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ON THE ASYMPTOTIC DISTRIBUTION OF SCRAMBLED NET QUADRATURE

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Recently, in a series of articles, Owen proposed the use of scrambled (t, m, s) nets and (t, s) sequences in high-dimensional numerical integration. These scrambled nets and sequences achieve the superior accuracy of equidistribution methods while allowing for the simpler error estimation techniques of Monte Carlo methods. The main aim of this article is to use Stein's method to study the asymptotic distribution of the scrambled $(0, m, s)$ net integral estimate. In particular, it is shown that, for suitably smooth integrands on the s-dimensional unit hypercube, the estimate has an asymptotic normal distribution.

1. Introduction. Let X be a random vector uniformly distributed on the s-dimensional unit hypercube $[0, 1)^s$ and let f be an integrable function from $[0, 1)^s$ to $\mathcal{R}$. An objective of many computer experiments [see, e.g., McKay, Conover and Beckman (1979), Stein (1987), Owen (1992a) and Tang (1993)] is to estimate

$$\mu = Ef(X) = \int_{[0,1)^s} f(x)\,dx \tag{1}$$

using a fixed number of function evaluations. It is well known [see Owen (1995)] that as the dimension s increases, Monte Carlo methods and (deterministic) equidistribution methods become competitive and ultimately dominant. Davis and Rabinowitz [(1984), Chapter 5.10] considered $s > 15$ to be a high enough dimensionality that sampling or equidistribution methods are indicated. Evans and Swartz [(2000), Chapter 1] gave an updated discussion of the dimension effect in numerical integration.

Among equidistribution methods, (t, m, s) nets and (t, s) sequences are one of the most popular and successful. They have been developed by Sobol', Faure, Niederreiter and Xing, among others. A comprehensive account of (t, m, s) nets and (t, s) sequences can be found in Niederreiter [(1992), Chapter 4; see also Niederreiter and Xing (1998) for more recent developments]. It is generally the case that equidistribution methods offer superior convergence rates and accuracy, while Monte Carlo methods allow for simpler error estimation techniques. With this as motivation, Owen (1995, 1997a, b, 1998), in a series of articles, proposed

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the use of scrambled (t, m, s) nets and (t, s) sequences in high-dimensional numerical integration. The scrambled net and sequence approach can be regarded as a hybrid of equidistribution and Monte Carlo methods in that it possesses the superior accuracy of the former as well as the simpler error estimation techniques of the latter. Owen further showed that the resulting scrambled net or sequence estimator for μ is unbiased and has a variance that is $o(1/n)$ along the sample size sequence $n = \lambda b^m$, $1 \le \lambda < b$, $0 \le m$, where b denotes the base of the net or sequence. Thus for any nonconstant f, the ratio of the scrambled net variance to the usual Monte Carlo variance tends to zero as $n \to \infty$. Generalizations of Owen's work can be found in Yue (1999), Yue and Mao (1999) and Hickernell and Yue (2000).

The main purpose of this article is to study the asymptotic distribution of the scrambled net estimator. For instance, such a result will be useful in the construction of asymptotically valid confidence intervals for μ. The remainder of this article proceeds as follows. Section 2 gives a brief review of nets, (t, s) sequences and scrambled (t, m, s) nets. Following Owen (1997a), Section 3 develops a nested analysis of variance (ANOVA) type decomposition of f. This facilitates the computation of the variance of the scrambled net estimator. In particular, Theorem 1 shows that for a suitably smooth integrand f on $[0, 1)^s$, the variance of the scrambled $(0, m, s)$ net estimator $\hat{\mu}_{0,m,s}$ is exactly of order $O(m^{s-1} b^{-3m})$ as $m \to \infty$.

Stein (1972) introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation to the distribution of a sum of dependent random variables. Since then, Stein's method has found considerable application in combinatorics, probability and statistics [see, e.g., Stein (1986), Bolthausen and Götze (1993) and Shorack (2000), Chapter 11]. In Section 4, Stein's method is used to determine the asymptotic distribution of the scrambled $(0, m, s)$ net estimator $\hat{\mu}_{0,m,s}$. In particular, Theorem 2 establishes the asymptotic normality of $\hat{\mu}_{0,m,s}$ under weak smoothness conditions on f. We note that Hong, Hickernell and Wei (2001) exhibited additional empirical evidence that the central limit effect for scrambled nets can take place for reasonable sample sizes.

The Appendix consists of technical results that are needed in the proofs of Proposition 1 and Theorem 3. We end this Introduction with a remark on the notation used in this article. The indicator function is given by $\mathcal{I}\{\cdot\}$ and $\|\cdot\|_s$ denotes the Euclidean norm in $\mathcal{R}^s$. If $g : \mathcal{R} \to \mathcal{R}$ is a differentiable function, we write $g^{(1)}$ as its first derivative and if x is a vector, then x' is its transpose. Finally, $\lfloor \cdot \rfloor$ denotes the greatest integer function, while $\#\Omega$ denotes the cardinality of a set Ω.

2. Nets, (t, s) sequences and scrambled (t, m, s) nets. Let $b \ge 2$ and $s \ge 1$ be integers. An elementary interval in base b is a subset of $[0, 1)^s$ of the form

$$\mathcal{E} = \prod_{j=1}^{s} \left[\frac{c_j}{b^{k_j}}, \frac{c_j + 1}{b^{k_j}} \right)$$

for integers c_j, k_j with $k_j \geq 0$ and $0 \leq c_j \leq b^{k_j} - 1$.

DEFINITION. Let $0 \leq t \leq m$ be integers. A finite sequence $\{A_i : i = 1, \ldots, b^m\}$ of points from $[0, 1)^s$ is a (t, m, s) net in base b if every elementary interval $\mathcal{E}$ in base b of s-dimensional Lebesgue measure b^{t-m} satisfies

$$\sum_{i=1}^{b^m} \mathcal{I}\{A_i \in \mathcal{E}\} = b^t.$$

DEFINITION. For $t \geq 0$, an infinite sequence $\{A_i : i = 1, 2, \ldots\}$ of points from $[0, 1)^s$ is a (t, s) sequence in base b if for all integers $k \geq 0$ and $m \geq t$, the finite sequence $\{A_i : i = kb^m + 1, \ldots, (k+1)b^m\}$ is a (t, m, s) net in base b.

The usual (t, m, s) net or (t, s) sequence estimate for μ is given by

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} f(A_i).$$

If f is of finite total variation in the sense of Hardy and Krause, it follows from the Koksma–Hlawka inequality that $|\hat{\mu} - \mu| = O((\log_b n)^{s-1}/n)$ if $\{A_i : i = 1, \ldots, n\}$ is a (t, m, s) net in base b and $|\hat{\mu} - \mu| = O((\log_b n)^s/n)$ if $\{A_i : i \geq 1\}$ is a (t, s) sequence in base b. For more precise statements as well as details of the above statements and results, refer to Owen [(1997a), page 1887] and Niederreiter [(1992), Theorems 4.10 and 4.17].

Owen (1995) introduced the idea of scrambled (t, m, s) nets as follows. Suppose that $\{A_i = (A_{i,1}, \ldots, A_{i,s})' : i = 1, \ldots, b^m\}$ is a (t, m, s) net in base b. We observe that $A_{i,j}$ can be expressed as

$$A_{i,j} = \sum_{k=1}^{\infty} a_{i,j,k} b^{-k} \tag{2}$$

for suitable integers $0 \leq a_{i,j,k} \leq b - 1$. Let

$$\{\pi_j, \pi_{j;a_1}, \pi_{j;a_1,a_2}, \pi_{j;a_1,a_2,a_3}, \ldots : 1 \leq j \leq s, 0 \leq a_k \leq b-1, k = 1, 2, \ldots\} \tag{3}$$

be a set of mutually independent random permutations of $\{0, 1, \ldots, b-1\}$, where each of these permutations is uniformly distributed over its $b!$ possible values. Now a scrambled (t, m, s) net in base b has the form $\{X_i = (X_{i,1}, \ldots, X_{i,s})' : i = 1, \ldots, b^m\}$, where

$$X_{i,j} = \sum_{k=1}^{\infty} x_{i,j,k} b^{-k}$$

and, for $1 \leq i \leq b^m$, $1 \leq j \leq s$,

$$\begin{aligned} x_{i,j,1} &= \pi_j(a_{i,j,1}), \\ x_{i,j,k} &= \pi_{j;a_{i,j,1},\ldots,a_{i,j,k-1}}(a_{i,j,k}), \qquad \forall k \geq 2. \end{aligned} \tag{4}$$

Owen [(1995), pages 307–308] further showed that $\{X_i : i = 1, \dots, b^m\}$ is also a (t, m, s) net in base b with probability 1 and that for each i, X_i has the uniform distribution on $[0, 1)^s$.

For the remainder of this article, we shall assume that the integrand f in (1) is smooth in the following sense.

DEFINITION. A real-valued function f on $[0, 1)^s$ is smooth and has a Lipschitz continuous mixed partial of order s if there exist finite constants $B \geq 0$ and $\beta \in (0, 1]$ such that

$$\left| \frac{\partial^s}{\partial x_1 \cdots \partial x_s} f(x) - \frac{\partial^s}{\partial x_1 \cdots \partial x_s} f(y) \right| \leq B\|x - y\|_s^\beta \qquad \forall x, y \in [0, 1)^s.$$

The estimator for μ that we are concerned with in this article is based on the scrambled (t, m, s) net $\{X_i : i = 1, \dots, b^m\}$, namely

$$\hat{\mu}_{t,m,s} = \frac{1}{b^m} \sum_{i=1}^{b^m} f(X_i). \tag{5}$$

Since X_i is uniformly distributed over $[0, 1)^s$, $\hat{\mu}_{t,m,s}$ is an unbiased estimator for μ.

3. Nested ANOVA decomposition. Owen (1997a, b, 1998) investigated the variance of the scrambled (t, m, s) net estimator $\hat{\mu}_{t,m,s}$ by applying an s-dimensional base b Haar multiresolution analysis to f. More precisely, for any integer $k \geq 0$, let $\mathcal{Y}_k$ be the linear span of the functions $\{\psi_{k,t,c} : t \geq 0, 0 \leq c \leq b - 1\}$, where

$$\psi_{k,t,c}(x) = b^{(k+1)/2} \mathcal{I}\{\lfloor b^{k+1} x \rfloor = bt + c\} - b^{(k-1)/2} \mathcal{I}\{\lfloor b^k x \rfloor = t\} \qquad \forall x \in [0, 1)$$

for integers $t \geq 0$ and $0 \leq c \leq b - 1$. We observe that for arbitrary (but fixed) integers t_1 and t_2, the functions in $\mathcal{Y}_k$ are constant on $[t_1 b^{-k-1}, (t_1 + 1)b^{-k-1})$ and integrate to zero over $[t_2 b^{-k}, (t_2 + 1)b^{-k})$. Next let $\mathcal{U}_0$ denote the space of functions that are constant on $[0, 1)$ and

$$\mathcal{U}_k = \{g + g_0 + \cdots + g_{k-1} : g \in \mathcal{U}_0,\ g_j \in \mathcal{Y}_j,\ j = 0, \dots, k - 1\} \qquad \forall k \geq 1.$$

Then it is well known that $\bigcup_{k=0}^\infty \mathcal{U}_k$ is dense in $L^2([0, 1))$ and $\bigcap_{k=0}^\infty \mathcal{U}_k = \mathcal{U}_0$. We further observe from Owen [(1997a), page 1897] that a typical basis function for $L^2([0, 1)^s)$ is of the form

$$\prod_{r=1}^{l} \psi_{k_{j_r}, t_{j_r}, c_{j_r}}(x_{j_r}) \qquad \forall (x_1, \dots, x_s)' \in [0, 1)^s, \tag{6}$$

where $1 \leq j_1 < \cdots < j_l \leq s$, and $k_{j_r} \geq 0$, $0 \leq t_{j_r} \leq b^{k_{j_r}} - 1$, $0 \leq c_{j_r} \leq b - 1$ whenever $1 \leq r \leq l$. Here by convention, an empty product (i.e., $l = 0$) is taken to be 1. Hence for a smooth function f, it follows from (6.6) of Owen [(1997a), page 1898] that

$$f(x) = \mu + \sum_{l=1}^{s} \sum_{1 \le j_1 < \cdots < j_l \le s} \sum_{k_{j_1}=0}^{\infty} \sum_{t_{j_1}=0}^{b^{k_{j_1}}-1} \sum_{c_{j_1}=0}^{b-1} \cdots \sum_{k_{j_l}=0}^{\infty} \sum_{t_{j_l}=0}^{b^{k_{j_l}}-1} \sum_{c_{j_l}=0}^{b-1} \left\langle f, \prod_{r=1}^{l} \psi_{k_{j_r},t_{j_r},c_{j_r}} \right\rangle \prod_{r=1}^{l} \psi_{k_{j_r},t_{j_r},c_{j_r}}(x_{j_r}) \qquad \forall x \in [0,1)^s, \tag{7}$$

where μ is as in (1) and the inner product $\langle \cdot, \cdot \rangle$ is defined as

$$\left\langle f, \prod_{r=1}^{l} \psi_{k_{j_r},t_{j_r},c_{j_r}} \right\rangle = \int_{[0,1)^s} f(x) \left[\prod_{r=1}^{l} \psi_{k_{j_r},t_{j_r},c_{j_r}}(x_{j_r}) \right] dx. \tag{8}$$

REMARK. As noted by Owen [(1997a), page 1895], the basis functions in (6) are not mutually orthogonal, but Lemma 3 of Owen (1997a) shows that (7) is still valid.

