldsymbol{\theta}} E_{\mathbf{Y}}(\mathbf{LY} - \boldsymbol{\theta})'(\mathbf{LY} - \boldsymbol{\theta}) = \operatorname{Tr} \sigma^2\mathbf{LVL}' + \operatorname{Tr} \sigma^2(\mathbf{L} - \mathbf{I})\mathbf{W}(\mathbf{L} - \mathbf{I})' \tag{4.2}$$

where $\sigma^2\mathbf{W} = E(\boldsymbol{\theta\theta}')$. The expression (4.2) attains a minimum when $\mathbf{L}$ is

$$\mathbf{L}_* = \mathbf{W}(\mathbf{V} + \mathbf{W})^{-1} \tag{4.3}$$

which involves only the function $E(\boldsymbol{\theta\theta}') = \sigma^2\mathbf{W}$ of the distribution of $\boldsymbol{\theta}$. We shall call

$$\mathbf{W}(\mathbf{V} + \mathbf{W})^{-1}\mathbf{Y} \tag{4.4}$$

the BHLE of $\boldsymbol{\theta}$ with respect to $\mathbf{W}$ (see Rao, 1971).

Let $\mathbf{D}$ be a diagonal matrix with the roots of the equation $|\mathbf{W} - d\mathbf{V}| = 0$ as diagonal elements. Then (4.4) can be written in the form

$$\mathbf{QD}(\mathbf{I} + \mathbf{D})^{-1}\mathbf{Q}'\mathbf{Y} \tag{4.5}$$

where $\mathbf{Q}$ is such that $\mathbf{W} = \mathbf{QDQ}'$ and $\mathbf{V} = \mathbf{QQ}'$, or in the form

$$\mathbf{V}^{\frac{1}{2}}\mathbf{RGR}'\mathbf{V}^{-\frac{1}{2}}\mathbf{Y} \tag{4.6}$$

where $\mathbf{R}$ is an orthogonal matrix and $\mathbf{G} = \mathbf{D}(\mathbf{I} + \mathbf{D})^{-1}$, with diagonal elements in the semiopen interval [0, 1). The expression (4.6) can then be written as

$$\mathbf{V}^{\frac{1}{2}}\mathbf{MV}^{-\frac{1}{2}}\mathbf{Y}\,, \tag{4.7}$$

where $\mathbf{M}$ is a symmetric matrix with eigenvalues in the semiopen interval [0, 1).

Given such an $\mathbf{M}$ we can find $\mathbf{W}$ to satisfy the equation

$$\mathbf{W}(\mathbf{V} + \mathbf{W})^{-1} = \mathbf{V}^{\frac{1}{2}}\mathbf{M}\mathbf{V}^{-\frac{1}{2}} \tag{4.8}$$

so that the complete class of BHLE's is characterized by the expression (4.7). This is precisely the class obtained by the author in the paper (Rao, 1971).

It may be seen that an expression of the form $\mathbf{V}^{\frac{1}{2}}\mathbf{A}\mathbf{V}^{-\frac{1}{2}}\mathbf{Y}$ where $\mathbf{A}$ is a symmetric matrix with eigenvalues in the closed interval [0, 1] is a limit point of the set (4.7) for varying $\mathbf{M}$, but may not belong to the set (4.7), since if any of the eigenvalues of $\mathbf{A}$ is unity there does not exist any finite $\mathbf{W}$ satisfying the equation

$$\mathbf{W}(\mathbf{V} + \mathbf{W})^{-1} = \mathbf{V}^{\frac{1}{2}}\mathbf{A}\mathbf{V}^{-\frac{1}{2}} . \tag{4.9}$$

Thus the limit points correspond to the situation where some linear functions of the parameter $\boldsymbol{\theta}$ have infinite variance. We shall call the extended set

$$\{\mathbf{V}^{\frac{1}{2}}\mathbf{M}\mathbf{V}^{-\frac{1}{2}}\mathbf{Y}\} , \tag{4.10}$$

where $\mathbf{M}$ is a symmetric matrix with eigenvalues in the closed interval [0, 1], the complete class of generalized BHLE's.

