The Annals of Statistics
2003, Vol. 31, No. 5, 1537–1560

SINGULAR WISHART AND MULTIVARIATE BETA DISTRIBUTIONS[1]

BY M. S. SRIVASTAVA

University of Toronto

In this article, we consider the case when the number of observations n is less than the dimension p of the random vectors which are assumed to be independent and identically distributed as normal with nonsingular covariance matrix. The central and noncentral distributions of the singular Wishart matrix $S = XX'$, where X is the $p \times n$ matrix of observations are derived with respect to Lebesgue measure. Properties of this distribution are given. When the covariance matrix is singular, pseudo singular Wishart distribution is also derived. The result is extended to any distribution of the type $f(XX')$ for the central case. Singular multivariate beta distributions with respect to Lebesgue measure are also given.

1. Introduction. Singular Wishart and multivariate beta distributions were well defined by Mitra (1970), Khatri (1970) and Srivastava and Khatri (1979), among others. However, no practical applications were foreseen. Recently, Uhlig (1994) clearly demonstrated the need for such distributions in his Bayesian analysis of some interesting problems.

In this article, we derive the probability density functions of singular Wishart and multivariate beta distributions with respect to Lebesgue measure. To motivate it, we consider the simplest case when we have only one observation vector $\mathbf{x}_1 = (x_{11}, x_{21})'$ on the two-dimensional random vector $\mathbf{x}$ distributed as multivariate normal with mean vector zero and 2×2 positive definite covariance matrix Σ, written as $\mathbf{x} \sim N_2(\mathbf{0}, \Sigma)$, $\Sigma > 0$. The distribution of $S = \mathbf{x}_1\mathbf{x}_1'$ is called singular Wishart distribution with 1 degree of freedom. The p.d.f. of $\mathbf{x}_1$ is given by

$$c\big(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}\mathbf{x}_1\mathbf{x}_1'\big),$$

where

$$c = (2\pi)^{-1}|\Sigma|^{-1/2}$$

and $(\operatorname{etr} A)$ stands for the exponential of the trace of the matrix A. Let $\mathbf{h}_1' = (\cos\theta, \sin\theta)$ and $\mathbf{h}_2' = (-\sin\theta, \cos\theta)$. Then $\mathbf{h}_1'\mathbf{h}_2 = 0$ and $H = (\mathbf{h}_1, \mathbf{h}_2)$ is an orthogonal matrix. We also note that

$$(\mathbf{h}_2'\, d\mathbf{h}_1) = d\theta.$$

Received June 2001; revised August 2002.

[1]Supported in part by the Natural Sciences and Engineering Research Council of Canada.

AMS 2000 subject classifications. Primary 62H10; secondary 62E15.

Key words and phrases. Jacobian of transformations, normal distribution, pseudo Wishart, singular noncentral Wishart, Stiefel manifold.

Consider the transformation

$$\mathbf{x}_1 = r\mathbf{h}_1, \qquad r > 0,\ 0 < \theta < 2\pi.$$

The Jacobian of the transformation from (x_{11}, x_{21}) to (r, θ) is simply r. Hence, the probability density (p.d.) of (r, θ) is

$$cr\big(\operatorname{etr} -\tfrac{1}{2}r^2\Sigma^{-1}\mathbf{h}_1\mathbf{h}_1'\big)\,dr\,d\theta.$$

Letting $r^2 = l_1$ and noting that $J(r \to l_1) = (2r)^{-1}$, the p.d. of (l_1, θ) is

$$\tfrac{1}{2}c\big(\operatorname{etr} -\tfrac{1}{2}l_1\,\Sigma^{-1}\mathbf{h}_1\mathbf{h}_1'\big)\,dl_1\,d\theta. \tag{1.1}$$

Equivalently, we can also write it as the joint p.d. of $(l_1, \mathbf{h}_1)$ as

$$\tfrac{1}{2}c(l_1)^{-1}\big(\operatorname{etr} -\tfrac{1}{2}l_1\,\Sigma^{-1}\mathbf{h}_1\mathbf{h}_1'\big)(l_1\mathbf{h}_2'\,d\mathbf{h}_1)\,dl_1. \tag{1.2}$$

Let

$$S = l_1\mathbf{h}_1\mathbf{h}_1' \tag{1.3}$$

and

$$(dS) = l_1(\mathbf{h}_2'\,d\mathbf{h}_1)\,dl_1.$$

Then Uhlig (1994) writes the p.d.f. of S with respect to the volume (dS) as

$$\tfrac{1}{2}c(l_1)^{-1}\big(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}S\big). \tag{1.4}$$

For practical applications, however, one needs to evaluate the volume (dS). It is rather difficult to evaluate it without specifying the functionally independent elements of S. This leads to the p.d.f. with respect to Lebesgue measure. For this, we consider the transformation (1.3) in terms of functionally independent elements as

$$\begin{aligned} S &= \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} \\ &= l_1 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta, \sin\theta) \\ &= l_1 \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \end{aligned}$$

As we can see, there are only two independent elements in S. We can choose either (S_{11}, S_{12}) or (S_{12}, S_{22}). Choosing S_{11} and S_{12} we find that

$$\begin{aligned} S_{11} &= l_1\cos^2\theta, \\ S_{12} &= l_1\cos\theta\sin\theta. \end{aligned}$$

The Jacobian of the transformation from (l_1, θ) to (S_{11}, S_{12}) is $(l_1 \cos^2\theta)^{-1} = S_{11}^{-1}$. Hence, the joint p.d.f. of (S_{11}, S_{12}) with respect to Lebesgue measure is given by

$$\tfrac{1}{2} c S_{11}^{-1} (\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1} S),$$

where $S_{22} = S_{12}^2/S_{11}$.

From the above discussion, it is clear that for the general case, we need to consider the singular value decomposition of $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n) : p \times n$, $n < p$, namely $X = H_1' F L$, $H_1 : n \times p$, $H_1 H_1' = I_n$, $L : n \times n$, $LL' = I_n$, $F = \operatorname{diag}(f_1, \ldots, f_n)$, $f_1 > \cdots > f_n > 0$ and $S = XX' = H_1' F^2 H_1 = H_1' D H_1$. For these transformations, we need to obtain the Jacobian of the transformations. Similarly, we give the p.d.f. of multivariate beta, considered earlier by Díaz-García and Gutiérrez-Jáimez (1997). The p.d.f. of pseudo Wishart, considered earlier by Díaz-García, Gutiérrez-Jáimez and Mardia (1997) is also given with respect to Lebesgue measure. The organization of the article is as follows.

In Section 2, we develop the needed Jacobians of the transformations and some connected results, such as the distribution of a subset of a semiorthogonal matrix. Section 3 gives the derivation of the central and noncentral singular Wishart distributions along with properties of this distribution as well as marginal and conditional distributions. The singular multivariate beta and F-distributions are considered in Section 4. The case when the covariance matrix is also singular, the pseudo singular Wishart case, is considered in Section 5.

2. Jacobians of transformations. In this section, we derive the relevant Jacobians of the transformations needed to derive the results of this article. We write $\mathcal{L}_{p,n}(q)$ for the linear space of all real $p \times n$ matrices of rank q. The set of matrices $H_1 \in \mathcal{L}_{p,n}(p)$ such that $H_1 H_1' = I_p$ is a manifold, called the Stiefel manifold and denoted by $\mathcal{H}_{p,n}$; it will also be called semiorthogonal matrices. The set of $p \times p$ orthogonal matrices H will be denoted by $\mathcal{H}_p$; $HH' = H'H = I_p$. The set of $p \times p$ lower triangular matrices with positive diagonal elements will be denoted by $\underline{\mathcal{T}}_+(p)$. The set of $p \times p$ symmetric positive semidefinite matrices of rank q will be denoted by $S_p^+(q)$. The Jacobian of the transformation is always from functionally independent variables to the same number of functionally independent variables. For example, if $Y \in \mathcal{L}_{p,n}(n)$ and $Y = BX$, where B is a $p \times p$ nonsingular matrix of constants, then this transformation is valid only if $X \in \mathcal{L}_{p,n}(n)$ also. In this case, the Jacobian of the transformation, denoted by $J(Y \to X)$, is well known to be $|B|_+^n$, where $|B|_+$ denotes the positive value of the determinant of the $p \times p$ nonsingular matrix B. However, if $Y \in \mathcal{L}_{p,n}(q)$, then we need to define which of the $q(p + n - q)$ functionally independent elements of Y is transformed to the same number $q(p + n - q)$ of functionally independent elements of X. Thus, whenever it is feasible, a subscript I has been added to the variables to indicate this fact in the derivation of the Jacobians of the transformations. Before we derive these results, we first give some known results in the full rank case.