Next let $\hat{\mu}_{0,m,s}$ be as in (5) with $t = 0$ and let $\{X_i : i = 1, \ldots, b^m\}$ be a scrambled $(0, m, s)$ net in base b. Assuming that $\mathrm{Var}(\hat{\mu}_{0,m,s}) = \sigma_{0,m,s}^2 > 0$, we define

$$W = \sigma_{0,m,s}^{-1}(\hat{\mu}_{0,m,s} - \mu). \tag{9}$$

For simplicity let

$$\{U[\tilde{c}_{j,1}, \ldots, \tilde{c}_{j,u_j} : 1 \le j \le s] : 0 \le \tilde{c}_{j,1}, \ldots, \tilde{c}_{j,u_j} \le b-1,\ u_j \ge 0,\ 1 \le j \le s\}$$

be a set of mutually independent random vectors, where each $U[\tilde{c}_{j,1}, \ldots, \tilde{c}_{j,u_j} : 1 \le j \le s]$ has the uniform distribution on the elementary interval

$$\prod_{j=1}^{s} \left[\sum_{k=1}^{u_j} \tilde{c}_{j,k} b^{-k}, b^{-u_j} + \sum_{k=1}^{u_j} \tilde{c}_{j,k} b^{-k} \right).$$

Furthermore we assume that the U's are independent of the π's [defined as in (3)]. For nonnegative integers $u_1^*, u_1, \ldots, u_s^*, u_s$, we write:

(i) $(u_1^*, \ldots, u_s^*) \preceq (u_1, \ldots, u_s)$ if and only if $u_j^* \le u_j$ for all $1 \le j \le s$;
(ii) $(u_1^*, \ldots, u_s^*) \prec (u_1, \ldots, u_s)$ if and only if $u_j^* \le u_j$ for all $1 \le j \le s$ with at least one strict inequality.

With (7) as motivation, the following construction establishes a nested ANOVA decomposition of $Ef(U)$. For integers $u_j \geq 0$, $0 \leq \tilde{c}_{j,k} \leq b-1$, $1 \leq j \leq s$, $k \geq 1$, define recursively

$$\begin{aligned}
&\nu_{u_1,\dots,u_s}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j} : 1 \leq j \leq s] \\
&\quad = Ef(U[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j} : 1 \leq j \leq s]) \\
&\qquad - \sum_{u_1^*,\dots,u_s^* : (0,\dots,0) \preceq (u_1^*,\dots,u_s^*) \prec (u_1,\dots,u_s)} \nu_{u_1^*,\dots,u_s^*}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j^*} : 1 \leq j \leq s]
\end{aligned}$$

and hence

$$\begin{aligned}
&\lim_{u_1,\dots,u_s \to \infty} Ef(U[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j} : 1 \leq j \leq s]) \\
&\qquad = \sum_{u_1,\dots,u_s \geq 0} \nu_{u_1,\dots,u_s}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j} : 1 \leq j \leq s]. \tag{10}
\end{aligned}$$

Writing $|u| = \sum_{j=1}^{s} \mathcal{I}\{u_j \geq 1\}$ such that $1 \leq j_1 < \cdots < j_{|u|} \leq s$ and $u_j \geq 1$ if and only if $j \in \{j_1,\dots,j_{|u|}\}$, it follows from (7) that $\nu_{u_1,\dots,u_s}$ can be written down explicitly as

$$\nu_{0,\dots,0}[\cdot] = \mu \qquad \text{if } |u| = 0$$

and

$$\begin{aligned}
&\nu_{u_1,\dots,u_s}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j} : 1 \leq j \leq s] \\
&\qquad = \sum_{t_{j_1}=0}^{b^{u_{j_1}-1}-1} \sum_{c_{j_1}=0}^{b-1} \cdots \sum_{t_{j_{|u|}}=0}^{b^{u_{j_{|u|}}-1}-1} \sum_{c_{j_{|u|}}=0}^{b-1} \left\langle f, \prod_{l=1}^{|u|} \psi_{u_{j_l}-1,t_{j_l},c_{j_l}} \right\rangle \\
&\qquad\quad \times E\left\{ \prod_{l=1}^{|u|} \psi_{u_{j_l}-1,t_{j_l},c_{j_l}} \circ U_{j_l}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j} : 1 \leq j \leq s] \right\} \\
&\qquad\qquad \text{if } |u| \geq 1. \tag{11}
\end{aligned}$$

Here U_{j_l} denotes the j_lth coordinate of U. Another useful consequence of the ANOVA decomposition is

$$\sum_{0 \leq \tilde{c}_{k,l} \leq b-1 : (k,l) = (j,u_j^*+1),\dots,(j,u_j),\ 1 \leq j \leq s} \nu_{u_1,\dots,u_s}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j} : 1 \leq j \leq s] = 0 \tag{12}$$

whenever $(u_1^*,\dots,u_s^*) \prec (u_1,\dots,u_s)$.

REMARK. Since $\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}$ is constant on $[t_1 b^{-u_{j_l}}, (t_1+1)b^{-u_{j_l}})$ for an arbitrary but fixed integer t_1, we note that in (11),

$$\begin{aligned} E\left\{\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\circ U_{j_l}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j}:1\le j\le s]\right\} \\ =\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\circ U_{j_l}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j}:1\le j\le s]. \end{aligned} \tag{13}$$

Let $\{A_i=(A_{i,1},\dots,A_{i,s})':i=1,\dots,b^m\}$ be the deterministic $(0,m,s)$ net in base b from which the scrambled net $\{X_i:i=1,\dots,b^m\}$ is derived. We observe from (2), (4), (10) and the smoothness of f that

$$\begin{aligned} &\hat{\mu}_{0,m,s}-\mu \\ &\quad=\frac{1}{b^m}\sum_{i=1}^{b^m}[f(X_i)-\mu] \\ &\quad=\frac{1}{b^m}\sum_{i=1}^{b^m}\sum_{u_1,\dots,u_s:(0,\dots,0)\prec(u_1,\dots,u_s)} \\ &\qquad\qquad\times\nu_{u_1,\dots,u_s}\big[\pi_j(a_{i,j,1}),\dots, \\ &\qquad\qquad\qquad\pi_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}):1\le j\le s\big] \\ &\quad=\frac{1}{b^m}\sum_{i=1}^{b^m}\sum_{u_1,\dots,u_s\ge 0:\,u_1+\cdots+u_s\ge m+1} \\ &\qquad\qquad\times\nu_{u_1,\dots,u_s}\big[\pi_j(a_{i,j,1}),\dots, \\ &\qquad\qquad\qquad\pi_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}):1\le j\le s\big]. \end{aligned}$$

The last equality uses (12) and the definition of a $(0,m,s)$ net. Hence we conclude from (9) that

$$\begin{aligned} W&=\frac{1}{b^m\sigma_{0,m,s}} \\ &\quad\times\sum_{i=1}^{b^m}\sum_{u_1,\dots,u_s\ge 0:\,u_1+\cdots+u_s\ge m+1} \\ &\qquad\qquad\times\nu_{u_1,\dots,u_s}\big[\pi_j(a_{i,j,1}),\dots, \\ &\qquad\qquad\qquad\pi_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}):1\le j\le s\big]. \end{aligned} \tag{14}$$

REMARK. Related (although admittedly simpler) ANOVA-type decompositions have been applied to other sampling designs in computer experiments such as randomized orthogonal arrays [see, e.g., Owen (1992a) and Loh (1996a)] and Latin hypercube sampling [see Owen (1992b) and Loh (1996b)].

The following theorem, which is from Owen (1997b), gives the exact rate of convergence for $\sigma^2_{0,m,s}$.

THEOREM 1. *Let* $b \geq \max\{s, 2\}$ *and* $f : [0, 1)^s \to \mathcal{R}$ *be smooth and have a Lipschitz continuous mixed partial of order* s *such that*

$$\int_{[0,1)^s} \left[\frac{\partial^s}{\partial x_1 \cdots \partial x_s} f(x) \right]^2 dx > 0.$$

Then there exist positive constants c, C *such that*

$$cm^{s-1} b^{-3m} \leq \sigma^2_{0,m,s} \leq Cm^{s-1} b^{-3m}$$

as $m \to \infty$.

Theorem 2 in Owen (1997b) explicitly gives the upper bound, while the lower bound can be obtained easily from the proof of the same theorem. We note that Owen's proof essentially makes use of the previously discussed nested ANOVA decomposition.

REMARK. Owen [(1997b), page 1555] noted that it is an open question whether the rate in Theorem 1 holds under weaker smoothness conditions.

We end this section with a lemma on $(0, m, s)$ nets and $(0, s)$ sequences that complements Theorem 1.

LEMMA 1. *For* $m \geq 2$, *a* $(0, m, s)$ *net in base* b *can exist only if* $s \leq b + 1$. *Furthermore a* $(0, s)$ *sequence in base* b *can exist only if* $s \leq b$.

The reader is referred to Corollaries 4.21 and 4.24 of Niederreiter (1992) for the proof of Lemma 1.

4. Stein's method. In this section, we shall use Stein's method to determine the asymptotic distribution of $\hat{\mu}_{0,m,s}$, where $\hat{\mu}_{0,m,s}$ is as in (5) with $t = 0$ and $\{X_i : i = 1, \ldots, b^m\}$ is a scrambled $(0, m, s)$ net in base b. Central to this normal approximation technique is the following lemma.

LEMMA 2 (Stein). *Let* $z \in \mathcal{R}$ *and let* Φ *and* ϕ *denote the cumulative distribution function and probability density function of the standard normal* *distribution, respectively. The unique bounded solution* $g_z : \mathcal{R} \to \mathcal{R}$ *of the differential equation*

$$g^{(1)}(w) - wg(w) = \mathcal{I}\{w \le z\} - \Phi(z) \qquad \forall\, w \in \mathcal{R}$$

is given by

$$g_z(w) = \begin{cases} \Phi(w)[1-\Phi(z)]/\phi(w), & \text{if } w \le z,\\ \Phi(z)[1-\Phi(w)]/\phi(w), & \text{if } w > z.\end{cases}$$

Furthermore, $0 \le g_z(w) \le 1$ *and* $|g_z^{(1)}(w)| \le 1$ *for all* $w \in \mathcal{R}$.

Lemma 2 is from Stein (1972); refer to his paper for a proof.
The following theorem is the main result of this article.

THEOREM 2. *Let* $b \ge \max\{s, 2\}$, $f : [0,1)^s \to \mathcal{R}$ *be smooth and have a Lipschitz continuous mixed partial of order* s *such that*

$$\int_{[0,1)^s} \left[\frac{\partial^s}{\partial x_1 \cdots \partial x_s} f(x)\right]^2 dx > 0$$

and let W *be as in* (9). *Then* $W \to \Phi$ *in distribution as* $m \to \infty$.

To prove Theorem 2, it is convenient first to define $\tilde{m} = \lfloor 2s \log_b m \rfloor$ and

$$\begin{aligned}\tilde{W} &= \frac{1}{b^m \sigma_{0,m,s}}\\ &\quad \times \sum_{i=1}^{b^m} \sum_{u_1,\dots,u_s \ge \tilde{m} : u_1+\cdots+u_s \ge m+1} \times \nu_{u_1,\dots,u_s}\big[\pi_j(a_{i,j,1}), \dots,\\ &\qquad \pi_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}) : 1 \le j \le s\big].\end{aligned} \tag{15}$$

PROPOSITION 1. *Let* $b \ge \max\{s, 2\}$ *and let* $f : [0,1)^s \to \mathcal{R}$ *be smooth and have a Lipschitz continuous mixed partial of order* s *such that*

$$\int_{[0,1)^s} \left[\frac{\partial^s}{\partial x_1 \cdots \partial x_s} f(x)\right]^2 dx > 0.$$

Then $E[(W - \tilde{W})^2] = O(\tilde{m}/m)$ *and hence* $W - \tilde{W} \to 0$ *in probability as* $m \to \infty$, *where* W *and* $\tilde{W}$ *are as in* (14) *and* (15), *respectively*.

The proof of Proposition 1 is deferred to the Appendix. Hence it follows from Proposition 1 and Slutsky's theorem that to prove Theorem 2, it suffices to establish the asymptotic normality of $\tilde{W}$ via Stein's method.

