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INADMISSIBILITY RESULTS FOR GENERALIZED BAYES ESTIMATORS OF COORDINATES OF A LOCATION VECTOR[1]

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Let $X = (X_1, \cdots, X_n)$ be an n-dimensional random vector with density $f(x - \theta)$. Assume the loss incurred in estimating θ by d is $L(d - \theta)$ and is convex. Let F be a generalized prior density. The question of inadmissibility of generalized Bayes estimators of θ is considered.

As in Brown (1974c), the crucial role played by the "moment structure" in determining inadmissibility results is indicated (moments defined as the quantities $m_{i,j(1),\cdots,j(l)} = \int \prod_{k=1}^{l} x_{j(k)} \cdot [(\partial/\partial x_i)L(x)]f(x)\,dx$).

Detailed inadmissibility results are given for a particular moment structure, one which arises most naturally in trying to estimate a single coordinate or a linear combination of coordinates of θ. For example, suppose θ_1 is to be estimated by the generalized Bayes estimator δ_F. (Thus $\theta_2, \cdots, \theta_n$ are nuisance parameters.) Under certain conditions, the most important being that the moment structure be of a certain form, it is shown that if there exist constants $\lambda > 0$ and $T > 0$ such that

$$\delta_F(x) \geqq x_1 - (n - 3 - \lambda)\Big/\left[2x_1 \int \left(\frac{\partial^2}{\partial x_1^2} L(x)\right) f(x)\,dx\right] \quad \text{for} \quad x_1 > T,$$

then δ_F is inadmissible. Thus, for example, under certain quite general assumptions, the best invariant estimator, $\delta_0(x) = x_1$, of the first coordinate of a location vector is inadmissible if $n \geqq 4$.

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1. Introduction.

1.1. *Summary of results.* The question of admissibility of estimators of loca-

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tion vectors has been important for many years in mathematical statistics. This paper is written with two important objectives. The first is to partially answer a long outstanding question in location parameter theory—namely, when is an estimator of a single coordinate of a location vector inadmissible? (This question was posed by James and Stein (1960), whose conjectured answer for the best invariant estimator turns out to be essentially correct.)

The second objective of this paper has to do with methodology. Recently, Brown (1974c) has developed a powerful new method of solving inadmissibility problems. Its essential idea is to reduce the study of inadmissibility in a given situation to the analysis of an associated nonlinear partial differential inequality. This paper is based on an extensive application of Brown's method. As it is one of the first to use this method, an attempt has been made to clearly outline the needed steps, and to discuss the various techniques needed.

An exact statement of the problem discussed in this paper, along with its relationship to known admissibility results, is given in Section 1.2, after the necessary notation has been developed. Section 1.3 gives a heuristic development of the methods used in this paper. Chapters 2 and 3 present the formal results. Summaries of assumptions and results are given in Section 2.1 and the beginning of Chapter 3. The assumptions, though numerous, are usually trivial to verify, with the exception of the "moment structure" assumption. Section 2.4 gives relatively simple situations in which this assumption is satisfied, along with applications of the theory. It should be mentioned that the theory will not apply to estimating a normal mean. It can, however, be applied in certain situations of estimating normal variances, and to estimating location parameters of densities such as the multivariate lognormal. For details, see Section 2.4.

1.2. *Preliminaries.* Let $X = (X_1, X_2, \cdots, X_n)$ be an n dimensional random variable with density $f(x - \theta)$ with respect to Lebesgue measure ($\theta \in R^n$). Let F be a bounded generalized prior density w.r.t. Lebesgue measure (i.e. $0 \leqq F(\theta) \leqq B < \infty$, while possibly $\int F(\theta)\, d\theta = \infty$). Assume the loss incurred in estimating θ by d is of the form $L(d - \theta)$, where L is a nonnegative convex function and $L(0) = 0$.

In much of this paper (Chapter 2), the special situation of estimating θ_1 (with $\theta_2, \cdots, \theta_n$ being nuisance parameters) will be considered. The loss will then be assumed to be of the form $L(d_1 - \theta_1)$. (Note that here L is a function only of one coordinate, while above it is a function of n coordinates. Though a slight logical inconsistency is introduced, it seems worthwhile to keep the same symbol for the loss throughout the paper. It will be clear from context which version is being considered.)

For convenience, the notation $h^{(i)}(x) = (\partial/\partial x_i)h(x)$, $h^{(i,j)}(x) = (\partial^2/\partial x_i \partial x_j)h(x)$, etc., will be adopted for any function h with the appropriate number of derivatives. Let $|\xi|$ denote the usual Euclidean norm of the vector ξ. Denote the $r \times r$ identity matrix by I_r. Define for any vector y, the vector $y^* = (y_2, y_3, \cdots, y_n)$. (Thus $X = (X_1, X^*)$ and $\theta = (\theta_1, \theta^*)$.) For notational convenience let K be a generic constant. Finally, define a differential operator "$\mathscr{D}$" on twice differentiable functions $G: R^n \to R^1$, by

$$\mathscr{D}G(x) = -[G^{(1)}(x) - \tfrac{1}{2}\textstyle\sum_{i=2}^{n} G^{(i,i)}(x)]\,.$$

The following conditions on L, f, and F will be needed throughout the paper:

(i) All third order partial derivatives of L and F exist.

(ii) $E_0 L(X + c) < \infty$, $\forall c \in R^n$, and $E_0|L^{(i)}(X + c)| < \infty$, $\forall c \in R^n$ and $i = 1, \cdots, n$. (E_θ is, of course, the expectation under θ.)

(iii) $E_0 L^{(i)}(X) = 0$, $i = 1, \cdots, n$.

(iv) $\int f(x - \theta)F(\theta)\, d\theta > 0$, $\forall x \in R^n$.

Assumption (iii) can really be made without loss of generality since using (i), (ii), and the convexity of L, it is easy to see that there exists $c \in R^n$ for which $E_0 L^{(i)}(X + c) = 0$, $i = 1, \cdots, n$. A simple translation of the density f will now ensure that (iii) is satisfied, while leaving admissibility considerations unchanged.

Denote the generalized Bayes estimator for θ, w.r.t. the generalized prior density F, by $\delta_F = (\delta_{F,1}, \delta_{F,2}, \cdots, \delta_{F,n})$. Under the above assumptions, it can easily be checked that δ_F satisfies $\int_{R^n} L^{(i)}(\delta_F(x) - \theta)f(x - \theta)F(\theta)\, d\theta = 0$, for $i = 1, \cdots, n$.

For given F and any estimator $\delta = (\delta_1, \cdots, \delta_n)$, define

$$\gamma_F(x) = (\gamma_{F,1}(x), \cdots, \gamma_{F,n}(x)) = \delta_F(x) - x\,,$$
$$\gamma(x) = (\gamma_1(x), \cdots, \gamma_n(x)) = \delta(x) - \delta_F(x)\,.$$

As usual, define the risk of an estimator δ by $R(\delta, \theta) = \int_{R^n} L(\delta(x) - \theta)f(x - \theta)\, dx$. (Only estimators for which the risk is defined and finite for every θ will be considered.) Also, define $\Delta_\delta^F(\theta) = R(\delta_F, \theta) - R(\delta, \theta)$.

As usual, δ_F is said to be admissible if $\Delta_\delta^F(\theta) \geqq 0$ for all θ, implies that $\Delta_\delta^F(\theta) \equiv 0$. Thus δ_F is inadmissible if there exists an estimator δ such that $\Delta_\delta^F(\theta) \geqq 0$ for all θ, with strict inequality for some θ.

The following quantities play a crucial and relatively unheralded role in questions of admissibility. Define $b_{ij} = E_0 L^{(i,j)}(X)$, and $m_{i,j(1),j(2),\cdots,j(l)} = E_0[(\prod_{k=1}^{l} X_{j(k)})L^{(i)}(X)]$ for $l \geqq 1$. The above quantities will be called the "moment structure" of the problem. Of particular interest will be the $n \times n$ matrix M_1 with elements $m_{i,j}$. Denote the rank of M_1 by r_1.

The notation has now been developed far enough to take a brief digression into what is already known about admissibility in location parameter problems. Only the latest, most inclusive results will be mentioned. References to earlier works are contained in papers cited below and in the bibliography.

For $n = 1$, the major result known is from Brown (1966), that for general loss and general density f (subject mainly to a moment condition), the best invariant estimator is admissible. Farrell (1964) proved admissibility of generalized Bayes estimators satisfying certain conditions, but mainly for squared error loss. No general result has been obtained for generalized Bayes estimators, general density, and general loss.

For $n = 2$, the major result known is from Brown and Fox (1974a), which shows that for general loss and density, the best invariant estimator is admissible.

For $n \geqq 3$, the major result is by Brown (1966), showing that for general loss and density, the best invariant estimator is inadmissible if $r_1 = n$. A little more work would prove the result for $r_1 \geqq 3$.

There are two major gaps in the above theory. First, general results exist mainly for the best invariant estimator. It will become clear that similar general results can be obtained for generalized Bayes estimators using the methods of Brown (1974c) and of this paper.

The second and perhaps more important gap in the theory is that results are essentially known only for the case $n \leqq 2$ and the case $r_1 \geqq 3$. The two essentially unsolved situations are (i) $r_1 = 1$, $n \geqq 3$, and (ii) $r_1 = 2$, $n \geqq 3$. Inadmissibility results for Case (i) will be the subject of this paper. Subsequent papers will deal with admissibility in Case (i) and with Case (ii).

It should be noted that the above classification of problems by n and r_1 is only a first order classification in that the rest of the moment structure can play a crucial role.

Two further comments about "history" are needed before we proceed. First is a comment about Brown (1971). This paper considers the question of admissibility of generalized Bayes estimators for the mean of an n dimensional normal random variable, using quadratic loss. It contains a very complete analysis of the problem, with F, n, and the moment structure all having evident roles. (Though this was the first paper to make extensive use of reducing admissibility to the study of an associated partial differential inequality, the idea itself goes back in part to Stein (1965).)

Secondly, it should be noted that the problem $r_1 = 1$, $n \geqq 3$ has been partially solved by Portnoy (1975). He has obtained results for the best invariant estimator, using squared error loss, and for a class of distributions with mass on the $(n - 1)$ planes determined by $x_1 = \pm m$ (m an integer).

The remainder of this paper will be devoted to consideration of the problem defined by $r_1 = 1$. (The case $n \geqq 4$ will be of primary interest, but the assumption will not be explicitly made as new results will be obtained for smaller n also.) The most obvious situation in which $r_1 = 1$ is when we are interested in estimating only one coordinate of θ, or some linear combination $\sum c_i \theta_i$. It is straightforward to see that through linear transformations this could be transformed into the problem of estimating θ_1. This indeed will be the situation considered in Chapter 2.

Unfortunately, the assumption $r_1 = 1$ does not completely specify the problem. One needs to make additional assumptions on the moment structure to isolate a particular problem. The exact moment structure that will be considered in this paper is given in the following lines. Assume

$$m_{1,1} = 1\,, \qquad m_{i,j} = 0 \quad \text{otherwise}, \tag{1.2.1}$$

$$(1.2.2) \qquad m_{1,i,j} = -1 \quad \text{if} \quad 2 \leqq i = j \leqq n$$
$$= 0 \quad \text{if} \quad i \neq j\,, \quad i \geqq 2\,, \quad j \geqq 2\,.$$

The $m_{k,i,j}$, $k \geqq 2$, turn out to be unimportant for our purposes and hence need not be specified. Indeed, note that in the special situation of estimating only θ_1, where L is a function only of the first coordinate, $L^{(k)} = 0$ for $k \geqq 2$. By definition, it is then clear that $m_{k,i,j} = 0$ for $k \geqq 2$.

A large number of problems with $r_1 = 1$ can be reduced to a problem with the above "canonical form" of the moment structure. Indeed we will now prove the following two lemmas.

LEMMA 1.2.1. *If M_1 has rank* 1 *and a positive characteristic root, then the original problem can be transformed into an equivalent admissibility problem with a moment structure having the canonical form* (1.2.1).

LEMMA 1.2.2. *Assume M_1 is in the canonical form* (1.2.1). *Let M_2 be the $(n-1) \times (n-1)$ matrix with elements $m_{1,i,j}$, $2 \leqq i \leqq n$ and $2 \leqq j \leqq n$. If M_2 is positive or negative definite, then the original problem can be transformed into an equivalent admissibility problem with a moment structure having the form* (1.2.1) *and* (1.2.2).

The following lemma provides the key to the proofs of the above lemmas, namely that linear transformations of the problem preserve admissibility properties.

LEMMA 1.2.3. *If Q is a nonsingular $n \times n$ matrix, admissibility of $\delta_F(x)$ for estimating θ is equivalent to admissibility of $\delta_{F^*}(y)$ for estimating η in the transformed problem (Y, η, L^*, f^*, F^*), where $Y = XQ^{-1}$, $\eta = \theta Q^{-1}$, $L^*(y) = L(yQ)$, $f^*(y - \eta) = |Q| f(yQ - \eta Q)$, and $F^*(\eta) = |Q| F(\eta Q)$. (Here $|Q|$ denotes the determinant of Q.)*

PROOF. Straightforward. □

Fortunately, while linear transformations leave admissibility properties unchanged, they do change the moment structure. Thus we can now prove Lemmas 1.2.1 and 1.2.2.

PROOF OF LEMMA 1.2.1. The first step is to note that since M_1 has rank 1 and a positive characteristic root, the Jordan canonical form theorem states that for some nonsingular $n \times n$ matrix Q, the matrix QM_1Q^{-1} has all zero elements except for a constant $C > 0$ in the (1, 1) position. Consider the transformed * problem defined by $Y = XQ^{-1}$ (see Lemma 1.2.3). Noting that M_1 can be written as $M_1 = E_0[(\nabla L(X))^t X]$ (where ∇ denotes the gradient and the expectation is taken componentwise), calculation gives

$$M_1^* = E_0[(\nabla L^*(Y))^t Y] = E_0[(\nabla L(YQ))^t Y]$$
$$= E_0[(\nabla L(X)Q^t)^t XQ^{-1}] = QE_0[(\nabla L(X))^t X]Q^{-1} = QM_1Q^{-1}\,.$$

Finally, consider the above transformed problem with loss multiplied by C^{-1}. Admissibility considerations are clearly unchanged, but now $M_1^* = C^{-1}QM_1Q^{-1}$, which by construction is of the form (1.2.1). □

PROOF OF LEMMA 1.2.2. Recalling that $m_{1,i,j} = E_0[L^{(1)}(X)X_i X_j]$, it is clear that M_2 is symmetric. If M_2 is negative definite, it is well known that there exists a nonsingular matrix P for which $PM_2P^t = -I_{n-1}$. Let Q be the matrix with elements $q_{1,1} = 1$, $q_{i,1} = q_{1,i} = 0$ for $i = 2, \cdots, n$, and $q_{i,j} = p_{j-1,i-1}$ for $2 \leqq i \leqq n$, $2 \leqq j \leqq n$. In the transformed * problem defined by $Y = XQ$, it is easy to check that $M_1{}^* = M_1$ and $M_2{}^* = PM_2P^t = -I_{n-1}$. But $M_1{}^*$ and $M_2{}^*$ thus have the canonical form given by (1.2.1) and (1.2.2).