LEMMA 2.1. *Let $X \in \mathcal{L}_{p,n}(p)$ and $X = TH_1$, where $T \in \underline{\mathcal{T}}_+(p)$ and $H_1 \in \mathcal{H}_{p,n}$. Then the Jacobian of the transformation is*

$$J(X \to T, H_1) = \prod_{i=1}^{p} t_{ii}^{n-i} g_{p,n}(H_1),$$

where $H' = (H_1', H_2') : H \in \mathcal{H}_n$, $g_{p,n}(H_1) = J(H(dH_1') \to dH_1')$.

The proof can be obtained along the lines of Theorem 1.11.5 (page 31) and Corollary 3.2.1 (page 75) in Srivastava and Khatri (1979), hereafter referred to as S&K. Let

$$C(p,n) = \int_{H_1'H_1 = I_p} g_{p,n}(H_1)\, dH_1 = 2^p \pi^{pn/2} \Gamma_p\left(\frac{n}{2}\right), \tag{2.1}$$

where

$$\Gamma_p\left(\frac{n}{2}\right) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{n-i+1}{2}\right).$$

LEMMA 2.2. *Suppose we write the $p \times n$ semiorthogonal matrix $H_1 = (H_{11}, H_{12})$, where H_{11} is a $p \times r$, $r \geq p$, matrix containing all the restrictions that arise out of the condition $H_1 H_1' = I_p$ and all the elements of the $p \times (n-r)$ matrix H_{12} are functionally independent random variables of H_1. Then the p.d.f. of H_{12} is given by*

$$\frac{C(p,r)}{C(p,n)} |I_p - H_{12}H_{12}'|^{(r-p-1)/2}, \qquad H_{12}H_{12}' < I_p.$$

This is Lemma 2 in Khatri (1970).

COROLLARY 2.1. *Let $L_1 = (I_p - H_{12}H_{12}')^{-1/2} H_{11}$. Then*

$$J(H_{11}, H_{12} \to L_1, H_{12}) = |I_p - H_{12}H_{12}'|^{(r-p-1)/2} \frac{g_{p,r}(L_1)}{g_{p,n}(H_1)}.$$

LEMMA 2.3. *Let X be a $p \times n$ matrix of rank $p \leq n$. Suppose that for any $n \times n$ orthogonal matrix P, X and XP have the same distribution. Then for any nonsingular factorization of $XX' = CC'$, $C : p \times p$, $H = C^{-1}X$ and XX' are independently distributed. The p.d.f. of H is given by*

$$(C(p,n))^{-1} g_{p,n}(H).$$

This result has been known for some time, but does appear in Khatri (1970). When the p.d.f. of a $p \times n$ random matrix X is given by

$$[(2\pi)^p |\Sigma|]^{-n/2} |A|^{-p/2} \left[\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}(X-\eta)A^{-1}(X-\eta)'\right],$$

we write it as $X \sim N_{p,n}(\eta, \Sigma, A)$; see S&K, pages 54 and 55.

REMARK 2.1. A consequence of Lemma 2.3 is that if $X \sim N_{p,n}(0, \Sigma, I_n)$, $n \geq p$, the distribution of $H_1 = (XX')^{-1/2}X$, where $(XX')^{1/2}((XX')^{1/2})' = XX'$, is independent of Σ and for $(XX')^{1/2}$ we may use the triangular factorization of XX'.

LEMMA 2.4. *Let $X \in \mathcal{L}_{p,n}(n)$ and $X = H_1' F L$, where $H_1 \in \mathcal{H}_{n,p}$, $L \in \mathcal{H}_n$ and $F = \operatorname{diag}(f_1, \dots, f_n)$, $f_1 > \cdots > f_n > 0$. Then, for $n \geq 2$,*

$$J(X \to H_1, F, L) = 2^{-n} |F|^{p-n} \left[\prod_{i<j}^{n} (f_i^2 - f_j^2) g_{n,n}(L)\right] g_{n,p}(H_1).$$

The proof can be obtained from Theorem 1.15 of Olkin (1951).

We now generalize these results to the nonfull-rank case. That is, let X be a $p \times n$ matrix of rank $q \leq \min(p, n)$, $X \in \mathcal{L}_{p,n}(q)$. Then, without loss of generality, we may assume that

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{21}X_{11}^{-1}X_{12} \end{pmatrix} = \begin{pmatrix} I_q \\ X_{21}X_{11}^{-1} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \end{pmatrix}, \tag{2.2}$$

where X_{11} is a $q \times q$ nonsingular matrix; see S&K, page 11, Theorem 1.5.3. Since $(X_{11}\ X_{12})$ is a $q \times n$ matrix of rank q, we can write it as $T_1 L_1$, where $T_1 \in \mathcal{T}_+(q)$ and $L_1 \in \mathcal{H}_{q,n}$. Writing $L_1 = (L_{11}\ L_{12})$, where L_{11} is a $q \times q$ nonsingular matrix, we find that $X_{11} = T_1 L_{11}$. Hence,

$$X = \begin{pmatrix} I_q \\ X_{21}L_{11}^{-1}T_1^{-1} \end{pmatrix} T_1 L_1 = \begin{pmatrix} T_1 \\ X_{21}L_{11}^{-1} \end{pmatrix} L_1.$$

Making the transformation $T_2 = X_{21}L_{11}^{-1}$, the Jacobian of the transformation from

$$X = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} L_1$$

is given by

$$J(X_{11}, X_{12} \to T_1, L_1) J(X_{21} \to T_2) = \prod_{i=1}^{q} t_{ii}^{n-i} g_{q,n}(L_1) |L_{11}|_+^{p-q},$$

where $|A|_+$ denotes the positive value of the determinant of A. Thus, we have the following theorem.

THEOREM 2.1. *Let X be a $p \times n$ matrix of rank $q \leq \min(p, n)$. Then the Jacobian of the transformation $X' = L_1'(T_1', T_2') \equiv L_1'T'$, where $L_1 = (L_{11}, L_{12}) \in \mathcal{H}_{q,n}$ and $T_1 \in \mathcal{T}_+(q)$, is given by*

$$|L_{11}|_+^{p-q} \prod_{i=1}^{q} t_{ii}^{n-i} g_{q,n}(L_1).$$

COROLLARY 2.2. *Let $X \in \mathcal{L}_{p,n}(q)$ and write $X = H_1'T$, where $H_1 \in \mathcal{H}_{q,p}$, $T = (T_1, T_2)$, $T_1 \in \mathcal{T}_+(q)$ and T_2 is a $q \times (n-q)$ matrix. Then*

$$J(X_I \to H_1, T) = |H_{11}|_+^{n-q} \prod_{i=1}^{q} t_{ii}^{p-q+i-1} g_{q,p}(H_1),$$

where $H_1 = (H_{11}, H_{12})$, $H_{11} : q \times q$.

THEOREM 2.2. *Let $X \in \mathcal{L}_{p,n}(q)$ and $Y = AX$, where A is a $p \times p$ nonsingular matrix. Then*

$$J(Y_I \to X_I) = |A|^q |A_{11}|^{n-q},$$

where $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $A_{11} : q \times q$, and X_I and Y_I denote the functionally independent elements of X and Y, respectively.

PROOF. As in (2.2), we can write

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where $X_{11} : q \times q$ and $X_{22} = X_{21}X_{11}^{-1}X_{12}$. Hence

$$\begin{aligned} Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} &= AX \\ &= \left(A \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix} A \begin{pmatrix} X_{12} \\ X_{22} \end{pmatrix} \right). \end{aligned}$$

Thus, the Jacobian of the transformation

$$J(Y_{11}', Y_{21}' \to X_{11}', X_{21}') = |A|^q.$$

Now

$$A \begin{pmatrix} X_{12} \\ X_{22} \end{pmatrix} = \begin{pmatrix} Y_{12} \\ Y_{22} \end{pmatrix}$$

gives $A_{11}X_{12} + A_{12}X_{22} = Y_{12}$ and $A_{21}X_{12} + A_{22}X_{22} = Y_{22}$. From the first equation we get $A_{21}X_{12} = A_{21}A_{11}^{-1}Y_{12} - A_{21}A_{11}^{-1}A_{12}X_{22}$. Substituting in the second equation, we get $(A_{22} - A_{21}A_{11}^{-1}A_{12})X_{22} = Y_{22} - A_{21}A_{11}^{-1}Y_{12}$. Thus, given X_{22} and Y_{12}, Y_{22} is fixed. Hence $J(Y_{12} \to X_{12}) = |A_{11}|^{n-q}$. Combining the two, we get the result. $\square$

THEOREM 2.3. *Let $p \geq q$ be integers and let S be a $p \times p$ matrix of rank q with distinct positive eigenvalues in the space of $S_p^+(q)$ of $p \times p$ positive semidefinite matrices. Then S can be written as $S = H_1' D H_1$, where $H_1 \in \mathcal{H}_{q,p}$ and $D = \operatorname{diag}(d_1, \dots, d_q)$, $d_1 > \cdots > d_q > 0$. The Jacobian of the transformation of functionally independent elements of S, denoted by S_I, to H_1 and D is given by*

$$J(S_I \to H_1, D) = 2^{-q} |H_{11}|_+^{(p-q+1)} |D|^{p-q} \prod_{i<j}^{q} (d_i - d_j) g_{q,p}(H_1),$$

where $H_1 = (H_{11}, H_{12})$, $H_{11} : q \times q$ is a nonsingular matrix and S_I denotes the functionally independent elements of S.