THEOREM 3. *Let* $b \geq \max\{s, 2\}$, *let* $f : [0, 1)^s \to \mathcal{R}$ *be smooth and have a Lipschitz continuous mixed partial of order* s *such that*

$$\int_{[0,1)^s} \left[\frac{\partial^s}{\partial x_1 \cdots \partial x_s} f(x) \right]^2 dx > 0,$$

and let $\tilde{W}$ *be as in* (15). *Then as* $m \to \infty$,

$$\sup\{|P(\tilde{W} \leq w) - \Phi(w)| : -\infty < w < \infty\} = O\left(\left(\frac{\log_b m}{m} \right)^{1/2} \right).$$

PROOF. Let I and J be random variables uniformly distributed over $\{1, \ldots, b^m\}$ and $\{1, \ldots, s\}$, respectively. Furthermore let

$$\{\pi_j^*, \pi_{j;a_1}^*, \pi_{j;a_1,a_2}^*, \pi_{j;a_1,a_2,a_3}^*, \ldots : 1 \leq j \leq s,\ 0 \leq a_k \leq b-1,\ k = 1, 2, \ldots\}$$

be an independent replication of the π's as defined in (3). We assume that I, J and the π^*'s are independent and that they are also independent of all previously defined random quantities. Next define for $1 \leq i \leq b^m$, $1 \leq j \leq s$,

$$\tilde{\pi}_{j;a_{i,j,1},\ldots,a_{i,j,u_j-1}} = \begin{cases} \pi^*_{j;a_{i,j,1},\ldots,a_{i,j,u_j-1}}, & \text{if } J = j,\ (a_{I,j,1}, \ldots, a_{I,j,\tilde{m}-1}) \\ & \qquad = (a_{i,j,1}, \ldots, a_{i,j,\tilde{m}-1}) \text{ and } u_j \geq \tilde{m}, \\ \pi_{j;a_{i,j,1},\ldots,a_{i,j,u_j-1}}, & \text{otherwise,} \end{cases}$$

where the $a_{i,j,k}$'s are as in (2) for the $(0, m, s)$ net $\{A_i : i = 1, \ldots, b^m\}$. With the representation of $\tilde{W}$ in (15) as motivation, define

$$\begin{aligned} \tilde{W}^* = {} & \frac{1}{b^m \sigma_{0,m,s}} \\ & \times \sum_{i=1}^{b^m} \sum_{u_1,\ldots,u_s \geq \tilde{m} : u_1+\cdots+u_s \geq m+1} \\ & \qquad \times \nu_{u_1,\ldots,u_s}\big[\tilde{\pi}_j(a_{i,j,1}), \ldots, \\ & \qquad\qquad \tilde{\pi}_{j;a_{i,j,1},\ldots,a_{i,j,u_j-1}}(a_{i,j,u_j}) : 1 \leq j \leq s\big]. \end{aligned}$$

From symmetry, we observe that $(\tilde{W}, \tilde{W}^*)$ is an exchangeable pair of random variables in that $(\tilde{W}, \tilde{W}^*)$ and $(\tilde{W}^*, \tilde{W})$ possess the same bivariate distribution. We now write

$$\begin{aligned} \tilde{W}^* - \tilde{W} &= \tilde{S} - S, \\ V &= \tilde{W} - S, \end{aligned} \tag{16}$$

where

$$\tilde{S} = \frac{1}{b^m \sigma_{0,m,s}} \times \sum_{i=1}^{b^m} \sum_{u_1,\dots,u_s \geq \tilde{m} : u_1+\cdots+u_s \geq m+1} \mathcal{I}\{\Omega_{i,I;J;\tilde{m}}\} \times \nu_{u_1,\dots,u_s}[\tilde{\pi}_j(a_{i,j,1}), \dots, \tilde{\pi}_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}) : 1 \leq j \leq s],$$

$$S = \frac{1}{b^m \sigma_{0,m,s}} \times \sum_{i=1}^{b^m} \sum_{u_1,\dots,u_s \geq \tilde{m} : u_1+\cdots+u_s \geq m+1} \mathcal{I}\{\Omega_{i,I;J;\tilde{m}}\} \times \nu_{u_1,\dots,u_s}[\pi_j(a_{i,j,1}), \dots, \pi_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}) : 1 \leq j \leq s]$$

and $\Omega_{i,I;J;\tilde{m}}$ denotes the event that $(a_{I,J,1}, \dots, a_{I,J,\tilde{m}-1}) = (a_{i,J,1}, \dots, a_{i,J,\tilde{m}-1})$. Let $\mathcal{W}$ be the σ-field generated by the random quantities

$$\{\pi_j(a_{i,j,1}), \pi_{j;a_{i,j,1}}(a_{i,j,2}), \pi_{j;a_{i,j,1},a_{i,j,2}}(a_{i,j,3}), \dots : 1 \leq i \leq b^m, 1 \leq j \leq s\} \tag{17}$$

and let $E^{\mathcal{W}}$ denote conditional expectation given $\mathcal{W}$. Next let $z \in \mathcal{R}$ and $g_z : \mathcal{R} \to \mathcal{R}$ be as in Lemma 2. From the exchangeability of $(\tilde{W}, \tilde{W}^*)$, we have

$$\begin{aligned} 0 &= E\{(\tilde{W}^* - \tilde{W})[g_z(\tilde{W}) + g_z(\tilde{W}^*)]\} \\ &= 2E[g_z(\tilde{W}) E^{\mathcal{W}}(\tilde{W}^* - \tilde{W})] + E\{(\tilde{W}^* - \tilde{W})[g_z(\tilde{W}^*) - g_z(\tilde{W})]\}. \end{aligned}$$

Consequently we observe from Proposition 2 (see the Appendix) that

$$\begin{aligned} E[\tilde{W} g_z(\tilde{W})] &= \frac{b^{\tilde{m}-1}}{2} E\{(\tilde{W}^* - \tilde{W})[g_z(\tilde{W}^*) - g_z(\tilde{W})]\} \\ &= E\left[\int g_z^{(1)}(V + w) K_{\tilde{W},\tilde{W}^*}(w)\, dw\right], \end{aligned} \tag{18}$$

where for all $w \in \mathcal{R}$,

$$K_{\tilde{W},\tilde{W}^*}(w) = \begin{cases} (b^{\tilde{m}-1}/2)(\tilde{W}^* - \tilde{W}), & \text{if } S < w \leq \tilde{S}, \\ (b^{\tilde{m}-1}/2)(\tilde{W} - \tilde{W}^*), & \text{if } \tilde{S} < w \leq S, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Now we observe from Lemma 2 and (18) that

$$
\begin{aligned}
&P(\tilde{W} \le z) - \Phi(z) \\
&\quad = E\big[g_z^{(1)}(\tilde{W}) - \tilde{W} g_z(\tilde{W})\big] \\
&\quad = E\int \big[g_z^{(1)}(\tilde{W}) - g_z^{(1)}(V+w)\big] K_{\tilde{W},\tilde{W}^*}(w)\,dw \\
&\qquad + E\big[g_z^{(1)}(\tilde{W})\big] E\bigg[\int K_{\tilde{W},\tilde{W}^*}(w)\,dw\bigg] \\
&\qquad - E\bigg[g_z^{(1)}(\tilde{W}) \int K_{\tilde{W},\tilde{W}^*}(w)\,dw\bigg] \\
&\qquad + \big[E g_z^{(1)}(\tilde{W})\big]\bigg[1 - E\int K_{\tilde{W},\tilde{W}^*}(w)\,dw\bigg].
\end{aligned} \tag{20}
$$

Thus to prove Theorem 3, it suffices to obtain appropriate bounds for the terms on the right-hand side of (20). This is achieved by Propositions 3–5 in the Appendix. Hence we conclude that as $m \to \infty$,

$$
\begin{aligned}
&\sup\big\{|P(\tilde{W} \le w) - \Phi(w)| : -\infty < w < \infty\big\} \\
&\quad = O\bigg(\Big(\frac{\tilde{m}}{m}\Big)^{1/2}\bigg) + O(b^{-\tilde{m}/3}) \\
&\quad = O\bigg(\Big(\frac{\log_b m}{m}\Big)^{1/2}\bigg).
\end{aligned}
$$

This proves Theorem 3. $\square$

REMARK. The first two terms on the right-hand side of (20) have been studied in some detail by Ho and Chen (1978) in the context of investigating the convergence rate of Hoeffding's combinatorial central limit theorem and the proof of Proposition 4 was motivated by their results. A Berry–Esseen type bound eventually was obtained by Bolthausen (1984) for Hoeffding's combinatorial central limit theorem using mathematical induction in conjunction with Stein's method.

REMARK. The reason for using $\tilde{W}$ as a proxy for W is that it is possible then to obtain a sharp bound for the fourth moment of S (see Lemma 4 in the Appendix) even though the proof is rather tedious and involved. It can be shown (by taking explicit note of the error terms) that the proofs of this article give a logarithmic upper bound (with respect to sample size n) on the convergence rate of the law of W to normality. However, because the somewhat crude proxy $\tilde{W}$ is used for W, we believe it is unlikely that the above convergence rate is optimal. As we noted in the Introduction, numerical studies reported in Hong, Hickernell and Wei (2001), for example, indicate that the central limit effect can take place for reasonable sample sizes.

APPENDIX

PROOF OF PROPOSITION 1. We observe from (14) and (15) that

$$W - \tilde{W} = \sum_{r=1}^{s-1} \sum_{1 \le j_1 < \cdots < j_r \le s} \Delta_{j_1,\ldots,j_r}, \tag{21}$$

where

$$\begin{aligned}\Delta_{j_1,\ldots,j_r} = {} & \frac{1}{b^m \sigma_{0,m,s}} \\ & \times \sum_{i=1}^{b^m} \sum_{\substack{u_1,\ldots,u_s \ge 0 : u_1+\cdots+u_s \ge m+1, \\ u_k \ge \tilde{m} \Leftrightarrow k \in \{j_1,\ldots,j_r\}}} \\ & \qquad \times \nu_{u_1,\ldots,u_s}[\pi_j(a_{i,j,1}), \ldots, \\ & \qquad\qquad \pi_{j;a_{i,j,1},\ldots,a_{i,j,u_j-1}}(a_{i,j,u_j}) : 1 \le j \le s].\end{aligned}$$

Here $\Leftrightarrow$ denotes if and only if. Next as in (11), we write $|u| = \sum_{j=1}^{s} \mathcal{I}\{u_j \ge 1\}$ and define

$$\begin{aligned}\Upsilon_{i_1,i_2;j,u_j} = {} & \frac{1}{b-1}\big[b\mathcal{I}\{(a_{i_1,j,1}, \ldots, a_{i_1,j,u_j}) = (a_{i_2,j,1}, \ldots, a_{i_2,j,u_j})\} \\ & - \mathcal{I}\{(a_{i_1,j,1}, \ldots, a_{i_1,j,u_j-1}) = (a_{i_2,j,1}, \ldots, a_{i_2,j,u_j-1})\}\big].\end{aligned}$$

Consequently $\Upsilon_{i_1,i_2;j,0} = 1$ and from the definition of a $(0, m, s)$ net [see also Corollary 4 of Owen (1997a)], we obtain

$$\begin{aligned}& \sum_{i_2=1}^{b^m} \prod_{j=1}^{s} \Upsilon_{i_1,i_2;j,u_j} \\ & \quad = \frac{1}{(b-1)^{|u|}} \sum_{l=0}^{|u|} \binom{|u|}{l} b^l (-1)^{|u|-l} \max\{1, b^{m-u_1-\cdots-u_s+|u|-l}\} \\ & \quad = \begin{cases} 0, & \text{if } u_1 + \cdots + u_s \le m, \\ \Gamma_{u_1,\ldots,u_s}, & \text{if } m+1 \le u_1 + \cdots + u_s \le m + |u| - 1, \\ 1, & \text{if } u_1 + \cdots + u_s \ge m + |u|, \end{cases}\end{aligned} \tag{22}$$

where

$$\begin{aligned}\Gamma_{u_1,\ldots,u_s} = 1 + {} & \frac{(-1)^{|u|}}{(b-1)^{|u|}} \\ & \times \bigg[b^{m-u_1-\cdots-u_s+|u|} \binom{|u|-1}{m-u_1-\cdots-u_s+|u|} (-1)^{m-u_1-\cdots-u_s+|u|} \\ & \qquad - \sum_{l=0}^{m-u_1-\cdots-u_s+|u|} \binom{|u|}{l} (-b)^l\bigg].\end{aligned}$$

