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EMPIRICAL BAYES ESTIMATION WITH CONVERGENCE RATES IN NONCONTINUOUS LEBESGUE EXPONENTIAL FAMILIES

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Empirical Bayes estimators, asymptotically optimal with rates, are proposed. In the component problem there is a pair (X, ω) of real valued random variables. The Lebesgue density of X, conditional on ω, is of the form $u(x)C(\omega)e^{\omega x}$. Based on a realization of X, the problem is squared error loss estimation of ω. Let G be a prior distribution on ω, and $R(G)$ be the Bayes optimal risk wrt G.

Using $(X_1, \cdots, X_n)$, the observations in the past n such problems, mean square consistent estimators of the derivative of $\log(\int C(\omega)e^{\omega x}\,dG(\omega))$ are proposed. Then these statistics and the present observation X are used to exhibit estimators ψ_n for the present problem whose risks R_n converge to the Bayes optimal risk $R(G)$ as $n \to \infty$. In particular, with no assumption on the smoothness or on the form of u, a ψ_n for each γ in [0,2) is exhibited. Sufficient conditions are given under which $c_1 n^{-4/(4+3\gamma)} \leqq R_n - R(G) \leqq c_2 n^{-2\gamma/(4+3\gamma)}$, where c_1 and c_2 are positive constants. The rhs inequality holds *uniformly* in G with support in a bounded interval of the real line, while the other holds for a G degenerate at a point and for all n sufficiently large. (Thus with γ close to 2, ψ_n achieves almost the exact rate.) Examples of families, including one whose u function has infinitely many discontinuities, are given where conditions for the above inequalities are satisfied for γ arbitrarily close to 2.

1. Introduction. Empirical Bayes (EB) problems have been described and discussed in great detail in the literature; for examples see Robbins [12], [13], Johns [5], and Johns and Van Ryzin [6]. (Examples where EB solutions are applicable are discussed in the highly illustrative paper by Neyman [10].) Johns [5], Robbins [13], Samuel [14] and Hannan and Macky [3] have exhibited EB procedures for certain problems which are *asymptotically optimal* (a.o.) in the sense that the risk for the nth decision problem converges to the Bayes optimal risk (which would have been obtained if the prior distribution involved were *known* and the best procedure based on this knowledge were used). *If the rate of such convergence is of order $O(n^{-\delta})$ for a $\delta > 0$ we will say that the* EB *procedure is* a.o. *with a rate* δ. The usefulness of an EB procedure clearly depends on the rate of the asymptotic optimality of the procedure. With this view in mind, in some of the recent papers on the subject attention has been paid to the problem of exhibiting EB procedures a.o. with rates (e.g., see Johns and Van Ryzin ([6], [7]) and Yu ([17], Chapters 1 and 2)).

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Yu ([17], Chapter 2) exhibited EB estimators in squared error loss estimation (SELE) of the natural parameter in Lebesgue-exponential families. His estimators are a.o. with a rate. But his rate depends on what order of the derivative of the Lebesgue-density involved exists, and is *less than* $\frac{1}{3}$ no matter how smooth is this density. Moreover, in the absence of the existence of the second order derivative of this density, his estimators may not even be a.o. Very recently, O'Bryan and Susarla [11] exhibited estimators in EB SELE (with varying sample size) of the mean in the normal family with variance unity, which are a.o. with rates *less than* $\frac{1}{3}$ under the assumption that the support of the prior distribution is in [0, 1]. No EB estimator a.o. with a rate $\frac{1}{3}$ as yet is exhibited in a SELE problem even in a very special family of Lebesgue-densities.

The object of this paper is to exhibit EB estimators in SELE in Lebesgue-exponential families, and to show, with no assumption on the smoothness of the Lebesgue-density involved, that the estimators are a.o. with a rate arbitrarily close to $\frac{2}{5}$. In particular, for each $0 \leqq \gamma < 2$ an EB estimator is exhibited. Theorem 4.1 gives sufficient conditions under which the estimator is a.o. with a rate $2\gamma/(4+3\gamma)$ *uniformly* over the class of all priors whose supports are in a bounded interval of the *real line* R. Examples of families, including one whose Lebesgue-density has infinitely many discontinuity points, are given where conditions of the theorem are satisfied for γ arbitrarily close to 2. Theorem 5.1 shows that the best possible rate of the asymptotic optimality of the estimator is $4/(4+3\gamma)$.