If the original M_2 is instead positive definite, first consider the transformed * problem determined by $Y = (-X_1, X_2, \cdots, X_n)$. Clearly $L^{*(1)}(Y) = -L^{(1)}(X)$. Calculation gives

$$m_{1,1}^* = E_0[L^{*(1)}(Y)Y_1] = E_0[L^{(1)}(X)X_1] = m_{1,1} = 1 ,$$
$$m_{1,i,j}^* = E_0[L^{*(1)}(Y)Y_i Y_j] = -m_{1,i,j} \qquad \text{for} \quad 2 \leqq i \leqq n , \quad 2 \leqq j \leqq n .$$

Thus $M_1{}^* = M_1$, and $M_2{}^* = -M_2$ which is negative definite. The analysis then continues as before. □

In determining whether or not a problem with $r_1 = 1$ has a moment structure which can be reduced to the proper canonical form, verification of the condition of Lemma 1.2.1 usually presents no difficulty. As mentioned earlier, the common situation in which $r_1 = 1$ is when only θ_1 is to be estimated. The loss incurred in estimating θ_1 by d_1 is then $L(d_1 - \theta_1)$, a function of only the first coordinate. It is thus clear that $m_{i,j} = 0$ for $i > 1$ (since $L^{(i)}(x_1) = 0$ for $i > 1$). If L is also strictly convex, then $m_{1,1} = E_0[L^{(1)}(X_1)X_1] > 0$. M_1 is thus upper diagonal of rank 1 with a positive characteristic root, so Lemma 1.2.1 can be applied.

The condition of Lemma 1.2.2 is, however, much more restrictive and must be checked with care. For example, an easy calculation shows that if f is the multivariate normal density, the reduction to canonical form results in a moment structure where M_2 is the $(n - 1) \times (n - 1)$ zero matrix. Hence (1.2.2) will be violated and the theory will not apply to estimating a normal mean. (This difficulty is due to the unusual property of the multivariate normal distribution that a linear transformation $Y = XQ$ can be made for which Y_1 is independent of $Y_2, \cdots, Y_n$.) For situations in which the conditions of Lemma 1.2.2 (or line (1.2.2)) can be verified, see Section 2.4.

It should be emphasized that the restriction on M_2, while unpleasant, is necessary to completely define the problem. If M_2 is assumed to be of a different form, the admissibility results will be different. A rough indication of this is given in the next section.

Finally, it is worthwhile noting that a slight weakening of the condition on M_2 is sometimes possible. For example, consider the special situation where only θ_1 is to be estimated and the loss is only a function of the first coordinate. It is often possible that an inadmissibility result can be proven for a subproblem (a problem where only some of the nuisance coordinates are considered). It is then clear that the inadmissibility conclusion follows for the original problem.

Thus, if for a subproblem the matrix M_2 is positive or negative definite, inadmissibility results can often be obtained.

1.3. *Heuristic approach to results.* As the technical aspects of the theoretical development are somewhat intimidating, it seems desirable to briefly sketch the important arguments. The main ideas are patterned after Brown (1974c), to which the reader is referred for further discussion.

It is desired to obtain an approximation for

$$\begin{aligned}\Delta_\delta^F(\theta) &= R(\delta_F, \theta) - R(\delta, \theta)\\ &= \int_{R^n} [L(\delta_F(x) - \theta) - L(\delta(x) - \theta)]f(x-\theta)\,dx \\ &= \int_{R^n} [L(\gamma_F(x) + x - \theta) - L(\gamma_F(x) + \gamma(x) + x - \theta)]f(x-\theta)\,dx\,.\end{aligned} \tag{1.3.1}$$

To do this, first expand $L(\gamma_F(x) + x - \theta)$ and $L(\gamma_F(x) + \gamma(x) + x - \theta)$ in Taylor expansions about $(x - \theta)$, and then expand the $\gamma_i(x)$ and $\gamma_{F,i}(x)$ in Taylor expansions about θ. Inserting these expansions into the last integral of (1.3.1) and wading through considerable calculation, gives

$$\begin{aligned}\Delta_\delta^F(\theta) = &-\textstyle\sum_{i=1}^n \sum_{j=1}^n m_{i,j}\gamma_i^{(j)}(\theta) - \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n m_{i,j,k}\gamma_i^{(j,k)}(\theta) \\ &- \textstyle\sum_{i=1}^n \sum_{j=1}^n b_{ij}\gamma_i(\theta)[\gamma_{F,j}(\theta) + \gamma_j(\theta)/2] + \mathscr{R}\,.\end{aligned} \tag{1.3.2}$$

Here $\mathscr{R}$ consists of all the higher order terms resulting from the Taylor expansions, while the $m_{i,j}$, $m_{i,j,k}$, and b_{ij} are elements of the moment structure. It is obviously desirable to make $\mathscr{R}$ small in some sense. $\mathscr{R}$ can be thought of as consisting of higher order derivatives and higher powers of the γ_i and $\gamma_{F,i}$. If the $\gamma_i(\theta)$ and $\gamma_{F,i}(\theta)$ are smooth enough and small for large θ, it can be shown that their higher order derivatives and powers go to zero faster than their lower order derivatives and powers. Thus for large enough θ, $\mathscr{R}$ can be ignored. (Attempting to justify such statements is often quite difficult as we shall see in Chapters 2 and 3.)

The general formula (1.3.2) encompasses a broad range of location vector problems. At this point the moment structure assumptions (1.2.1) and (1.2.2) will be employed to specify the problem. Also, the choice of the estimator δ will be up to us. (δ is the competitor to δ_F.) The situation is considerably simplified if δ is chosen so that $\gamma_i = 0$ for $i > 1$. Using this choice, together with (1.2.1) and (1.2.2) gives

$$\begin{aligned}\Delta_\delta^F(\theta) = &-\gamma_1^{(1)}(\theta) + \tfrac{1}{2}\textstyle\sum_{j=2}^n \gamma_1^{(j,j)}(\theta) - \tfrac{1}{2}m_{1,1,1}\gamma_1^{(1,1)}(\theta) \\ &- \textstyle\sum_{k=2}^n m_{1,1,k}\gamma_1^{(1,k)}(\theta) - \gamma_1^2(\theta)b_{11}/2 \\ &- \textstyle\sum_{j=1}^n b_{1j}\gamma_1(\theta)\gamma_{F,j}(\theta) + \mathscr{R}\,.\end{aligned} \tag{1.3.3}$$

The sense in which $\mathscr{R}$ can be made small in this setting is that as $\theta_1 \to \infty$, $\mathscr{R}$ will go to zero faster than the other terms. Since γ_1 will be chosen to be smooth and flat, it will also follow that as $\theta_1 \to \infty$, $\gamma_1^{(1,k)}(\theta)$ $(k \geqq 1)$ will go to zero faster than $\gamma_1^{(1)}(\theta)$. This, together with the definition of the differential operator $\mathscr{D}$, finally gives

$$\Delta_\delta^F(\theta) \simeq \mathscr{D}\gamma_1(\theta) - \gamma_1^2(\theta)b_{11}/2 - \textstyle\sum_{j=1}^n b_{1j}\gamma_1(\theta)\gamma_{F,j}(\theta) \tag{1.3.4}$$

for large θ_1. (Here $\simeq$ is to be understood as approximate equality, where the error goes to zero faster than the right hand side, as $\theta_1 \to \infty$.)

In demonstrating, heuristically, the use to which the differential approximation (1.3.4) will be put, the case $\gamma_F = 0$ will be considered for simplicity. The problem is then one of determining when the best invariant estimator is inadmissible. (1.3.4) thus implies that for large θ_1,

$$\Delta_\delta^F(\theta) \simeq \mathscr{D}\gamma_1(\theta) - \gamma_1^2(\theta)b_{11}/2 . \tag{1.3.5}$$

To prove that the best invariant estimator is inadmissible, a solution γ_1 to $\Delta_\delta^F > 0$ must be found. We look first for a solution for large θ_1, and hence can use (1.3.5). Note that $\mathscr{D}\gamma_1(\theta) = 0$ is the $(n-1)$ dimensional heat equation, which has as a solution for $\theta_1 > 0$, $\gamma_1(\theta) = -\theta_1^{-(n-1)/2}\exp[-|\theta^*|^2/2\theta_1]$. This suggests looking at estimators of the form

$$\gamma_1(\theta) = -\theta_1^{-a}\exp[-|\theta^*|^2/2\theta_1] . \tag{1.3.6}$$

Inserting this estimator into (1.3.5) and going through some calculation, gives that for large θ_1,

$$\begin{aligned}\Delta_\delta^F(\theta) \simeq \left(\frac{n-1}{2} - a\right)\theta_1^{-(a+1)}\exp[-|\theta^*|^2/2\theta_1] \\ - \theta_1^{-2a}\exp[-|\theta^*|^2/\theta_1]b_{11}/2 .\end{aligned} \tag{1.3.7}$$

For this to be positive for large θ_1, one needs $a < (n-1)/2$ and $(a+1) < 2a$. In conclusion, if γ_1 is defined by (1.3.6) and $1 < a < (n-1)/2$, it should be possible to conclude that $\Delta_\delta^F(\theta) > 0$ for large enough θ_1. If $n \geqq 4$, such a choice of "a" is possible, and it is then reasonable to expect that the best invariant estimator is inadmissible. An estimator which would appear to be better is

$$\begin{aligned}\delta(x) &= (x_1 - x_1^{-a}\exp[-|x^*|^2/2x_1], x^*) \qquad &&\text{for } x_1 > A \\ &= x &&\text{otherwise,}\end{aligned} \tag{1.3.8}$$

where $1 < a < (n-1)/2$ and A is appropriately large. Unfortunately, this estimator does not work in our analysis, since the error terms involved in the approximation (1.3.5) are not small enough. The reason for this is that the estimator is not appropriately smooth, as the terms $\gamma_1^{(i,j)}$ are too big. One can smooth it out, however, by defining

$$\begin{aligned}\gamma_1(x) &= -\textstyle\int_{R^{n-1}} x_1^{-a}\exp[-|\xi - x^*|^2/2x_1](1+|\xi|^n)^{-1}d\xi \qquad &&\text{for } x_1 > A \\ &= 0 &&\text{otherwise.}\end{aligned} \tag{1.3.9}$$

The estimator $\delta(x) = (x_1 + \gamma_1(x), x^*)$ is actually shown in Chapter 2 to be better than the best invariant estimator (for appropriate choices of A and a). It would be interesting to know if the simpler estimator (1.3.8) leads to an improvement itself.

Passing from estimators which improve upon the risk for large θ_1, to estimators which improve upon the risk everywhere, requires additional arguments. Two different types of such arguments are given in Chapters 2 and 3. They both use the basic idea that an estimator, δ, such as (1.3.8) is better than the best invariant estimator for $\theta_1 < A - 1$. This is clear since if $x_1 < A$, then $\delta(x)$ is the best invariant estimator x, while if $x_1 > A$, then $\delta(x)$ "corrects" the best invariant estimator x by moving it closer to θ.

To deal with generalized Bayes estimators, one uses the approximation (1.3.4). An extra term is involved in the calculations, but it is usually fairly easy to compensate for it.

Before proceeding to the rigorous theory, one final observation is in order. Hopefully, some feeling has been given of the importance of the moment structure. Had we started with a different moment structure, the differential approximation resulting from (1.3.2) would have been different and it is believable that quite different results would have been obtained. To see that this is true, one can look at examples of other moment structures in Brown (1974c).

The conclusion that can be drawn from this observation is that the question of admissibility of generalized Bayes estimators of location parameters will probably not admit nice elegant solutions. There can be no way to avoid bringing the "messy" $m_{i,j(1),\cdots,j(l)}$, b_{ij}, and perhaps even worse terms into the analysis, because the ultimate answers depend upon them so profoundly. Hopefully, of course, the problems of practical interest will have relatively simple moment structures which can be handled.

2. Inadmissibility of generalized Bayes estimators. This chapter will deal rigorously with the question of when a generalized Bayes estimator of θ_1 is inadmissible. (Thus $\theta_2, \cdots, \theta_n$ are nuisance location parameters.) As mentioned in Chapter 1, it will be assumed that the loss in estimating θ_1 by d_1 is $L(d_1 - \theta_1)$. Since the loss is then a function on R^1, the notation $L'(y) = (d/dy)L(y)$, $L''(y) = (d^2/dy^2)L(y)$, etc., will be adopted for the derivatives of L. For convenience in stating assumptions, the notation $L^{\langle i\rangle}(y) = (d^i/dy^i)L(y)$ will also be used. A large amount of notation will be saved later on by adopting the convention that $L^{\langle 0\rangle}(y) \equiv 1$.

Since just θ_1 is to be estimated, only $\delta_{F,1}$, δ_1, $\gamma_{F,1}$, and γ_1 will be of interest. Thus, for convenience, drop the subscripts "1" above. Likewise, for simplicity, denote $b = b_{11} = E_0 L''(X_1)$.

2.1. *Assumptions and results.* It will be assumed that the assumptions of Section 1.2 hold (the most important of which are the moment structure assumptions (1.2.1) and (1.2.2)). In addition, the following assumptions are needed:

(1) If $|y_1| < D < \infty$, then there exist finite constants K_D and C_D for which

$$|L^{\langle i\rangle}(x_1 + y_1)| \leqq K_D|L^{\langle i\rangle}(x_1)| + C_D\,, \qquad i = 2 \text{ or } 3.$$

(2) $E_0[|X|^\alpha |L^{\langle i\rangle}(X_1)|] < \infty$ for $i = 0, 1, 2, 3$ and $\alpha = \max(n + 6, 2n + 3)$.

(3) There exist $K > 0$ and $\tau > 0$ such that if $|x_1| < \tau$, then

$$\int_{R^{n-1}} (1 + |x|^n) f(x_1, x^*)\, dx^* \leqq K\,.$$

(4) $|\gamma_F| < B < \infty$.
(5) There exist $C > 0$ and $0 < \lambda < \frac{1}{2}$ such that if $x_1 > C$, then

$$\gamma_F(x) \geqq -(n - 3 - 4\lambda)/(2bx_1) .$$

Now define

$$\begin{aligned} \gamma_{A,\varepsilon}(x) &= -\textstyle\int_{R^{n-1}} x_1^{-(1+\varepsilon)} \exp[-|x^* - \xi|^2/2x_1](1 + |\xi|^n)^{-1}\, d\xi \\ &\qquad\qquad \text{if } x_1 > A > C \qquad (2.1.1) \\ &= 0 \qquad\qquad \text{if } x_1 \leqq A . \end{aligned}$$

This chapter is devoted to the proof of the following theorem:

THEOREM 1. *Under the above assumptions, δ_F is inadmissible for estimating θ_1. In fact, there exists a constant A such that $\delta_{A,\lambda} = \delta_F + \gamma_{A,\lambda}$ is better than δ_F.*

Discussion of assumptions.

(1) Assumption 1 requires that L be in some sense smooth and increasing slower than exponentially.

(2) Thus if $n = 4$ and L is squared error, we require 12 absolute moments of our density. This is probably unnecessarily restrictive, but is needed for technical reasons in the proof.

(3) This is another fairly weak technical assumption. It could actually be eliminated, but the argument would become considerably more involved.

(4) Again this is more restrictive than absolutely necessary, but does admit consideration of usual generalized Bayes estimators.

(5) This last assumption is the important one. Essentially, the dividing line between admissibility and inadmissibility is $\gamma_F(x) = -(n - 3)/(2bx_1)$. Indeed in Berger (1974) it is shown that if $\gamma_F(x) < -(n - 3 + \varepsilon)/(2bx_1)$ for some $\varepsilon > 0$ and large x_1, then δ_F is admissible.