We observe that the left-hand side of (22) does not depend on i_1. Owen [(1997b), page 1547] showed that $0 \le \Gamma_{u_1,\dots,u_s} \le e$ since $s \le b$. We further observe from (11) and (12) that for $1 \le j_1 < \cdots < j_r \le s$,

$$
\begin{aligned}
E(\Delta^2_{j_1,\dots,j_r})
&= \frac{1}{b^{2m}\sigma^2_{0,m,s}} \\
&\quad \times \sum_{i_1=1}^{b^m}\sum_{i_2=1}^{b^m} \sum_{\substack{u_1,\dots,u_s\ge 0:\, u_1+\cdots+u_s\ge m+1\\ u_k\ge \tilde m \Leftrightarrow k\in\{j_1,\dots,j_r\}}} \\
&\qquad \times E\{\nu_{u_1,\dots,u_s}[\pi_j(a_{i_1,j,1}),\dots, \\
&\qquad\qquad \pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_j-1}}(a_{i_1,j,u_j}) : 1\le j\le s] \\
&\qquad \times \nu_{u_1,\dots,u_s}[\pi_j(a_{i_2,j,1}),\dots, \\
&\qquad\qquad \pi_{j;a_{i_2,j,1},\dots,a_{i_2,j,u_j-1}}(a_{i_2,j,u_j}) : 1\le j\le s]\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{b^{2m}\sigma^2_{0,m,s}} \\
&\quad \times \sum_{\substack{u_1,\dots,u_s\ge 0:\, u_1+\cdots+u_s\ge m+1\\ u_k\ge \tilde m \Leftrightarrow k\in\{j_1,\dots,j_r\}}} \sum_{i_1=1}^{b^m}\left(\sum_{i_2=1}^{b^m}\prod_{j=1}^{s}\Upsilon_{i_1,i_2;j,u_j}\right) \\
&\qquad \times E\{\nu_{u_1,\dots,u_s}[\pi_j(a_{i_1,j,1}),\dots, \\
&\qquad\qquad \pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_j-1}}(a_{i_1,j,u_j}) : 1\le j\le s]^2\}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\le \frac{e}{b^{m}\sigma^2_{0,m,s}} \\
&\quad \times \sum_{\substack{u_1,\dots,u_s\ge 0:\, u_1+\cdots+u_s\ge m+1\\ u_k\ge \tilde m \Leftrightarrow k\in\{j_1,\dots,j_r\}}} \\
&\qquad \times E\{\nu_{u_1,\dots,u_s}[\pi_j(a_{1,j,1}),\dots, \\
&\qquad\qquad \pi_{j;a_{1,j,1},\dots,a_{1,j,u_j-1}}(a_{1,j,u_j}) : 1\le j\le s]^2\}.
\end{aligned}
$$

For simplicity, we write $1 \le j_1^* < \cdots < j_{s-r}^* \le s$ such that $\{j_1,\dots,j_r,j_1^*,\dots,j_{s-r}^*\} = \{1,\dots,s\}$. Now it follows from Theorem 1, (23) and Lemma 3 that as $m \to \infty$,

$$
\begin{aligned}
E(\Delta^2_{j_1,\dots,j_r})
&= \frac{O(1)}{b^m \sigma^2_{0,m,s}} \sum_{u_{j_1^*}=0}^{\tilde{m}-1} \cdots \sum_{u_{j_{s-r}^*}=0}^{\tilde{m}-1} \sum_{u_{j_1},\dots,u_{j_r} \geq \tilde{m} : u_1+\cdots+u_s \geq m+1} b^{-2(u_1+\cdots+u_s)} \\
&= \frac{O(1)}{b^m \sigma^2_{0,m,s}} \\
&\quad \times \sum_{u_{j_1^*}=0}^{\tilde{m}-1} \cdots \sum_{u_{j_{s-r}^*}=0}^{\tilde{m}-1} b^{-2(u_{j_1^*}+\cdots+u_{j_{s-r}^*})} \sum_{l=m+1-u_{j_1^*}-\cdots-u_{j_{s-r}^*}}^{\infty} b^{-2l} \binom{l-1}{r-1} \\
&= \frac{O(1)}{b^{3m} \sigma^2_{0,m,s}} \sum_{u_{j_1^*}=0}^{\tilde{m}-1} \cdots \sum_{u_{j_{s-r}^*}=0}^{\tilde{m}-1} \sum_{l=0}^{\infty} b^{-2l} \binom{l+m}{r-1} \\
&= O\left(\left(\frac{\tilde{m}}{m}\right)^{s-r}\right).
\end{aligned}
$$

This implies that $E[(W - \tilde{W})^2] = O(\tilde{m}/m)$ and consequently, using Markov's inequality, we obtain $W - \tilde{W} \to 0$ in probability as $m \to \infty$. This proves Proposition 1. $\square$

PROPOSITION 2. *With the notation and assumptions of Theorem* 3,

$$
E^{\mathcal{W}}(\tilde{W}^* - \tilde{W}) = -\frac{1}{b^{\tilde{m}-1}} \tilde{W}.
$$

PROOF. First we observe from (17) that

$$
\begin{aligned}
E^{\mathcal{W}}(S) &= \frac{1}{s b^{m+\tilde{m}-1} \sigma_{0,m,s}} \\
&\quad \times \sum_{i=1}^{b^m} \sum_{k=1}^{s} \sum_{u_1,\dots,u_s \geq \tilde{m} : u_1+\cdots+u_s \geq m+1} \\
&\qquad \times \nu_{u_1,\dots,u_s}[\pi_j(a_{i,j,1}), \dots, \\
&\qquad\qquad \pi_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}) : 1 \leq j \leq s] \\
&= \frac{1}{b^{\tilde{m}-1}} \tilde{W}.
\end{aligned} \tag{24}
$$

Next we observe from (12) and (17) that

$$E^{\mathcal{W},I,J}(\tilde{S}) = 0 \quad \Longrightarrow \quad E^{\mathcal{W}}(\tilde{S}) = 0, \tag{25}$$

where $E^{\mathcal{W},I,J}$ denotes conditional expectation given $\mathcal{W}$, I and J. Since $\tilde{W}^* - \tilde{W} = \tilde{S} - S$, Proposition 2 follows directly from (24) and (25). □

LEMMA 3. *With the notation and assumptions of Proposition* 1 *or Theorem* 3,

$$\begin{aligned}
&\left\langle f, \prod_{l=1}^{|u|} \psi_{u_{j_l}-1,t_{j_l},c_{j_l}} \right\rangle \\
&\quad = b^{-(3\sum_{l=1}^{|u|} u_{j_l} - 2|u|)/2} \\
&\qquad \times \left[\frac{\partial^{|u|}}{\partial x_{j_1} \cdots \partial x_{j_{|u|}}} \int_{[0,1)^{s-|u|}} f(x_{u_{j_1},\dots,u_{j_l};t_{j_1},\dots,t_{j_l}}) \prod_{1 \le j \le s : u_j = 0} dx_j \right] \\
&\qquad \times \left(\prod_{l=1}^{|u|} \frac{2c_{j_l} - b + 1}{2b} \right) \left(1 + O\left(\sum_{l=1}^{|u|} b^{-(1+\beta)u_{j_l}} \right) \right),
\end{aligned} \tag{26}$$

where $1 \le j_1 < \cdots < j_{|u|} \le s$ *and* $u_j \ge 1$ *if and only if* $j \in \{j_1, \dots, j_{|u|}\}$. *Here the* j*th coordinate of the point* $x_{u_{j_1},\dots,u_{j_l};t_{j_1},\dots,t_{j_l}}$ *equals* $b^{-u_j+1}(t_j + 0.5)$ *if* $j \in \{j_1, \dots, j_l\}$ *and equals* x_j *otherwise. Furthermore we have*

$$\begin{aligned}
&E\{\nu_{u_1,\dots,u_s}[\pi_j(a_{1,j,1}), \dots, \pi_{j;a_{1,j,1},\dots,a_{1,j,u_j-1}}(a_{1,j,u_j}) : 1 \le j \le s]^2\} \\
&\quad = b^{-2(u_1+\cdots+u_s)} \left(\frac{b^2-1}{12} \right)^{|u|} \left(1 + O\left(\sum_{l=1}^{|u|} b^{-(1+\beta)u_{j_l}} \right) \right) \\
&\qquad \times \int_{[0,1)^{|u|}} \left[\frac{\partial^{|u|}}{\partial x_{j_1} \cdots \partial x_{j_{|u|}}} \right. \\
&\qquad\qquad \left. \times \int_{[0,1)^{s-|u|}} f(x_1, \dots, x_s) \prod_{1 \le j \le s : u_j = 0} dx_j \right]^2 dx_{j_1} \cdots dx_{j_{|u|}}
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&E\{\nu_{u_1,\dots,u_s}[\pi_j(a_{1,j,1}), \dots, \pi_{j;a_{1,j,1},\dots,a_{1,j,u_j-1}}(a_{1,j,u_j}) : 1 \le j \le s]^4\} \\
&\quad = O(b^{-4(u_1+\cdots+u_s)}).
\end{aligned}$$

PROOF. For the proofs of (26) and (27), we refer the reader to Lemmas 1 and 2 of Owen [(1997b), pages 1552 and 1553]. Next we observe from (11) and (13) that

$$
\begin{aligned}
&\nu_{u_1,\dots,u_s}[\pi_j(a_{1,j,1}),\dots,\pi_{j;a_{1,j,1},\dots,a_{1,j,u_j-1}}(a_{1,j,u_j}):1\le j\le s]^4\\
&\quad=\sum_{t_{j_1}=0}^{b^{u_{j_1}-1}-1}\cdots\sum_{t_{j_{|u|}}=0}^{b^{u_{j_{|u|}}-1}-1}\\
&\qquad\times\prod_{i=1}^{4}\Biggl\{\sum_{c_{i,j_1}=0}^{b-1}\cdots\sum_{c_{i,j_{|u|}}=0}^{b-1}\Biggl\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{i,j_l}}\Biggr\rangle\\
&\qquad\qquad\times\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{i,j_l}}(U_{j_l}[\pi_j(a_{1,j,1}),\dots,\\
&\qquad\qquad\qquad\pi_{j;a_{1,j,1},\dots,a_{1,j,u_j-1}}(a_{1,j,u_j}):1\le j\le s])\Biggr\},
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
&E\Biggl\{\prod_{i=1}^{4}\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{i,j_l}}(U_{j_l}[\pi_j(a_{1,j,1}),\dots,\\
&\qquad\qquad\pi_{j;a_{1,j,1},\dots,a_{1,j,u_j-1}}(a_{1,j,u_j}):1\le j\le s])\Biggr\}\\
&\quad=\prod_{l=1}^{|u|}\int_0^1\Biggl[\prod_{i=1}^{4}\psi_{u_{j_l}-1,t_{j_l},c_{i,j_l}}(x_{j_l})\Biggr]dx_{j_l}\\
&\quad=O\bigl(b^{u_{j_1}+\cdots+u_{j_{|u|}}}\bigr).
\end{aligned}
\tag{29}
$$

The final statement of Lemma 3 follows from (26), (28) and (29). □

PROPOSITION 3. *With the notation and assumptions of Theorem* 3,

$$
\left|E\int K_{\tilde W,\tilde W^*}(w)\,dw-1\right|=O\left(\frac{\tilde m}{m}\right)
$$

as $m\to\infty$.

PROOF. We observe from (21) and the proof of Proposition 1 that

$$E(\tilde{W}^2) = E(W^2) - 2\sum_{r=1}^{s-1} \sum_{1\le j_1<\cdots<j_r\le s} E(W\Delta_{j_1,\dots,j_r}) + \sum_{r=1}^{s-1} \sum_{1\le j_1<\cdots<j_r\le s} E(\Delta^2_{j_1,\dots,j_r}) = 1 + O\left(\frac{\tilde{m}}{m}\right) \tag{30}$$

as $m \to \infty$. By replacing $g_z(\tilde{W})$ by $\tilde{W}$ in (18), we have $E\int K_{\tilde{W},\tilde{W}^*}(w)\,dw = E(\tilde{W}^2)$ and this proves Proposition 3. □

PROPOSITION 4. *With the notation and assumptions of Theorem* 3,

$$\left|E\int [g_z^{(1)}(\tilde{W}) - g_z^{(1)}(V+w)]K_{\tilde{W},\tilde{W}^*}(w)\,dw\right| = O(b^{-\tilde{m}/3})$$

as $m \to \infty$.