Similarly we can define Bayes linear estimators (BLE's) and generalized BLE's by considering the class of estimators of the form $\mathbf{A} = \mathbf{LY} + \mathbf{a}$ where $\mathbf{a}$ is a vector of constants. In such a case the BLE is of the form

$$\mathbf{V}(\mathbf{V} + \mathbf{W})^{-1}\boldsymbol{\delta} + \mathbf{W}(\mathbf{V} + \mathbf{W})^{-1}\mathbf{Y} = \boldsymbol{\delta} + \mathbf{W}(\mathbf{V} + \mathbf{W})^{-1}(\mathbf{Y} - \boldsymbol{\delta}) \tag{4.11}$$

where $\boldsymbol{\delta} = E(\boldsymbol{\theta})$ and $\sigma^2\mathbf{W} = D(\boldsymbol{\theta})$. As in the case of BHLE, the set of BLE's is defined by

$$\{\boldsymbol{\delta} + \mathbf{V}^{\frac{1}{2}}\mathbf{M}\mathbf{V}^{-\frac{1}{2}}(\mathbf{Y} - \boldsymbol{\delta})\} \tag{4.12}$$

for any vector $\boldsymbol{\delta}$ and a symmetric matrix $\mathbf{M}$ with eigenvalues in [0, 1). The generalized BLE's constitute the set

$$\{\boldsymbol{\delta} + \mathbf{V}^{\frac{1}{2}}\mathbf{M}\mathbf{V}^{-\frac{1}{2}}(\mathbf{Y} - \boldsymbol{\delta})\} \tag{4.13}$$

where $\boldsymbol{\delta}$ is as in (4.12) and $\mathbf{M}$ is a symmetric matrix with eigenvalues in the interval [0, 1].

But (4.10) and (4.13) are precisely the classes of admissible linear estimators considered in Section 3. Thus we have the following theorem:

THEOREM 4.1. *The class of* ALE'*s is precisely the class of generalized* BLE'*s.*

5. Minimax estimation. Let Y be a k-vector such that $E(\mathbf{Y}) = \boldsymbol{\theta}$ and $D(\mathbf{Y}) = \sigma^2\mathbf{V}$ with the parameter $\boldsymbol{\theta}$ subject to the restriction $\boldsymbol{\theta} \in \mathscr{U} = \{\boldsymbol{\theta} : \boldsymbol{\theta}'\mathbf{H}\boldsymbol{\theta} \leqq \sigma^2\}$ where $\mathbf{H}$ is a p.d. matrix.

DEFINITION 5.1. A linear function $\mathbf{q}_*'\mathbf{Y}$ is said to be a minimax linear estimator of the parametric function $\mathbf{p}'\boldsymbol{\theta}$ iff

$$E(\mathbf{q}_*'\mathbf{Y} - \mathbf{p}'\boldsymbol{\theta})^2 = \min_{\mathbf{q}} \max_{\boldsymbol{\theta} \in \mathscr{U}} E(\mathbf{q}'\mathbf{Y} - \mathbf{p}'\boldsymbol{\theta})^2 . \tag{5.1}$$

DEFINITION 5.2. A vector function **LY** is said to be a minimax estimator of $\boldsymbol{\theta}$ iff

$$E(\mathbf{LY}-\boldsymbol{\theta})'\mathbf{B}(\mathbf{LY}-\boldsymbol{\theta}) = \min_{\mathbf{L}} \max_{\boldsymbol{\theta}\in\mathscr{U}} E(\mathbf{LY}-\boldsymbol{\theta})'\mathbf{B}(\mathbf{LY}-\boldsymbol{\theta}), \tag{5.2}$$

where **B** is an n.n.d. matrix.