The model we will be considering is the following. Let μ be a σ-finite measure dominated by the Lebesgue measure on R. With $\Omega = \{\omega \in R \mid C(\omega) = (\int e^{\omega x}\, d\mu(x))^{-1} > 0\}$, let $\mathscr{P} = \{P_\omega \mid \omega \in \Omega\}$ be a family of probability measures on Borel subsets of R such that $\mathscr{P} \ll \mu$. For $P_\omega \in \mathscr{P}$, let

$$f_\omega(x) = C(\omega)e^{\omega x} \tag{1.1}$$

be a fixed determination of $dP_\omega/d\mu$. With G (an unknown) prior distribution on Ω, let the component problem be SELE of $\omega \sim G$ based on a realization of a random variable $X \sim P_\omega \in \mathscr{P}$. Hence the unconditional μ-density of X is

$$f(x) = \int f_\omega(x)\, dG(\omega) \tag{1.2}$$

and the Bayes estimate against G is

$$\psi_G(X) = \frac{\int \omega f_\omega(X)\, dG}{f(X)}\,. \tag{1.3}$$

2. The empirical Bayes approach. In the EB estimation case an estimator ψ_n for the present problem is based on a sequence of past observations $\mathbf{X}_n = (X_1, \cdots, X_n)$ and the present observation $X_{n+1} = X$, where $X_j \sim P_{\omega_j} \in \mathscr{P}$ are independent and ω_j are i.i.d. according to G. (Thus the present problem is essentially the $(n+1)$st problem. Note that X_j are i.i.d. with μ-density $f(x)$.) Abbreviating ω_{n+1} to ω, let P stand for the product measure on $(X_1, \omega_1), \cdots, (X_n, \omega_n), (X, \omega)$. For a measure ξ, denote $\int \cdot\, d\xi$ by $\xi\cdot$ or $\xi(\cdot)$. Then the risk of ψ_n (i.e. the risk in the $(n+1)$st problem) is

$$R_n = P(\psi_n(X) - \omega)^2 . \tag{2.1}$$

For a function $g(\omega)$, we note that $P_{\mathbf{X}_n, X}(g(\omega)) = P_X(g(\omega))$, where $P_X(g(\omega))$ stands for the conditional (on X) expectation of $g(\omega)$. Thus ψ_G continues to be Bayes in the EB problem, and if $\psi_n - \psi_G \in L_2(P)$, then $P(\psi_n - \psi_G)(\psi_G - \omega) = 0$ and

$$0 \leqq R_n - R(G) = P(\psi_n(X) - \psi_G(X))^2 , \tag{2.2}$$

where $R(G)$ is the Bayes risk (against G) in the component problem. Thus the problem of exhibiting an a.o. EB estimator is equivalent to the problem of exhibiting a ψ_n such that $P(\psi_n - \psi_G)^2 \to 0$.

Hereinafter, we restrict μ by requiring that $\exists$ a determination u of $d\mu/dx$ and an $a \geqq -\infty$ such that

$$u(x) > 0 \qquad \text{iff} \quad x > a . \tag{2.3}$$

For $\alpha \leqq \beta$ finite numbers in Ω (note that the log convexity of $\int e^{\omega x} \, d\mu(x)$ implies that $[\alpha, \beta] \subseteq \Omega$), let $\mathscr{G}$ be the class of all priors on Ω with supports in $[\alpha, \beta]$. We restrict G by requiring that $G \in \mathscr{G}$. Since $P[X_j \leqq a] = 0 = P\{\omega_j \notin [\alpha, \beta]\}$ arguments of f and u below are in $[x > a]$ and $\omega_j \in [\alpha, \beta]$.

3. The proposed EB estimator. For brevity in writing, the indicator function of a set A will be denoted by A itself. Let $0 < h = h_n \leqq 1$ be a sequence of numbers such that $h_n \downarrow 0$. Define a real valued function Q on the space of all real valued nonnegative functions t on R by

$$Q(t)(x) = h^{-1}\left(\log \frac{t(x+h)}{t(x)}\right)[t(x+h) + t(x) > 0] . \tag{3.1}$$

Let $\delta_j(\cdot) = [\cdot \leqq X_j \leqq \cdot + h]/u(X_j)$. Note that $\delta_j(y)$ are well defined with probability one, and are i.i.d. rv's with expectation $\delta(y) = \int_y^{y+h} f$.