PROOF. Consider the transformation

$$S_{p\times p} = H'_{1_{p\times q}} D_{q\times q} H_{1_{q\times p}},$$

where $H_1 : q \times p$, $H_1 H_1' = I_q$. Let

$$H' = (H_1', H_2') : p \times p \quad \text{such that} \quad HH' = I_p.$$

Then

$$S = \begin{pmatrix} H_1' & H_2' \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = H' \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} H.$$

Taking differentials, we get

$$dS = (dH') \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} H + H' \begin{pmatrix} dD & 0 \\ 0 & 0 \end{pmatrix} H + H' \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} (dH).$$

Hence,

$$H(dS)H' = H(dH)' \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} dD & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} (dH)H'.$$

Since $HH' = I$, $(dH)H' + H(dH)' = 0$. Thus $R = H(dH)'$ is a skew-symmetric matrix. We write

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} H_1(dH_1)' & H_1(dH_2)' \\ H_2(dH_1)' & H_2(dH_2)' \end{pmatrix}.$$

Let

$$W = H(dS)H' = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}.$$

Then

$$\begin{aligned} W &= R \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} dD & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} R \\ &= \begin{pmatrix} R_{11}D - DR_{11} + dR & -DR_{12} \\ R_{21}D & 0 \end{pmatrix}. \end{aligned} \tag{2.3}$$

There are only $pq - \frac{q(q-1)}{2}$ elements on the right-hand side of (2.3), whereas dS has $\frac{p(p+1)}{2}$ elements, out of which only $pq - \frac{q(q-1)}{2}$ elements are functionally independent. Thus we need to find the Jacobian of the transformation from (S_{11}, S_{12}) to (H_1, D). From above, we note that

$$\begin{aligned}
W_{11} &= R_{11}D + dD - DR_{11},\\
W_{12} &= -DR_{12},\\
W_{21} &= R_{21}D,\\
W_{22} &= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&J(S_I \to H_1, D)\\
&\quad = J(S_{11}, S_{12} \to H_1, D)\\
&\quad = J(dS_{11}, dS_{12} \to dH_1, dD)\\
&\quad = J(dS_{11}, dS_{12} \to W_{11}, W_{12})J(W_{11} \to R_{11}, dD)\\
&\qquad \times J(W_{21} \to R_{21})J(R_{11}, R_{21}, dD \to dH_1, dD)\\
&\quad = J_1|D|^{p-q}\prod_{i<j}^{q}(d_i - d_j)J(H\,dH_1' \to dH_1'),
\end{aligned} \tag{2.4}$$

where $J_1 = J(dS_{11}, dS_{12} \to W_{11}, W_{12})$. To find J_1, let us define

$$H_1{}'_{p\times q} = \begin{pmatrix} H'_{11\,q\times q} \\ H'_{12\,(p-q)\times q} \end{pmatrix}$$

and

$$H_2' = \begin{pmatrix} H'_{21\,q\times(p-q)} \\ H'_{22\,(p-q)\times(p-q)} \end{pmatrix}.$$

Then

$$\begin{aligned}
dS &= H'WH\\
&= (H_1', H_2')\begin{pmatrix} W_{11} & W_{12} \\ W_{12}' & 0 \end{pmatrix}\begin{pmatrix} H_1 \\ H_2 \end{pmatrix}\\
&= H_1'W_{11}H_1 + H_2'W_{12}'H_1 + H_1'W_{12}H_2\\
&\equiv (1) + (2) + (2)'.
\end{aligned}$$

Expanding, we find that

$$(1) = \begin{pmatrix} H_{11}'W_{11}H_{11} & H_{11}'W_{11}H_{12} \\ H_{12}'W_{11}H_{11} & H_{12}'W_{11}H_{12} \end{pmatrix}$$

and

$$(2) = \begin{pmatrix} H_{21}'W_{12}'H_{11} & H_{21}'W_{12}'H_{12} \\ H_{22}'W_{12}'H_{11} & H_{22}'W_{12}'H_{12} \end{pmatrix}.$$

Hence,

$$dS_{11q\times q} = H_{11}'W_{11}H_{11} + H_{21}'W_{12}'H_{11} + H_{11}'W_{12}H_{21}$$

and

$$\begin{aligned} dS_{12q\times(p-q)} &= H_{11}'W_{11}H_{12} + H_{21}'W_{12}'H_{12} + H_{11}'W_{12}H_{22} \\ &= (dS_{11}H_{11}^{-1} - H_{21}'W_{12}' - H_{11}'W_{12}H_{21}H_{11}^{-1})H_{12} \\ &\quad + H_{21}'W_{12}'H_{12} + H_{11}'W_{12}H_{22} \\ &= dS_{11}H_{11}^{-1}H_{12} + H_{11}'W_{12}(H_{22} - H_{21}H_{11}^{-1}H_{12}). \end{aligned}$$

Hence, from Theorem 1.11.2, page 29, of S&K,

$$\begin{aligned} &J(dS_{11}, dS_{12} \to W_{11}, W_{12}) \\ &\quad = J(dS_{11} \to W_{11})J(dS_{12} \to W_{12} \mid S_{11}) \\ &\quad = |H_{11}|_+^{q+1}|H_{11}|_+^{p-q}|H_{22} - H_{21}H_{11}^{-1}H_{12}|_+^q \\ &\quad = |H_{11}|_+^{q+1}|H_{11}|_+^{p-q}|H_{11}|_+^{-q} \\ &\quad = |H_{11}|_+^{p-q+1}. \end{aligned}$$

Thus, the Jacobian of the transformation $S = H_1'DH_1$ is given by

$$J(S \to H_1, D) = 2^{-q}|H_{11}|_+^{p-q+1}|D|^{p-q}\prod_{i<j}^{q}(d_i - d_j)g_{q,p}(H_1),$$

since the transformation is 1 to 2^q. $\square$

THEOREM 2.4. *Let $U, V \in S_p^+(q)$ be related by $U = \Gamma V\Gamma' = \Gamma H_1'DH_1\Gamma'$, where $\Gamma \in \mathcal{H}_p$ and $H_1 \in \mathcal{H}_{q,p}$. Then the Jacobian of the transformation from $U_I \to V_I$ is given by*

$$J(U_I \to V_I) = |L_{11}|_+^{p+q-1}/|H_{11}|_+^{p+q-1}$$

where $\Gamma' = (\Gamma_1', \Gamma_2')$, $\Gamma_1' : p \times q$, $H_1 = (H_{11}, H_{12})$ and $H_{11} : q \times q$ is nonsingular.

PROOF. Let $V = H_1'DH_1$. Then

$$\begin{aligned} U &= \Gamma V\Gamma' = \Gamma H_1'DH_1\Gamma' \\ &\equiv L_1'DL_1, \end{aligned}$$

where $L_1 = H_1\Gamma' = H_1(\Gamma_1', \Gamma_2') \equiv (L_{11}, L_{12})$. Thus, $L_{11} = H_1\Gamma_1'$ and

$$J(U_I \to V_I) = J(U_I \to L_1, D)J(L_1, D \to H_1, D)J(H_1, D \to V_I).$$

From Theorem 2.3, $J(U_I \to L_1, D)J(H_1, D \to V_I) = (|L_{11}|_+/|H_{11}|_+)^{p-q+1} \times (g_{q,p}(L_1))/(g_{q,p}(H_1))$. It remains to show that $J(L_1, D \to H_1, D) = (g_{q,p}(H_1))/(g_{q,p}(L_1))$. With $H' = (H_1', H_2')$, $HH' = I_p$ and $L' = (L_1', L_2')$, $LL' = I_p$, we find from Roy (1957) that

$$\begin{aligned} &J(L_1, D \to H_1, D) \\ &\quad = \left(\frac{J(L_1, D \to H_1, D, \text{no restrictions})}{J(L_1L_1' \to H_1H_1')}\right)\left(\frac{J(H\,dH_1' \to dH_1')}{J(L\,dL_1' \to dL_1')}\right) \\ &\quad = \frac{g_{q,p}(H_1)}{g_{q,p}(L_1)}, \end{aligned}$$

since $J(L_1 \to H_1 \mid \text{no restriction}) = |\Gamma_+|^q = 1$ and $J(L_1L_1' \to H_1H_1') = 1$. Thus, $J(U_I \to V_I) = |L_{11}|_+^{p+q-1}/|H_{11}|_+^{p+q-1}$. $\square$

This result can also be obtained from Theorem 2.5, which is presented next, but the proof given here may be of independent interest.