THEOREM 5.1. *The minimax estimator of* $\mathbf{p}'\boldsymbol{\theta}$ *is* $\mathbf{p}'\boldsymbol{\theta}^{(m)}$ *where*

$$\boldsymbol{\theta}^{(m)} = \mathbf{H}^{-1}(\mathbf{V}+\mathbf{H}^{-1})\mathbf{Y} = (\mathbf{V}^{-1}+\mathbf{H})^{-1}\mathbf{Y} \tag{5.3}$$

i.e., $\boldsymbol{\theta}^{(m)}$ *is the* BHLE *of* $\boldsymbol{\theta}$ *with* $\mathbf{W}=\mathbf{H}^{-1}$. *The minimax loss is*

$$\mathbf{p}'\mathbf{K}\mathbf{p}, \qquad \mathbf{K}=\sigma^2(\mathbf{V}^{-1}+\mathbf{H})^{-1}. \tag{5.4}$$

Consider

$$E(\mathbf{q}'\mathbf{Y}-\mathbf{p}'\boldsymbol{\theta})^2 = \sigma^2\mathbf{q}'\mathbf{V}\mathbf{q} + [(\mathbf{q}'-\mathbf{p}')\boldsymbol{\theta}]^2. \tag{5.5}$$

It is easy to see that

$$\max_{\boldsymbol{\theta}\in\mathscr{U}} [\sigma^2\mathbf{q}'\mathbf{V}\mathbf{q} + [(\mathbf{q}'-\mathbf{p}')\boldsymbol{\theta}]^2] = \sigma^2[\mathbf{q}'\mathbf{V}\mathbf{q} + (\mathbf{q}'-\mathbf{p}')\mathbf{H}^{-1}(\mathbf{q}-\mathbf{p})]. \tag{5.6}$$

The minimum of (5.6) is attained at

$$(\mathbf{V}+\mathbf{H}^{-1})\mathbf{q}_* = \mathbf{H}^{-1}\mathbf{p}, \quad \text{i.e.,} \quad \mathbf{q}_* = (\mathbf{V}+\mathbf{H}^{-1})^{-1}\mathbf{H}^{-1}\mathbf{p} \tag{5.7}$$

so that the minimax estimator of $\mathbf{p}'\boldsymbol{\theta}$ is $\mathbf{p}'\boldsymbol{\theta}^{(m)}$ where $\boldsymbol{\theta}^{(m)}$ is as defined in (5.3).

The result of Theorem 5.1 can also be expressed in the following form due to Kuks and Olman (1972): $\boldsymbol{\theta}^{(m)}$ *as defined in* (5.3) *is the minimax estimator of* $\boldsymbol{\theta}$ *in the sense of definition on* 5.2 *for the choice* $\mathbf{B}=\mathbf{pp}'$ *or any* n.n.d. *matrix* **B** *with rank unity.*

Suppose that **MY** is an estimator of $\boldsymbol{\theta}$. Then

$$E(\mathbf{p}'\mathbf{MY}-\mathbf{p}'\boldsymbol{\theta})^2 = \sigma^2\mathbf{p}'\mathbf{MVM}'\mathbf{p} + [\mathbf{p}'(\mathbf{M}-\mathbf{I})\boldsymbol{\theta}]^2 \tag{5.8}$$

$$\begin{aligned} \sup_{\boldsymbol{\theta}\in\mathscr{U}} (5.8) &= \sigma^2[\mathbf{p}'\mathbf{MVM}'\mathbf{p} + \mathbf{p}'(\mathbf{M}-\mathbf{I})\mathbf{H}^{-1}(\mathbf{M}-\mathbf{I})'\mathbf{p}] \\ &= \sigma^2\mathbf{p}'[\mathbf{MVM}+(\mathbf{M}-\mathbf{I})\mathbf{H}^{-1}(\mathbf{M}-\mathbf{I})']\mathbf{p} \\ &\geqq \sigma^2\mathbf{p}'\mathbf{K}(\mathbf{MY},\boldsymbol{\theta})\mathbf{p} \end{aligned} \tag{5.9}$$

where $\mathbf{K}(\mathbf{MY},\boldsymbol{\theta}) = E(\mathbf{MY}-\boldsymbol{\theta})(\mathbf{MY}-\boldsymbol{\theta})'$. Since (5.9) is true for all **p**,

$$\mathbf{MVM}+(\mathbf{M}-\mathbf{I})\mathbf{H}^{-1}(\mathbf{M}-\mathbf{I}) \geqq \mathbf{K}(\mathbf{MY},\boldsymbol{\theta}) \tag{5.10}$$

so that the left-hand side of (5.10) is the supremum of $\mathbf{K}(\mathbf{MY},\boldsymbol{\theta})$ over $\boldsymbol{\theta}\in\mathscr{U}$ and may be written as $\sup \mathbf{K}(\mathbf{MY},\boldsymbol{\theta})$. From the minimax property of $\boldsymbol{\theta}^{(m)}$, it is seen that

$$\sup(\boldsymbol{\theta}^{(m)},\boldsymbol{\theta}) \leqq \sup \mathbf{K}(\mathbf{MY},\boldsymbol{\theta}). \tag{5.11}$$