Let $(b)_{\alpha,\beta} = \alpha$, b or β according as $b < \alpha$, $\alpha \leqq b \leqq \beta$ or $b > \beta$. Also, for $a_j \in R$, define $\bar{a} = n^{-1} \sum_1^n a_j$. Our proposed EB estimator in the first problem is an arbitrary function of X_1 (with values in $[\alpha, \beta]$), and in the present problem (which is in fact the $(n+1)$st problem) is

$$\psi_n(X) = (Q(\bar{\delta})(X))_{\alpha,\beta} . \tag{3.2}$$

Intuitively one expects ψ_n to be an estimator of $\psi_G = (\log f)^{(1)}$. We confirm this in the next section by showing that, under certain conditions, $P(\psi_n - \psi_G)^2 \to 0$ with a rate.

Estimator ψ_n is similar to the compound estimator in the $(n+1)$st problem exhibited by Singh in [15] and in Chapter 2 of [16], which in turn is (partially) motivated by Gilliland ([1], Chapter 3), Yu ([17], Section 2 of the appendix) and Hannan and Macky [3].

4. An upper bound for $R_n - R(G)$ and rates of asymptotic optimality of the estimator. We will obtain an explicit upper bound for the excess risk $R_n - R(G)$

and give sufficient conditions under which this bound is arbitrarily close to $O(n^{-\frac{2}{5}})$. Examples of families (including one whose Lebesgue-density has infinitely many discontinuity points) are given where such rates are achieved.

Let $c = |\alpha| \vee |\beta|$ and $\eta = e^{2hc}$. Abbreviate $Q(\hat{\delta})(\cdot)$ and $Q(\delta)(\cdot)$ to $\hat{Q}(\cdot)$ and $Q(\cdot)$ respectively. Further, for $x > a$, let $u_*(x)$ and $u^*(x)$ be respectively *Lebesgue-inf* and *Lebesgue-sup* of the restriction to $[x, x + 2h)$ of u. In Lemmas 4.1, 4.2 and 4.3 below and in their proofs $\hat{Q}$, Q, u_*, u^* and f all are evaluated at a fixed point $x > a$.

LEMMA 4.1. *For every* $\gamma > 0$,

$$P(|\hat{Q} - Q| \wedge 2c)^\gamma \leqq k_0(\gamma)(nh^3 f u_*^2/u^*)^{-\gamma/2}, \tag{4.1}$$

where $k_0(\gamma) = \gamma\Gamma(\gamma/2)\{8\eta^3(1 + \eta^2)/3k^+\}^{\gamma/2}$ *with* $k = 1 - h\eta u^* f$.

PROOF. The lhs of (4.1) is

$$\textstyle\int_0^{2c} P[|\hat{Q} - Q| > v]\, d(v^\gamma) = \int_0^{2c} (p_1(v) + p_2(v))\, d(v^\gamma), \tag{4.2}$$

where $p_1(v) = P[(\hat{Q} - Q) > v]$ and $p_2(v) = P[(Q - \hat{Q}) > v]$. Our method of the proof involves obtaining an exponential bound for $p_1(v) + p_2(v)$ with $0 < v < 2c$.

Fix $0 < v < 2c$ until stated otherwise. With $r = \delta(x + h)/\delta(x)$, $Y_j = \delta_j(x + h) - re^{hv}\delta_j(x)$ and $\nu = P(Y_1)$, we get

$$-\eta\delta(x + h) \leqq \nu = (1 - e^{hv})\delta(x + h) \leqq -hv\delta(x + h). \tag{4.3}$$

Also, since $\sigma^2 = \operatorname{Var}(Y_1) \leqq P(Y_1^2)$, $\delta_j \geqq 0$ and $P(\delta_1^2(y)) \leqq \delta(y)/u_*$ for $y = x$, $x + h$,

$$u_*\sigma^2 \leqq (1 + r\eta^2)\delta(x + h) = ((\delta(x + h))^{-1} + \eta^2(\delta(x))^{-1})\delta^2(x + h). \tag{4.4}$$

But, since for $y = x$, $x + h$,

$$h\eta^{-1} \leqq \delta(y) = \textstyle\int_y^{y+h} \int C(\omega)e^{\omega t}\, dG(\omega)\, dt \leqq h\eta f, \tag{4.5}$$

the first inequality in (4.5), and (4.4) lead to $hu_* f\sigma^2 \leqq (1 + \eta^2)\eta\delta^2(x + h)$. This and the second inequality in (4.3) give

$$\frac{(-\nu)^2}{\sigma^2} \geqq \frac{h^3v^2 f u_*}{(1 + \eta^2)\eta}. \tag{4.6}$$