Given a generalized prior F, it is necessary to estimate γ_F to verify Condition (5). For this purpose, we refer to a result in Berger (1974). (See Theorem A in Section 1.5 of that article.) Define $\mathscr{D}^*$ as the "adjoint" operator to $\mathscr{D}$, namely $\mathscr{D}^*G = -[G^{(1)} + \frac{1}{2}\sum_{i=2}^n G^{(i,i)}]$. Theorem A essentially says that if F is smooth, bounded, and nonzero, then for large x_1,

$$\gamma_F(x) = -\mathscr{D}^*F(x)/(bF(x)) + o(|\mathscr{D}^*F(x)|/(bF(x))) .$$

(Here we are using the familiar "little oh" notation.)

As an example, suppose F is given by

$$\begin{aligned} F(\theta) &= 1 &&\text{for } |\theta_1| \leqq 1 \\ &= |\theta_1|^{-a} &&\text{for } |\theta_1| > 1 . \end{aligned}$$

Then $\mathscr{D}^*F(x) = -ax_1^{-(a+1)}$ for $x_1 > 1$. Hence by the theorem, if x_1 is large, then $\gamma_F(x) = -a/(bx_1) + o(x_1^{-1})$. Therefore if $0 < a < (n - 3)/2$, Condition (5) will be satisfied.

It should be noted that Theorem 1 does not really depend upon the fact that δ_F is generalized Bayes. As long as the estimator under consideration is measurable and satisfies Assumptions (4) and (5), Theorem 1 can be applied to find a better estimator.

2.2. *Preliminary lemmas.* As indicated in Section 2.1, the competitor to δ_F will be $\delta_{A,\lambda} = \delta_F + \gamma_{A,\lambda}$. For notational convenience, we will henceforth just write γ for $\gamma_{A,\lambda}$ (or γ_A if the A is to be stressed).

We will be dealing with a Taylor expansion of γ. Hence, information will be needed about the partial derivatives of γ and about the remainder term of the Taylor expansion. This section gives a number of lemmas useful in this regard.

When dealing with derivatives of γ_A, we will be working with expressions of the form

$$\int_{R^{n-1}} \frac{|x^* - \xi|^m \exp[-|x^* - \xi|^2/2x_1]\, d\xi}{x_1^{(a+m/2)}(1 + |\xi|^n)} .$$

By simple changes of variables, we get that the above expression is equal to

$$\int_{R^{n-1}} \frac{|\xi|^m \exp[-|\xi|^2/2x_1]\, d\xi}{x_1^{(a+m/2)}(1 + |x^* + \xi|^n)} = \int_{R^{n-1}} \frac{|\xi|^m \exp[-|\xi|^2/2]\, d\xi}{x_1^{[a-(n-1)/2]}(1 + |x^* + \xi x_1^{\frac{1}{2}}|^n)} .$$

All three expressions will be used in this section. For notational convenience in the following lemma, define

$$\varphi(\xi, x) = \frac{\exp[-|\xi|^2/2x_1]}{x_1^{(1+\lambda)}(1 + |x^* + \xi|^n)} .$$

LEMMA 2.2.1. *There exist constants A and K, such that for $x_1 > A$ and $i = 2, \cdots, n$, $j = 2, \cdots, n$,*

(a) $\gamma_A^{(1)}(x) = (1 + \lambda)x_1^{-1} \int \varphi\, d\xi - (x_1^{-2}/2) \int |\xi|^2\varphi\, d\xi$,
(b) $\gamma^{(i,i)}(x) = x_1^{-1} \int \varphi\, d\xi - x_1^{-2} \int \xi_i^2\varphi\, d\xi$,
(c) $-\mathscr{D}\gamma(x) = (n - 3 - 2\lambda)\gamma(x)/(2x_1)$,
(d) $|\gamma^{(i)}(x)| \leqq Kx_1^{-1} \int |\xi|\varphi\, d\xi$,
(e) $|\gamma^{(i,j)}(x)| \leqq Kx_1^{-1} \int (1 + x_1^{-1}|\xi|^2)\varphi\, d\xi$,
(f) $|\gamma^{(1,j)}(x)| \leqq Kx_1^{-\frac{3}{2}} \int (\sum_{l=0}^{4} x_1^{-l/2}|\xi|^l)\varphi\, d\xi$,
(g) $|\gamma^{(i,j,k)}(x)| \leqq Kx_1^{-\frac{3}{2}} \int (\sum_{l=0}^{6} x_1^{-l/2}|\xi|^l)\varphi\, d\xi$.

PROOF. Straightforward calculation. □

For use in the following lemmas, define

$$H_{m,A}(x) = \int_{R^{n-1}} \frac{|\xi|^m \exp[-|\xi|^2/2x_1]\, d\xi}{x_1^{[m/2+(n-1)/2]}(1 + |x^* + \xi|^n)} \quad \text{if } x_1 \geqq A$$
$$= 0 \quad \text{if } x_1 < A .$$

The next three lemmas are of a highly technical nature Their proofs, though involved, add little insight into the problem. Therefore, the proofs have been put into an appendix.

LEMMA 2.2.2. *There exist constants* A_0 *and* K, *such that if* $A > A_0$, *then* $H_{m,A}(x) \leqq K x_1^{\lambda/4} H_{0,A}(x)$.

The next lemma will be used later on to handle the remainder term in a Taylor expansion. Let t_i, $i = 1, 2, 3$, be measurable functions from $R^n \times R^n \to R^1$, satisfying $0 \leqq t_i \leqq 1$. Define $\nu(x, \theta) = t_1(x, \theta)x + [1 - t_1(x, \theta)]\theta$, $\omega(x, \theta) = t_2 x + (1 - t_2)\theta$, and $\tau(x, \theta) = t_3 x + (1 - t_3)\theta$. (Note ν, ω, and τ are row vectors. Hence ν_1 is the first component of ν, $\nu^* = (\nu_2, \cdots, \nu_n)$, etc.) Define $Q_A(\theta)$ as the integral

$$\int_{\{x: x_1 > A\}} \left[\int_{R^{n-1}} \frac{|\xi|^m \exp[-|\xi|^2/2\omega_1]\, d\xi}{\nu_1^{[2(1+\lambda)+m/2]}(1 + |\tau^* + \xi|^n)} \right] \times |L^{(i)}(x_1 - \theta_1)|\, |x - \theta|^j f(x - \theta)\, dx\,, \tag{2.2.9}$$

where $0 \leqq j \leqq 3$, $0 \leqq m \leqq 6$, and $0 \leqq i \leqq 3$. (More explicitly we could write $Q_{A,i,j,\lambda,m,t_1,t_2,t_3}(\theta)$, but this would be a bit cumbersome.)

LEMMA 2.2.3. *Assume that either* $m = 0$, *or that* $m > 0$ *with* $t_1 = t_2 = t_3$. *There exist constants* A_0 *and* K *such that if* $A > A_0$ *and* $\theta_1 > A$, *then* $Q_A(\theta) \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)}$.

LEMMA 2.2.4. *For fixed* $c > 0$, $c' > 0$, *and* $a > 0$, *there exist constants* A_0 *and* K *such that if* $A > A_0$ *and* $4A - c/A \leqq \theta_1 \leqq 4A$, *then*

$$|\gamma_A((4A - c'(4A)^{-a}, \theta^*))| \leqq K|\gamma_A(\theta)|\,.$$

2.3. *Proof of Theorem* 1. To prove that there exists an A for which $\delta_{A,\lambda}$ is better than δ_F, it is necessary to show that

$$\begin{aligned} \Delta^F_{\delta_{A,\lambda}} &= R(\delta_F, \theta) - R(\delta_{A,\lambda}, \theta) \\ &= \int [L(\gamma_F(x) + x_1 - \theta_1) \\ &\quad - L(\gamma_F(x) + \gamma_A(x) + x_1 - \theta_1)] f(x - \theta)\, dx \end{aligned} \tag{2.3.1}$$

is greater than or equal to zero with strict inequality for some θ. For notational convenience, define $\Delta_A = \Delta^F_{\delta_{A,\lambda}}$, and $\alpha = -(n - 3 - 4\lambda)/2$.

Expanding $L(\gamma_F + \gamma_A + x_1 - \theta_1)$ in a Taylor expansion about $(\gamma_F + x_1 - \theta_1)$ gives

$$\begin{aligned} &L(\gamma_F + \gamma_A + x_1 - \theta_1) \\ &\quad = L(\gamma_F + x_1 - \theta_1) + \gamma_A L'(\gamma_F + x_1 - \theta_1) \\ &\qquad + \int_{(\gamma_F + x_1 - \theta_1)}^{(\gamma_F + \gamma + x_1 - \theta_1)} L''(\eta)[\gamma_F + \gamma + x_1 - \theta_1 - \eta]\, d\eta\,. \end{aligned} \tag{2.3.2}$$

Define

$$S_A(\theta) = -\int \gamma_A(x) L'(\gamma_F(x) + x_1 - \theta_1) f(x - \theta)\, dx\,,$$

$$\varepsilon_1(\theta) = -\int_{R^n} \int_{(\gamma_F + x_1 - \theta_1)}^{(\gamma_F + \gamma + x_1 - \theta_1)} L''(\eta)[\gamma_F + \gamma + x_1 - \theta_1 - \eta]\, d\eta f(x - \theta)\, dx\,.$$

Inserting (2.3.2) into (2.3.1) gives

$$\Delta_A(\theta) = S_A(\theta) + \varepsilon_1(\theta)\,. \tag{2.3.3}$$

LEMMA 2.3.1. *There exist* A_0 *and* K *such that if* $A > A_0$ *and* $\theta_1 > A$, *then* $|\varepsilon_1(\theta)| \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)}$.

PROOF. Using the fact that $L'' \geqq 0$ and assumptions (1) and (4), we get

$$\begin{aligned}|\varepsilon_1(\theta)| &\leqq \int_{R^n}\int_{|\eta|<|\gamma|} L''(\gamma_F + x_1 - \theta_1 + \eta)|\gamma|\, d\eta f(x-\theta)\, dx \\ &\leqq \int_{R^n}\int_{|\eta|<|\gamma|} [K_{B+1}L''(x_1-\theta_1) + C_{B+1}]|\gamma| f(x-\theta)\, d\eta\, dx \\ &\leqq K \int \gamma^2[L''(x_1-\theta_1)+1]f(x-\theta)\, dx\,.\end{aligned}$$

For $x_1 > A$, Jensen's inequality gives

$$\begin{aligned}\gamma^2(x) &= \left(-\int \frac{\exp[-|\xi|^2/2x_1]\, d\xi}{x_1^{(1+\lambda)}(1+|x^*+\xi|^n)}\right)^2 \\ &\leqq K \int \frac{\exp[-|\xi|^2/x_1]\, d\xi}{x_1^{2(1+\lambda)}(1+|x^*+\xi|^n)} \\ &\leqq K \int \frac{\exp[-|\xi|^2/2x_1]\, d\xi}{x_1^{2(1+\lambda)}(1+|x^*+\xi|^n)}\,.\end{aligned}$$

Thus

$$\begin{aligned}|\varepsilon_1(\theta)| &\leqq K \int_{R^n}\int_{R^{n-1}} \frac{\exp[-|\xi|^2/2x_1]\, d\xi}{x_1^{2(1+\lambda)}(1+|x^*+\xi|^n)} [L''(x_1-\theta_1)+1]f(x-\theta)\, dx \\ &\leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)} \quad \text{(by Lemma 2.2.3).} \ \square\end{aligned}$$

The next lemma can be considered the heart of the proof, as it analyzes $S_A(\theta)$, the dominant term in the risk expansion.

LEMMA 2.3.2. *There exists an A_0 such that if $A > A_0$ and $\theta_1 > 2A$, then $S_A(\theta) \geqq -\lambda\gamma_A(\theta)/(2\theta_1)$.*

PROOF. Note that $(-\gamma) \geqq 0$, $\gamma_F(x) > \alpha/(bx_1)$ for $x_1 > A$, and L' is nondecreasing. Hence

$$\begin{aligned}S_A(\theta) &= \int -\gamma(x)L'(\gamma_F(x) + x_1 - \theta_1)f(x-\theta)\, dx \\ &\geqq \int -\gamma(x)L'(\alpha/(bx_1) + x_1 - \theta_1)f(x-\theta)\, dx\,.\end{aligned}$$

Expanding $L'(\alpha/(bx_1) + x_1 - \theta_1)$ above, in a Taylor expansion about $(x_1 - \theta_1)$, yields

$$\begin{aligned}S_A(\theta) \geqq \int -\gamma(x)[&L'(x_1-\theta_1) + L''(x_1-\theta_1)\alpha/(bx_1) \\ &+ \int_{(x_1-\theta_1)}^{(\alpha/(bx_1)+x_1-\theta_1)} L'''(\eta)(\alpha/(bx_1) \\ &+ x_1 - \theta_1 - \eta)\, d\eta] f(x-\theta)\, dx\,.\end{aligned} \tag{2.3.4}$$

Define

$$h(\theta) = \int -\gamma(x)\alpha(bx_1)^{-1}L''(x_1-\theta_1)f(x-\theta)\, dx\,,$$

$$r(\theta) = \int -\gamma(x)L'(x_1-\theta_1)f(x-\theta)\, dx = \int_{\{x:x_1>A\}} -\gamma(x)L'(x_1-\theta_1)f(x-\theta)\, dx\,,$$

$$\varepsilon_2(\theta) = \int -\gamma(x)\int_{(x_1-\theta_1)}^{(\alpha/(bx_1)+x_1-\theta_1)} L'''(\eta)[\alpha/(bx_1) + x_1 - \theta_1 - \eta]\, d\eta f(x-\theta)\, dx\,.$$

Then (2.3.4) becomes

$$S_A(\theta) \geqq r(\theta) + h(\theta) + \varepsilon_2(\theta)\,. \tag{2.3.5}$$

As in Lemma 2.3.1 we get the bound

$$|\varepsilon_2(\theta)| \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)}\,. \tag{2.3.6}$$

Next, consider $h(\theta)$. Note it is only necessary to integrate over $\{x: x_1 > A\}$, since $\gamma_A(x) = 0$ for $x_1 \leqq A$. If $x_1 > A$ and $\theta_1 > A$, we can expand x_1^{-1} in the Taylor expansion

$$x_1^{-1} = \theta_1^{-1} + (x_1 - \theta_1)[t(x_1, \theta_1)x_1 + \{1 - t(x_1, \theta_1)\}\theta_1]^{-2}, \quad \text{where} \quad 0 \leqq t(x_1, \theta_1) \leqq 1 .$$

Thus, defining

$$\varepsilon_3(\theta) = -\int \alpha\gamma(x)L''(x_1 - \theta_1)(x_1 - \theta_1)b^{-1}[tx_1 + (1 - t)\theta_1]^{-2}f(x - \theta)\,dx ,$$
$$g(\theta) = \int -\alpha\gamma(x)(b\theta_1)^{-1}L''(x_1 - \theta_1)f(x - \theta)\,dx ,$$

we get

$$h(\theta) = g(\theta) + \varepsilon_3(\theta) . \tag{2.3.7}$$

To bound $\varepsilon_3(\theta)$, note that

$$\frac{|\gamma(x)|}{[tx_1 + (1 - t)\theta_1]^2} \leqq \int \frac{\exp[-|\xi|^2/2x_1]\,d\xi}{x_1^{(3+\lambda)}(1 + |x^* + \xi|^n)} + \int \frac{\exp[-|\xi|^2/2x_1]\,d\xi}{[tx_1 + (1 - t)\theta_1]^{(3+\lambda)}(1 + |x^* + \xi|^n)} .$$