Thus the estimator $\boldsymbol{\theta}^{(m)}$ which is minimax under the loss function (5.2) for the special choice of **B** with rank unity has the optimum property (5.11), which is pointed out by Bunke (1975).

The problem of obtaining the minimax estimator in the sense of Definition 5.2

for any choice of $\mathbf{B}$ has been recently solved by Läuter (1975). The solution is somewhat complicated.

6. The Gauss–Markoff model. We shall now consider the estimation of parameters in the general Gauss–Markoff model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma})$, where $\mathbf{Y}$ is an n-vector of random variables such that $E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$, $D(\mathbf{Y}) = \sigma^2\boldsymbol{\Sigma}$, and $\boldsymbol{\beta}$, σ^2 are unknown parameters.

6.1. *Some comments on the model.* The Gauss–Markoff model is well known and has been studied extensively. However, some of the inherent restrictions in the model when $\mathbf{V}$ is singular are not so well known. We shall consider them before discussing the estimation of parameters.

(a) $\mathbf{Y} \in \mathscr{M}(\boldsymbol{\Sigma} : \mathbf{X}) = \mathscr{M}(\boldsymbol{\Sigma}\mathbf{Z} : \mathbf{X})$ with probability 1.

(b) Let $\mathbf{K} = \boldsymbol{\Sigma}^{\perp}$, i.e., a matrix of maximum rank such that $\mathbf{K}'\boldsymbol{\Sigma} = \mathbf{0}$. Then

(i) $\mathbf{K}'\mathbf{Y} = \mathbf{d}$ with probability 1

(ii) $\mathbf{K}'\mathbf{X}\boldsymbol{\beta} = \mathbf{d}$

where $\mathbf{d}$ is a constant vector.

(c) Let $\mathbf{N} = \mathbf{K}\mathbf{d}^{\perp}$ and $\mathbf{S} = (\mathbf{X}'\mathbf{N})^{\perp}$. Then

(i) $\mathbf{N}'\mathbf{Y} = \mathbf{0}$ with probability 1

(ii) $\mathbf{N}'\mathbf{X}\boldsymbol{\beta} = \mathbf{0}$

(iii) $\mathbf{Y} \in \mathscr{M}(\boldsymbol{\Sigma} : \mathbf{X}\mathbf{S})$ with probability 1.

Note that singularity of $\boldsymbol{\Sigma}$ imposes some restrictions on the random variable $\mathbf{Y}$ as well as on the unknown parameter $\boldsymbol{\beta}$.

(d) A necessary and sufficient condition for a given linear function $\mathbf{m}'\mathbf{Y}$ to be unbiased for a linear parametric function $\mathbf{p}'\boldsymbol{\beta}$ is

$$\mathbf{X}'\mathbf{m} - \mathbf{p} \in \mathscr{M}(\mathbf{X}'\mathbf{N}) \,. \tag{6.1}$$

Note that the condition quoted in the literature on linear estimation, $\mathbf{X}'\mathbf{m} = \mathbf{p}$, is only sufficient and not necessary.

(e) If $\mathbf{m}'\mathbf{Y}$ is unbiased for $\mathbf{p}'\boldsymbol{\beta}$ in the sense of (6.1) then there exists a vector $\mathbf{k}$ such that

$$\text{(6.2)} \qquad \begin{aligned} &\text{(i)} \quad \mathbf{m}'\mathbf{Y} = \mathbf{k}'\mathbf{Y} \quad \text{with probability 1} \\ &\text{(ii)} \quad \mathbf{k}'\mathbf{X} = \mathbf{p} \,. \end{aligned}$$

(f) We shall say that a parametric function $\mathbf{p}'\boldsymbol{\beta}$ is identifiable if it can be estimated unbiasedly by a linear function of $\mathbf{Y}$, or $\mathbf{p} \in \mathscr{M}(\mathbf{X}')$.