Next observe that for $y = x$, $x + h$, $\operatorname{Var}(\delta_1(y)) = \int_y^{y+h}(f/u) - \delta^2(y) \geqq \delta(y)(1 - u^*\delta(y))^+/u^*$. Thus by the second inequality in (4.5) $u^* \operatorname{Var}(\delta_1(y)) \geqq \delta(y)(1 - h\eta u^* f)^+ = \delta(y)k^+$, where k is as given in the lemma. Consequently, since $\delta_1 \geqq 0$ and $\delta_1(x)\delta_1(x + h) = 0$,

$$\begin{aligned} \sigma^2 &\geqq \operatorname{Var}(\delta_1(x + h)) + r^2 \operatorname{Var}(\delta_1(x)) \\ &\geqq \{(\delta(x + h) + r^2\delta(x))k^+/u^*\} = (1 + r)\delta(x + h)k^+/u^*. \end{aligned} \tag{4.7}$$

Thus, since $Y_1 \geqq -re^{hv}\delta_1(x) \geqq -r\eta/u_*$ implies $-\nu \leqq r\eta/u_*$, and since

$Y_j \leqq \delta_j(x+h) \leqq 1/u_*$, by (4.3) and (4.7) $(Y_j - \nu)(-\nu/\sigma^2) \leqq \{(1+\eta r)\eta u^*/((1+r)u_* k^+)\} \leqq \eta^2 u^*(u_* k^+)^{-1}$. This leads to

$$Y_j - \nu \leqq \frac{\eta^2 u^*}{u_* k^+}\left(-\frac{\sigma^2}{\nu}\right). \tag{4.8}$$

Finally, since the event in $p_1(v)$ is $[\bar{Y} > 0]$, (4.8) and the Bernstein inequality given in (2.13) of Hoeffding [4] give

$$\begin{aligned} p_1(v) = P[\bar{Y} - \nu > -\nu] &\leqq \exp\left\{-n(-\nu)^2\Big/\left(2\sigma^2\left(1+\frac{\eta^2 u^*}{3k^+u_*}\right)\right)\right\} \\ &\leqq \exp\left\{-\frac{3nk^+h^3v^2{u_*}^2 f}{8\eta^3(1+\eta^2)u^*}\right\}, \end{aligned} \tag{4.9}$$

where the last inequality follows from (4.6) and from the fact that $(1 + \eta^2 u^*/(3k^+u_*)) \leqq 4(3k^+u_*)^{-1}\eta^2 u^*$, since $\eta \geqq 1$, $k^+ \leqq 1$ and $u^* \geqq u_*$.

By interchanging x and $x+h$ in the definition of Y_j and by the technique leading to the bound (4.9) for $p_1(v)$ we get the same bound for $p_2(v)$.

Now substituting $p_1(v) + p_2(v)$ in (4.2) by its bound just obtained and then performing the integration there after extending the range of integration from $(0, 2c)$ to $(0, \infty)$ we get the desired conclusion. □

LEMMA 4.2. *Let $t^{(j)}$ denote the* jth *derivative of* t. *Then*

$$\sup_x |(Q(\delta) - (\log f)^{(1)})(x)| \leqq 3c^2h/2\,. \tag{4.10}$$

PROOF. Since $\delta(t) = \int_{-\infty}^{t+h} f - \int_{-\infty}^{t} f$, by the Cauchy-mean value theorem ([2], page 81) for some ε in $(0, 1)$

$$\frac{\delta(x+h)}{\delta(x)} = \frac{f(x+h+\varepsilon h)}{f(x+\varepsilon h)}. \tag{4.11}$$

Thus by the Taylor expansion with integral form of the remainder,

$$\begin{aligned} hQ(\delta)(x) &= \log f(x+h+\varepsilon h) - \log f(x+\varepsilon h) \\ &= \int_{x+\varepsilon h}^{x+h+\varepsilon h} (\log f)^{(1)}(t)\,dt\,. \end{aligned} \tag{4.12}$$

But, again by similar expansion, $(\log f)^{(1)}(t) - (\log f)^{(1)}(x) = \int_x^t (\log f)^{(2)}(v)\,dv$. This and (4.12) give

$$(Q(\delta) - (\log f)^{(1)})(x) = h^{-1}\int_{x+\varepsilon h}^{x+h+\varepsilon h}\int_x^t (\log f)^{(2)}(v)\,dv\,dt\,. \tag{4.13}$$