THEOREM 2.5. *Let $U, V \in S_p^+(q)$ be related by $U = BVB'$, where B is a $p \times p$ nonsingular matrix. Then the Jacobian of the transformation from U_I to V_I is given by*

$$J(U_I \to V_I) = \frac{|B|^q|B_1H_1'|_+^{p-q+1}}{|H_{11}|_+^{p-q+1}},$$

where $B' = (B_1', B_2')$, $B_1 : q \times p$, $V = H_1'DH_1$, $H_1H_1' = I_q$, $H_1 = (H_{11}, H_{12})$ and $H_{11} : q \times q$.

PROOF. Let $V = H_1'DH_1$ and $D = \operatorname{diag}(d_1, \dots, d_q)$, $d_1 > \cdots > d_q > 0$. Then

$$U = BVB' = BH_1'DH_1B'.$$

Let

$$C = B^{-1}.$$

Then

$$CUC' = V = H_1'DH_1.$$

Taking differentials and proceeding as in Theorem 2.3 with $H' = (H_1', H_2')$ and $HH' = I_p$, we find that

$$HC(dU)C'H' = H(dH)'\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} dD & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(dH)H'. \tag{2.5}$$

Let

$$W = HC(dU)C'H' = \begin{pmatrix} W_{11} & W_{12} \\ W'_{12} & W_{22} \end{pmatrix}.$$

Then $W_{22} = 0$ and we need only to find the Jacobian of the transformation $J(dU_{11}, dU_{12} \to dW_{11}, dW_{12})$. From (2.4) in the proof of Theorem 2.3, it follows from the relationship (2.5) that

$$J(W_{11}, W_{12} \to dH_1, dD) = |D|^{p-q} \prod_{i<j}^{q} (d_i - d_j) J(H_1 dH_1' \to dH_1'). \tag{2.6}$$

Let

$$G = (HC)^{-1} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \qquad G_{11} : q \times q.$$

Then $dU = GWG'$ and hence following as in Theorem 2.3 [see the derivation after (2.4)], we get

$$\begin{aligned} &J(dU_{11}, dU_{12} \to W_{11}, W_{12}) \\ &\quad = |G_{11}|_+^{p-q+1} |G|_+^q \\ &\quad = |B|_+^q |G_{11}|_+^{p-q+1}. \end{aligned}$$

Note that

$$\begin{aligned} G &= C^{-1}H' = BH' = (BH_1', BH_2') \\ &= \begin{pmatrix} B_1 H_1' & B_1 H_2' \\ B_2 H_1' & B_2 H_2' \end{pmatrix}, \qquad B_1 : q \times p. \end{aligned}$$

Hence $|G_{11}|_+ = |B_1 H_1'|_+$. Now, from (2.6) and Theorem 2.3,

$$\begin{aligned} J(U_I \to V_I) &= J(dU_I \to dV_I) \\ &= J(dU_I \to W_I) J(W_I \to dH_1, dD) J(dH_1, dD \to dV_I) \\ &= \frac{|B|_+^q |B_1 H_1'|_+^{p-q+1}}{|H_{11}|_+^{p-q+1}}. \end{aligned}$$

□

The next theorem combines the results of Lemma 2.4 and Theorem 2.3.

THEOREM 2.6. *Let $X \in \mathcal{L}_{p,n}(n)$, $X = H_1' F L$, $H_1 \in \mathcal{H}_{n,p}$, $L \in \mathcal{H}_n$ and $S = XX' = H_1' F^2 H_1 = H_1' D H_1$. Then, if we write*

$$H_1 = n \overset{\quad n \qquad p-n}{\begin{pmatrix} H_{11} & H_{12} \end{pmatrix}} \quad \textit{and} \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{pmatrix},$$

$S_{11} = H_{11}' D H_{11}$, $S_{12} = H_{11}' D H_{12}$ *and* $S_{22} = S_{12}' S_{11}^{-1} S_{12}$, *the Jacobian of the transformation from* X_I *to* S_{11}, S_{12} *and* L *is given by*

$$J(X_I \to S_{11}, S_{12}) = 2^{-n} |S_{11}|^{(n-p-1)/2} g_{n,n}(L).$$

PROOF. From the transformations

$$X = H_1' F L, \qquad XX' = H_1' F^2 H_1 = H_1' D H_1 = S,$$

we get

$$\begin{aligned} J(X_I &\to S_I, L) \\ &= J(X_I \to H_1, F, L) J(H_1, F, L \to H_1, D, L) J(H_1, D, L \to S_I, L) \\ &= 2^{-n} |F|^{p-n} \prod_{i<j}^{n} (f_i^2 - f_j^2) g_{n,n}(L) g_{n,p}(H_1) 2^{-n} |F|^{-1} 2^n \\ &\qquad \times |H_{11}|_+^{-(p-n+1)} |D|^{n-p} \big(g_{n,p}(H_1)\big)^{-1} \left[\prod_{i<j} (f_i^2 - f_j^2) \right]^{-1} \\ &= 2^{-n} |D|^{(n-p-1)/2} |H_{11}|^{n-p-1} g_{n,n}(L) \\ &= 2^{-n} |S_{11}|^{(n-p-1)/2} g_{n,n}(L). \qquad \square \end{aligned}$$

3. Singular Wishart distributions. Let $X = (x_1, \dots, x_n)$ be $\sim N_{p,n}(0, \Sigma, I_n)$. That is, the n columns of the $p \times n$ matrix X are i.i.d. $N_p(0, \Sigma)$. We assume that Σ is p.d. and $n < p$. In this case, the $p \times p$ matrix $S = XX'$ is said to have singular Wishart distribution. The p.d.f. of X with respect to Lebesgue measure is given by

$$\frac{1}{(2\pi)^{pn/2} |\Sigma|^{n/2}} \left(\operatorname{etr} -\frac{1}{2} \Sigma^{-1} X X' \right). \tag{3.1}$$

We first obtain the distribution by using the singular-value decomposition method. Consider the transformation

$$X = H_1' F L,$$

where $H_1 \in \mathcal{H}_{n,p}$, $L \in \mathcal{H}_p$ and $F = \operatorname{diag}(f_1, \dots, f_n)$, $f_i > 0$. Then using the Jacobian of the transformation given in Lemma 2.4, the joint p.d.f. of (H_1, F, L) with respect to Lebesgue measure is obtained from (3.1) as

$$\frac{2^{-n}}{(2\pi)^{pn/2} |\Sigma|^{n/2}} \left(\operatorname{etr} -\frac{1}{2} \Sigma^{-1} H_1' F^2 H_1 \right) |F|^{p-n} \prod (f_i^2 - f_j^2) g_{n,n}(L) g_{n,p}(H_1).$$

Integrating out L over the space $LL' = I_n$, we get from (2.1) the joint p.d.f. of F and H_1 with respect to Lebesgue measure as

$$\frac{2^{-n} C(n,n)}{(2\pi)^{pn/2} |\Sigma|^{n/2}} \left(\operatorname{etr} -\frac{1}{2} \Sigma^{-1} H_1' F^2 H_1 \right) |F|^{p-n} \prod_{i<j} (f_i^2 - f_j^2) g_{n,p}(H_1).$$

Making the transformation $f_i^2 = d_i$, the joint p.d.f. of H_1 and $D = \operatorname{diag}(d_1, \dots, d_n)$ with respect to Lebesgue measure is

$$\frac{2^{-n}C(n,n)}{2^n(2\pi)^{pn/2}|\Sigma|^{n/2}}\left(\operatorname{etr} -\frac{1}{2}\Sigma^{-1}H_1'DH_1\right)|D|^{(p-n-1)/2}\prod_{i<j}(d_i - d_j)g_{n,p}(H_1).$$

Now consider the transformation

$$S = H_1'DH_1,$$

where S is a $p \times p$ symmetric matrix. Writing $H_1 = (H_{11}, H_{12})$, where $H_{11} : n \times n$ and

$$S = \begin{array}{c c} & \begin{array}{cc} n & p-n \end{array} \\ \begin{array}{c} n \\ p-n \end{array} & \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix} \end{array} = \begin{pmatrix} H_{11}'DH_{11} & H_{11}'DH_{12} \\ H_{12}'DH_{11} & H_{12}'DH_{12} \end{pmatrix},$$

gives $S_{11} = H_{11}'DH_{11}$, $S_{12} = H_{11}'DH_{12}$ and $S_{22} = H_{12}'DH_{12}$.