6.2. *Unbiased estimation.* In view of (6.2), to find the minimum variance unbiased linear estimator (MVULE) of $\mathbf{p}'\boldsymbol{\beta}$ it is only necessary to find an $\mathbf{m}_*$ such that

$$\mathbf{m}_*'\mathbf{V}\mathbf{m}_* \leqq \mathbf{m}'\mathbf{V}\mathbf{m} \,, \quad \forall\, \mathbf{m} \ni \mathbf{X}'\mathbf{m} = \mathbf{p} \,. \tag{6.3}$$

Two approaches to the problem of computing $\mathbf{m}_*$ in the general case have been developed by the author. One is called the IPM (Inverse Partitioned Matrix) method (Rao, 1971, 1972) and another, the unified theory of the least squares (Rao, 1971, 1973a). We shall exhibit these results in terms of the projection operator described in Section 1 of this paper. [MVULE's which satisfy (6.1) and not (6.2) have been obtained in Rao (1973b)].

THEOREM 6.1. *Let* $\mathbf{m}$ *be any given vector such that* $\mathbf{X}'\mathbf{m} = \mathbf{p}$, i.e., *let* $\mathbf{m}'\mathbf{Y}$ *be any unbiased estimator of* $\mathbf{p}'\boldsymbol{\beta}$. *Then* $\mathbf{m}_*'\mathbf{Y}$ *is the* MVULE *of* $\mathbf{p}'\boldsymbol{\beta}$, *where*

$$\mathbf{m}^* = \mathbf{P}_{\mathbf{X}|\mathbf{VZ}}\mathbf{m} = (\mathbf{I} - \mathbf{P}_{\mathbf{ZV}})\mathbf{m} = (\mathbf{I} - \mathbf{P}'_{\mathbf{VZ}|\mathbf{X}})\mathbf{m} \,. \tag{6.4}$$

In (6.4) $\mathbf{P}_{\mathbf{X}|\mathbf{VZ}}$ is the projection operator onto $\mathscr{M}(\mathbf{X})$ parallel to $\mathscr{M}(\mathbf{VZ})$ as in (2.2) and $\mathbf{P}_{\mathbf{ZV}}$ is the projection operator into $\mathscr{M}(\mathbf{Z})$ based on the seminorm $||\mathbf{x}|| = (\mathbf{x}'\mathbf{V}\mathbf{x})^{\frac{1}{2}}$, as defined in Rao and Mitra (1971) and further examined in Mitra and Rao (1973). The results in (6.4) follow from the definitions of the operators $\mathbf{P}_{\mathbf{X}|\mathbf{VZ}}$ and $\mathbf{P}_{\mathbf{ZV}}$. (See Kolmogorov, 1946, 1947 for the case $|\mathbf{V}| \neq 0$.)

THEOREM 6.2. *The* MVULE *of an identifiable parametric function* $\mathbf{p}'\boldsymbol{\beta}$ *is* $\mathbf{p}\boldsymbol{\beta}^{(l)}$ *where*

$$\boldsymbol{\beta}^{(l)} = \mathbf{G}_{\mathbf{X}|\mathbf{VZ}}\mathbf{Y} \tag{6.5}$$

where $\mathbf{G}_{\mathbf{X}|\mathbf{VZ}}$ *is a g-inverse of* $\mathbf{X}$ *constrained by* $\mathbf{VZ}$ *as defined in* (2.4).

Note that $\mathbf{G}_{\mathbf{X}|\mathbf{VZ}}$ is any matrix satisfying the conditions

$$\mathbf{X}\mathbf{G}_{\mathbf{X}|\mathbf{VZ}}\mathbf{X} = \mathbf{X} \,, \qquad \mathbf{X}\mathbf{G}_{\mathbf{X}|\mathbf{VZ}}\mathbf{VZ} = \mathbf{0} \tag{6.6}$$

as defined in (2.4).