Notice that $(\log f)^{(2)}(v) = \mathrm{Var}_{X=v}(\omega) \leqq P'_{X=v}(\omega^2) \leqq c^2$. Thus, the rhs of (4.13) is positive and is at most $3c^2h/2$ uniformly in x. □

Notice that (4.11) implies $e^{-hc} \leqq (\delta(x+h)/\delta(x)) \leqq e^{hc}$ which in turn implies $|Q(\delta)| \leqq c$. Consequently, from the definition of ψ_n in (3.2) $|\psi_n - Q(\delta)| \leqq |Q(\bar{\delta}) - Q(\delta)| \wedge 2c$. Thus, triangle inequality followed by c_r-inequality ([9], page 155) gives

$$\begin{aligned} 2^{-1}|\psi_n - (\log f)^{(1)}|^2 &\leqq |\psi_n - Q(\delta)|^2 + |Q(\delta) - (\log f)^{(1)}|^2 \\ &\leqq (2c)^{2-\gamma}(|\psi_n - Q(\delta)| \wedge 2c)^\gamma + (3c^2h/2)^2 \end{aligned}$$

by (4.10), and $\forall\, 0 \leqq \gamma \leqq 2$. This last inequality and Lemma 4.1 lead to

LEMMA 4.3. *If ψ_n is given by* (3.2), *then* $\forall\, \gamma \in [0, 2]$,

$$(4.14)\qquad P|\psi_n - (\log f)^{(1)}|^2 \leqq 2\{(2c)^{2-\gamma}k_0(\gamma)(nh^3 f u_*{}^2/u^*)^{-\gamma/2} + (3c^2h/2)^2\}\,.$$

Thus ψ_n is a mean square consistent point-estimator of $(\log f)^{(1)}$, *and* (4.14) *gives the rate of convergence.*

LEMMA 4.4. *Let $k_1(\gamma) = k_0(\gamma)$ in Lemma 4.1 with k there replaced by* $1 - h\eta \sup_{x>a}(u^* f)(x)$. *If ψ_n is given by* (3.2), *then* $\forall\, \gamma \in [0, 2)$,

$$(4.15)\qquad R_n - R(G) \leqq 2\{(2c)^{2-\gamma}k_1(\gamma)(nh^3)^{-\gamma/2}\mu\left(f^{1-\gamma/2}\left(\frac{u^*}{u_*{}^2}\right)^{\gamma/2}\right) + \left(\frac{3c^2h}{2}\right)^2\}\,.$$

PROOF. By (2.2)

$$(4.16)\qquad R_n - R(G) = P(P_X|\psi_n(X) - \psi_G(X)|^2)\,.$$

Thus, since $\psi_G = (\log f)^{(1)}$, (4.15) follows from (4.16) and (4.14). □

Theorem 4.1 below is the first of the main results of this paper. It gives sufficient conditions under which our EB estimator is a.o. with rates. Let $c_0, c_1, \cdots$ below be absolute constants.

THEOREM 4.1. *Let*

$$(\text{A. }0)\qquad \sup_{x>a} u^*(x)f(x) \leqq c_0$$

and γ be a number in $[0, 2)$ *for which*

$$(\text{A. }1)\qquad \mu\left\{f^{1-\gamma/2}\left(\frac{u^*}{u_*{}^2}\right)^{\gamma/2}\right\} < \infty\,.$$

Let ψ_n be given by (3.2) *with* $h = c_1(n^{-\gamma/(4+3\gamma)})/(c_0e^c)$ *for some* $0 < c_1 < 1 \wedge (c_0e^c)$. *Then* $\exists$ *a* $c_2 = c_2(\gamma)$ *such that*

$$(4.17)\qquad R_n - R(G) \leqq c_2 n^{-2\gamma/(4+3\gamma)}\,.$$

PROOF. Since (A.0) and the hypothesis on h implies $k_1(\gamma)$ in (4.15) is finite, (4.17) follows by (4.15), (A.1) and the hypothesis on h. □

Note that no assumptions on the smoothness or on the form of u are made for (4.17). We now give examples where (4.17) holds for γ arbitrarily close to 2.