If we choose H_{11} such that it is nonsingular and we can do so, we find that

$$S_{22} = H_{12}'DH_{12} = H_{12}'DH_{11}(H_{11}'DH_{11})^{-1}H_{11}'DH_{12} = S_{12}'S_{11}^{-1}S_{12}.$$

Hence, S_{22} is functionally dependent on S_{12} and S_{11} and it is not a new transformation. Thus, we need the Jacobian of the transformation from (S_{11}, S_{12}) to (H_1, D) which is given in Theorem 2.3. Thus, the joint p.d.f. of S_{11} and S_{12} with respect to Lebesgue measure is

$$\frac{\pi^{n(n-p)/2}2^{-pn/2}}{\Gamma_n(\frac{n}{2})|\Sigma|^{n/2}}|S_{11}|^{(n-p-1)/2}\left(\operatorname{etr} -\frac{1}{2}\Sigma^{-1}S\right). \tag{3.2}$$

Among the many methods available in the literature (see S&K, page 73) for deriving the nonsingular Wishart distribution, the triangular factorization method appears to be the most popular. Thus, it would be appropriate to have a similar derivation in the singular case as well, which we do next.

Consider the transformation

$$X_{p\times n} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} L_1, \tag{3.3}$$

where T_1 is an $n \times n$ lower triangular matrix and T_2 is a $(p-n) \times n$ matrix. Since $q = n$, $L_1 \in \mathcal{H}_n$ and $|L_1|_+ = 1$, the joint p.d.f. of $T = (T_1', T_2')'$ and L_1, using Theorem 2.1, is given by

$$(2\pi)^{-pn/2}|\Sigma|^{-n/2}\left(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}TT'\right)\left(\prod_{i=1}^{n} t_{ii}^{n-i}\right)g_{n,n}(L_1). \tag{3.4}$$

Note that

$$TT' = \begin{pmatrix} T_1T_1' & T_1T_2' \\ T_2T_1' & T_2T_2' \end{pmatrix}. \tag{3.5}$$

Making the transformation

$$T_1T_1' = S_{11} \quad \text{and} \quad T_1T_2' = S_{12}, \tag{3.6}$$

we find that the Jacobian of these transformations is given by

$$J(T_1 \to S_{11})J(T_2 \to S_{12}) = \left(2^{-n}\prod_{i=1}^{n} t_{ii}^{-n+i-1}\right)|T_1|^{-(p-n)}.$$

Hence, the joint p.d.f. of S_{11} and S_{12} is given by

$$2^{-n}C(n,n)(2\pi)^{-pn/2}|\Sigma|^{-n/2}|S_{11}|^{(n-p-1)/2}\left(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}S\right), \tag{3.7}$$

where $S_{22} = S_{12}'S_{11}^{-1}S_{12}$. This may be called singular Wishart distribution. As usual, we denote it by $S \sim W_p(\Sigma, n)$, $n < p$. Thus, we get the following theorem.

THEOREM 3.1. *Let $X \sim N_{p,n}(0, \Sigma, I_n)$, $n < p$. Then the p.d.f. of the functionally independent elements of the matrix $S = XX'$ is given by* (3.7). *This is the joint p.d.f. of S_{11} and S_{12} and is the same as that obtained earlier by the singular-value decomposition method.*

The above result can easily be generalized to any $p \times n$ matrix X of rank n with p.d.f. given by $f(XX')$. Thus, we get the following corollary.

COROLLARY 3.1. *Let X be a $p \times n$ matrix of rank n with p.d.f. given by $f(XX')$. Let $X = TL_1$, as in* (3.3) *and $S = XX' = TT'$, where the upper left $n \times n$ submatrix of S is denoted by $S_{11} = T_1T_1'$. Then the p.d.f.'s of T and S are, respectively, given by*

$$\frac{2^n\pi^{n^2/2}}{\Gamma_n(\frac{n}{2})}\prod_{i=1}^{n} t_{ii}^{n-i} f(TT') \tag{3.8}$$

and

$$\frac{\pi^{n^2/2}}{\Gamma_n(\frac{n}{2})}|S_{11}|^{(n-p-1)/2} f(S). \tag{3.9}$$

COROLLARY 3.2. *Let X be a $p \times n$ matrix of rank n with p.d.f. given by $f(XX')$. Let $X = H_1'FL$, $H_1 \in \mathcal{H}_{n,p}$, $L \in \mathcal{H}_p$ and $S = XX' = H_1'F^2H_1 = H_1'DH_1$, where we write $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22}\end{pmatrix}$, $S_{22} = S_{12}'S_{11}^{-1}S_{12}$. Then the joint p.d.f. of S_{11} and S_{12} is given by* (3.9).

COROLLARY 3.3. *Let $S \sim W_p(\Sigma, n)$, $n \le p$, and $U = BSB'$, where B is a $p \times p$ nonsingular matrix. Then $U \sim W_p(B\Sigma B', n)$, $n < p$. That is, the p.d.f. of U is given by*

$$c|B\Sigma B'|^{-n/2}|U_{11}|^{(n-p-1)/2}\left(\operatorname{etr} -\tfrac{1}{2}(B\Sigma B')^{-1}U\right),$$

where

$$c = \frac{\pi^{n(n-p)/2} 2^{-pn/2}}{\Gamma_n(\frac{n}{2})},$$

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{12}' & U_{22} \end{pmatrix},$$

U_{11} *is an* $n \times n$ *nonsingular matrix and* $U_{22} = U_{12}' U_{11}^{-1} U_{12}$.

PROOF. Write $B' = (B_1', B_2')$, $B_1 : n \times p$, $S = H_1' D H_1$, $H_1 H_1' = I_n$ and $H_1 = (H_{11}, H_{12})$, where $H_{11} : n \times n$ and is nonsingular. Then

$$U_{11} = B_1 S B_1' = B_1 H_1' D H_1 B_1'$$

and, from Theorem 2.5,

$$J(S_I \to U_I) = |H_{11}|^{p-n+1} |B|^{-n} |B_1 H_1'|^{-(p-n+1)}. \tag{3.10}$$

Also

$$S_{11} = H_{11}' D H_{11} : n \times n.$$

Since, the p.d.f. of S is given by

$$\begin{aligned} &c|S_{11}|^{(n-p-1)/2} |\Sigma|^{-n/2} \big(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1} S\big) \\ &\quad = c|H_{11}' D H_{11}|^{(n-p-1)/2} |\Sigma|^{-n/2} \big(\operatorname{etr} -\tfrac{1}{2}(B\Sigma B')^{-1} B S B'\big), \end{aligned}$$

the p.d.f. of U is given by

$$c|B_1 H_1' D H_1 B_1'|^{(n-p-1)/2} |B \Sigma B'|^{-n/2} \big(\operatorname{etr} -\tfrac{1}{2}(B\Sigma B')^{-1} U\big).$$

In the next corollary we give the marginal distribution of S_{11} and the conditional distribution of S_{12} given S_{11}; the proof can be obtained along the lines of S&K, page 79. □

COROLLARY 3.4. *Let* $S \sim W_p(\Sigma, n)$, $n < p$ *and* $\Sigma > 0$, *where*

$$S = \begin{array}{c} \\ n \\ p-n \end{array} \!\!\begin{array}{c} \begin{array}{cc} n & p-n \end{array} \\ \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix} \end{array}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix},$$

$|S_{11}| \neq 0$ *and* $S_{22} = S_{12}' S_{11}^{-1} S_{12}$. *Then*:

(i) $S_{11} \sim W_n(\Sigma_{11}, n)$.

(ii) *The conditional distribution of* S_{12}' *given* S_{11} *is* $N_{p-n,n}(\beta S_{11}, \Sigma_{2.1}, S_{11})$, *where*

$$\Sigma_{2.1} = \Sigma_{22} - \Sigma_{12}' \Sigma_{11}^{-1} \Sigma_{12}, \qquad \beta = \Sigma_{12}' \Sigma_{11}^{-1}.$$

By using the results of Lemma 2.4, the p.d.f. of the nonzero eigenvalues of S in the case of $\Sigma = I$ can easily be obtained. Alternatively, we can use the fact that the nonzero eigenvalues of XX' are the same as those of $X'X$, which is nonsingular. Using either of the two methods, the p.d.f. of the nonzero eigenvalues $d_1 > \cdots > d_n$ of S when $\Sigma = I$ is given by

$$\frac{\pi^{n^2/2}}{2^{pn/2}\Gamma_n(\frac{n}{2})\Gamma_n(\frac{p}{2})}\left(\prod_{i=1}^{n} d_i^{(p-n-1)/2} e^{-d_i/2}\right)\prod_{i<j}^{n}(d_i - d_j). \tag{3.11}$$

Next, we consider the noncentral case; that is, let $X \sim N_{p,n}(\mu, \Sigma, I_n)$, where $n < p$. Then the p.d.f. of X is given by

$$\begin{aligned}&\big((2\pi)^{np/2}|\Sigma|^{n/2}\big)^{-1}\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}(X-\mu)(X-\mu)'\\ &\qquad\equiv k\big(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}XX'\big)(\operatorname{etr}\Sigma^{-1}X\mu'),\end{aligned}$$

where $k = ((2\pi)^{np/2}|\Sigma|^{n/2})^{-1}(\operatorname{etr} -\frac{1}{2}\Omega)$, $\Omega = \Sigma^{-1}\mu\mu'$. Make the transformation $X = TL$, where $T = (T_1{}', T_2{}')' : p \times n$ with $T_1 \in \mathcal{T}_+(n)$ and $L \in \mathcal{H}_n$. Then the joint p.d.f. of T and L is given by

$$k\prod_{i=1}^{n} t_{ii}^{n-i} g_{n,p}(L)\big(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}TT'\big)(\operatorname{etr}\Sigma^{-1}TL\mu').$$

Integrating out L, we get the p.d.f. of T as

$$kC(n,n)\prod_{i=1}^{n} t_{ii}^{n-i}\big(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-1}TT'\big)_0F_1\big(\tfrac{1}{2}n, \tfrac{1}{4}\Omega\Sigma^{-1}TT'\big),$$

from James (1964). Hence, the p.d.f. of $S = XX' = TT'$ is given by the following theorem.