Using Lemma 2.2.3, we thus see that there exists an A_0 such that if $A > A_0$ and $\theta_1 > A$, then

$$|\varepsilon_3(\theta)| \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)} . \tag{2.3.8}$$

Next, consider the term $g(\theta)$. Since $\theta_1 > A$ and $x_1 > A$, we can expand $\gamma(x)$ in a Taylor expansion about θ, getting

$$\gamma(x) = \gamma(\theta) + \sum_{i=1}^{n}(x_i - \theta_i)\gamma^{(i)}(tx + (1 - t)\theta) , \quad \text{where} \quad 0 \leqq t \leqq 1 .$$

Hence

$$g(\theta) = -\alpha\gamma(\theta)/\theta_1 + \int -\alpha(b\theta_1)^{-1}[\textstyle\sum_{i=1}^{n}(x_i - \theta_i)\gamma^{(i)}(\quad)]L''(x_1 - \theta_1)f(x - \theta)\,dx . \tag{2.3.9}$$

Call the last integral above $\varepsilon_4(\theta)$. Recalling the bound for $\gamma^{(i)}(tx + (1 - t)\theta)$ from Lemma 2.2.1, we can use an argument identical to that following (2.3.7) to conclude

$$|\varepsilon_4(\theta)| \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)} . \tag{2.3.10}$$

Combining (2.3.8), (2.3.9), and (2.3.10) shows that there exists an A_0 such that if $A > A_0$ and $\theta_1 > A$, then

$$h(\theta) = -\alpha\gamma(\theta)/\theta_1 + \varepsilon_3(\theta) + \varepsilon_4(\theta) , \quad \text{where} \quad |\varepsilon_i(\theta)| \leqq K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)},\ i = 3, 4 . \tag{2.3.11}$$

Finally, consider the term $r(\theta)$. Again, since $\theta_1 > A$ and $x_1 > A$, we can expand $\gamma(x)$ in the Taylor expansion

$$
\begin{aligned}
\gamma(x) &= \gamma(\theta) + \gamma^{(1)}(\theta)(x_1 - \theta_1) + \textstyle\sum_{i=2}^{n} (x_i - \theta_i)\gamma^{(i)}(\theta) \\
&\quad + \tfrac{1}{2} \textstyle\sum_{i=1}^{n} (x_i - \theta_i)^2 \gamma^{(i,i)}(\theta) \\
&\quad + \tfrac{1}{2} \textstyle\sum \sum_{i,j=2,i\neq j}^{n} (\ )_i(\ )_j \gamma^{(i,j)}(\theta) \\
&\quad + \textstyle\sum_{i=2}^{n} (x_1 - \theta_1)(x_i - \theta_i)\gamma^{(1,i)}(\theta) \\
&\quad + \tfrac{1}{6} \textstyle\sum_i \sum_j \sum_k (\ )_i(\ )_j(\ )_k \gamma^{(i,j,k)}(\omega)
\end{aligned} \tag{2.3.12}
$$

where $\omega = tx + (1 - t)\theta$, $0 \leqq t(x, \theta) \leqq 1$.

Insert this expansion into the expression for $r(\theta)$ and expand it into a sum of 7 integrals (corresponding to the 7 terms in (2.3.12)). We consider these integrals in order.

(i) To deal with the first integral, note that by assumption, $E_0 L'(X_1) = 0$. Hence,

$$
\begin{aligned}
|\textstyle\int_{\{x:x_1>A\}} -\gamma(\theta)L'(x_1 - \theta_1)f(x - \theta)\,dx| &= |\gamma(\theta) \textstyle\int_{\{x:x_1<A\}} L'(x_1 - \theta_1)f(x - \theta)\,dx| \\
&= |\gamma(\theta) \textstyle\int_{\{z:z_1<A-\theta_1\}} L'(z_1)f(z)\,dz|\,.
\end{aligned}
$$

Since we are assuming that $\theta_1 > 2A$, it is clear that $\{z: z_1 < A - \theta_1\} \subset \{z: z_1 < -\theta_1/2\}$. A simple Chebyshev argument then gives that the last term above is bounded by $K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)}$.

(ii) To deal with the second integral in the expansion of $r(\theta)$, note that by the moment structure assumptions, $E_0[X_1 L'(X_1)] = 1$. Hence,

$$
\begin{aligned}
&\textstyle\int_{\{x:x_1>A\}} -\gamma^{(1)}(\theta)L'(x_1 - \theta_1)(x_1 - \theta_1)f(x - \theta)\,dx \\
&\qquad = -\gamma^{(1)}(\theta) + \gamma^{(1)}(\theta) \textstyle\int_{\{x:x_1<A\}} (x_1 - \theta_1)L'(\ )f(\ )\,dx\,.
\end{aligned}
$$

Again the second term above can be bounded by $K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)}$ (using the Chebyshev argument and Lemmas 2.2.1 and 2.2.2 to bound $\gamma^{(1)}(\theta)$).

(iii)—(vii) The integrals corresponding to terms (iii), (iv), (v), (vi), and (vii) of (2.3.12) are analyzed analogously, using the particular form of the moment structure and Lemmas 2.2.1 and 2.2.2 (Lemma 2.2.3 is also needed to deal with the remainder term (vii)). The integral corresponding to term (iv) gives the dominant term $\frac{1}{2} \sum_{i=2}^{n} \gamma^{(i,i)}(\theta)$. The other integrals can all be shown to be bounded by $K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)}$.

Collecting everything together gives

$$
\begin{aligned}
r(\theta) &= -\gamma^{(1)}(\theta) + \tfrac{1}{2} \textstyle\sum_{i=2}^{n} \gamma^{(i,i)}(\theta) + \varepsilon_5(\theta) \\
&= \mathscr{D}\gamma(\theta) + \varepsilon_5(\theta) \\
&= -(n - 3 - 2\lambda)\gamma(\theta)/(2\theta_1) + \varepsilon_5(\theta)\,,
\end{aligned} \tag{2.3.13}
$$

where $|\varepsilon_5(\theta)| \leqq K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)}$. Combining (2.3.5), (2.3.6), (2.3.11), and (2.3.13) gives

$$
\begin{aligned}
S_A(\theta) &\geqq \varepsilon_2(\theta) - \alpha\gamma(\theta)/\theta_1 + \varepsilon_3(\theta) + \varepsilon_4(\theta) - (n - 3 - 2\lambda)\gamma(\theta)/(2\theta_1) + \varepsilon_5(\theta) \\
&= -\lambda\gamma(\theta)/\theta_1 + \varepsilon_6(\theta)\,,
\end{aligned}
$$

where $|\varepsilon_6(\theta)| \leqq K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)}$.

Choosing A_0 large enough (recall $\theta_1 > 2A > 2A_0$) so $|\varepsilon_6(\theta)| \leqq \lambda|\gamma(\theta)|/(2\theta_1)$, we have that if $A > A_0$ and $\theta_1 > 2A$, then $S_A(\theta) \geqq -\lambda\gamma_A(\theta)/(2\theta_1)$. □

LEMMA 2.3.3. *There exists an* A_0 *such that if* $A > A_0$ *and* $\theta_1 > 2A$, *then* $\Delta_A(\theta) \geqq -\lambda\gamma_A(\theta)/(4\theta_1) > 0$.

PROOF. Obvious, using (2.3.3) and Lemmas 2.3.1 and 2.3.2. (Recall $\gamma(\theta)$ is negative.) □

It has thus been established for large enough A and for $\theta_1 > 2A$, that $\delta_{A,\lambda}$ is better than δ_F. The rest of this section is devoted to showing that this improvement can be extended to included all θ. The following slightly more explicit version of Theorem 1 will show this.

THEOREM 1*. *Assume* A_0 *is chosen so Lemmas* 2.3.1, 2.3.2, *and* 2.3.3 *hold. Then there exists an* $A > A_0$ *such that* $\Delta_{4A}(\theta) \geqq 0 \;\forall\,\theta$, *with strict inequality for* $\theta_1 > 4A$.

PROOF. Denote $c = \int_{R^{n-1}} (1 + |\xi|^n)^{-1}\,d\xi$. Consider first

CASE 1. $\theta_1 \leqq 4A + \alpha(4Ab)^{-1} - c(4A)^{-(1+\lambda)}$. For convenience, denote $\beta = c(4A)^{-(1+\lambda)}$. By definition, $\gamma_{4A}(x) = 0$ for $x_1 < 4A$. Hence,

$$\delta_{4A,\lambda}(x) = \delta_F(x) + \gamma_{4A}(x) = \delta_F(x) \qquad \text{if } x_1 < 4A\,. \tag{2.3.14}$$

On the other hand, if $x_1 \geqq 4A$, an easy calculation shows that $|\gamma_{4A}(x)| \leqq \beta$. Furthermore, since $\gamma_F(x) \geqq \alpha/(4Ab)$, we can conclude

$$\delta_{4A,\lambda}(x) = \delta_F(x) + \gamma_{4A}(x) \geqq 4A + \alpha(4Ab)^{-1} - \beta \geqq \theta_1\,, \qquad \text{if } x_1 \geqq 4A\,.$$

Since $\gamma_{4A}(x)$ is negative for $x_1 \geqq 4A$, the estimate $\delta_{4A,\lambda}(x)$ can only be closer to θ_1 than $\delta_F(x)$. Combining this with (2.3.14), it is thus clear that $\Delta_{4A}(\theta) \geqq 0$ for $\theta_1 \leqq 4A + \alpha(4Ab)^{-1} - \beta$.

CASE 2. Assume $\theta_1 \geqq 4A + \alpha(4Ab)^{-1} - \beta$. As in (2.3.3), consider

$$\Delta_{4A}(\theta) = \varepsilon_1(\theta) + S_{4A}(\theta)\,. \tag{2.3.15}$$

By Lemma 2.3.1, it is clear that

$$|\varepsilon_1(\theta)| \leqq K|\gamma_{4A}(\theta)|\theta_1^{-(1+\lambda/4)} \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)}\,. \tag{2.3.16}$$

To deal with $S_{4A}(\theta)$, define $Q(\theta) = \{x.: \alpha(x_1 b)^{-1} + x_1 < \theta_1\}$.

$$\begin{aligned} S_{4A}(\theta) &= \int_{\{x:x_1>4A\}} -\gamma_{4A}(x)L'(\gamma_F + x_1 - \theta_1)f(x-\theta)\,dx \\ &\geqq \int_{\{\ \}} -\gamma_{4A}(x)L'(\alpha(x_1 b)^{-1} + x_1 - \theta_1)f(x-\theta)\,dx \\ &\geqq \int_{\{x:x_1>4A\}} -\gamma_{4A}(x)L'(\alpha(x_1 b)^{-1} + x_1 - \theta_1)f(x-\theta)\,dx \\ &\qquad + \int_{\{x:A<x_1<4A\}\cap Q(\theta)} -\gamma_A(x)L'(\alpha(x_1 b)^{-1} + x_1 - \theta_1)f(x-\theta)\,dx\,. \end{aligned}$$

The last step above follows since if $x \in Q(\theta)$, then $L'(\alpha(x_1 b)^{-1} + x_1 - \theta_1) \leqq 0$. Finally, noting that $\gamma_{4A}(x) = \gamma_A(x)$ if $x_1 \geqq 4A$, we get

$$\begin{aligned} S_{4A}(\theta) \geqq &\int_{\{x:x_1>A\}} -\gamma_A(x)L'(\alpha(x_1 b)^{-1} + x_1 - \theta_1)f(x-\theta)\,dx \\ &- \int_{\{x:A<x_1<4A\}\cap Q(\theta)^c} -\gamma_A(x)L'(\alpha(x_1 b)^{-1}+x_1-\theta_1)f(x-\theta)\,dx\,. \end{aligned} \tag{2.3.17}$$

The proof of Lemma 2.3.2 verifies that

(2.3.18) $$\int_{\{x:x_1>A\}} -\gamma_A(x)L'(\alpha(x_1b)^{-1} + x_1 - \theta_1)f(x-\theta)\,dx \geqq -\lambda\gamma_A(\theta)/(2\theta_1)\,.$$

Thus it is only necessary to deal with the second integral of (2.3.17). Denote this term $\varepsilon_7(\theta)$.

By the definition of $Q(\theta)$, if $x \in Q(\theta)^c$ and $\theta_1 \geqq 4A + \alpha/(4Ab)$, then $x_1 + \alpha/(4Ax_1) \geqq 4A + \alpha/(4Ab)$. Since $\alpha < 0$, this implies that $x_1 \geqq 4A$. But then $\{x: A < x_1 < 4A\} \cap Q(\theta)^c = \emptyset$. In conclusion,

(2.3.19) $$\varepsilon_7(\theta) = 0 \qquad \text{if} \quad \theta_1 \geqq 4A + \alpha/(4Ab)\,.$$

Finally, consider the region $P = \{\theta : 4A - \beta + \alpha/(4Ab) \leqq \theta_1 \leqq 4A + \alpha/(4Ab)\}$. Note first that if $\theta_1 \geqq 4A - \beta + \alpha/(4Ab)$, then

$$\begin{aligned}\{x: A < x_1 < 4A\} &\cap Q(\theta)^c \\ &= \{x: A < x_1 < 4A\} \cap \{x: x_1 + \alpha/(bx_1) \geqq \theta_1\} \\ &\subset \{x: A < x_1 < 4A\} \cap \{x: x_1 + \alpha/(bx_1) \geqq 4A - \beta + \alpha/(4Ab)\} \\ &\subset \{x: 4A - \beta \leqq x_1 \leqq 4A\}\,.\end{aligned}$$

To see the last step above, note that if $x_1 > A$ (large), then $x_1 + \alpha/(bx_1)$ is increasing in x_1. But $x_1 = 4A - \beta$ doesn't satisfy the inequality $x_1 + \alpha/(bx_1) \geqq 4A - \beta + \alpha/(4Ab)$ (recall $\alpha < 0$). Hence we must have $x_1 \geqq 4A - \beta$, and the result follows.