EXAMPLE 4.1 (Normal $N(\omega, 1)$-family). Let $u(x) = (2\pi)^{-\frac{1}{2}}e^{-x^2/2}[-\infty < x < \infty]$. (Thus $a = -\infty$ and $C(\omega) = e^{-\omega^2/2}$.) Let $0 < -\alpha = \beta = c$. Considering the upper and lower bounds for the ratio $u(t)/u(x)$ for $x \leqq t < x + 2h$, we get

$$(4.18)\qquad u^*(x) \leqq u(x)e^{2h|x|} \qquad \text{and} \qquad u_*(x) \geqq u(x)e^{-2h(|x|+h)}\,.$$

Therefore,

$$(4.19)\qquad \begin{aligned} u^*(x)f_\omega(x) &\leqq (2\pi)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}((|x| - \omega\,\text{sign}\,x)^2 - 4hx)\} \\ &\leqq \exp\{2h(h + \omega\,\text{sign}\,x)\} \leqq \exp\{2h(h + c)\} \qquad \text{a.e.}\quad G\,. \end{aligned}$$

Thus $\sup_x u^*(x)f(x) < \infty$ and (A.0) holds. Moreover, since (4.18) implies

$$(4.20) \qquad \left(\frac{u^*}{u_*^2}(x)\right)^{\frac{1}{2}} \leqq (2\pi)^{\frac{1}{4}} \exp(\tfrac{1}{4}x^2 + 3h|x| + 2h^2),$$

and since $f(x) \leqq e^{c|x|}$, (A.1) holds $\forall\, 0 \leqq \gamma < 2$.

EXAMPLE 4.2. Let $u(x) = x^\tau[x > 0]$ for a $\tau \geqq 0$. (Thus $a = 0$, $\Omega = (-\infty, 0)$ and $C(\omega) = (-\omega)^{\tau+1}/\Gamma(\tau + 1)$.) Let $-\infty < \alpha \leqq \beta < 0$. Then $c = -\alpha$. By c_r-inequality ([9], page 155) $u^*(x) \leqq 2^{(\tau-1)^+}(x^\tau + (2h)^\tau)[x > 0]$ and hence $u^*(x)f_\omega(x) \leqq 2^{(\tau-1)^+}(x^\tau + (2h)^\tau)(-\alpha)^{\tau+1}e^{\beta x}[x > 0]$ a.e. G. Consequently, since $\beta < 0$, (A.0) holds. Moreover, since by c_r-inequality $(u^*/u_*^2)^{\gamma/2}(x) \leqq 2^{(\tau-1)^+\gamma/2}(x^{-\tau\gamma/2} + (2h)^{\tau\gamma/2}x^{-\gamma\tau})[x > 0]\ \forall\, 0 \leqq \gamma < 2$, and since $f(x) \leqq (-\alpha)^{\tau+1}e^{\beta x}$, (A.1) holds $\forall\, \gamma \in [0, 2)$ if $\tau = 0$, and $\forall\, \gamma \in [0, 2)$ with $\gamma < (1 + \tau)/\tau$ if $\tau > 0$.

EXAMPLE 4.3. Let $u(x) = \sum_0^\infty (i + 1)[i < x \leqq i + 1]$. (Thus $a = 0$, $\Omega = (-\infty, 0)$ and $C(\omega) = \omega(e^\omega - 1)$.) Let $-\infty < \alpha \leqq \beta < 0$. In this example verification of (A.0) and (A.1) for any $\gamma \in [0, 2)$ is easy.

In Examples 4.1 and 4.3, ψ_n is a.o. with rates arbitrarily close to $\frac{2}{5}$. The same is true in Example 4.2 with $0 \leqq \tau \leqq 1$ (for $\tau > 1$, rates depend on the magnitude of τ). In the next section we will obtain the best possible rate of the asymptotic optimality of ψ_n which is, in these examples, arbitrarily close to (from above) $\frac{2}{5}$.

5. A lower bound for $R_n - R(G)$ and the best possible rate of asymptotic optimality of the estimator. Throughout this section, let G be degenerate at an arbitrary but a *fixed* point $\omega \in \Omega$ with $\alpha \leqq \omega < \beta$. (Thus f here is f_ω.) Let c_1, c_2, $\cdots$ denote absolute constants (c_j here are not necessarily the same as c_j in Section 4). We will show that, with h as in Theorem 4.1, $R_n - R(G) \geqq c_1(\omega)n^{-4/(4+3\gamma)}$ for all sufficiently large n.