THEOREM 3.2. *Let $X \sim N_{p,n}(\mu, \Sigma, I_n)$, $n < p$. Then the p.d.f. of $S = XX'$ is given by*

$$\frac{\pi^{n(n-p)/2}2^{-pn/2}}{\Gamma_n(\frac{n}{2})|\Sigma|^{n/2}}|S_{11}|^{(n-p-1)/2}\left(\operatorname{etr} -\frac{1}{2}\Sigma^{-1}S\right)_0 F_1\left(\frac{1}{2}n, \frac{1}{4}\Omega\Sigma^{-1}S\right).$$

We write it as $S \sim W_p(\Sigma, n, \Omega)$. Following the steps of Corollary 3.3, we obtain the following corollary.

COROLLARY 3.5. *Let $S \sim W_p(\Sigma, n, \Omega)$, $n < p$. Then for a $p \times p$ nonsingular matrix B, $U = BSB' \sim W_p(B\Sigma B', n, \Omega_1)$, $\Omega_1 = (B\Sigma B')^{-1}B\mu\mu'B'$.*

4. Singular multivariate beta distribution. We use the following definition of a multivariate beta distribution as given by Khatri (1970) and Mitra (1970).

DEFINITION 4.1. Let $X \sim N_{p,n_1}(0, \Sigma, I_{n_1})$ be independently distributed of $Y \sim N_{p,n_2}(0, \Sigma, I_{n_2})$ with $(n_1 + n_2) \geq p$. Let $Z = (XX' + YY')^{-1/2}X$, where $(XX' + YY')^{-1/2}$ is any nonsingular factorization of $(XX' + YY')$; $(XX' + YY')^{1/2}(XX' + YY')^{1/2'} = XX' + YY'$. Then $U = ZZ'$ is said to have a multivariate beta distribution, denoted by $B_p(n_1/2, n_2/2)$ with $n_1 + n_2 \geq p$; if $n_1 < p$, it is called a singular multivariate beta distribution.

An alternative definition in terms of Wishart distribution can also be given, namely

$$U = (V + W)^{-1/2} V (V + W)^{-1/2'},$$

where V and W are independently distributed as $W_p(\Sigma, n_1)$ and $W_p(\Sigma, n_2)$, respectively, with $n_1 + n_2 \geq p$; see Khatri (1970) or S&K, pages 93 and 96. However, we use the definition in terms of the normal random matrices. Recall that from Remark 2.1 in connection with Lemma 2.3, we may use the triangular factorization of $XX' + YY'$ or $V + W$ without any loss of generality and we do so in the following development.

It may be pointed out that Uhlig's (1994) Theorem 1 is Khatri's (1970) Theorem 2.

THEOREM 4.1. *Let $X \sim N_{p,m}(0, \Sigma, I_m)$ and $Y \sim N_{p,n}(0, \Sigma, I_n)$ be independently distributed with $\Sigma > 0$, $m \geq p$ and $n < p$. Let $XX' + YY' = TT'$, where T is a lower triangular matrix with positive diagonal elements $t_{ii} > 0$, $i = 1, \dots, p$. Then the distribution of $U = T^{-1}YY'T'^{-1}$ is given by*

$$\pi^{n(n-p)/2}\left[\Gamma_p\left(\frac{m+n}{2}\right)\Big/\Gamma_p\left(\frac{m}{2}\right)\Gamma_n\left(\frac{n}{2}\right)\right] \times |U_{11}|^{(n-p-1)/2}|I - U|^{(m-p-1)/2}, \tag{4.1}$$

where $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{11} & U_{12} \end{pmatrix}$, $U_{11} : n \times n$. We denote the p.d.f. given in (4.1) *as $M\beta_{\mathrm{I}}(p, n, m)$, $m \geq p$, $n < p$, and call it, as in S&K, singular multivariate beta Type* I *distribution.*

PROOF. From Lemma 2.3 and Remark 2.1, we may assume without loss of generality that $\Sigma = I$. The joint p.d.f. of X and Y in this case is given by

$$(2\pi)^{-pN/2}\left(\operatorname{etr} -\tfrac{1}{2}(XX' + YY')\right), \qquad N = m + n.$$

Let

$$(X, Y) = TH_1, \qquad H_1 \in \mathcal{H}_{p,N}, \qquad T \in \underline{\mathcal{T}}_+(p).$$

Using the Jacobian of the transformation from (X, Y) to (T, H_1) from Lemma 2.1, we get the joint p.d.f. of T and H_1 as

$$(2\pi)^{-pN/2} g_{p,N}(H_1) \prod_{i=1}^{p} t_{ii}^{N-i} (\operatorname{etr} -\tfrac{1}{2} T T').$$

Integrating out T, we get the p.d.f. of H_1 as

$$(C(p, N))^{-1} g_{p,N}(H_1).$$

Noting that

$$H_1 = T^{-1}(X, Y) = (H_{11}, H_{12}),$$

where $H_{11} : p \times m$ and $H_{12} : p \times n$, $m \geq p$, we find that

$$H_{12} = T^{-1} Y$$

and

$$H_{12} H_{12}' = T^{-1} Y Y' T^{-1'} = U.$$

From Lemma 2.2 with $r \to m$ and $n \to N$ the p.d.f. of H_{12} is given by

$$\frac{C(p, m)}{C(p, N)} |I_p - H_{12} H_{12}'|^{(m-p-1)/2}, \qquad H_{12} H_{12}' < I_p. \tag{4.2}$$

Using the singular-value decomposition of H_{12}, $H_{12} = M_1' F L$, $M_1 : n \times p$, $M_1 M_1' = I_n$ and $L : n \times n$, $L L' = I_n$ and then $U = H_{12} H_{12}' = M_1' D M_1$, we find from Theorem 2.6 that the p.d.f. of U is as given in the theorem after integrating out L.

Alternatively and more easily, we obtain the p.d.f. of U from Corollary 3.1. Note, as before, that

$$\begin{aligned} U &= H_{12} H_{12}' \\ &= M_1' D M_1 \\ &= \begin{pmatrix} M_{11}' \\ M_{12}' \end{pmatrix} D (M_{11}, \; M_{12}) \\ &= \begin{pmatrix} M_{11}' D M_{11} & M_{11}' D M_{12} \\ M_{12}' D M_{11} & M_{12}' D M_{12} \end{pmatrix} \\ &= \begin{pmatrix} U_{11} & U_{12} \\ U_{12}' & U_{22} \end{pmatrix}, \end{aligned}$$

where $M_1 = (M_{11}, M_{12})$, $M_{11} : n \times n$. Since

$$U_{22} = U_{12}' U_{11}^{-1} U_{12} = M_{12}' D M_{11} (M_{11}' D M_{11})^{-1} M_{11}' D M_{12} = M_{12}' D M_{12},$$

it follows that the p.d.f. given in the theorem is, in fact, the joint p.d.f. of U_{11} and U_{12}. $\square$

COROLLARY 4.1. *Let* $U \sim M\beta_{\mathrm{I}}(p, n, m)$, *the p.d.f. of which is given by* (4.1). *For any* $p \times p$ *orthogonal matrix* Γ, *let* $V = \Gamma U \Gamma'$. *Then* $V \sim M\beta_{\mathrm{I}}(p, n, m)$.