Using the above result, it is clear that

(2.3.20) $$|\varepsilon_7(\theta)| \leqq \int_{\{x:4A-\beta\leqq x_1\leqq 4A\}} |\gamma_A(x)|\,|L'(\alpha(x_1b)^{-1} + x_1 - \theta_1)|f(x-\theta)\,dx\,.$$

Note next that $|\alpha(x_1b)^{-1} + x_1 - \theta_1| \leqq K$ for x_1 in the above region of integration and θ in P. Hence $|L'(\alpha(x_1b)^{-1} + x_1 - \theta_1)| \leqq K$. Also, since $x_1 \geqq 4A - \beta$ and A is large, then $|\gamma_A(x)| \leqq |\gamma_A((4A-\beta), x^*))|$. (It can easily be checked by Lemma 2.2.1 (a) and Lemma 2.2.2 that $\gamma_A^{(1)}(x)$ is increasing in x_1 if A is large enough.) Thus (2.3.20) implies

(2.3.21) $$|\varepsilon_7(\theta)| \leqq K \int_{4A-\beta}^{4A} \int_{R^{n-1}} |\gamma_A((4A-\beta, x^*))| f(x-\theta)\,dx^*\,dx_1\,.$$

As a step in the proof of Lemma 2.2.3 (namely, the analysis following line (2.2.10) in the appendix), arguments were given showing that for a fixed constant d, there exists a K for which

$$\int_{R^{n-1}} |\gamma_A((d, z^*))| f(z-\theta)\,dz^* \leqq \int_{R^{n-1}} K|\gamma_A((d,\theta^*))|(1 + |z-\theta|^n) f(z-\theta)\,dz^*\,.$$

Applying this result to (2.3.21) gives

(2.3.22) $$\begin{aligned}|\varepsilon_7(\theta)| &\leqq K|\gamma_A((4A-\beta, \theta^*))| \int_{4A-\beta}^{4A} \int_{R^{n-1}} (1 + |x-\theta|^n) f(x-\theta)\,dx^*\,dx_1 \\ &= K|\gamma_A(\quad)| \int_{4A-\beta-\theta_1}^{4A-\theta_1} \int_{R^{n-1}} (1 + |x|^n) f(x)\,dx^*\,dx_1\,.\end{aligned}$$

Note that if θ is in P and $4A - \beta - \theta_1 \leqq x_1 \leqq 4A - \theta_1$, then $|x_1| \leqq -\alpha/(4Ab) + \beta$. Choose A large enough so $-\alpha/(4Ab) + \beta < \tau$ (where τ comes from assumption 3). Thus in the above region of integration, $|x_1| < \tau$, and hence by assumption

3 and (2.3.22)

$$(2.3.23) \qquad |\varepsilon_7(\theta)| \leqq K|\gamma_A((4A-\beta, \theta^*))| \int_{4A-\beta-\theta_1}^{4A-\theta_1} K'\, dx_1 = K|\gamma_A((4A-\beta, \theta^*))|\beta\,.$$

Note that if θ is in P, then $\theta_1/4A \leqq 2$. Hence $\beta = c(4A)^{-(1+\lambda)} \leqq K\theta_1^{-(1+\lambda)}$. Finally, by Lemma 2.2.4, $|\gamma_A((4A-\beta, \theta^*))| \leqq K|\gamma_A(\theta)|$. Hence (2.3.23) implies

$$(2.3.24) \qquad |\varepsilon_7(\theta)| \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda)} \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)} \quad \text{if } \theta \text{ is in } P\,.$$

Combining (2.3.15), (2.3.16), (2.3.17), (2.3.18), (2.3.19), and (2.3.24) it is clear that for $\theta_1 \geqq 4A - \beta + \alpha/(4Ab)$,

$$(2.3.25) \qquad \Delta_{4A}(\theta) \geqq -\lambda\gamma_A(\theta)/(2\theta_1) + \varepsilon_1(\theta) + \varepsilon_7(\theta)\,,$$

where $|\varepsilon_i(\theta)| \leqq K|\gamma_A(\theta)|\theta_1^{-(1+\lambda/4)}$. Choosing A large enough so that $KA^{-\lambda/4} \leqq \lambda/8$, we can conclude that $\Delta_{4A}(\theta) \geqq -\lambda\gamma_A(\theta)/(4\theta_1) > 0$. □

2.4. *Applications.* In attempting to apply Theorem 1, it is usually straightforward to check that the assumptions of Section 2.1 are satisfied. Verifying the moment structure assumption (1.2.2), however, can be fairly difficult. (Note that the moment structure assumption (1.2.1) can virtually always be satisfied for the situation of this chapter. See the discussion following Lemma 1.2.3.) The following theorem gives certain useful conditions under which (1.2.2) can be satisfied, and under which we can conclude that δ_F is inadmissible. The theorem is used for several applications.

THEOREM 2. *Assume* $E_0 L'(X_1) = 0$, $E_0[L'(X_1)X_1] = 1$, $E_0[L'(X_1)X_i] = k$ $(2 \leqq i \leqq n)$, *and*

$$E_0[L'(X_1)X_iX_j] = \begin{cases} a & \text{if } i=j=1 \\ r & \text{if } i=1 \text{ or } j=1\,,\quad i\neq j \\ c & \text{if } 2\leqq i=j\leqq n \\ d & \text{otherwise.}\end{cases}$$

Assume, also, that assumptions (i) *through* (iv) *of Section* 1.2 *and assumptions* (1) *through* (5) *of Section* 2.1 *hold.*

(a) *If* $(c-d)$ *and* $[(c-d)+(n-1)(d-2kr+k^2a)]$ *are nonzero and have the same sign, then the problem can be reduced to one with the correct moment structure, and* δ_F *is inadmissible.*

(b) *Assume* $(c-d)$ *is nonzero, and that there exist* $K > 0$ *and* $\lambda > 0$ *such that if* $x_1 > K$, *then* $\gamma_F(x) \geqq -(n-4-4\lambda)/(2bx_1)$. *Then* δ_F *is inadmissible.*

PROOF. It is necessary to show only that the problem can be reduced to the canonical form given by (1.2.1) and (1.2.2).

Define $Y_1 = X_1$ and $Y_i = X_i - kX_1$ for $2 \leqq i \leqq n$. In the transformed problem, $E_0[L'(Y_1)Y_1] = 1$, while $E_0[L'(Y_1)Y_i] = E_0[L'(X_1)(X_i - kX_1)] = 0$ for $2 \leqq i \leqq n$. Thus (1.2.1) is satisfied.

Next, note that for $2 \leqq i \leqq n$ and $2 \leqq j \leqq n$,

$$\begin{aligned} m_{1,i,j} &= E_0[L'(Y_1)Y_iY_j] \\ &= c - 2kr + k^2a \qquad \text{if} \quad i = j \\ &= d - 2kr + k^2a \qquad \text{if} \quad i \neq j . \end{aligned}$$

It is well known that an $(n-1) \times (n-1)$ matrix with diagonal elements α and off diagonal elements β, has characteristic roots $[\alpha + (n-2)\beta]$ and $(\alpha - \beta)$ (the latter with multiplicity $n-2$). Hence M_2 (the matrix with elements $m_{1,i,j}$ above) has characteristic roots $[(c-d) + (n-1)(d - 2kr + k^2a)]$ and $(c-d)$ (the latter with multiplicity $n-2$). If these roots are nonzero and have the same sign, Lemma 1.2.3 can be applied to conclude that the problem can be reduced to one with the canonical form (1.2.2). Hence δ_F is inadmissible and part (a) of the theorem is verified.

To prove part (b), consider the subproblem consisting of Y_1 and the $(n-2)$ random variables corresponding to the $(n-2)$ eigenvectors of $(c-d)$. Since $(c-d)$ is nonzero, one can again apply Lemma 1.2.3 to conclude that the subproblem has a moment structure which can be reduced to (1.2.2). Note that the effective dimension has been lowered by 1. Hence Condition (5) of Section 2.1 becomes the condition given in (b). Again, it can be concluded that δ_F is inadmissible. □

Theorem 2 is useful in situations where enough symmetry is present. Two such cases will now be presented.

EXAMPLE 1. Suppose $Y_i = \ln(X_i - \theta_i)$, $1 \leqq i \leqq n$, where $Y = (Y_1, \cdots, Y_n)$ has a multivariate normal density with mean 0 and covariance matrix $\sum$. Assume $\sum$ has elements $\sigma_{11} = \alpha_1$, $\sigma_{ii} = \alpha_2$ $(2 \leqq i \leqq n)$, $\sigma_{1j} = \sigma_{j1} = \alpha_3$ $(2 \leqq i \leqq n)$, and $\sigma_{ij} = \alpha_4$ otherwise. $X = (X_1, \cdots, X_n)$ thus has a density $f(x - \theta)$ which is a version of a multivariate lognormal density.

To apply Theorem 2, the problem must first be properly centered. Thus find c' so that $E_0 L'(X_1 + c') = 0$. Consider, then, the reparameterized problem of estimating $\eta = (\theta_1 - c', \theta^*)$ with the induced density and generalized prior. Clearly admissibility considerations are unchanged in the new problem, while $E_{\eta=0} L'(X_1) = \int L'(x_1) f(x_1 - c', x^*)\, dx = 0$.

Next, normalize L so that $E_0[L'(X_1)X_1] = 1$. Finally, due to the form of $\sum$, it is clear that $E_0[L'(X_1)X_i]$ and $E_0[L'(X_1)X_iX_j]$ will be of the form given in Theorem 2. Hence to apply the theorem it is only necessary to calculate $(c-d)$ and $[(c-d) + (n-1)(d - 2kr + k^2a)]$.

As a specific example, suppose $n = 4$, $F \equiv 1$, L is squared error, $\sigma_{ii} = 1$ $(1 \leqq i \leqq n)$, $\sigma_{1i} = \sigma_{i1} = .8$ $(i \leqq 2 \leqq n)$, and $\sigma_{ij} = .64$ otherwise. It is straightforward to verify that assumptions (i) through (iv) of Section 1.2 and assumptions (1) through (5) of Section 2.1 are satisfied. Numerical calculation gave $c' = -1.67$ (to the nearest hundredth). Note that the generalized Bayes estimator of θ_1 is thus (approximately) $\delta_F(x) = x_1 - 1.67$. Numerical calculation likewise showed that $(c-d) = 2.32$ and $[(c-d) + (n-1)(d-2kr+k^2a)] = 3.19$. Since both characteristic roots are positive, part (a) of Theorem 2 implies that δ_F is inadmissible.

It has already been mentioned that the theory will not apply to estimating a normal mean. It can be applied, however, in many situations of estimating a normal variance. Example 2 develops this idea through consideration of a more general multiplicative model.

EXAMPLE 2. Suppose $X = (c_1\varepsilon_1, c_2\varepsilon_2, \cdots, c_n\varepsilon_n)$, where $\varepsilon_1, \cdots, \varepsilon_n$ are positive random variables with a known joint density f, and $c_1, \cdots, c_n$ are unknown positive constants. This is a model where the errors ε_i are multiplicative. Note that X has density $(\prod_{i=1}^n c_i^{-1}) f(x_1/c_1, \cdots, x_n/c_n)$. Assume it is desired to estimate c_1, and that the loss incurred in estimating c_1 by d_1 is $L(\ln(d_1/c_1))$ (where $L(\cdot)$ satisfies our usual properties). Finally, assume the generalized prior density is of the form $G(c_1, \cdots, c_n) = F(\ln c_1, \ln c_2, \cdots, \ln c_n) \prod_{i=1}^n c_i^{-1}$.

The above problem can be transformed into a location parameter problem by the transformation $Y_i = \ln X_i$ and $\theta_i = \ln c_i$. It is easy to see that Y is a random variable with density $g(y-\theta) = \exp(\sum (y_i - \theta_i)) f(\exp(y_1 - \theta_1), \cdots, \exp(y_n - \theta_n))$. The loss incurred in estimating θ_1 by a_1 is $L(a_1 - \theta_1)$, and the generalized prior becomes $F(\theta_1, \theta_2, \cdots, \theta_n)$. It is easy to check that $\delta_F(y) = \ln \delta_G(x)$, and hence that admissibility of δ_G in the original problem is equivalent to admissibility of δ_F in the transformed problem. The transformed problem can now be handled by the theory of Chapter 2.

As an example, suppose $X = (X_1, \cdots, X_n)$ has a multivariate normal distribution with mean 0 and a known correlation matrix with elements $\rho_{ii} = 1$ $(1 \leqq i \leqq n)$, $\rho_{i1} = \rho_{1i} = \alpha$ $(2 \leqq i \leqq n)$, and $\rho_{ij} = \beta$ otherwise. Assume the standard deviations, σ_i, of the X_i are unknown, and that it is desired to estimate σ_1.

Usual estimates of σ_1 would be based solely on $|X_1|$. Hence, consider the reduced problem of observing $|X_1|, |X_2|, \cdots, |X_n|$ and estimating σ_1. If the usual estimator is inadmissible in this subproblem, it is clearly inadmissible in the original problem.

It is clear that we now have a multiplicative model with $c_i = \sigma_i$ and $\varepsilon_i = |X_i|/\sigma_i$. Hence, making the transformations $Y_i = \ln |X_i|$ and $\theta_i = \ln \sigma_i$, we get a location parameter problem. One can now go through an argument similar to that in Example 1, leading to application of the inadmissibility results.

As a specific example, suppose the loss is $(\ln(d_1/\sigma_1))^2$, the generalized prior is $G(\sigma_1, \cdots, \sigma_n) = \prod_{i=1}^n \sigma_i^{-1}$, and $n = 4$. In the transformed problem, the loss is squared error and the generalized prior is $F \equiv 1$. As in Example 1, the relevant constants were numerically calculated for $\alpha = .8$ and $\beta = .64$. The centering constant was $c' = .63$, while $(c-d) = -.20$ and $[(c-d) + (n-1)(d-2kr+k^2a)] = -.24$. Again, the characteristic roots have the same sign. Hence by Theorem 2, part (a), it can be concluded that the generalized Bayes estimator $Y_1 + .63$ is inadmissible for estimating θ_1. In terms of the original problem, this says that the generalized Bayes estimator $\exp[Y_1 + .63] = 1.88|X_1|$ is inadmissible for estimating σ_1.

3. Inadmissibility of the best invariant estimator. In this chapter we revert to the general setting of Chapter 1. Thus L need no longer be a function of just one coordinate. Of course, it is still assumed that the problem satisfies the moment structure assumptions (1.2.1) and (1.2.2). We are interested in the question of admissibility of the best invariant estimator for θ. It is easy to check that under the assumptions of Section 1.2, $\delta_0(x) = x$ is the best invariant estimator.

A word is in order as to how this chapter relates to Chapter 2. The fact that the moment structure assumptions, (1.2.1) and (1.2.2), are the same in both chapters, means that essentially the same problem is being considered. The difference is in the generality with which the problem is treated. To deal with generalized Bayes estimators in Chapter 2, it was technically necessary to assume that the loss was a function of only one coordinate. In this chapter this additional restriction will be dropped, though the price to be paid is that only the best invariant estimator will be discussed.

As indicated in Chapter 1, the common situation in which the moment structure assumptions (1.2.1) and (1.2.2) are satisfied, is when the loss is a function of only one coordinate. Thus Theorem 1 of Chapter 2 is really a more important result than the theorem of this chapter. The major purpose of this chapter is more related to the stated objective of describing a methodology. The particular technique of interest is the method by which one passes from conclusions about "large θ" (obtained by the differential approximation argument) to conclusions about all θ.

In Chapter 2, Lemma 2.3.3 established that the generalized Bayes estimator could be improved upon for large θ_1. Extending this result to all θ was done by a trick that was very dependent on the fact that the loss was a function of only one coordinate. As a general tool in dealing with other location vector problems, the trick seems to be of limited usefulness.

When dealing specifically with the best invariant estimator, however, a more general and elegant technique can be used to pass from risk improvement for large θ to risk improvement for all θ. This is the "randomization-of-the-origin" argument used in other contexts by Brown. (See Brown (1974c).) The basic idea of the technique is as follows. First, an estimator δ' is found which improves upon the best invariant estimator for $\theta_1 > A$. Defining $\Delta(\theta) = R(\delta_0, \theta) - R(\delta', \theta)$, it can be shown that $\int \Delta(\theta)\, d\theta = \infty$. Intuitively, this leads one to hope that the good effects of δ' (where $\Delta(\theta) > 0$) could be made to overpower the bad effects of δ' (where $\Delta(\theta) < 0$). Indeed, a properly randomized version of δ' is shown to do this. The details, as usual, get fairly involved.

While "randomizing-the-origin" appears to work only for the best invariant estimator, it is useful in a wide variety of location parameter problems. Because of the generality and attractiveness of the technique, it seems desirable to exhibit its use in a situation considerably more complicated than ones in which it has been previously used. Of course, the theorem obtained for the best invariant estimator will be more general than the corresponding result of Chapter 2.