PROOF. Let $\varepsilon > 0$. From Lemma 2, we observe that

$$\begin{aligned}
E\biggl[\int |g_z^{(1)}(\tilde{W}) &- g_z^{(1)}(V+w)|K_{\tilde{W},\tilde{W}^*}(w)\,dw\biggr] \\
&\le 2E\biggl[\int_{|w|>2\varepsilon} K_{\tilde{W},\tilde{W}^*}(w)\,dw\biggr] \\
&\quad + 2E\biggl[\int_{|w|\le 2\varepsilon} \mathcal{I}\{|S|>2\varepsilon\}K_{\tilde{W},\tilde{W}^*}(w)\,dw\biggr] \\
&\quad + E\biggl[\int_{|w|\le 2\varepsilon} |\mathcal{I}\{V+S\le z\} \\
&\qquad\qquad - \mathcal{I}\{V+w\le z\}|\mathcal{I}\{|S|\le 2\varepsilon\}K_{\tilde{W},\tilde{W}^*}(w)\,dw\biggr] \\
&\quad + E\biggl[\int_{|w|\le 2\varepsilon} |\tilde{W}g_z(\tilde{W}) \\
&\qquad\qquad - (V+w)g_z(V+w)|\mathcal{I}\{|S|\le 2\varepsilon\}K_{\tilde{W},\tilde{W}^*}(w)\,dw\biggr] \\
&\le 2E\biggl[\int_{|w|>2\varepsilon} K_{\tilde{W},\tilde{W}^*}(w)\,dw\biggr] \\
&\quad + E\biggl[\int_{|w|\le 2\varepsilon} \mathcal{I}\{z-2\varepsilon\le V\le z+2\varepsilon\}K_{\tilde{W},\tilde{W}^*}(w)\,dw\biggr]
\end{aligned} \tag{31}$$

$$+ 2E\biggl[\int_{|w|\le 2\varepsilon} \mathcal{I}\{|S| > 2\varepsilon\} K_{\tilde W, \tilde W^*}(w)\,dw\biggr]$$
$$+ 4\varepsilon E\biggl[\int_{|w|\le 2\varepsilon} (|\tilde W| + 1) K_{\tilde W, \tilde W^*}(w)\,dw\biggr].$$

Now it suffices to establish appropriate bounds for the terms on the right-hand side of (31). To do so we shall divide the remainder of this proof into four steps.

Step 1. First we observe that

$$\begin{aligned}
2E\biggl[&\int_{w>2\varepsilon} K_{\tilde W, \tilde W^*}(w)\,dw\biggr] \\
&= 2E\biggl[\int_{w>2\varepsilon} \bigl(\mathcal{I}\{S \le \tilde S\} + \mathcal{I}\{S > \tilde S\}\bigr) K_{\tilde W, \tilde W^*}(w)\,dw\biggr] \\
&\le 2E\biggl[\int \bigl(\mathcal{I}\{\tilde S \ge 2\varepsilon\} + \mathcal{I}\{S \ge 2\varepsilon\}\bigr) K_{\tilde W, \tilde W^*}(w)\,dw\biggr] \\
&= b^{\tilde m - 1} E\bigl[(\tilde S - S)^2\bigl(\mathcal{I}\{\tilde S \ge 2\varepsilon\} + \mathcal{I}\{S \ge 2\varepsilon\}\bigr)\bigr].
\end{aligned}$$

Using symmetry and Lemma 4, we have

$$\begin{aligned}
2E\biggl[&\int_{|w|>2\varepsilon} K_{\tilde W, \tilde W^*}(w)\,dw\biggr] \\
&\le b^{\tilde m - 1} E\bigl[(\tilde S - S)^2\bigl(\mathcal{I}\{|\tilde S| \ge 2\varepsilon\} + \mathcal{I}\{|S| \ge 2\varepsilon\}\bigr)\bigr] \\
&\le 8 b^{\tilde m - 1}\bigl[E(S^4)\bigr]^{1/2}\bigl[P(|S| \ge 2\varepsilon)\bigr]^{1/2} \\
&\le 2 b^{\tilde m - 1} \varepsilon^{-2} E(S^4) \\
&= O(\varepsilon^{-2} b^{-\tilde m})
\end{aligned}$$

as $m \to \infty$ uniformly over $\varepsilon > 0$.

Step 2. Using Hölder's and Markov's inequalities, we have

$$\begin{aligned}
2E\biggl[&\int_{|w|\le 2\varepsilon} \mathcal{I}\{|S| > 2\varepsilon\} K_{\tilde W, \tilde W^*}(w)\,dw\biggr] \\
&\le 2\varepsilon b^{\tilde m - 1} E\bigl[(|\tilde S| + |S|)\mathcal{I}\{|S| > 2\varepsilon\}\bigr] \\
&\le 4\varepsilon b^{\tilde m - 1}\bigl[E(S^4)\bigr]^{1/4}\bigl[P(|S| > 2\varepsilon)\bigr]^{3/4} \\
&\le b^{\tilde m - 1} \varepsilon^{-2} E(S^4) \\
&= O(\varepsilon^{-2} b^{-\tilde m})
\end{aligned}$$

as $m \to \infty$ uniformly over $\varepsilon > 0$.

Step 3. Next we observe from (16), (19) and (30) that

$$\begin{aligned} 4\varepsilon E\bigg[\int_{|w|\le 2\varepsilon} (|\tilde{W}|+1) K_{\tilde{W},\tilde{W}^*}(w)\,dw\bigg] &\le 2\varepsilon b^{\tilde{m}-1} E\big[(|\tilde{W}|+1)(\tilde{S}-S)^2\big] \\ &\le 8\varepsilon b^{\tilde{m}-1}\big\{\big[E(S^4)\big]^{1/2}\big[E(\tilde{W}^2)\big]^{1/2} + E(S^2)\big\} \\ &= O(\varepsilon) \end{aligned}$$

as $m \to \infty$ uniformly over $\varepsilon > 0$.

Step 4. Define

$$h_z(w) = \begin{cases} -4\varepsilon, & \text{if } w \le z - 4\varepsilon, \\ w - z, & \text{if } z - 4\varepsilon \le w \le z + 4\varepsilon, \\ 4\varepsilon, & \text{if } z + 4\varepsilon \le w. \end{cases}$$

Consequently,

$$\begin{aligned} E\bigg[\int h_z^{(1)}(V+w) K_{\tilde{W},\tilde{W}^*}(w)\,dw\bigg] \ge E\bigg[\int_{|w|\le 2\varepsilon} \mathcal{I}(z-2\varepsilon \le V \le z+2\varepsilon) K_{\tilde{W},\tilde{W}^*}(w)\,dw\bigg]. \end{aligned}$$

Thus we observe as in (18) and (30) that

$$\begin{aligned} \bigg|E\bigg[\int_{|w|\le 2\varepsilon} \mathcal{I}(z-2\varepsilon \le V \le z+2\varepsilon) K_{\tilde{W},\tilde{W}^*}(w)\,dw\bigg]\bigg| &\le E|\tilde{W} h_z(\tilde{W})| \\ &\le 4\varepsilon E|\tilde{W}| \\ &= O(\varepsilon) \end{aligned}$$

as $m \to \infty$ uniformly over $\varepsilon > 0$. Now Proposition 4 follows from (31) and Steps 1–4 by taking $\varepsilon = b^{-\tilde{m}/3}$. □

LEMMA 4. *With the notation and assumptions of Theorem* 3,

$$b^{\tilde{m}-1} E(S^2) = 1 + O\left(\frac{\tilde{m}}{m}\right),$$

$$E(S^4) = O(b^{-2\tilde{m}})$$

as $m \to \infty$.

PROOF. We observe from the definition of S in (16) that

$$
\begin{aligned}
b^{\tilde m-1}&E^{\mathcal W}(S^2)\\
&=\frac{b^{\tilde m-1}}{b^{2m}\sigma_{0,m,s}^2}\\
&\quad\times E^{\mathcal W}\Biggl\{\Biggl[\sum_{i=1}^{b^m}\sum_{u_1,\dots,u_s\ge\tilde m\,:\,u_1+\cdots+u_s\ge m+1}\mathcal I\{\Omega_{i,I;J;\tilde m}\}\\
&\qquad\qquad\times\nu_{u_1,\dots,u_s}\bigl[\pi_j(a_{i,j,1}),\dots,\\
&\qquad\qquad\qquad\pi_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}):1\le j\le s\bigr]\Biggr]^2\Biggr\}\\
&=\frac{b^{\tilde m-1}}{sb^{2m}\sigma_{0,m,s}^2}\\
&\quad\times\sum_{k=1}^{s}E^{\mathcal W}\Biggl\{\Biggl[\sum_{i=1}^{b^m}\sum_{u_1,\dots,u_s\ge\tilde m\,:\,u_1+\cdots+u_s\ge m+1}\mathcal I\{\Omega_{i,I;k;\tilde m}\}\\
&\qquad\qquad\times\nu_{u_1,\dots,u_s}\bigl[\pi_j(a_{i,j,1}),\dots,\\
&\qquad\qquad\qquad\pi_{j;a_{i,j,1},\dots,a_{i,j,u_j-1}}(a_{i,j,u_j}):1\le j\le s\bigr]\Biggr]^2\Biggr\} \qquad (32)\\
&=\frac{1}{sb^{2m}\sigma_{0,m,s}^2}\\
&\quad\times\sum_{k=1}^{s}\sum_{i_1=1}^{b^m}\sum_{u_{1,1},\dots,u_{1,s}\ge\tilde m\,:\,u_{1,1}+\cdots+u_{1,s}\ge m+1}\\
&\qquad\qquad\times\nu_{u_{1,1},\dots,u_{1,s}}\bigl[\pi_j(a_{i_1,j,1}),\dots,\\
&\qquad\qquad\qquad\pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_{1,j}-1}}(a_{i_1,j,u_{1,j}}):1\le j\le s\bigr]\\
&\quad\times\sum_{i_2=1}^{b^m}\sum_{u_{2,1},\dots,u_{2,s}\ge\tilde m\,:\,u_{2,1}+\cdots+u_{2,s}\ge m+1}\mathcal I\{\Omega_{i_1,i_2;k;\tilde m}\}\\
&\qquad\qquad\times\nu_{u_{2,1},\dots,u_{2,s}}\bigl[\pi_j(a_{i_2,j,1}),\dots,\\
&\qquad\qquad\qquad\pi_{j;a_{i_2,j,1},\dots,a_{i_2,j,u_{2,j}-1}}(a_{i_2,j,u_{2,j}}):1\le j\le s\bigr].
\end{aligned}
$$

Hence it follows from (15) and (30) that

$$
\begin{aligned}
b^{\tilde m-1}&E(S^2)\\
&=E\bigl[b^{\tilde m-1}E^{\mathcal W}(S^2)\bigr]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{b^{2m}\sigma_{0,m,s}^2} \\
&\quad \times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{u_1,\dots,u_s \geq \tilde{m} : u_1+\cdots+u_s \geq m+1} \\
&\qquad \times E\{\nu_{u_1,\dots,u_s}[\pi_j(a_{i_1,j,1}),\dots, \\
&\qquad\qquad \pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_j-1}}(a_{i_1,j,u_j}) : 1 \leq j \leq s] \\
&\qquad\quad \times \nu_{u_1,\dots,u_s}[\pi_j(a_{i_2,j,1}),\dots, \\
&\qquad\qquad \pi_{j;a_{i_2,j,1},\dots,a_{i_2,j,u_j-1}}(a_{i_2,j,u_j}) : 1 \leq j \leq s]\} \\
&= E(\tilde{W}^2) \\
&= 1 + O\left(\frac{\tilde{m}}{m}\right)
\end{aligned}
$$

as $m \to \infty$. Next without loss of generality, we shall suppose for the remainder of this proof that $s \geq 2$ since the case for $s = 1$ is similar though more straightforward. We observe that

$$
\begin{aligned}
&b^{2(\tilde{m}-1)} E(S^4) \\
&\quad = \frac{b^{2(\tilde{m}-1)}}{b^{4m}\sigma_{0,m,s}^4} \\
&\qquad \times E\left\{\prod_{l=1}^{4}\left[\sum_{i_l=1}^{b^m} \sum_{u_{l,1},\dots,u_{l,s} \geq \tilde{m} : u_{l,1}+\cdots+u_{l,s} \geq m+1} \mathcal{I}\{\Omega_{i_l,I;J;\tilde{m}}\}\right.\right. \\
&\qquad\qquad \times \nu_{u_{l,1},\dots,u_{l,s}}[\pi_j(a_{i_l,j,1}),\dots, \\
&\qquad\qquad\quad \left.\left. \pi_{j;a_{i_l,j,1},\dots,a_{i_l,j,u_{l,j}-1}}(a_{i_l,j,u_{l,j}}) : 1 \leq j \leq s]\right]\right\} \qquad (33) \\
&\quad = \frac{b^{2(\tilde{m}-1)}}{s b^{4m}\sigma_{0,m,s}^4} \\
&\qquad \times \sum_{k=1}^{s} E\left\{\prod_{l=1}^{4}\left[\sum_{i_l=1}^{b^m} \sum_{u_{l,1},\dots,u_{l,s} \geq \tilde{m} : u_{l,1}+\cdots+u_{l,s} \geq m+1} \mathcal{I}\{\Omega_{i_l,I;k;\tilde{m}}\}\right.\right. \\
&\qquad\qquad \times \nu_{u_{l,1},\dots,u_{l,s}}[\pi_j(a_{i_l,j,1}),\dots, \\
&\qquad\qquad\quad \left.\left. \pi_{j;a_{i_l,j,1},\dots,a_{i_l,j,u_{l,j}-1}}(a_{i_l,j,u_{l,j}}) : 1 \leq j \leq s]\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{b^{\tilde{m}-1}}{s b^{4m} \sigma_{0,m,s}^4} \\
&\quad \times \sum_{k=1}^{s} E\Biggl\{ \prod_{l=1}^{4} \Biggl[\sum_{i_l=1}^{b^m} \sum_{u_{l,1},\dots,u_{l,s} \ge \tilde{m} : u_{l,1}+\dots+u_{l,s} \ge m+1} \\
&\qquad\qquad \times \nu_{u_{l,1},\dots,u_{l,s}} \bigl[\pi_j(a_{i_l,j,1}), \dots, \\
&\qquad\qquad\qquad \pi_{j;a_{i_l,j,1},\dots,a_{i_l,j,u_{l,j}-1}}(a_{i_l,j,u_{l,j}}) : 1 \le j \le s \bigr] \Biggr] \Biggr\} \\
&\quad \times \mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\},
\end{aligned}
$$

where $\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}$ denotes the event that

$$
(a_{i_1,k,1}, \dots, a_{i_1,k,\tilde{m}-1}) = \dots = (a_{i_4,k,1}, \dots, a_{i_4,k,\tilde{m}-1}).
$$