PROOF. Let $V = \Gamma U \Gamma'$. Then from Theorem 2.4, the Jacobian of the transformation from U_I to V_I is given by

$$J(U_I \to V_I) = |H_{11}|_+^{p-n+1} / |H_1 \Gamma_1'|_+^{p-n+1},$$

where

$$\begin{aligned} U &= H_1' D H_1, & H_1' H_1 &= I_n, \\ H_1 &= (H_{11}, H_{12}), & H_{11} &: n \times n,\ |H_{11}| \neq 0, \\ \Gamma' &= (\Gamma_1', \Gamma_2'), & \Gamma_1' &: p \times n. \end{aligned}$$

Hence,

$$\begin{aligned} U_{11} &= H_{11}' D H_{11}, \\ V_{11} &= \Gamma_1 H_1' D H_1 \Gamma_1'. \end{aligned}$$

Thus, the p.d.f. of V is given by

$$\begin{aligned} &c_1 |H_{11}|^{p-n+1} |H_1 \Gamma_1'|^{-(p-n+1)} |H_{11}' D H_{11}|^{(n-p-1)/2} |I - V|^{(m-p-1)/2} \\ &\quad = c_1 |\Gamma_1 H_1' D H_1 \Gamma_1'|^{(n-p-1)/2} |I - V|^{(m-p-1)/2} \\ &\quad = c_1 |V_{11}|^{(n-p-1/2} |I - V|^{(m-p-1)/2}, \end{aligned}$$

where

$$c_1 = \pi^{n(n-p)/2} \left[\Gamma_p\left(\frac{m+n}{2}\right) \Big/ \Gamma_p\left(\frac{m}{2}\right) \Gamma_n\left(\frac{n}{2}\right) \right]. \qquad \square$$

COROLLARY 4.2. *Let* $U \sim M\beta_{\mathrm{I}}(p, n, m)$, *where*

$$U = \begin{array}{c c} & \begin{array}{cc} n & p-n \end{array} \\ \begin{array}{c} n \\ p-n \end{array} & \begin{pmatrix} U_{11} & U_{12} \\ U_{12}' & U_{22} \end{pmatrix} \end{array},$$

$|U_{11}| \neq 0$, $U_{22} = U_{12}' U_{11}^{-1} U_{12}$ *and where the p.d.f. of* U *is given by* (4.1). *Then the p.d.f. of* U_{11} *is given by*

$$c_2 |U_{11}|^{1/2} |I - U_{11}|^{(m-n-1)/2}, \qquad c_2 = \Gamma_n\left(\frac{m+n}{2}\right) \Big/ \Gamma_n\left(\frac{n}{2}\right) \Gamma_n\left(\frac{m}{2}\right).$$

PROOF. We first note that

$$\begin{aligned} |I - U| &= |I - U_{11}| \big| I - U_{22} - U_{12}'(I - U_{11})^{-1} U_{12} \big| \\ &= |I - U_{11}| \big| I - U_{12}' [U_{11}^{-1} + (I - U_{11})^{-1}] U_{12} \big| \\ &= |I - U_{11}| \big| I - U_{12}' U_{11}^{-1/2} (I - U_{11})^{-1} U_{11}^{-1/2} U_{12} \big|. \end{aligned}$$

Let

$$W = (I - U_{11})^{-1/2} U_{11}^{-1/2} U_{12}.$$

Then

$$J(U_{12} \to W) = |U_{11}|^{(p-n)/2} |I - U_{11}|^{(p-n)/2}.$$

Hence, from (4.1) the joint p.d.f. of U_{11} and W is given by

$$\text{Const} \cdot |U_{11}|^{-1/2} |I - U_{11}|^{(m-n-1)/2} |I - WW'|^{(m-p-1)/2}.$$

Integrating out W, we get the p.d.f. of U_{11} as given in the corollary. $\square$

The joint p.d.f. of the nonzero eigenvalues d_i of $H_{12}H_{12}'$ can be obtained from Lemma 2.4 or directly from (4.2) by using Lemma 3.2.3 (page 76) and Theorem 1.11.5 (page 31) of S&K, and the fact that

$$|I_p + AB| = |I_q + BA|$$

for $A : p \times q$ and $B : q \times p$. The p.d.f. of d_i is given by

$$\frac{\Gamma_p(\frac{m+n}{2})\pi^{n^2/2}}{\Gamma_p(\frac{m}{2})\Gamma_n(\frac{n}{2})\Gamma_n(\frac{p}{2})} \left(\prod_{i=1}^{n} d_i^{(p-n-1)/2} (1 - d_i)^{(m-p-1)/2} \right) \prod_{i<j} (d_i - d_j). \tag{4.3}$$

The above p.d.f. differs from the one given by Díaz-García and Gutiérrez-Jáimez (1997); they used $\Gamma_p(\frac{n}{2})$ in place of $\Gamma_n(\frac{p}{2})$ in the denominator.

THEOREM 4.2. *Let $X \sim N_{p,m}(0, I, I)$ and $Y \sim N_{p,n}(0, I, I)$ be independently distributed with $m \geq p$ and $n \leq p$. Let $XX' = W^{1/2}W^{1/2'}$, where $W^{1/2}$ is any nonsingular factorization of XX'. Define*

$$G = W^{-1/2'}(YY')W^{-1/2}.$$

Then the p.d.f. of G is given by

$$\frac{\pi^{n(n-p)/2}\Gamma_p(\frac{m+n}{2})}{\Gamma_p(\frac{m}{2})\Gamma_n(\frac{n}{2})} |G_{11}|^{(n-p-1)/2} |I + G|^{-(m+n)/2}, \tag{4.4}$$

where $G = \binom{G_{11}\ G_{12}}{G_{12}'\ G_{22}}$. We denote the p.d.f. given in (4.4) *as $M\beta_{\mathrm{II}}(p, n, m)$, $m \geq p$, $n < p$, and, as in S&K, call it singular multivariate beta Type* II *distribution.*

PROOF. Let $W = XX'$. Then the joint p.d.f. of W and Y is given by

$$(2\pi)^{-pn/2} C_1(p, m) |W|^{(m-p-1)/2} \left(\operatorname{etr} -\tfrac{1}{2}(W + YY')\right),$$

where

$$C_1(p, m) = \left(2^{pm/2}\Gamma_p\left(\frac{m}{2}\right)\right)^{-1}.$$

Making the transformation

$$Z = (W^{-1/2})'Y$$

and integrating out W, we find that the p.d.f. of Z is given by

$$(2\pi)^{-pn/2}\frac{C_1(p,m)}{C_1(p,m+n)}|I+ZZ'|^{-(m+n)/2}.$$

Hence from Theorem 2.6, the p.d.f. of G is given as in the theorem after integrating out L_1. $\square$

COROLLARY 4.3. *Let $G \sim M\beta_{\text{II}}(p,n,m)$, the p.d.f. of which is given in* (4.4). *Then for any $p \times p$ orthogonal matrix Γ, the p.d.f. of $P = \Gamma G \Gamma'$ is again $M\beta_{\text{II}}(p,n,m)$.*

COROLLARY 4.4. *Let $G \sim M\beta_{\text{II}}(p,n,m)$, where*

$$G = \begin{array}{c} \\ n \\ p-n \end{array}\begin{array}{c} \begin{array}{cc} n & p-n \end{array} \\ \begin{pmatrix} G_{11} & G_{12} \\ G_{12}' & G_{22} \end{pmatrix} \end{array},$$

$|G_{11}| \neq 0$, $G_{22} = G_{12}'G_{11}^{-1}G_{12}$ *and its p.d.f. is given by* (4.4). *Then the p.d.f. of G_{11} is given by*

$$\frac{\Gamma_n(\frac{m+n}{2})}{\Gamma_n(\frac{m}{2})\Gamma_n(\frac{n}{2})}|G_{11}|^{-1/2}|I+G_{11}|^{-(m+n)/2}.$$

THEOREM 4.3. *Let $X \sim N_{p,m}(0,I,I)$ and $Y \sim N_{p,n}(0,I,I)$ be independently distributed. Let $d_1 > \cdots > d_n > 0$ be the nonzero eigenvalues of $(XX')^{-1}YY'$ for $m \geq p$, $n < p$. Then the joint p.d.f. of $d_1,\dots,d_n$ is given by*

$$\frac{\pi^{n^2/2}\Gamma_p(\frac{n+m}{2})}{\Gamma_p(\frac{m}{2})\Gamma_n(\frac{n}{2})\Gamma_n(\frac{p}{2})}\prod_{i=1}^{n} d_i^{(p-n-1)/2}(1+d_i)^{-(n+m)/2}\prod_{i<j}^{n}(d_i-d_j).$$