It will be shown that if $n \geqq 4$, then the best invariant estimator, δ_0, is inadmissible. The assumptions needed, in addition to those of Chapter 1, are

(1) If $|y_1| < D$, then there exist K_D and C_D such that

$$|L^{(1,1)}(x_1 + y_1, x^*)| \leqq K_D|L^{(1,1)}(x)| + C_D\,, \qquad \text{and}$$
$$|L^{(1,1,1)}(x_1 + y_1, x^*)| \leqq K_D|L^{(1,1,1)}(x)| + C_D\,.$$

(2) $$E_0|X|^\alpha < \infty\,, \quad E_0[|X|^\alpha|L^{(1)}(X)|] < \infty\,, \quad E_0[|X|^\alpha|L^{(1,1)}(X)|] < \infty\,, \text{ and}$$
$$E_0[|X|^\alpha|L^{(1,1,1)}(X)|] < \infty\,, \qquad \text{where } \alpha \doteq \max(n + 6, 2n + 3)\,.$$

These assumptions are familiar and relatively weak.

Define the estimator $\delta'(x) = (x_1 + \gamma_{A,\frac{1}{8}}(x), x^*)$. ($\gamma_{A,\frac{1}{8}}$ is defined in (2.1.1).) Also denote $\Delta_A(\theta) = R(\delta_0, \theta) - R(\delta', \theta)$. We begin by proving an analogue of Lemma 2.3.3.

LEMMA 3.1. *Assume $n \geqq 4$. There exists an A_0 such that if $A > A_0$ and $\theta_1 > 2A$, then $\Delta_A(\theta) \geqq -\gamma_{A,\frac{1}{8}}(\theta)/(8\theta_1) > 0$.*

PROOF. Denote $\gamma(x) = \gamma_{A,\frac{1}{8}}(x)$ for convenience. An expansion of $L(\gamma(x) + x_1 - \theta_1, x^* - \theta^*)$ about $(x_1 - \theta_1)$ gives

$$\begin{aligned}\Delta_A(\theta) &= \int [L(x - \theta) - L(\gamma(x) + x_1 - \theta_1, x^* - \theta^*)]f(x - \theta)\,dx \\ &= -\int \gamma(x)L^{(1)}(x - \theta)f(x - \theta)\,dx \\ &\qquad - \int \Big[\int_{(x_1-\theta_1)}^{(\gamma+x_1-\theta_1)} L^{(1,1)}(\eta, x^* - \theta^*)[\gamma + x_1 - \theta_1 - \eta]\,d\eta\Big]f(x - \theta)\,dx \\ &= S_A(\theta) + \varepsilon_1(\theta) \quad \text{(definition).}\end{aligned}$$

The rest of the proof is analogous to the corresponding proof in Section 2.3. The dominant term in the analysis will be $-(n - 3 - \frac{1}{4})\gamma(\theta)/(2\theta_1) \geqq -\gamma(\theta)/(4\theta_1)$ since $n \geqq 4$. □

We now proceed with the "randomization-of-the-origin" argument. Let $\eta = (\eta_1, \cdots, \eta_n)$ be an n-dimensional random variable, where η_1 has a Cauchy density $\rho^{\frac{1}{2}}/[\pi(1 + \rho\eta_1^2)]$, and $\eta^* = (\eta_2, \cdots, \eta_n)$ has a uniform density on $|\eta^*| \leqq D$ (independent of η_1). Define $\delta_{\rho,D}(x) = E^{\eta}_{\rho,D}[\delta'(x + \eta) - \eta]$. (The η in $E^{\eta}_{\rho,D}$ is to emphasize the variable with respect to which the expectation is to be taken.)

THEOREM 3. *If $n \geqq 4$ and the previous assumptions are satisfied, then the best invariant estimator, δ_0 is inadmissible. Indeed, there exist constants A, ρ, and D such that $R(\delta_0, \theta) - R(\delta_{\rho,D}, \theta) > 0$ for every θ.*

PROOF. Define $\Delta' = R(\delta_0, \theta) - R(\delta_{\rho,D}, \theta) = E_\theta[L(X - \theta) - L(\delta_{\rho,D}(X) - \theta)]$. Clearly $E_\theta L(X - \theta) = E_\theta E^{\eta}_{\rho,D} L([X + \eta] - [\theta + \eta])$, while

$$\begin{aligned}E_\theta L(\delta_{\rho,D}(X) - \theta) &= E_\theta L(E^{\eta}_{\rho,D}[\delta'(X + \eta) - (\theta + \eta)]) \\ &\leqq E_\theta E^{\eta}_{\rho,D} L(\delta'(X + \eta) - (\theta + \eta))\,.\end{aligned}$$

(The last step follows from Jensen's inequality since L is convex.) Hence,

$$\begin{aligned}\Delta'(\theta) &\geqq E_\theta E^{\eta}_{\rho,D}[L([X+\eta]-[\theta+\eta])-L(\delta'(X+\eta)-(\theta+\eta))]\\ &= E^{\eta}_{\rho,D}E_{(\theta+\eta)}[L(\delta_0(X)-(\theta+\eta))-L(\delta'(X)-(\theta+\eta))]\\ &= E^{\eta}_{\rho,D}\Delta_A(\theta+\eta)=E_D{}^{\eta^*}E_\rho{}^{\eta_1}\Delta_A(\theta+\eta)\,.\end{aligned}$$

Breaking up the inner expectation above, gives

$$\Delta'(\theta)\geqq E_D{}^{\eta^*}\left[\int_{-\infty}^{A-1-\theta_1}\frac{\Delta_A(\theta+\eta)\rho^{\frac12}\,d\eta_1}{\pi(1+\rho\eta_1^2)} + \int_{(A-1-\theta_1)}^{(2A-\theta_1)}(\quad)\,d\eta_1+\int_{(2A-\theta_1)}^{\infty}(\quad)\,d\eta_1\right]. \tag{3.1}$$

It is desired to find A, D, and ρ such that the expression in (3.1) is positive for every θ. First note that if $\theta_1 < A-1$, then $\Delta_A(\theta)>0$. This is because $\delta'=\delta_0$ for $x_1<A$, while for $x_1\geqq A$, δ' moves the estimate $\delta_0(x)=x$ closer to θ. (Recall $\gamma_A(\theta)\leqq 0$ and $|\gamma_A(\theta)|<1$). Hence,

$$\int_{-\infty}^{A-1-\theta_1}\frac{\Delta_A(\theta+\eta)\rho^{\frac12}\,d\eta_1}{\pi(1+\rho\eta_1^2)}>0\,. \tag{3.2}$$

By Lemma 3.1, $\Delta_A(\theta+\eta)>0$ for $2A-\theta_1\leqq\eta_1<\infty$. Using this together with (3.1) and (3.2), it is clear that to show that $\Delta'(\theta)>0$, we need only find A, ρ, and D for which

$$E_D{}^{\eta^*}\left[\int_{A-1-\theta_1}^{2A-\theta_1}\frac{|\Delta_A(\theta+\eta)|\rho^{\frac12}\,d\eta_1}{\pi(1+\rho\eta_1^2)}\right]\leqq E_D{}^{\eta^*}\left[\int_{2A-\theta_1}^{\infty}\frac{\Delta_A(\theta+\eta)\rho^{\frac12}\,d\eta_1}{\pi(1+\rho\eta_1^2)}\right].$$

Defining $V=\{\eta^*:|\eta^*-\theta^*|\leqq D\}$, it is clear that the above expression is equivalent to

$$\int_V\int_{(A-1)}^{2A}\frac{|\Delta_A(\eta)|\,d\eta_1\,d\eta^*}{[1+\rho(\eta_1-\theta_1)^2]}\leqq\int_V\int_{2A}^{\infty}\frac{\Delta_A(\eta)\,d\eta_1\,d\eta^*}{[1+\rho(\eta_1-\theta_1)^2]}\,. \tag{3.3}$$

Define

$$\begin{aligned}H_D(\theta)&=\int_V\int_{(A-1)}^{2A}|\Delta_A(\eta)|\,d\eta_1\,d\eta^*\,,\\ G_D{}^T(\theta)&=\int_V\int_{2A}^{T}\Delta_A(\eta)\,d\eta_1\,d\eta^*\,.\end{aligned}$$

It is clear that (3.3) will be verified if we find T, ρ, D, and A for which

$$H_D(\theta)\leqq\frac{\inf_{\eta_1\in(A-1,2A)}[1+\rho(\eta_1-\theta_1)^2]}{\sup_{\eta_1\in(2A,T)}[1+\rho(\eta_1-\theta_1)^2]}G_D{}^T(\theta)\,. \tag{3.4}$$

At this point, the following lemma is needed:

LEMMA 3.2. *There exists a $T>4A$ such that if $D=T^{\frac12}$, then $H_D(\theta)<G_D{}^T(\theta)/6$ for every θ.*

PROOF OF LEMMA. Using the notation of Lemma 3.1, it is clear that $|\Delta_A(\eta)|\leqq|S_A(\eta)|+|\varepsilon_1(\eta)|$. An easy modification of Lemma 2.2.3 gives that if $\eta_1>A-1$, then $|S_A(\eta)|\leqq K\eta_1{}^{\frac{1}{8}}|\gamma(\eta)|$. The same bound can be obtained for $|\varepsilon_1(\eta)|$. Recalling the definition of $\gamma(\eta)$, it is thus clear that

$$H_D(\theta) = \int_V \int_{A-1}^{2A} |\Delta_A(\eta)|\, d\eta_1\, d\eta^* \leqq \int_V \int_{(A-1)}^{2A} \int_{R^{n-1}} \frac{\exp[-|\xi - \eta^*|^2/2\eta_1]\, d\xi\, d\eta_1\, d\eta^*}{\eta_1^{[1+\frac{1}{16}]}(1 + |\xi|^n)}$$
$$= \int_V \int_{(A-1)}^{2A} \int_{R^{n-1}} \frac{\exp[-|\xi|^2/2]\, d\xi\, d\eta_1\, d\eta^*}{\eta_1^{[\frac{17}{16}-(n-1)/2]}(1 + |\xi\eta_1^{\frac{1}{2}} + \eta^*|^n)}\,.$$

For convenience, denote the above integral $\bar{H}_D(\theta)$. By Lemma 3.1, $\Delta_A(\eta) \geqq -\gamma(\eta)/(8\eta_1)$ if $\eta_1 > 2A$. Thus

$$G_D{}^T(\theta) = \int_V \int_{2A}^{T} \Delta_A(\eta)\, d\eta_1\, d\eta^* \geqq \tfrac{1}{8} \int_V \int_{2A}^{T} \int_{R^{n-1}} \frac{\exp[-|\xi|^2/2]\, d\xi\, d\eta_1\, d\eta^*}{\eta_1^{[\frac{17}{8}-(n-1)/2]}(1 + |\xi\eta_1^{\frac{1}{2}} + \eta^*|^n)}\,.$$

Denote the last integral above by $\bar{G}_D(\theta)$. Clearly, to complete the proof of the lemma, it is only necessary to show that $\bar{H}_D(\theta) \leqq \bar{G}_D(\theta)/6$. There are two cases to consider.

CASE 1. $|\theta^*| \leqq T^{\frac{1}{2}} \ln\ln T$. Clearly

$$\bar{H}_D(\theta) \leqq \int_{(A-1)}^{2A} \int_{R^{n-1}} \int_{R^{n-1}} \frac{\exp[-|\xi|^2/2]\, d\eta^*\, d\xi\, d\eta_1}{\eta_1^{[\frac{17}{16}-(n-1)/2]}(1 + |\xi\eta_1^{\frac{1}{2}} + \eta^*|^n)} = K_1 < \infty\,. \tag{3.5}$$

Defining $Q = \{\xi : |\xi\eta_1^{\frac{1}{2}} + \eta^*| \leqq 1\} = \{\xi : |\xi + \eta^*\eta_1^{-\frac{1}{2}}| \leqq \eta_1^{-\frac{1}{2}}\}$, it is clear that since $T > 4A$,

$$\bar{G}_D(\theta) \geqq \tfrac{1}{8} \int_V \int_{T/2}^{T} \int_Q \frac{\exp[-|\xi|^2/2]\, d\xi\, d\eta_1\, d\eta^*}{\eta_1^{[\frac{17}{8}-(n-1)/2]}(1 + |\xi\eta_1^{\frac{1}{2}} + \eta^*|^n)}\,. \tag{3.6}$$

The measure of Q is $K\eta_1^{-(n-1)/2}$ (the volume of the $(n-1)$ sphere of radius $\eta_1^{-\frac{1}{2}}$), which is greater than $KT^{-(n-1)/2}$ over the given region of integration. Furthermore, since $\eta_1 \geqq T/2$ and $|\eta^*| \leqq |\theta^*| + T^{\frac{1}{2}} \leqq T^{\frac{1}{2}}(\ln\ln T + 1)$, it is clear that $\sup_{\xi\in Q} |\xi| \leqq (|\eta^*| + 1)\eta_1^{-\frac{1}{2}} \leqq K' \ln\ln T$. This last fact implies that over the region of integration in (3.6), $\exp[-|\xi|^2/2] \geqq (\ln T)^{-K'/2}$. Hence, recalling the definition of V, it is clear that if $n \geqq 4$, then

$$\bar{G}_D(\theta) \geqq \tfrac{1}{8} \int_V \int_{T/2}^{T} \frac{KT^{-(n-1)/2}(\ln T)^{-K'/2}\, d\eta_1\, d\eta^*}{2\eta_1^{[\frac{17}{8}-(n-1)/2]}} \tag{3.7}$$
$$\geqq K''T^{\frac{3}{8}}(\ln T)^{-K'/2}\,.$$

Combining (3.5) and (3.7), it is clear that if T is large enough, then $\bar{H}_D(\theta) \leqq \bar{G}_D(\theta)/6$, as was to be shown.

Finally, we need to consider

CASE 2. $|\theta^*| \geqq T^{\frac{1}{2}} \ln\ln T$.