THEOREM 6.3. *The* MVULE *of an identifiable parametric function* $\mathbf{p}'\boldsymbol{\beta}$ *is* $\mathbf{p}'\boldsymbol{\beta}^{(l)}$ *where* $\boldsymbol{\beta}^{(l)}$ *is any solution of the equation*

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{P}_{\mathbf{X}|\mathbf{VZ}}\mathbf{Y} \,. \tag{6.7}$$

The results (6.5) and (6.7) follow from the definitions of $\mathbf{G}_{\mathbf{X}|\mathbf{VZ}}$ and $\mathbf{P}_{\mathbf{X}|\mathbf{VZ}}$. The following theorem is easy to prove.

THEOREM 6.4. *Let* $\mathbf{K}\boldsymbol{\beta}$ (*where* $\mathbf{K}$ *is of order* $r \times m$) *be a set of* r *identifiable parametric functions and* $\mathbf{LY}$ *be any unbiased estimator of* $\mathbf{K}\boldsymbol{\beta}$. *Then*

$$\mathbf{M}(\mathbf{LY}, \mathbf{K}\boldsymbol{\beta}) \geqq \mathbf{M}(\mathbf{K}\boldsymbol{\beta}^{(l)}, \mathbf{K}\boldsymbol{\beta}) \tag{6.8}$$

where $\mathbf{M}$ *denotes the matrix loss in the estimation of* $\mathbf{K}\boldsymbol{\beta}$, *and* $\boldsymbol{\beta}^{(l)}$ *is as defined in* (6.5) *or* (6.7).

6.3. *Admissible estimation.* We shall first prove that $\boldsymbol{\beta}^{(l)}$ (the least squares estimator) as defined in (6.5) or (6.7) is sufficient for admissible estimation of linear functions of $\boldsymbol{\beta}$ under a quadratic loss function as observed by Shinozaki (1975). The result may be stated explicitly in a more general form as follows.

THEOREM 6.5. *Let* $\mathbf{LY}$ *be any estimator of* $\mathbf{K}\boldsymbol{\beta}$. *Then*

$$\mathbf{M}(\mathbf{LY}, \boldsymbol{\beta}) \geqq \mathbf{M}(\mathbf{LX}\boldsymbol{\beta}^{(l)}, \boldsymbol{\beta}) \tag{6.9}$$

where $\mathbf{M}$ *denotes the matrix loss function as defined in* (3.2).

The result (6.9) is established by observing that

$$E(\mathbf{LY}-\mathbf{K}\boldsymbol{\beta})(\mathbf{LY}-\mathbf{K}\boldsymbol{\beta})' = E(\mathbf{LY}-\mathbf{LX}\boldsymbol{\beta}^{(l)})(\mathbf{LY}-\mathbf{LX}\boldsymbol{\beta}^{(l)})' + E(\mathbf{LX}\boldsymbol{\beta}^{(l)}-\boldsymbol{\beta})(\mathbf{LX}\boldsymbol{\beta}^{(l)}-\boldsymbol{\beta})' . \tag{6.10}$$

Theorem 6.5 shows that in order to consider admissible estimators of $\mathbf{K}\boldsymbol{\beta}$ under a quadratic loss function of the type

$$E(\mathbf{LY}-\boldsymbol{\beta})'\mathbf{G}(\mathbf{LY}-\boldsymbol{\beta}) \tag{6.11}$$

where $\mathbf{G}$ is an n.n.d. matrix, we need consider only linear functions of $\mathbf{X}\boldsymbol{\beta}^{(l)}$.

We shall now discuss the problem of admissible estimation of s independent identifiable parametric functions $\mathbf{K}\boldsymbol{\beta}$. Without such a restriction of identifiability there is a possibility of any set of linear functions being admissible for $\mathbf{K}\boldsymbol{\beta}$.