THEOREM 5.1. *Let $\varepsilon > 0$ and $l > a$ be finite numbers $\ni$ Lebesgue-*inf *and Lebesgue-*sup *of the restriction to $(l, l + \varepsilon)$ of u are, respectively, positive and finite. Let h be as given in Theorem* 4.1 (i.e. *for a $\gamma \in [0, 2)$, $h = h_n = c_2n^{-\gamma/(4+3\gamma)}$*). *Then*

$$(5.1) \qquad R_n - R(G) \geqq c_1n^{-4/(4+3\gamma)} \qquad \forall \textit{ sufficiently large } n.$$

PROOF. By our hypothesis, there are c_3 and c_4 such that

$$(5.2) \qquad c_3 \leqq \mu(l < x < l + \varepsilon) \leqq c_4,$$

and since $\alpha \leqq \omega < \beta$,

$$(5.3) \qquad c_3 \leqq \inf_{l<t<l+\varepsilon} f(t) \leqq \sup_{l<t<l+\varepsilon} f(t) \leqq c_4.$$

Since G is degenerate at ω, $\psi_G \equiv \omega$, and by (2.2) $R_n - R(G) = P(\psi_n(X) - \omega)^2 \geqq P^2|\psi_n(X) - \omega|$. Thus, since $\omega < \beta$ and by (3.2) ψ_n is the retraction of $Q(\bar{\delta})$ to $[\alpha, \beta]$,

$$(5.4) \qquad \begin{aligned}(R_n - R(G))^{\frac{1}{2}} &\geqq P(P_X|\psi_n(X) - \omega|) \geqq P(\textstyle\int_0^{\beta-\omega} P_X[\psi_n(X) - \omega > v]\,dv)\\ &\geqq P\{[l < X < l + \varepsilon/2] \textstyle\int_0^{\beta-\omega} P_X[Q(\bar{\delta})(X) > v + \omega]\,dv\}.\end{aligned}$$

Fix $X \in (l, l + (\varepsilon/2))$ and $v \in (0, \beta - \omega)$ until stated otherwise. Recall from Section 3 that $\delta_j(\cdot) = [\cdot \leqq X_j < \cdot + h]/u(X_j)$ and $\delta(y) = \int_y^{y+h} f$. (Thus $\delta(y) = f(y)(e^{\omega h} - 1)$.) As in the second paragraph of the proof of Lemma 4.1, let $Y_j = \delta_j(X + h) - e^{h(v+\omega)}\delta_j(X)$. Let n be large enough to make $h \leqq \varepsilon/4$. Since $(\delta(\cdot + h)/\delta(\cdot)) = e^{h\omega}$, with $\nu = P(Y_1) = \delta(X + h) - e^{h(v+\omega)}\delta(X)$, we have

$$\nu = \delta(X)e^{h\omega}(1 - e^{hv}) \geqq -hv\delta(X)e^{h(\omega+v)} \geqq -c_5 h^2 v \tag{5.5}$$

where the last inequality follows by (5.3). Since r in (4.7) is (here) $e^{h\omega}$, (4.7) followed by the hypothesis of the theorem and (5.3) gives

$$\operatorname{Var}(Y_1) = \sigma^2 \geqq c_6 \delta(X + h) \geqq c_7 h\,. \tag{5.6}$$

Notice that in view of our hypothesis, Y_j are uniformly bounded. Thus, since $P_X[Q(\bar{\delta})(X) > v + \omega] = P_X[\sum_1^n Y_j > 0] \geqq P_X[\sum_1^n (Y_j - \nu) > c_5 nh^2 v]$ by (5.5), Lemma 3 on page 47 of Lamperty [8] gives, for n sufficiently large and for a $\zeta > 0$,

$$\begin{aligned} P_X[Q(\bar{\delta})(X) > v + \omega] &> \exp\left\{-\frac{nh^4(c_5 v)^2}{2\sigma^2}(1 + \zeta)\right\} \\ &\geqq \exp\{-c_8 nh^3 v^2\} \end{aligned} \tag{5.7}$$

by (5.6). Thus making the transformation $(c_8 nh^3)^{\frac{1}{2}} v = t$ we get from (5.4) and (5.7)