We first obtain an upper bound for $\bar{H}_D(\theta)$. Break $\bar{H}_D(\theta)$ up into two integrals, $\bar{H}_{D,1}(\theta)$ and $\bar{H}_{D,2}(\theta)$, consisting of the integrals over $|\xi| \leqq T^{\frac{1}{4}}$ and $|\xi| \geqq T^{\frac{1}{4}}$ respectively. Thus

$$\bar{H}_{D,1}(\theta) = \int_V \int_{(A-1)}^{2A} \int_{|\xi| \leqq T^{\frac{1}{4}}} \frac{\exp[-|\xi|^2/2]\, d\xi\, d\eta_1\, d\eta^*}{\eta_1^{[\frac{17}{16}-(n-1)/2]}(1 + |\xi\eta_1^{\frac{1}{2}} + \eta^*|^n)}\,. \tag{3.8}$$

Note that over the above region of integration, η_1 is bounded, $|\xi| \leqq T^{\frac{1}{4}}$, and

$|\eta^*| \geqq |\theta^*| - T^{\frac{1}{2}} \geqq T^{\frac{1}{2}}(\ln \ln T - 1) \geqq (T^{\frac{1}{2}} \ln \ln T)/2$. Hence, if T is large enough, then $|\xi\eta_1^{\frac{1}{2}} + \eta^*|^n \geqq KT^{n/2}$. Using this bound in (3.8) and integrating gives $\bar{H}_{D,1}(\theta) \leqq KT^{-\frac{1}{2}}$. A simple calculation also shows that $\bar{H}_{D,2}(\theta) \leqq KT^{(n-1)/2} \exp[-T^{\frac{1}{4}}/2]$. Thus for large enough T, we can conclude that

$$\bar{H}_D(\theta) \leqq KT^{-\frac{1}{2}} . \tag{3.9}$$

To obtain a lower bound for $\bar{G}_D(\theta)$, restrict the region of integration to $|\xi| \leqq 1$. Then, since $\eta_1 < T$ and $|\eta^*| \geqq (T^{\frac{1}{2}} \ln \ln T)/2$, it is clear that $|\xi\eta_1^{\frac{1}{2}} + \eta^*|^n \leqq KT^{n/2}$. Using this bound in the expression for $\bar{G}_D(\theta)$ and integrating, gives for $n \geqq 4$,

$$\bar{G}_D(\theta) \geqq KT^{(n-1)/2}T^{-n/2}T^{\frac{3}{8}} = KT^{-\frac{1}{8}} . \tag{3.10}$$

Combining (3.9) and (3.10), it is thus clear that if T is large enough, then $\bar{H}_D(\theta) < \bar{G}_D(\theta)/6$. □

The final step of the proof of Theorem 3 is to prove

LEMMA 3.3. *If* $\rho = T^{-2}$, *then*

$$W = \frac{\inf_{\eta_1 \in (A-1,2A)} [1 + \rho(\eta_1 - \theta_1)^2]}{\sup_{\eta_1 \in (2A,T)} [1 + \rho(\eta_1 - \theta_1)^2]} \geqq \tfrac{1}{6} .$$

PROOF OF LEMMA. Three cases must be considered.

CASE 1. $\theta_1 \leqq 0$. Clearly $\inf_{\eta_1 \in (A-1,2A)} [1 + \rho(\eta_1 - \theta_1)^2] = 1 + \rho(A - 1 - \theta_1)^2$, while $\sup_{\eta_1 \in (2A,T)} [1 + \rho(\eta_1 - \theta_1)^2] \leqq 1 + \rho(T - \theta_1)^2$. Hence,

$$W \geqq \frac{1 + \rho(A - 1 - \theta_1)^2}{1 + \rho(T - \theta_1)^2} \geqq \frac{1 + \rho(A - 1)^2}{1 + \rho T^2} \geqq \frac{1}{1 + T^{-2}T^2} = \tfrac{1}{2} .$$

CASE 2. $0 < \theta_1 < 4A$. Clearly $\inf_{\eta_1 \in (A-1,2A)} [1 + \rho(\eta_1 - \theta_1)^2] \geqq 1$, while $\sup_{\eta_1 \in (2A,T)} [1 + \rho(\eta_1 - \theta_1)^2] \leqq 1 + \rho(T - \theta_1)^2 \leqq 1 + \rho T^2 = 2$. Hence $W \geqq \frac{1}{2}$.

CASE 3. $\theta_1 \geqq 4A$. Clearly $\inf_{\eta_1 \in (A-1,2A)} [1 + \rho(\eta_1 - \theta_1)^2] \geqq 1 + \rho\theta_1^2/4$, while $\sup_{\eta_1 \in (2A,T)} [1 + \rho(\eta_1 - \theta_1)^2] \leqq 1 + \rho\theta_1^2 + \rho\eta_1^2 \leqq 1 + \rho\theta_1^2 + T^{-2}T^2 \leqq 2 + \rho\theta_1^2$. Hence, $W \geqq (1 + \rho\theta_1^2/4)/(2 + \rho\theta_1^2) \geqq \frac{1}{4}$. □

In conclusion, Lemmas 3.2 and 3.3 establish that if T is large enough, $D = T^{\frac{1}{2}}$, and $\rho = T^{-2}$, then

$$H_D(\theta) \leqq G_D{}^T(\theta)/6 \leqq WG_D{}^T(\theta) .$$

This establishes line (3.4) and hence proves the theorem. □

4. Generalizations and conclusions.

1. The obvious generalization of interest is the generalization to the multi-observational situation. It should be possible to do this by the usual method of conditioning on the maximal invariant. (See Farrell (1964) or Brown (1966).) One proves inadmissibility in the conditional problems, and hence inadmissibility in the full problem (modulo the not too difficult task of showing that the conditional "improved estimators" can be combined in a measurable way).

2. Most of the results should carry over to the case where the distribution of X does not have a density w.r.t. Lebesgue measure.

3. Generalizations to nonconvex loss are harder than the above generalizations. For the best invariant estimator, we believe nonconvex loss could be handled along the lines of Chapter 3, using a somewhat more involved "randomization-of-the-origin" argument. For generalized Bayes estimators, however, it is not clear how to deal with convex loss.

4. The need for so many moments (12 in the squared error loss, best invariant estimator case) is unattractive. Most of these moments were needed for the proofs of the technical lemmas in Section 2.2. One would hope to be able to significantly reduce the number of moments needed by using more delicate arguments in the proofs of these lemmas.

5. From a practical point of view, the estimators we found which improve upon δ_F are not very significant, since they are complicated and yield such a small level of improvement. Hopefully, however, they will provide insight in the search for significantly better estimators.

APPENDIX

PROOF OF LEMMA 2.2.2. The inequality is obvious for $x_1 \leqq A$. Hence assume that $x_1 > A$. Note first that if $x_1 > 0$ and $r > 0$, then

$$(2.2.1)\qquad \begin{array}{lll} r^m \exp[-r^2/2x_1] & \text{is increasing in } r & \text{for } r < (mx_1)^{\frac{1}{2}} \\ & \text{is decreasing in } r & \text{for } r > (mx_1)^{\frac{1}{2}}, \end{array}$$

and hence is maximized at $r = (mx_1)^{\frac{1}{2}}$ with a maximum value of $(mx_1)^{m/2}e^{-m/2}$.

For notational convenience, define (for fixed $0 < \lambda_1 < \frac{1}{2}$)

$$\begin{aligned} Q_1 &= \{x : |x^*| \leqq 3[(m+1)x_1]^{\frac{1}{2}}\}, \\ Q_2 &= \{x : |x^*| \geqq [(1+\lambda_1)(x_1-1)\ln x_1]^{\frac{1}{2}}\}, \\ Q_3 &= \{x : 2[(m+1)x_1]^{\frac{1}{2}} \leqq |x^*| \leqq [(1+\lambda_1)(x_1+1)\ln x_1]^{\frac{1}{2}}\}, \\ \varphi_m(\xi, x) &= |\xi|^m \exp[-|\xi|^2/2x_1]/\{x_1^{[m/2+(n-1)/2]}(1+|x^*+\xi|^n)\}. \end{aligned}$$

STEP 1. We first show that there exist A_0 and $K > 0$ such that if $A > A_0$, then

$$(2.2.2)\qquad \begin{array}{ll} H_{0,A}(x) \geqq Kx_1^{-(n-1)/2} & \text{if } x \in Q_1 \\ H_{0,A}(x) \geqq K|x^*|^{-n} & \text{if } x \in Q_2 \\ H_{0,A}(x) \geqq Kx_1^{-(n-1)/2} \exp[-|x^*|^2/2x_1] & \text{if } x \in Q_3. \end{array}$$

To verify this, note that

$$\begin{aligned} H_{0,A}(x) = \int \varphi_0(\xi, x)\, d\xi &\geqq \int_{\{\xi : |\xi+x^*|<\frac{1}{2}\}} \varphi_0(\xi, x)\, d\xi \\ &\geqq 2^{-1} \exp[-(|x^*|+\tfrac{1}{2})^2/2x_1]x_1^{-(n-1)/2}. \end{aligned}$$

The bounds in (2.2.2) for the regions Q_1 and Q_3 follow immediately from this.

For the region Q_2, note that

$$\begin{aligned} H_{0,A}(x) &\geqq \int_{\{\xi:|\xi|<|x^*|\}} \varphi_0(\xi, x)\, d\xi \\ &\geqq [1 + (2|x^*|)^n]^{-1} \int_{\{\xi:|\xi|<|x^*|\}} \exp[-|\xi|^2/2x_1] x_1^{-(n-1)/2}\, d\xi \\ &\geqq K|x^*|^{-n} \int_{\{\xi:|\xi|<|x^*|x_1^{-\frac{1}{2}}\}} \exp[-|\xi|^2/2]\, d\xi \\ &\geqq K|x^*|^{-n} \int_{\{\xi:|\xi|<(\ln x_1)^{\frac{1}{2}}\}} \exp[-|\xi|^2/2]\, d\xi \geqq K'|x^*|^{-n} . \end{aligned}$$

Clearly K and K' above can be chosen positive. Thus step 1 is verified.

STEP 2. Let $0 < \lambda_2 < \frac{1}{2}$ be fixed. We show that there exist A_0 and $K > 0$ such that if $A > A_0$, then

$$(2.2.3) \qquad \begin{aligned} H_{m,A}(x) &\leqq K x_1^{-(n-1)/2} && \text{if } x \in Q_1 \\ H_{m,A}(x) &\leqq K|x^*|^{-n} && \text{if } x \in Q_2 \\ H_{m,A}(x) &\leqq \frac{K}{x_1^{n/2}} + \frac{K|x^*|^m \exp[-|x^*|^2(1-\lambda_2)/2x_1]}{x_1^{[m/2+(n-1)/2]}} && \text{if } x \in Q_3 . \end{aligned}$$

To verify this, consider first the region Q_1. From (2.2.1) it is clear that

$$H_{m,A}(x) \leqq \frac{(mx_1)^{m/2} e^{-m/2}}{x_1^{[m/2+(n-1)/2]}} \int \frac{d\xi}{(1 + |x^* + \xi|^n)} \leqq K x_1^{-(n-1)/2} .$$

Consider next the region Q_3. Break up $H_{m,A}(x)$ into the integrals over the regions $P = \{\xi : |\xi| < (1-\lambda_2)^{\frac{1}{2}}|x^*|\}$ and P^c. Consider first the integral over P. If ξ is in P, there clearly exists $K > 0$ such that $|x^* + \xi| \geqq K|x^*|$. Hence,

$$(2.2.4) \qquad \begin{aligned} \int_P \varphi_m(\xi, x)\, d\xi &\leqq \frac{1}{1 + (K|x^*|)^n} \int \frac{|\xi|^m \exp[-|\xi|^2/2x_1]\, d\xi}{x_1^{[m/2+(n-1)/2]}} \\ &\leqq K'|x^*|^{-n} \int |\xi|^m \exp[-|\xi|^2/2]\, d\xi \leqq K'' x_1^{-n/2} . \end{aligned}$$

Next, consider the integral over P^c. If $\xi \in P^c$ and $x \in Q_3$, then $|\xi| > (1 - \lambda_2)^{\frac{1}{2}} 2[(m+1)x_1]^{\frac{1}{2}} > [mx_1]^{\frac{1}{2}}$. Hence, $|\xi|^m \exp[-|\xi|^2/2x_1]$ is decreasing in $|\xi|$ (by 2.2.1). Therefore,

$$(2.2.5) \qquad \begin{aligned} &\int_{P^c} \varphi_m(\xi, x)\, d\xi \\ &\quad\leqq \frac{[(1-\lambda_2)^{\frac{1}{2}}|x^*|]^m \exp[-|x^*|^2(1-\lambda_2)/2x_1]}{x_1^{[m/2+(n-1)/2]}} \int \frac{d\xi}{1 + |x^* + \xi|^n} . \end{aligned}$$

Notice that the last integral above is a finite constant. Hence (2.2.4) and (2.2.5) establish that (2.2.3) is true if $x \in Q_3$.

Finally, consider the region Q_2. Break up $H_{m,A}(x)$ into the integrals over the regions $T = \{\xi : |\xi| < (1 - \lambda_1/2)^{\frac{1}{2}}|x^*|\}$ and T^c. As in line (2.2.4), one obtains

$$(2.2.6) \qquad \int_T \varphi_m(\xi, x)\, d\xi \leqq K|x^*|^{-n} .$$

A calculation similar to (2.2.5) gives

$$(2.2.7) \qquad \int_{T^c} \varphi_m(\xi, x)\, d\xi \leqq \frac{K|x^*|^m \exp[-|x^*|^2(1-\lambda_1/2)/2x_1]}{x_1^{[m/2+(n-1)/2]}} .$$

Note next that if $x \in Q_2$ and if A_0 is chosen large enough (recall $x_1 > A > A_0$), then $|x^*| \geqq [(x_1 - 1)\ln x_1]^{\frac{1}{2}} \geqq [(m+n)x_1]^{\frac{1}{2}}$. Hence, by line (2.2.1) it is clear that

$$|x^*|^{(m+n)} \exp[-|x^*|^2(1-\lambda_1/2)/2x_1] \leqq K[(\ln x_1)x_1]^{(m+n)/2} \exp[-(\ln x_1)(1+\lambda_1)(1-\lambda_1/2)/2] .$$

Using this together with (2.2.7) gives

$$\int_{T^c} \varphi_m(\xi, x)\, d\xi \leqq \frac{K(\ln x_1)^{(m+n)/2} x_1^{-[(1+\lambda_1)(1-\lambda_1/2)/2]}}{|x^*|^n x_1^{-\frac{1}{2}}} \leqq K|x^*|^{-n} . \tag{2.2.8}$$

(The last step follows since $(1+\lambda_1)(1-\lambda_1/2) > 1$.) Combining (2.2.6) and (2.2.8) completes the verification of Step 2.

STEP 3. From (2.2.2) and (2.2.3), it is clear that $H_{m,A}(x) \leqq KH_{0,A}(x)$ for x in Q_1 or Q_2. For x in Q_3, (2.2.2) and (2.2.3) give

$$\begin{aligned}\frac{H_{m,A}(x)}{H_{0,A}(x)} &\leqq \frac{Kx_1^{-n/2} + K|x^*|^m \exp[-|x^*|^2(1-\lambda_2)/2x_1]x_1^{-[m/2+(n-1)/2]}}{x_1^{-(n-1)/2}\exp[-|x^*|^2/2x_1]} \\ &= Kx_1^{-\frac{1}{2}}\exp[|x^*|^2/2x_1] + K|x^*|^m x_1^{-m/2}\exp[|x^*|^2\lambda_2/2x_1] \\ &\leqq K'x_1^{-\frac{1}{2}}\exp[(\ln x_1)x_1(1+\lambda_1)/2x_1] \\ &\quad + K'x_1^{-m/2}[(1+\lambda_1)x_1 \ln x_1]^{m/2}\exp[(1+\lambda_1)\lambda_2 x_1(\ln x_1)/2x_1] \\ &\leqq K''[x_1^{\lambda_1/2} + x_1^{\lambda_2}] .\end{aligned}$$

Choosing $\lambda_1 = \lambda/2$ and $\lambda_2 = \lambda/4$, the proof of the lemma is complete. □

PROOF OF LEMMA 2.2.3. If $m > 0$, then $t_1 = t_2 = t_3$, and hence $\nu = \omega = \tau$. Note also that if $\theta_1 > A$ and $x_1 > A$, then $\omega = t_2 x_1 + (1-t_2)\theta_1 > A$. Under these conditions, and using Lemma 2.2.2, it is clear that

$$\begin{aligned}\int_{R^{n-1}} \frac{|\xi|^m \exp[-|\xi|^2/2\omega_1]\, d\xi}{\nu_1^{[2(1+\lambda)+m/2]}(1+|\tau^*+\xi|^n)} &= H_{m,A}(\omega)\omega_1^{-[2(1+\lambda)-(n-1)/2]} \\ &\leqq K\omega_1^{\lambda/4} H_{0,A}(\omega)\omega_1^{-[2(1+\lambda)-(n-1)/2]} \\ &= K\int_{R^{n-1}} \frac{\exp[-|\xi|^2/2\omega_1]\, d\xi}{\omega_1^{[2+7\lambda/4]}(1+|\omega^*+\xi|^n)} .\end{aligned}$$

Letting $a = 2 + 7\lambda/4$, it is thus clear that under both situations of the lemma,

$$Q_A(\theta) \leqq K\int_A^\infty \int_{R^{n-1}} \int_{R^{n-1}} \frac{\exp[-|\xi|^2/2\omega_1]\, d\xi}{\nu_1^a(1+|\tau^*+\xi|^n)} |L^{\langle i\rangle}(x_1-\theta_1)|\,|x-\theta|^j f(x-\theta)\, dx^*\, dx_1 .$$

Changing orders of integration and making the change of variables $z = x - \theta$ gives

$$Q_A(\theta) \leqq K\int_{R^{n-1}} \int_{(A-\theta_1)}^\infty \int_{R^{n-1}} \frac{e^{-|\xi|^2/2(\theta_1+t_2 z_1)}|L^{\langle i\rangle}(z_1)|\,|z|^j f(z)\, dz^*\, dz_1\, d\xi}{[\theta_1+t_1 z_1]^a(1+|\theta^*+t_3 z^*+\xi|^n)} . \tag{2.2.10}$$

Consider first the inner integral above. Break this up into integrals over U and U^c, where $U = \{z^* : t_3|z^*| \leqq (1+|\theta^*+\xi|)/2\}$. Call these integrals I_1 and I_2 respectively.