THEOREM 6.6. *Let* $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$ *be a Gauss–Markoff model with* $\mathbf{V}$ *nonsingular. Then* $\mathbf{LY} \sim \mathbf{K}\boldsymbol{\beta}$ *(identifiable)* iff

(i) $\mathbf{VL}' \subset \mathscr{M}(\mathbf{X})$,
(ii) $\mathbf{LXT}^{-}\mathbf{K}'$ *is symmetric, and*
(iii) $\mathbf{LXT}^{-}\mathbf{X}'\mathbf{L}' \leqq \mathbf{LXT}^{-}\mathbf{K}'$,

where $\mathbf{T} = \mathbf{X}'\mathbf{V}^{-1}\mathbf{X}$ *and* $\mathbf{T}^{-}$ *is any g-inverse of* $\mathbf{T}$.

Consider $\boldsymbol{\beta}^{(l)} = \mathbf{T}^{-}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}$ which is a solution of (6.7) with

$$E(\boldsymbol{\beta}^{(l)}) = \mathbf{T}^{-}\mathbf{T}\boldsymbol{\beta} , \qquad D(\boldsymbol{\beta}^{(l)}) = \sigma^2\mathbf{T}^{-}\mathbf{T}\mathbf{T}^{-1} . \tag{6.12}$$

If $\mathbf{LY} \sim \mathbf{K}\boldsymbol{\beta}$, then using (6.9) it is easily shown that $\mathbf{VL}' \subset \mathscr{M}(\mathbf{X})$ for otherwise $\mathbf{LY}$ can be improved, which establishes the necessity of (i). Then $\mathbf{L} = \mathbf{BX}'\mathbf{V}^{-1}$ for some matrix $\mathbf{B}$, and $\mathbf{BX}'\mathbf{V}^{-1}\mathbf{Y} = \mathbf{BT}\boldsymbol{\beta}^{(l)}$, $\mathbf{K}\boldsymbol{\beta} = \mathbf{KT}^{-}\mathbf{T}\boldsymbol{\beta}$. Then applying Theorem 3.3 the necessity of (ii) and (iii) follows observing that $\mathbf{TT}^{-}\mathbf{T} = \mathbf{T}$ and $\mathbf{X}'(\mathbf{T}^{-})'\mathbf{K} = \mathbf{XT}^{-}\mathbf{K}'$.

To prove sufficiency, note that (i) implies $\mathbf{L} = \mathbf{BX}'\mathbf{V}^{-1}$ and then (ii) and (iii) imply by Theorem 3.3 that $\mathbf{LY} \sim \mathbf{K}\boldsymbol{\beta}$.

If we write $\mathbf{K} = \mathbf{C}'\mathbf{X}$ (which is the condition for identifiability of $\mathbf{K}\boldsymbol{\beta}$), then the n.s. conditions of Theorem 6.6 can be written as

(i) $\mathbf{VL}' \subset \mathscr{M}(\mathbf{X})$,
(ii) $\mathbf{LVC}$ is symmetric, and
(iii) $\mathbf{LVL}' \leqq \mathbf{LVC}$.

COROLLARY 6.1. *Let* $\mathbf{X}$ *in Theorem* 6.6 *be of full rank in which case* $\boldsymbol{\beta}^{(l)}$ *is uniquely defined with* $E(\boldsymbol{\beta}^{(l)}) = \boldsymbol{\beta}$ *and* $D(\boldsymbol{\beta}^{(l)}) = \sigma^2\mathbf{T}^{-1}$. *Then* $\mathbf{A}\boldsymbol{\beta}^{(l)} \sim \mathbf{K}\boldsymbol{\beta}$ iff

(i) $\mathbf{AT}^{-1}\mathbf{K}'$ *is symmetric and*
(ii) $\mathbf{AT}^{-1}\mathbf{A}' \leqq \mathbf{AT}^{-1}\mathbf{K}'$.

To sum up, the different classes of estimators considered with reference to the model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$ and their relationships are as given below when both $\mathbf{X}$ and $\mathbf{V}$ have full rank. Similar expressions can be written down when $\mathbf{V}$ and/or $\mathbf{X}$ are not of full rank. They are somewhat complicated.