To evaluate the right-hand side of (33), more notation is needed. Writing $\{l_1, l_2, l_3, l_4\} = \{1, 2, 3, 4\}$, we define the following subsets of $\{1, 2, \dots, s\}$:

(a) $j \in S_1[\{1, 2, 3, 4\}]$ if and only if

$$
(a_{i_1,j,1}, \dots, a_{i_1,j,u_{1,j}-1}) = \dots = (a_{i_4,j,1}, \dots, a_{i_4,j,u_{4,j}-1}).
$$

(b) $j \in S_2[\{l_1, l_2\}, \{l_3, l_4\}]$ if and only if $u_{l_1,j} = u_{l_2,j}$, $u_{l_3,j} = u_{l_4,j}$,

$$
\begin{aligned}
(a_{i_{l_1},j,1}, \dots, a_{i_{l_1},j,u_{l_1,j}-1}) &= (a_{i_{l_2},j,1}, \dots, a_{i_{l_2},j,u_{l_2,j}-1}), \\
(a_{i_{l_3},j,1}, \dots, a_{i_{l_3},j,u_{l_3,j}-1}) &= (a_{i_{l_4},j,1}, \dots, a_{i_{l_4},j,u_{l_4,j}-1}), \\
(a_{i_{l_1},j,1}, \dots, a_{i_{l_1},j,u_{l_1,j}\wedge u_{l_3,j}-1}) &\ne (a_{i_{l_3},j,1}, \dots, a_{i_{l_3},j,u_{l_1,j}\wedge u_{l_3,j}-1}).
\end{aligned}
$$

(c) $j \in S_3[\{l_1, l_2, l_3\}, \{l_4\}]$ if and only if $u_{l_1,j} = u_{l_2,j} = u_{l_3,j} > u_{l_4,j}$,

$$
\begin{aligned}
(a_{i_{l_1},j,1}, \dots, a_{i_{l_1},j,u_{l_1,j}-1}) &= \dots = (a_{i_{l_3},j,1}, \dots, a_{i_{l_3},j,u_{l_3,j}-1}), \\
(a_{i_{l_4},j,1}, \dots, a_{i_{l_4},j,u_{l_4,j}-1}) &= (a_{i_{l_1},j,1}, \dots, a_{i_{l_1},j,u_{l_4,j}-1}).
\end{aligned}
$$

(d) $j \in S_4[\{l_1, l_2\}, \{l_3\}, \{l_4\}]$ if and only if $u_{l_1,j} = u_{l_2,j} > u_{l_3,j} \vee u_{l_4,j}$,

$$
\begin{aligned}
(a_{i_{l_1},j,1}, \dots, a_{i_{l_1},j,u_{l_1,j}-1}) &= (a_{i_{l_2},j,1}, \dots, a_{i_{l_2},j,u_{l_2,j}-1}), \\
(a_{i_{l_1},j,1}, \dots, a_{i_{l_1},j,u_{l_3,j}-1}) &= (a_{i_{l_3},j,1}, \dots, a_{i_{l_3},j,u_{l_3,j}-1}), \qquad (34) \\
(a_{i_{l_1},j,1}, \dots, a_{i_{l_1},j,u_{l_4,j}-1}) &= (a_{i_{l_4},j,1}, \dots, a_{i_{l_4},j,u_{l_4,j}-1}).
\end{aligned}
$$

REMARK. We observe that if there is at least one k not contained in any one of the above mentioned subsets $S_i[\cdot]$, $1 \le i \le 4$, then

$$
E\Biggl\{ \prod_{l=1}^{4} \nu_{u_{l,1},\dots,u_{l,s}} \bigl[\pi_k(a_{i_l,k,1}), \dots, \pi_{k;a_{i_l,k,1},\dots,a_{i_l,k,u_{l,k}-1}}(a_{i_l,k,u_{l,k}}) : 1 \le k \le s \bigr] \Biggr\} = 0.
$$

Hence it follows from (33) that it suffices to consider only the subsets $S_i[\cdot]$, $1 \le i \le 4$.

Given integers $s_1, \dots, s_{14} \ge 0$ such that $s_1 + \cdots + s_{14} = s$, define

$$
\begin{aligned}
\xi_{s_1,\dots,s_{14}} = \mathcal{I}\{&\{1, \dots, s_1\} = S_1[\{1,2,3,4\}],\\
&\{s_1+1, \dots, s_1+s_2\} = S_2[\{1,2\},\{3,4\}],\\
&\{s_1+s_2+1, \dots, s_1+\cdots+s_3\} = S_2[\{1,3\},\{2,4\}],\\
&\{s_1+\cdots+s_3+1, \dots, s_1+\cdots+s_4\} = S_2[\{1,4\},\{2,3\}],\\
&\{s_1+\cdots+s_4+1, \dots, s_1+\cdots+s_5\} = S_3[\{2,3,4\},\{1\}],\\
&\{s_1+\cdots+s_5+1, \dots, s_1+\cdots+s_6\} = S_3[\{1,3,4\},\{2\}],\\
&\{s_1+\cdots+s_6+1, \dots, s_1+\cdots+s_7\} = S_3[\{1,2,4\},\{3\}],\\
&\{s_1+\cdots+s_7+1, \dots, s_1+\cdots+s_8\} = S_3[\{1,2,3\},\{4\}],\\
&\{s_1+\cdots+s_8+1, \dots, s_1+\cdots+s_9\} = S_4[\{1,2\},\{3\},\{4\}],\\
&\{s_1+\cdots+s_9+1, \dots, s_1+\cdots+s_{10}\} = S_4[\{2,3\},\{1\},\{4\}],\\
&\{s_1+\cdots+s_{10}+1, \dots, s_1+\cdots+s_{11}\} = S_4[\{1,3\},\{2\},\{4\}],\\
&\{s_1+\cdots+s_{11}+1, \dots, s_1+\cdots+s_{12}\} = S_4[\{2,4\},\{1\},\{3\}],\\
&\{s_1+\cdots+s_{12}+1, \dots, s_1+\cdots+s_{13}\} = S_4[\{1,4\},\{2\},\{3\}],\\
&\{s_1+\cdots+s_{13}+1, \dots, s_1+\cdots+s_{14}\} = S_4[\{3,4\},\{1\},\{2\}]\}.
\end{aligned}
$$

For $\{l_1, l_2, l_3, l_4\} = \{1,2,3,4\}$, we further define

$$
\begin{aligned}
&\Xi_{l_1,l_2,l_3,l_4;l_2}\\
&\quad = S_2[\{l_1,l_3\},\{l_2,l_4\}] \cup S_2[\{l_1,l_4\},\{l_2,l_3\}] \cup S_3[\{l_2,l_3,l_4\},\{l_1\}]\\
&\qquad \cup S_3[\{l_1,l_3,l_4\},\{l_2\}] \cup S_4[\{l_2,l_3\},\{l_1\},\{l_4\}] \cup S_4[\{l_1,l_3\},\{l_2\},\{l_4\}]\\
&\qquad \cup S_4[\{l_2,l_4\},\{l_1\},\{l_3\}] \cup S_4[\{l_1,l_4\},\{l_2\},\{l_3\}] \cup S_4[\{l_3,l_4\},\{l_1\},\{l_2\}],\\
&\Xi_{l_1,l_2,l_3,l_4;l_3}\\
&\quad = S_2[\{l_1,l_2\},\{l_3,l_4\}] \cup S_3[\{l_1,l_2,l_4\},\{l_3\}] \cup S_4[\{l_1,l_2\},\{l_3\},\{l_4\}]\\
&\qquad \cup S_4[\{l_2,l_4\},\{l_1\},\{l_3\}] \cup S_4[\{l_1,l_4\},\{l_2\},\{l_3\}] \cup S_4[\{l_3,l_4\},\{l_1\},\{l_2\}],\\
&\Xi_{l_1,l_2,l_3,l_4;l_4}\\
&\quad = S_3[\{l_1,l_2,l_3\},\{l_4\}] \cup S_4[\{l_1,l_2\},\{l_3\},\{l_4\}] \cup S_4[\{l_2,l_3\},\{l_1\},\{l_4\}]\\
&\qquad \cup S_4[\{l_1,l_3\},\{l_2\},\{l_4\}]
\end{aligned}
$$

and

$$\begin{aligned}
\tilde{u}_{l_1,l_2,l_3,l_4;l_1,j} &= u_{l_1,j} && \forall\, j = 1,\dots,s,\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_2,j} &= u_{l_1,j} && \forall\, j \in S_1[\{1,2,3,4\}] \cup S_2[\{l_1,l_2\},\{l_3,l_4\}]\\
&&& \quad \cup S_3[\{l_1,l_2,l_4\},\{l_3\}] \cup S_3[\{l_1,l_2,l_3\},\{l_4\}]\\
&&& \quad \cup S_4[\{l_1,l_2\},\{l_3\},\{l_4\}],\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_2,j} &= u_{l_2,j} && \forall\, j \in \Xi_{l_1,l_2,l_3,l_4;l_2},\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_3,j} &= u_{l_1,j} && \forall\, j \in S_1[\{1,2,3,4\}] \cup S_2[\{l_1,l_3\},\{l_2,l_4\}]\\
&&& \quad \cup S_3[\{l_1,l_3,l_4\},\{l_2\}] \cup S_3[\{l_1,l_2,l_3\},\{l_4\}]\\
&&& \quad \cup S_4[\{l_1,l_3\},\{l_2\},\{l_4\}],\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_3,j} &= u_{l_2,j} && \forall\, j \in S_2[\{l_1,l_4\},\{l_2,l_3\}] \cup S_3[\{l_2,l_3,l_4\},\{l_1\}]\\
&&& \quad \cup S_4[\{l_2,l_3\},\{l_1\},\{l_4\}],\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_3,j} &= u_{l_3,j} && \forall\, j \in \Xi_{l_1,l_2,l_3,l_4;l_3},\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_4,j} &= u_{l_1,j} && \forall\, j \in S_1[\{1,2,3,4\}] \cup S_2[\{l_1,l_4\},\{l_2,l_3\}]\\
&&& \quad \cup S_3[\{l_1,l_3,l_4\},\{l_2\}] \cup S_3[\{l_1,l_2,l_4\},\{l_3\}]\\
&&& \quad \cup S_4[\{l_1,l_4\},\{l_2\},\{l_3\}],\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_4,j} &= u_{l_2,j} && \forall\, j \in S_2[\{l_1,l_3\},\{l_2,l_4\}] \cup S_3[\{l_2,l_3,l_4\},\{l_1\}]\\
&&& \quad \cup S_4[\{l_2,l_4\},\{l_1\},\{l_3\}],\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_4,j} &= u_{l_3,j} && \forall\, j \in S_2[\{l_1,l_2\},\{l_3,l_4\}] \cup S_4[\{l_3,l_4\},\{l_1\},\{l_2\}],\\
\tilde{u}_{l_1,l_2,l_3,l_4;l_4,j} &= u_{l_4,j} && \forall\, j \in \Xi_{l_1,l_2,l_3,l_4;l_4}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{u}^{*}_{l_1,l_2,l_3,l_4} &= \sum_{j \in S_3[\{l_2,l_3,l_4\},\{l_1\}] \cup S_4[\{l_2,l_3\},\{l_1\},\{l_4\}] \cup S_4[\{l_2,l_4\},\{l_1\},\{l_3\}]} \tilde{u}_{l_1,l_2,l_3,l_4;l_1,j},\\
\tilde{u}^{**}_{l_1,l_2,l_3,l_4} &= \sum_{j \in S_1[\{1,2,3,4\}] \cup S_4[\{l_1,l_3\},\{l_2\},\{l_4\}] \cup S_4[\{l_1,l_4\},\{l_2\},\{l_3\}]} \tilde{u}_{l_1,l_2,l_3,l_4;l_2,j}\\
&\quad + \sum_{j \in S_3[\{l_1,l_3,l_4\},\{l_2\}] \cup S_3[\{l_1,l_2,l_4\},\{l_3\}] \cup S_3[\{l_1,l_2,l_3\},\{l_4\}]} \tilde{u}_{l_1,l_2,l_3,l_4;l_2,j},\\
\tilde{u}^{\dagger}_{l_1,l_2,l_3,l_4} &= \sum_{j \in S_2[\{l_1,l_2\},\{l_3,l_4\}] \cup S_4[\{l_1,l_2\},\{l_3\},\{l_4\}]} \tilde{u}_{l_1,l_2,l_3,l_4;l_2,j},\\
\tilde{u}^{\ddagger}_{l_1,l_2,l_3,l_4} &= \sum_{j \in S_4[\{l_3,l_4\},\{l_1\},\{l_2\}]} \big(\tilde{u}_{l_1,l_2,l_3,l_4;l_1,j} \vee \tilde{u}_{l_1,l_2,l_3,l_4;l_2,j}\big),\\
\tilde{u}^{\ddagger\ddagger}_{l_1,l_2,l_3,l_4} &= \sum_{j \in S_4[\{l_3,l_4\},\{l_1\},\{l_2\}]} \big(\tilde{u}_{l_1,l_2,l_3,l_4;l_1,j} \wedge \tilde{u}_{l_1,l_2,l_3,l_4;l_2,j}\big).
\end{aligned}$$