$$\begin{aligned} (R_n - R(G))^{\frac{1}{2}} &\geqq c_9(nh^3)^{-\frac{1}{2}} P\{[l < X < l + \varepsilon/2] \textstyle\int_0^{c_{10}(nh^3)^{\frac{1}{2}}} e^{-t^2}\,dt\} \\ &\geqq c_1^{\frac{1}{2}} n^{-2/(4+3\gamma)} \end{aligned}$$

by (5.2), (5.3) and the hypothesis that $h = c_2 n^{-\gamma/(4+3\gamma)}$. □

6. Remarks. If it is known that u is twice differentiable on (a, ∞), then taking

$$\psi_n{}^*(X) = (Q(\bar{\delta}^*)(X))_{\alpha,\beta} \tag{6.1}$$

(in place of $\psi_n(X)$), where $\delta_j{}^*(X) = [X \leqq X_j < X + h]$, it is expected that the analysis would become simpler; and perhaps (A.0) could be eliminated (provided a suitable lower bound for "σ^2" in (4.7) is used), and (A.1) could be weakened (to $\int (uf)^{1-\gamma/2} < \infty$). Nevertheless, the rate of asymptotic optimality with $\psi_n{}^*$ is the same as with ψ_n.

Hannan and Macky [3] treat a little more general problem (in the sense that the support of their prior G could be the whole real line), and deal with an estimator of the type (6.1). Under certain conditions (e.g. $G(\omega^2) < \infty$) they have proved the asymptotic optimality of their estimator without asserting any rate. The importance of our estimator is clear since for most of the practical problems parameter spaces are not unbounded and the usefulness of an EB estimator depends on the rate of its asymptotic optimality.

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REFERENCES

[1] GILLILAND, DENNIS C. (1966). Approximation to Bayes risk in sequences of nonfinite decision problems. RM-162, Department of Statistics and Probability, Michigan State Univ.

[2] GRAVES, LAWRENCE M. (1956). *The Theory of Functions of Real Variables*, 2nd ed. Macmillan, New York.

[3] HANNAN, JAMES and MACKY, DAVID W. (1971). Empirical Bayes squared error loss estimation of unbounded functionals in exponential families. RM-290, Department of Statistics and Probability, Michigan State Univ.

[4] HOEFFDING, WASSILY (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58** 13-30.

[5] JOHNS, M. V., JR. (1957). Nonparametric empirical Bayes procedures. *Ann. Math. Statist.* **28** 649-669.

[6] JOHNS, M. V., JR. and VAN RYZIN, J. R. (1971). Convergence rates for empirical Bayes two-action problems I. Discrete case. *Ann. Math. Statist.* **42** 1521-1539.

[7] JOHNS, M. V., JR. and VAN RYZIN, J. R. (1972). Convergence rates for empirical Bayes two-action problems II. Continuous case. *Ann. Math. Statist.* **43** 934-947.

[8] LAMPERTY, JOHN (1966). *Probability*. W. A. Benjamin, New York.

[9] LOÈVE, MICHEL (1963). *Probability Theory*, 3rd ed. Van Nostrand, Princeton.

[10] NEYMAN, J. (1962). Two breakthroughs in the theory of statistical decision making. *Rev. Inst. Internat. Statist.* **30** 11-27.

[11] O'BRYAN, THOMAS E. and SUSARLA, V. (1973). Rates in the empirical Bayes estimation problem with nonidentical components. Case of normal distributions. Technical report, Department of Mathematics, Univ. of Wisconsin, Milwaukee.

[12] ROBBINS, HERBERT (1955). An empirical Bayes approach to statistics. *Proc. Third Berkeley Symp. Math. Statist. Prob.* **1** 157-163. Univ. of California Press.

[13] ROBBINS, HERBERT (1963). The empirical Bayes approach to testing statistical hypothesis. *Rev. Inst. Internat. Statist.* **31** 195-208.

[14] SAMUEL, E. (1963). An empirical Bayes approach to the testing of certain parametric hypothesis. *Ann. Math. Statist.* **34** 1370-1385.

[15] SINGH, R. S. (1976). Sequence-compound estimation with rates in noncontinuous Lebesgue exponential families. *Sankhyā Ser. A*. To appear.

[16] SINGH, R. S. (1974). Estimation of derivatives of average of μ-densities and sequence-compound estimation in exponential families. RM-318, Department of Statistics and Probability, Michigan State Univ.

[17] YU, BENITO (1971). Rates of convergence in empirical Bayes two-action and estimation problems and in sequence-compound estimation problems. RM-279, Department of Statistics and Probability, Michigan State Univ.

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