To deal with I_1, we first show

$$(1 + |\theta^* + t_3 z^* + \xi|^n)^{-1} \leqq K(1 + |\theta^* + \xi|^n)^{-1} \quad \text{if} \quad z^* \in U . \tag{2.2.11}$$

To see this, note that if $|\theta^* + \xi| < 2$, then

$$(1 + |\theta^* + t_3 z^* + \xi|^n)^{-1} \leqq 1 \leqq (1 + 2^n)(1 + |\theta^* + \xi|^n)^{-1} . \tag{2.2.12}$$

On the other hand, if $|\theta^* + \xi| \geqq 2$ and $z^* \in U$, then $t_3|z^*| \leqq |\theta^* + \xi|$. Hence

$$\begin{aligned} |\theta^* + t_3 z^* + \xi|^n &\geqq (|\theta^* + \xi| - t_3|z^*|)^n \\ &\geqq [|\theta^* + \xi| - (1 + |\theta^* + \xi|)/2]^n \\ &= [(|\theta^* + \xi| - 1)/2]^n \geqq |\theta^* + \xi|^n/4^n . \end{aligned} \tag{2.2.13}$$

Combining (2.2.12) and (2.2.13) verifies (2.2.11). Using (2.2.11) gives

$$I_1 \leqq K \int_{R^{n-1}} \frac{\exp[-|\xi|^2/2(\theta_1 + t_2 z_1)]|L^{\langle i \rangle}(z_1)|\,|z|^j f(z)\,dz^*}{[\theta_1 + t_1 z_1]^a(1 + |\theta^* + \xi|^n)} .$$

For I_2, a Chebyshev argument gives the above bound with $|z|^{j+n}$ in the integrand in place of $|z|^j$. Using this bound in (2.2.10) and interchanging orders of integration gives that $Q_A(\theta)$ is bounded by

$$K \int\int\int_{(A-\theta_1)}^{\infty} \frac{\exp[-|\xi|^2/2(\theta_1 + t_2 z_1)]}{(\theta_1 + t_1 z_1)^a(1 + |\theta^* + \xi|^n)} (|z|^j + |z|^{j+n})|L^{\langle i \rangle}(z_1)| f(z)\,dz_1\,d\xi\,dz^* .$$

Break the above integral up into $Q_{A1} + Q_{A2}$, where Q_{A1} is the integral over the region $V = \{z : A - \theta_1 \leqq z_1 \leqq 0\}$, and Q_{A2} is the integral over the region $\{z : z_1 > 0\}$.

Consider first $Q_{A1}(\theta)$. If $z_1 < 0$, then $\theta_1 + t_2 z_1 < \theta_1$. Hence for $z \in V$, it is clear that $\exp[-|\xi|^2/2(\theta_1 + t_2 z_1)] \leqq \exp[-|\xi|^2/2\theta_1]$. This implies that

$$Q_{A1}(\theta) \leqq \int\int\int_{(A-\theta_1)}^{0} \frac{\exp[-|\xi|^2/2\theta_1]|L^{\langle i \rangle}(z_1)|(|z|^j + |z|^{j+n}) f(z)\,dz_1\,d\xi\,dz^*}{(\theta_1 + t_1 z_1)^a(1 + |\theta^* + \xi|^n)} . \tag{2.2.14}$$

Break this last integral up further into two integrals over the regions $V_1 = V \cap \{z : t_1|z_1| < \theta_1/2\}$, and $V_2 = V \cap \{z : t_1|z_1| \geqq \theta_1/2\}$. A similar argument to that following (2.2.10) gives that these integrals, and hence $Q_{A1}(\theta)$, are bounded by

$$K \int_{R^{n-1}} \frac{\exp[-|\xi|^2/2\theta_1]\,d\xi}{\theta_1^a(1 + |\theta^* + \xi|^n)} \int_{R^n} (|z|^j + |z|^{j+n})(1 + |z|^a)|L^{\langle i \rangle}(z_1)| f(z)\,dz .$$

The integral over R^n, above, is finite by assumption (2) of Section 2.1. Hence, recalling the definition of $\gamma_{A,\lambda}(\theta)$, it is clear that

$$Q_{A1}(\theta) \leqq K|\gamma_{A,\lambda}(\theta)|\theta_1^{-(1+3\lambda/4)} \leqq K|\gamma_{A,\lambda}(\theta)|\theta_1^{-(1+\lambda/4)} . \tag{2.2.15}$$

Finally, we must consider $Q_{A2}(\theta)$. By definition, the region of integration w.r.t. z_1 is $\{z_1 > 0\}$. Hence $(\theta_1 + t_1 z_1)^{-a} < \theta_1^{-a}$ and $\exp[-|\xi|^2/2(\theta_1 + t_2 z_1)] \leqq \exp[-|\xi|^2/2(\theta_1 + z_1)]$. Thus $Q_{A2}(\theta)$ is bounded by

$$\frac{K}{\theta_1^a} \int_{R^{n-1}} \int_{R^{n-1}} \int_0^{\infty} \frac{\exp[-|\xi|^2/2(\theta_1 + z_1)]}{(1 + |\theta^* + \xi|^n)} |L^{\langle i \rangle}(z_1)|(|z|^j + |z|^{j+n}) f(z)\,dz_1\,dz^*\,d\xi .$$

Making the change of variables $\eta = \xi(\theta_1 + z_1)^{-\frac{1}{2}}$, it is clear that the above integral equals

$$\frac{K}{\theta_1^a} \int_{R^{n-1}} \int_{R^{n-1}} \int_0^\infty \frac{\exp[-|\eta|^2/2](\theta_1 + z_1)^{(n-1)/2}}{(1 + |\theta^* + \eta(\theta_1 + z_1)^{\frac{1}{2}}|^n)} |L^{\langle i \rangle}(z_1)|(|z|^j + |z|^{j+n})f(z)\, dz_1\, dz^*\, d\xi .$$

Break this up into $Q_{A3} + Q_{A4}$, where Q_{A3} is the integral over the region $W_1 = \{z_1 : 0 \leqq z_1^{\frac{1}{2}} \leqq (1 + |\eta\theta_1^{\frac{1}{2}} + \theta^*|)/2|\eta|\}$, and Q_{A4} is the integral over $W_2 = \{z_1 : z_1^{\frac{1}{2}} > (1 + |\eta\theta_1^{\frac{1}{2}} + \theta^*|)/2|\eta|\}$.

Consider Q_{A4} first. A simple Chebyshev argument shows that

$$\text{(2.2.16)} \qquad Q_{A4}(\theta) \leqq \frac{K}{\theta_1^a} \int_{R^{n-1}} \int_{R^1} \int_{R^{n-1}} \frac{|\eta|^n \exp[-|\eta|^2/2]\, d\eta}{(1 + |\theta^* + \eta\theta_1^{\frac{1}{2}}|^n)} (\theta_1 + z_1)^{(n-1)/2} |L^{\langle i \rangle}(z_1)| \times [|z|^{(j+n/2)} + |z|^{(j+3n/2)}] f(z)\, dz_1\, dz^* .$$

Note that a change of variables and Lemma 2.2.2 verify that

$$\text{(2.2.17)} \qquad \begin{aligned} &\int_{R^{n-1}} \frac{|\eta|^n \exp[-|\eta|^2/2]\, d\eta}{(1 + |\theta^* + \eta\theta_1^{\frac{1}{2}}|^n)} \\ &\quad = \int_{R^{n-1}} \frac{|\xi|^n \exp[-|\xi|^2/2\theta_1]\, d\xi}{\theta_1^{(n-\frac{1}{2})}(1 + |\theta^* + \xi|^n)} \\ &\quad \leqq K \int_{R^{n-1}} \frac{\exp[-|\xi|^2/2\theta_1]\, d\xi}{\theta_1^{[(n-1)/2 - \lambda/4]}(1 + |\theta^* + \xi|^n)} = K|\gamma(\theta)|\theta_1^{-[(n-3)/2 - 5\lambda/4]} . \end{aligned}$$

Also, since $\theta_1 > A$, it is clear that

$$\text{(2.2.18)} \qquad (\theta_1 + z_1)^{(n-1)/2}\theta_1^{-(n-1)/2} \leqq K(1 + z_1^{(n-1)/2}) .$$

Applying the bounds (2.2.17) and (2.2.18) to the expression (2.2.16), it is clear that

$$Q_{A4}(\theta) \leqq K|\gamma(\theta)|\theta_1^{-[(n-3)/2 - 5\lambda/4 + a]} \int_{R^n} (1 + z_1^{(n-1)/2})|L^{\langle i \rangle}(z_1)| \times [|z|^{(j+n/2)} + |z|^{(j+3n/2)}] f(z)\, dz .$$

Recalling that $a = 2 + 7\lambda/4$, and again using assumption (2) of Section 2.1 to bound the integral over R^n above, it is clear that $Q_{A4}(\theta) \leqq K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)}$.

We finally have to consider $Q_{A3}(\theta)$. To do so, it is necessary to prove that

$$\text{(2.2.19)} \qquad (1 + |\theta^* + (\theta_1 + z_1)^{\frac{1}{2}}\eta|^n)^{-1} \leqq K(1 + |\theta^* + \theta_1^{\frac{1}{2}}\eta|^n)^{-1} \qquad \text{if} \quad z_1 \in W_1 .$$

Note that

$$\text{(2.2.20)} \qquad |\theta^* + (\theta_1 + z_1)^{\frac{1}{2}}\eta| = |(\theta^* + \theta_1^{\frac{1}{2}}\eta) + [(1 + \theta_1/z_1)^{\frac{1}{2}} - (\theta_1/z_1)^{\frac{1}{2}}]z_1^{\frac{1}{2}}\eta| .$$

Since $(1 + c)^{\frac{1}{2}} - c^{\frac{1}{2}} \leqq 1$ if $c > 0$, and since $z_1 \in W_1$, it is clear that

$$\text{(2.2.21)} \qquad |[(1 + \theta_1/z_1)^{\frac{1}{2}} - (\theta_1/z_1)^{\frac{1}{2}}]z_1^{\frac{1}{2}}\eta| \leqq z_1^{\frac{1}{2}}|\eta| \leqq (1 + |\theta_1^{\frac{1}{2}}\eta + \theta^*|)/2 .$$

Using (2.2.20) and (2.2.21), an argument exactly analogous to that following (2.2.11) verifies (2.2.19).

Using (2.2.19) to bound $Q_{A3}(\theta)$ and proceeding with by now familiar arguments, gives $Q_{A3}(\theta) \leqq K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)}$.

In conclusion,

$$Q_A(\theta) = Q_{A1}(\theta) + Q_{A3}(\theta) + Q_{A4}(\theta) \leqq K|\gamma(\theta)|\theta_1^{-(1+\lambda/4)} . \qquad \square$$

PROOF OF LEMMA 2.2.4. Note that $|\gamma_A(\theta)| = H_{0,A}(\theta)\theta_1^{[(n-1)/2-(1+\lambda)]}$. Defining $\beta = c'(4A)^{-\alpha}$, it is clear that if A_0 is large enough, then

$$\rho(\theta) = \frac{|\gamma_A((4A - \beta, \theta^*))|}{|\gamma_A(\theta)|} = \frac{H_{0,A}((4A - \beta, \theta^*))}{H_{0,A}(\theta)}\left[\frac{4A - \beta}{\theta_1}\right]^{[(n-1)/2-(1+\lambda)]} \leqq \frac{2H_{0,A}((4A - \beta, \theta^*))}{H_{0,A}(\theta)} . \tag{2.2.22}$$

CASE 1. Assume $|\theta^*| < 5A^{\frac{1}{2}}$. Then $|\theta^*| \leqq 3(4A - \beta)^{\frac{1}{2}}$ and $|\theta^*| \leqq 3\theta_1^{\frac{1}{2}}$. (Recall $\theta_1 \geqq 4A - c/A$.) Hence (2.2.2) and (2.2.3) can be applied to $H_{0,A}(\theta)$ and $H_{0,A}((4A - \beta, \theta^*))$ in (2.2.22) to get $\rho(\theta) \leqq K[\theta_1/(4A - \beta)]^{(n-1)/2}$. The result follows.

CASE 2. Assume $5A^{\frac{1}{2}} \leqq |\theta^*| \leqq [(1 + \lambda_1)4A \ln (4A)]^{\frac{1}{2}}$. (Here λ_1 is from Lemma 2.2.2.) Again, it is easy to check that (2.2.2) and (2.2.3) can be applied to (2.2.22). The result is

$$\rho(\theta) \leqq K[\theta_1/(4A - \beta)]^{(n-1)/2} \exp [-|\theta^*|^2((4A - \beta)^{-1} - \theta_1^{-1})/2] . \tag{2.2.23}$$

Note next that $(4A-\beta)^{-1} = (4A)^{-1}[1 + (\beta/4A) + (\beta/4A)^2 + \cdots] \geqq 1/4A + \beta/(16A^2)$. Similarly, if A_0 (and hence A) is large enough, $\theta_1^{-1} \leqq (4A - c/A)^{-1} \leqq 1/4A + c/(8A^3)$. Hence,

$$[(4A - \beta)^{-1} - \theta_1^{-1}] \geqq (8A^2)^{-1}(\beta/2 - c/A) . \tag{2.2.24}$$

Note also, that for θ^* in the given region,

$$|\theta^*|^2/(4A^2) \leqq [K \ln (4A)]/(4A) \to 0 \qquad \text{as } A_0 \to \infty . \tag{2.2.25}$$

Combining (2.2.23), (2.2.24), and (2.2.25), the result follows.

CASE 3. Assume $|\theta^*| \geqq [(1 + \lambda_1)4A \ln (4A)]^{\frac{1}{2}}$. Again (2.2.2) and (2.2.3) apply, and yield the desired conclusion directly. $\square$

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