(i) *least squares estimator*

$$\boldsymbol{\beta}^{(l)} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}$$

(ii) *Bayes homogeneous linear estimator* (**W** *is* n.n.d.)

$$\begin{aligned}\boldsymbol{\beta}_{\mathbf{W}}{}^{(b)} &= \mathbf{W}\mathbf{X}'(\mathbf{V} + \mathbf{X}\mathbf{W}\mathbf{X}')^{-1}\mathbf{Y} \\ &= \mathbf{W}[(\mathbf{X}'\mathbf{V}\mathbf{X})^{-1} + \mathbf{W}]^{-1}\boldsymbol{\beta}^{(l)} \\ &= (\mathbf{W}^{-1} + \mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}\,, \qquad \text{if} \quad |\mathbf{W}| \neq 0\,.\end{aligned}$$

(iii) *general ridge estimator* (**G** *is* n.n.d.)

$$\begin{aligned}\boldsymbol{\beta}_{\mathbf{G}}{}^{(r)} &= (\mathbf{G} + \mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y} = (\mathbf{G} + \mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\boldsymbol{\beta}^{(l)} \\ &= \mathbf{G}^{-1}[(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1} + \mathbf{G}^{-1}]^{-1}\boldsymbol{\beta}^{(l)} \qquad \text{if} \quad |\mathbf{G}| \neq 0\,, \\ &= \boldsymbol{\beta}_{\mathbf{G}^{-1}}^{(b)} \qquad \text{if} \quad |\mathbf{G}| \neq 0\,.\end{aligned}$$

Note that a general ridge estimator (see Rao, 1975c) is not necessarily a Bayes estimator, but can be exhibited as a limit of Bayes estimators.

(iv) *minimax estimator* (*with* $\boldsymbol{\beta}'\mathbf{H}\boldsymbol{\beta} \leqq \sigma^2$, **H** *is* p.d.)

$$\begin{aligned}\boldsymbol{\beta}^{(m)} &= \mathbf{H}^{-1}\mathbf{X}'(\mathbf{V} + \mathbf{X}\mathbf{H}^{-1}\mathbf{X}')^{-1}\mathbf{Y} = \boldsymbol{\beta}_{\mathbf{H}^{-1}}^{(b)} \\ &= (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X} + \mathbf{H})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y} = \boldsymbol{\beta}_{\mathbf{H}}{}^{(r)} \\ &= \mathbf{H}^{-1}[(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1} + \mathbf{H}^{-1}]^{-1}\beta^{(l)} \\ &\to \boldsymbol{\beta}^{(l)} \qquad \text{as} \quad \mathbf{H} \to \mathbf{0}\,.\end{aligned}$$

(v) *admissible estimator* ($\mathbf{T} = \mathbf{X}'\mathbf{V}^{-1}\mathbf{X}$)

$$\begin{aligned}\boldsymbol{\beta}^{(a)} &= \mathbf{B}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y} \quad (\mathbf{B} \text{ sym. and } \mathbf{B}\mathbf{T}\mathbf{B} \leqq \mathbf{B}) \\ &= \mathbf{C}\boldsymbol{\beta}^{(l)} \quad (\mathbf{C}\mathbf{T}^{-1} \text{ sym. and } \mathbf{C}\mathbf{T}^{-1}\mathbf{C}' \leqq \mathbf{C}\mathbf{T}^{-1})\,.\end{aligned}$$

It may be noted that the estimators (i)—(iv) are subclasses of (v). However, the Bayes class (ii) plays a special role since $\boldsymbol{\beta}^{(l)}$, $\boldsymbol{\beta}^{(r)}$ and $\boldsymbol{\beta}^{(m)}$ are either members of this class or its limit points, and the extended Bayes class including the limit points is precisely the class of admissible estimators. It appears that for a study of biased but admissible estimators one should start with the class $\mathbf{C}\boldsymbol{\beta}^{(l)}$ and examine the behaviour of subclasses defined by special choices of **C** such that $\mathbf{C}\mathbf{T}^{-1}$ is symmetric and $\mathbf{C}\mathbf{T}^{-1}\mathbf{C}' \leqq \mathbf{C}\mathbf{T}^{-1}$.

It may be noted that we have considered only linear estimators. If we admit nonlinear estimators and have some knowledge of the distribution of the observation vector **Y**, then the linear estimators may become inadmissible as shown by James and Stein (1961).

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