From (33), Theorem 1 and Lemma 3, we have via symmetry,

$$
\begin{aligned}
&b^{2(\tilde{m}-1)}E(S^4)\\
&\quad = O(1)m^{-2(s-1)}b^{2m+\tilde{m}}\\
&\qquad \times \sum_{s_1,\ldots,s_{14}\geq 0\,:\,s_1+\cdots+s_{14}=s}\\
&\qquad\qquad \times \left\{\prod_{l=1}^{4}\left[\sum_{u_{l,1},\ldots,u_{l,s}\geq\tilde{m}\,:\,u_{l,1}+\cdots+u_{l,s}\geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})}\right]\right\} \qquad (35)\\
&\qquad\qquad \times \sum_{k=1}^{s}\sum_{i_1=1}^{b^m}\sum_{i_2=1}^{b^m}\sum_{i_3=1}^{b^m}\sum_{i_4=1}^{b^m} \mathbb{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\}\xi_{s_1,\ldots,s_{14}}.
\end{aligned}
$$

Next we consider four cases that enable an evaluation of the right-hand side of (35).

Case 1. Suppose that $k \in \{1,\ldots,s_1\}$. Then if $s_1 + s_5 + \cdots + s_8 + s_{10} + \cdots + s_{13} \geq 2$ and using the definition of a $(0,m,s)$ net,

$$
\begin{aligned}
&m^{-2(s-1)}b^{2m+\tilde{m}}\left\{\prod_{l=1}^{4}\left[\sum_{u_{l,1},\ldots,u_{l,s}\geq\tilde{m}\,:\,u_{l,1}+\cdots+u_{l,s}\geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})}\right]\right\}\\
&\quad \times \sum_{i_1=1}^{b^m}\sum_{i_2=1}^{b^m}\sum_{i_3=1}^{b^m}\sum_{i_4=1}^{b^m} \mathbb{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\}\xi_{s_1,\ldots,s_{14}}\\
&\qquad = m^{-2(s-1)}b^{2m+\tilde{m}}\left\{\prod_{l=1}^{4}\left[\sum_{u_{l,1},\ldots,u_{l,s}\geq\tilde{m}\,:\,u_{l,1}+\cdots+u_{l,s}\geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})}\right]\right\}\\
&\qquad\qquad \times \mathbb{I}\{\tilde{u}^{\dagger}_{4,3,2,1} \leq \tilde{u}^{\dagger}_{1,2,3,4}\}\sum_{i_1=1}^{b^m}\sum_{i_2=1}^{b^m}\sum_{i_3=1}^{b^m}\sum_{i_4=1}^{b^m} \mathbb{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\}\xi_{s_1,\ldots,s_{14}}\\
&\qquad\quad + m^{-2(s-1)}b^{2m+\tilde{m}}\left\{\prod_{l=1}^{4}\left[\sum_{u_{l,1},\ldots,u_{l,s}\geq\tilde{m}\,:\,u_{l,1}+\cdots+u_{l,s}\geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})}\right]\right\}\\
&\qquad\qquad \times \mathbb{I}\{\tilde{u}^{\dagger}_{4,3,2,1} > \tilde{u}^{\dagger}_{1,2,3,4}\}\sum_{i_4=1}^{b^m}\sum_{i_3=1}^{b^m}\sum_{i_2=1}^{b^m}\sum_{i_1=1}^{b^m} \mathbb{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\}\xi_{s_1,\ldots,s_{14}}\\
&\qquad = O(1)m^{-2(s-1)}b^{2m+\tilde{m}}\xi_{s_1,\ldots,s_{14}}\\
&\qquad\quad \times \left\{\prod_{l=1}^{4}\left[\sum_{\tilde{u}_{1,2,3,4;l,1},\ldots,\tilde{u}_{1,2,3,4;l,s}\geq\tilde{m}\,:\,\tilde{u}_{1,2,3,4;l,1}+\cdots+\tilde{u}_{1,2,3,4;l,s}\geq m+1}\right.\right.\\
&\qquad\qquad\qquad \left.\left. \times b^{-(\tilde{u}_{1,2,3,4;l,1}+\cdots+\tilde{u}_{1,2,3,4;l,s})}\right]\right\} \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathcal{I}\{\tilde u^{\dagger}_{4,3,2,1} \le \tilde u^{\dagger}_{1,2,3,4}\} b^m \biggl(\frac{b^m}{b^{\tilde u^{*}_{1,2,3,4}+\tilde u^{**}_{1,2,3,4}+\tilde u^{\dagger}_{1,2,3,4}+\tilde u^{\ddagger\ddagger}_{1,2,3,4}}} \vee 1\biggr)\\
&\times \biggl(\frac{b^{m+\tilde u^{\dagger}_{4,3,2,1}}}{b^{\tilde u_{1,2,3,4;3,1}+\cdots+\tilde u_{1,2,3,4;3,s}+\tilde u^{\ddagger}_{1,2,3,4}}} \vee 1\biggr)\biggl(\frac{b^m}{b^{\tilde u_{1,2,3,4;4,1}+\cdots+\tilde u_{1,2,3,4;4,s}}} \vee 1\biggr)\\
&+ O(1) m^{-2(s-1)} b^{2m+\tilde m} \xi_{s_1,\dots,s_{14}}\\
&\times \Biggl\{\prod_{l=1}^{4}\Biggl[\sum_{\tilde u_{4,3,2,1;l,1},\dots,\tilde u_{4,3,2,1;l,s}\ge \tilde m\,:\,\tilde u_{4,3,2,1;l,1}+\cdots+\tilde u_{4,3,2,1;l,s}\ge m+1}\\
&\qquad\qquad \times b^{-(\tilde u_{4,3,2,1;l,1}+\cdots+\tilde u_{4,3,2,1;l,s})}\Biggr]\Biggr\}\\
&\times \mathcal{I}\{\tilde u^{\dagger}_{4,3,2,1} > \tilde u^{\dagger}_{1,2,3,4}\} b^m \biggl(\frac{b^m}{b^{\tilde u^{*}_{4,3,2,1}+\tilde u^{**}_{4,3,2,1}+\tilde u^{\dagger}_{4,3,2,1}+\tilde u^{\ddagger\ddagger}_{4,3,2,1}}} \vee 1\biggr)\\
&\times \biggl(\frac{b^{m+\tilde u^{\dagger}_{1,2,3,4}}}{b^{\tilde u_{4,3,2,1;2,1}+\cdots+\tilde u_{4,3,2,1;2,s}+\tilde u^{\ddagger}_{4,3,2,1}}} \vee 1\biggr)\biggl(\frac{b^m}{b^{\tilde u_{4,3,2,1;1,1}+\cdots+\tilde u_{4,3,2,1;1,s}}} \vee 1\biggr)\\
&= O\biggl(\frac{1}{m^2}\biggr)
\end{aligned}
$$

as $m \to \infty$. The second from last equality uses the fact that given $\xi_{s_1,\dots,s_{14}}$, there is a great deal of redundancy in the notation for $u_{l,k}$, $1 \le l \le 4$, $1 \le k \le s$. For example, if $k \in S_1[\{1,2,3,4\}]$, then $u_{1,k} = \cdots = u_{4,k}$. The additional notation used there helps to eliminate that type of redundancy and facilitate the use of a counting argument to obtain the appropriate bound. The last equality uses the observations that as $m \to \infty$,

$$
\begin{aligned}
\frac{b^m}{b^{\tilde u^{*}_{1,2,3,4}+\tilde u^{**}_{1,2,3,4}+\tilde u^{\ddagger\ddagger}_{1,2,3,4}}} &= O(b^{m-2\tilde m}),\\
\frac{b^{m+\tilde u^{\dagger}_{4,3,2,1}}}{b^{\tilde u_{1,2,3,4;3,1}+\cdots+\tilde u_{1,2,3,4;3,s}+\tilde u^{\ddagger}_{1,2,3,4}}} &= O(b^{m-2\tilde m}),\\
\frac{b^m}{b^{\tilde u_{1,2,3,4;4,1}+\cdots+\tilde u_{1,2,3,4;4,s}}} \vee 1 &= O(1)
\end{aligned}
$$

[incidentally, this is the main point for introducing the proxy $\tilde W$ for W in (15)] and

$$
\begin{aligned}
&\prod_{l=1}^{4}\Biggl[\sum_{\tilde u_{1,2,3,4;l,1},\dots,\tilde u_{1,2,3,4;l,s}\ge \tilde m\,:\,\tilde u_{1,2,3,4;l,1}+\cdots+\tilde u_{1,2,3,4;l,s}\ge m+1} b^{-(\tilde u_{1,2,3,4;l,1}+\cdots+\tilde u_{1,2,3,4;l,s})}\Biggr]\\
&\quad = O(1) m^{4(s-1)} b^{-4m}.
\end{aligned}
$$

On the other hand, if $s_1 = 1, s_5 + \cdots + s_8 + s_{10} + \cdots + s_{13} = 0$ and $s_9 + s_{14} \geq 1$, the above argument can be applied with the order of summations for i_1, i_2, i_3, i_4 suitably altered to obtain the bound in (36). If $s_1 = 1$ and $s_2 + s_3 + s_4 = s - 1$, then following (36) we have

$$\begin{aligned}
& m^{-2(s-1)} b^{2m+\tilde{m}} \left\{ \prod_{l=1}^{4} \left[\sum_{u_{l,1},\ldots,u_{l,s} \geq \tilde{m} :\, u_{l,1}+\cdots+u_{l,s} \geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})} \right] \right\} \\
& \quad \times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{i_3=1}^{b^m} \sum_{i_4=1}^{b^m} \ell\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\} \xi_{s_1,\ldots,s_{14}} \\
& \quad = O(1) m^{-2(s-1)} b^{3m+\tilde{m}} \xi_{s_1,\ldots,s_{14}} \\
& \qquad \times \sum_{\tilde{u}_{1,2,3,4;1,1},\ldots,\tilde{u}_{1,2,3,4;1,s} \geq \tilde{m} :\, \tilde{u}_{1,2,3,4;1,1}+\cdots+\tilde{u}_{1,2,3,4;1,s} \geq m+1} b^{-\tilde{u}_{1,2,3,4;1,k}} \\
& \qquad\qquad \times b^{-(\tilde{u}_{1,2,3,4;1,1}+\cdots+\tilde{u}_{1,2,3,4;1,s})} \\
& \qquad \times \sum_{\tilde{u}_{1,2,3,4;2,j} \geq \tilde{m} :\, j \in \Xi_{1,2,3,4;2}, \tilde{u}_{1,2,3,4;2,1}+\cdots+\tilde{u}_{1,2,3,4;2,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{1,2,3,4;2,1}+\cdots+\tilde{u}_{1,2,3,4;2,s})} \\
& \qquad \times \sum_{\tilde{u}_{1,2,3,4;3,j} \geq \tilde{m} :\, j \in \Xi_{1,2,3,4;3}, \tilde{u}_{1,2,3,4;3,1}+\cdots+\tilde{u}_{1,2,3,4;3,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{1,2,3,4;3,1}+\cdots+\tilde{u}_{1,2,3,4;3,s})} \qquad (37) \\
& \qquad + O(1) m^{-2(s-1)} b^{3m+\tilde{m}} \xi_{s_1,\ldots,s_{14}} \\
& \qquad \times \sum_{\tilde{u}_{4,3,2,1;4,1},\ldots,\tilde{u}_{4,3,2,1;4,s} \geq \tilde{m} :\, \tilde{u}_{4,3,2,1;4,1}+\cdots+\tilde{u}_{4,3,2,1;4,s} \geq m+1} b^{-\tilde{u}_{4,3,2,1;4,k}} \\
& \qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;4,1}+\cdots+\tilde{u}_{4,3,2,1;4,s})} \\
& \qquad \times \sum_{\tilde{u}_{4,3,2,1;3,j} \geq \tilde{m} :\, j \in \Xi_{4,3,2,1;3}, \tilde{u}_{4,3,2,1;3,1}+\cdots+\tilde{u}_{4,3,2,1;3,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;3,1}+\cdots+\tilde{u}_{4,3,2,1;3,s})} \\
& \qquad \times \sum_{\tilde{u}_{4,3,2,1;2,j} \geq \tilde{m} :\, j \in \Xi_{4,3,2,1;2}, \tilde{u}_{4,3,2,1;2,1}+\cdots+\tilde{u}_{4,3,2,1;2,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;2,1}+\cdots+\tilde{u}_{4,3,2,1;2,s})} \\
& \quad = O(1) m^{s-1+(s_2-1\vee 0)+(s_3+s_4-1\vee 0)} m^{-2(s-1)} \\
& \quad = O\left(\frac{1}{m}\right)
\end{aligned}$$

as $m \to \infty$. The second from last equality is a consequence of the observations that $\#\Xi_{1,2,3,4;2} = \#\Xi_{4,3,2,1;3} = s_3 + s_4$, $\#\Xi_{1,2,3,4;3} = \#\Xi_{4,3,2,1;2} = s_2$ and $b^{-\tilde{u}_{1,2,3,4;1,k}} = b^{-u_{1,k}} \le b^{-\tilde{m}}$. Here we adopt the convention that

$$\sum_{\tilde{u}_{1,2,3,4;3,j} \ge \tilde{m} :\, j \in \Xi_{1,2,3,4;3}, \tilde{u}_{1,2,3,4;3,1} + \cdots + \tilde{u}_{1,2,3,4;3,s} \ge m+1} b^{-(\tilde{u}_{1,2,3,4;3,1} + \cdots + \tilde{u}_{1,2,3,4;3,s})} = b^{-(\tilde{u}_{1,2,3,4;3,1} + \cdots + \tilde{u}_{1,2,3,4;3,s})}$$

if $\Xi_{1,2,3,4;3} = \phi$ and so forth.