5. Pseudo Wishart distribution. In this section we consider the case when $\rho(\Sigma) = r < p$, where $\rho(\Sigma)$ denotes the rank of Σ. Let $X \sim N_{p,n}(0, \Sigma, I_n)$, where $\rho(\Sigma) = r < p$, but $r > n$. Then from Khatri (1968) or S&K, page 43, the p.d.f. of X is given by

$$(2\pi)^{-rn/2}\prod_{i=1}^{r}\lambda_i^{-n/2}\left(\operatorname{etr} -\tfrac{1}{2}\Sigma^{-}XX'\right),$$

with respect to Lebesgue measure on the hyperplane $\Lambda_2 X = 0$ (with probability one), where Λ_2 is defined below, and where Σ^- is a generalized inverse of Σ, $\Sigma\Sigma^-\Sigma = \Sigma$ and λ_i's are the nonzero eigenvalues of Σ. Consider an orthogonal matrix $\Lambda' = (\Lambda_1', \Lambda_2')$, where Λ_1' is a $p \times r$ matrix such that $\Lambda_1\Lambda_1' = I_r$ and

$$\Sigma = \Lambda'\begin{pmatrix} D_\lambda & 0 \\ 0 & 0 \end{pmatrix}\Lambda = \Lambda_1' D_\lambda \Lambda_1, \qquad D_\lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_r).$$

Then, it follows that

$$\Lambda\Sigma^{-}\Lambda' = \Lambda(\Lambda_1' D_\lambda \Lambda_1)^{-}\Lambda' = \begin{pmatrix} D_\lambda^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, the p.d.f. of $\Lambda X = Y$ is given by

$$\left((2\pi)^{-rn/2}\prod_{i=1}^{r}\lambda_i^{-n/2}\right)\operatorname{etr} -\tfrac{1}{2}D_\lambda^{-1}Y_1Y_1',$$

where $Y' = (Y_1', Y_2')$, $Y_1 \sim N_{r,n}(0, D_\lambda, I_n)$ and $Y_2 = 0$ with probability 1. Hence, from Theorem 3.1, the p.d.f. of $V = Y_1Y_1'$ is given by

$$\frac{\pi^{n(n-r)/2}2^{-rn/2}}{\Gamma_n(\frac{n}{2})\prod_{i=1}^{r}\lambda_i^{n/2}}|V_{11}|^{(n-r-1)/2}\left(\operatorname{etr} -\frac{1}{2}D_\lambda^{-1}V\right),$$

where

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}' & V_{22} \end{pmatrix}, \qquad V_{11} : n \times n$$

and $V_{22} = V_{12}'V_{11}^{-1}V_{12}$. Alternatively, we can write $Y_1 = M_1'FL$, $Y_1Y_1' = V = M_1'DM_1$ where $L \in \mathcal{H}_p$, $F^2 = D$, $M_1 : n \times r$ and $M_1M_1' = I_n$. Hence, $V_{11} = M_{11}'DM_{11}$ with $M_1 = (M_{11}, M_{12})$, $M_{11} \times n \times n$. Use of Theorem 2.6 gives the result.

To write the p.d.f. in terms of $S = XX'$, we can use either Corollary 3.1 or 3.2. To use Corollary 3.2, we write $X = H_1'FL$, $H_1 \in \mathcal{H}_{p,n}$, $L \in \mathcal{H}_n$ and $F = \operatorname{diag}(f_1, \dots, f_n)$, $f_i > 0$, $S = XX' = H_1'F^2H_1 = H_1'DH_1$, giving $S_{11} = H_{11}'DH_{11}$ and $H_1 = (H_{11}, H_{12})$. Hence, we get the following theorem.

THEOREM 5.1. *Let $X \sim N_{p,n}(0, \Sigma, I_n)$, $\rho(\Sigma) = r > n$. Then the p.d.f. of $S = XX'$ is given by*

$$\frac{\pi^{n(n-r)/2}2^{-rn/2}}{\Gamma_n(\frac{n}{2})\prod_{i=1}^{r}\lambda_i^{n/2}}|S_{11}|^{(n-r-1)/2}\left(\operatorname{etr} -\frac{1}{2}\Sigma^{-}S\right).$$

The distribution of $S = XX'$ is called a pseudo Wishart distribution as defined by S&K, page 72.

We now obtain results analogous to the one given in (3.8), when $X \sim N_{p,n}(0, \Sigma, I_n)$, $\rho(\Sigma) = r \le p$ and $\rho(X) = q = \min(r, n)$. We can write

$$\begin{aligned} X &= \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} L_1 \\ &\equiv TL_1, \end{aligned}$$

where $T_1 : q \times q$ is a triangular matrix with positive diagonal elements t_{ii}, $T_2 : (p - q) \times q$ matrix and $L_1 \in \mathcal{H}_{q,n}$. Hence, the p.d.f. of T and L_1 is given by

$$\left[(2\pi)^{-rn/2}\prod_{i=1}^{r}\lambda_i^{-n/2}\right]\left(\prod_{i=1}^{q}t_{ii}^{n-i}\right)g_{q,n}(L_1)|L_{11}|_+^{p-q}(\operatorname{etr} -\tfrac{1}{2}\Sigma^- TT'),$$

where the $q \times n$ matrix $L_1 = (L_{11}, L_{12})$, $L_{11} : q \times q$ and $L_{12} : q \times (n - q)$. Integrating out L_1, the p.d.f. of T is given by

$$C(q,n)\left[(2\pi)^{-rn/2}\prod_{i=1}^{r}\lambda_i^{-n/2}\right]K\left(\prod_{i=1}^{q}t_{ii}^{n-i}\right)(\operatorname{etr} -\tfrac{1}{2}\Sigma^- TT'),$$

where, from Lemma 2.2 and Corollary 2.1 with $p \to q$ and $r \to q$,

$$\begin{aligned} K &= [C(q,n)]^{-1}\int_{L_1\in\mathcal{H}_{q,n}}|L_{11}L_{11}'|^{(p-q)/2}g_{q,n}(L_1)\,dL_1 \\ &= \frac{C(q,q)}{C(q,n)}\int_{L_{12}L_{12}'<I_q}|I_q - L_{12}L_{12}'|^{(p-q-1)/2}\,dL_{12} \\ &= \frac{C(q,q)}{C(q,n)}\frac{C(q,n+\alpha)}{C(q,q+\alpha)}, \qquad \alpha = p - q. \end{aligned}$$

Hence, we get the following theorem.

THEOREM 5.2. *Let $X \sim N_{p,n}(0, \Sigma, I_n)$, $\rho(\Sigma) = r \le p$ and $\rho(X) = q = \min(r, n)$. Consider the transformation $X' = L_1'T' = L_1'(T_1', T_2')$, where $L_1 \in \mathcal{H}_{q,n}$ and T_1 is a $q \times q$ lower triangular matrix with positive diagonal elements t_{ii} and T_2 is a $(p - q) \times q$ matrix. Then the p.d.f. of T is given by*

$$\frac{C(q,q)C(q,n+\alpha)}{C(q,q+\alpha)}\left[(2\pi)^{-rn/2}\prod_{i=1}^{r}\lambda_i^{-n/2}\right]\left[\prod_{i=1}^{q}t_{ii}^{n-i}\right]\left[\operatorname{etr}\left(-\frac{1}{2}\Sigma^- TT'\right)\right],$$
$$\alpha = (p - q).$$

Acknowledgment. I wish to thank the referees and the Associate Editor who handled this article for their very helpful suggestions.

REFERENCES

DÍAZ-GARCÍA, J. A. and GUTIÉRREZ-JÁIMEZ, R. (1997). Proof of the conjectures of H. Uhlig on the singular multivariate beta and the Jacobian of a certain matrix transformation. *Ann. Statist.* **25** 2018–2023.

DÍAZ-GARCÍA, J. A., GUTIÉRREZ-JÁIMEZ, R. and MARDIA, K. V. (1997). Wishart and pseudo-Wishart distributions and some applications to shape theory. *J. Multivariate Anal.* **63** 73–87.

JAMES, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

KHATRI, C. G. (1968). Some results for the singular normal multivariate regression models. *Sankhyā Ser. A* **30** 267–280.

KHATRI, C. G. (1970). A note on Mitra's paper "A density free approach to the matrix variate beta distribution." *Sankhyā Ser. A* **32** 311–317.

MITRA, S. K. (1970). A density free approach to the matrix variate beta distribution. *Sankhyā Ser. A* **32** 81–88.

OLKIN, I. (1951). On distribution problems in multivariate analysis. *Mimeograph Series* **43** 1–126. Institute of Statistics, Univ. North Carolina, Chapel Hill.

ROY, S. N. (1957). *Some Aspects of Multivariate Analysis*. Wiley, New York.

SRIVASTAVA, M. S. and KHATRI, C. G. (1979). *An Introduction to Multivariate Statistics*. North-Holland, New York.

UHLIG, H. (1994). On singular Wishart and singular multivariate beta distributions. *Ann. Statist.* **22** 395–405.

DEPARTMENT OF STATISTICS
UNIVERSITY OF TORONTO
100 ST. GEORGE STREET
TORONTO, ONTARIO
CANADA M5S 3G3
E-MAIL: srivasta@utstat.Toronto.edu