Case 2. Suppose that $k \in \{s_1 + s_2 + 1, \ldots, s_1 + s_2 + s_3\}$. Then if $s_1 + s_5 + \cdots + s_8 + s_{10} + \cdots + s_{13} \ge 1$, we have as in (36), but keeping in mind the term $\mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\}$,

$$\begin{aligned}
& m^{-2(s-1)} b^{2m+\tilde{m}} \left\{ \prod_{l=1}^{4} \left[\sum_{u_{l,1},\ldots,u_{l,s} \ge \tilde{m} :\, u_{l,1} + \cdots + u_{l,s} \ge m+1} b^{-(u_{l,1} + \cdots + u_{l,s})} \right] \right\} \\
& \quad \times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{i_3=1}^{b^m} \sum_{i_4=1}^{b^m} \mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\} \xi_{s_1,\ldots,s_{14}} \\
& \quad = O(1) m^{-2(s-1)} b^{3m+\tilde{m}} \xi_{s_1,\ldots,s_{14}} \\
& \qquad \times \left\{ \prod_{l=1}^{4} \left[\sum_{\tilde{u}_{1,2,3,4;l,1},\ldots,\tilde{u}_{1,2,3,4;l,s} \ge \tilde{m} :\, \tilde{u}_{1,2,3,4;l,1} + \cdots + \tilde{u}_{1,2,3,4;l,s} \ge m+1} \times b^{-(\tilde{u}_{1,2,3,4;l,1} + \cdots + \tilde{u}_{1,2,3,4;l,s})} \right] \right\} \\
& \qquad \times \mathcal{I}\{\tilde{u}^{\dagger}_{4,3,2,1} \le \tilde{u}^{\dagger}_{1,2,3,4}\} \left(\frac{b^m}{b^{\tilde{u}^*_{1,2,3,4} + \tilde{u}^{**}_{1,2,3,4} + \tilde{u}^{\dagger}_{1,2,3,4} + \tilde{u}^{\ddagger\ddagger}_{1,2,3,4} + \tilde{m}}} \vee 1 \right) \\
& \qquad \times \left(\frac{b^{m + \tilde{u}^{\dagger}_{4,3,2,1}}}{b^{\tilde{u}_{1,2,3,4;3,1} + \cdots + \tilde{u}_{1,2,3,4;3,s} + \tilde{u}^{\ddagger}_{1,2,3,4}}} \vee 1 \right) \left(\frac{b^m}{b^{\tilde{u}_{1,2,3,4;4,1} + \cdots + \tilde{u}_{1,2,3,4;4,s}}} \vee 1 \right) \\
& \qquad + O(1) m^{-2(s-1)} b^{3m+\tilde{m}} \xi_{s_1,\ldots,s_{14}} \\
& \qquad \times \left\{ \prod_{l=1}^{4} \left[\sum_{\tilde{u}_{4,3,2,1;l,1},\ldots,\tilde{u}_{4,3,2,1;l,s} \ge \tilde{m} :\, \tilde{u}_{4,3,2,1;l,1} + \cdots + \tilde{u}_{4,3,2,1;l,s} \ge m+1} \times b^{-(\tilde{u}_{4,3,2,1;l,1} + \cdots + \tilde{u}_{4,3,2,1;l,s})} \right] \right\} \\
& \qquad \times \mathcal{I}\{\tilde{u}^{\dagger}_{4,3,2,1} > \tilde{u}^{\dagger}_{1,2,3,4}\} \left(\frac{b^m}{b^{\tilde{u}^*_{4,3,2,1} + \tilde{u}^{**}_{4,3,2,1} + \tilde{u}^{\dagger}_{4,3,2,1} + \tilde{u}^{\ddagger\ddagger}_{4,3,2,1} + \tilde{m}}} \vee 1 \right)
\end{aligned} \tag{38}$$

$$\times \left(\frac{b^{m+\tilde{u}^{\dagger}_{1,2,3,4}}}{b^{\tilde{u}_{4,3,2,1;2,1}+\cdots+\tilde{u}_{4,3,2,1;2,s}+\tilde{u}^{\ddagger}_{4,3,2,1}}} \vee 1\right)\left(\frac{b^m}{b^{\tilde{u}_{4,3,2,1;1,1}+\cdots+\tilde{u}_{4,3,2,1;1,s}}} \vee 1\right)$$
$$= O\left(\frac{1}{m^2}\right)$$

as $m \to \infty$. On the other hand, if $s_1 + s_5 + \cdots + s_8 + s_{10} + \cdots + s_{13} = 0$ and $s_9 + s_{14} \geq 1$, the above argument can be applied with the order of the summations for i_1, i_2, i_3, i_4 appropriately altered to obtain (38). This implies that (38) holds whenever $s_2 + s_3 + s_4 \leq s - 1$. Finally, if $s_2 + s_3 + s_4 = s$, by simplifying (38) further we have

$$m^{-2(s-1)} b^{2m+\tilde{m}} \left\{ \prod_{l=1}^{4} \left[\sum_{u_{l,1},\ldots,u_{l,s} \geq \tilde{m} : u_{l,1}+\cdots+u_{l,s} \geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})} \right] \right\}$$
$$\times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{i_3=1}^{b^m} \sum_{i_4=1}^{b^m} \mathbb{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\} \xi_{s_1,\ldots,s_{14}}$$
$$= O(1) m^{-2(s-1)} b^{3m} \xi_{s_1,\ldots,s_{14}}$$
$$\times \sum_{\tilde{u}_{1,3,2,4;1,1},\ldots,\tilde{u}_{1,3,2,4;1,s} \geq \tilde{m} : \tilde{u}_{1,3,2,4;1,1}+\cdots+\tilde{u}_{1,3,2,4;1,s} \geq m+1} \times b^{-(\tilde{u}_{1,3,2,4;1,1}+\cdots+\tilde{u}_{1,3,2,4;1,s})}$$
$$\times \sum_{\tilde{u}_{1,3,2,4;3,j} \geq \tilde{m} : j \in \Xi_{1,3,2,4;3}, \tilde{u}_{1,3,2,4;3,1}+\cdots+\tilde{u}_{1,3,2,4;3,s} \geq m+1} \times b^{-(\tilde{u}_{1,3,2,4;3,1}+\cdots+\tilde{u}_{1,3,2,4;3,s})}$$
$$\times \sum_{\tilde{u}_{1,3,2,4;2,j} \geq \tilde{m} : j \in \Xi_{1,3,2,4;2}, \tilde{u}_{1,3,2,4;2,1}+\cdots+\tilde{u}_{1,3,2,4;2,s} \geq m+1} \times b^{-(\tilde{u}_{1,3,2,4;2,1}+\cdots+\tilde{u}_{1,3,2,4;2,s})}$$
$$+ O(1) m^{-2(s-1)} b^{3m} \xi_{s_1,\ldots,s_{14}}$$
$$\times \sum_{\tilde{u}_{4,2,3,1;4,1},\ldots,\tilde{u}_{4,2,3,1;4,s} \geq \tilde{m} : \tilde{u}_{4,2,3,1;4,1}+\cdots+\tilde{u}_{4,2,3,1;4,s} \geq m+1} \times b^{-(\tilde{u}_{4,2,3,1;4,1}+\cdots+\tilde{u}_{4,2,3,1;4,s})}$$
$$\times \sum_{\tilde{u}_{4,2,3,1;2,j} \geq \tilde{m} : j \in \Xi_{4,2,3,1;2}, \tilde{u}_{4,2,3,1;2,1}+\cdots+\tilde{u}_{4,2,3,1;2,s} \geq m+1} \times b^{-(\tilde{u}_{4,2,3,1;2,1}+\cdots+\tilde{u}_{4,2,3,1;2,s})}$$
$$\times \sum_{\tilde{u}_{4,2,3,1;3,j} \geq \tilde{m} : j \in \Xi_{4,2,3,1;3}, \tilde{u}_{4,2,3,1;3,1}+\cdots+\tilde{u}_{4,2,3,1;3,s} \geq m+1}$$

$$\times b^{-(\tilde{u}_{4,2,3,1;3,1}+\cdots+\tilde{u}_{4,2,3,1;3,s})}$$
$$= O(1)m^{s-1+(s_3-1\vee 0)+(s_2+s_4-1\vee 0)}m^{-2(s-1)}$$

as $m\to\infty$. The last equality follows from the observation that $\#\Xi_{1,3,2,4;3} = \#\Xi_{4,2,3,1;2} = s_2+s_4$ and $\#\Xi_{1,3,2,4;2} = \#\Xi_{4,2,3,1;3} = s_3$. Since $s_3\geq 1$, we conclude that

$$m^{-2(s-1)}b^{2m+\tilde{m}}\left\{\prod_{l=1}^{4}\left[\sum_{u_{l,1},\dots,u_{l,s}\geq\tilde{m}:\,u_{l,1}+\cdots+u_{l,s}\geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})}\right]\right\}$$
$$\times\sum_{i_1=1}^{b^m}\sum_{i_2=1}^{b^m}\sum_{i_3=1}^{b^m}\sum_{i_4=1}^{b^m}\mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\}\xi_{s_1,\dots,s_{14}}$$
$$= O(1)[\mathcal{I}\{s_3=s\}+m^{-1}]$$

as $m\to\infty$.

Case 3. Suppose that $k\in\{s_1+\cdots+s_5+1,\dots,s_1+\cdots+s_6\}$. Then if $s_1+s_5+\cdots+s_8+s_{10}+\cdots+s_{13}\geq 2$,

$$m^{-2(s-1)}b^{2m+\tilde{m}}\left\{\prod_{l=1}^{4}\left[\sum_{u_{l,1},\dots,u_{l,s}\geq\tilde{m}:\,u_{l,1}+\cdots+u_{l,s}\geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})}\right]\right\}$$
$$\times\sum_{i_1=1}^{b^m}\sum_{i_2=1}^{b^m}\sum_{i_3=1}^{b^m}\sum_{i_4=1}^{b^m}\mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\}\xi_{s_1,\dots,s_{14}}$$
$$= O(1)m^{-2(s-1)}b^{3m+\tilde{m}}\xi_{s_1,\dots,s_{14}}$$
$$\times\left\{\prod_{l=1}^{4}\left[\sum_{\tilde{u}_{1,2,3,4;l,1},\dots,\tilde{u}_{1,2,3,4;l,s}\geq\tilde{m}:\,\tilde{u}_{1,2,3,4;l,1}+\cdots+\tilde{u}_{1,2,3,4;l,s}\geq m+1}\right.\right.$$
$$\left.\left.\times b^{-(\tilde{u}_{1,2,3,4;l,1}+\cdots+\tilde{u}_{1,2,3,4;l,s})}\right]\right\}$$
$$\times\mathcal{I}\{\tilde{u}^{\dagger}_{4,3,2,1}<\tilde{u}^{\dagger}_{1,2,3,4}\}\left(\frac{b^m}{b^{\tilde{u}^{*}_{1,2,3,4}+\tilde{u}^{**}_{1,2,3,4}+\tilde{u}^{\dagger}_{1,2,3,4}+\tilde{u}^{\ddagger\ddagger}_{1,2,3,4}}}\vee 1\right)$$
$$\times\left(\frac{b^{m+\tilde{u}^{\dagger}_{4,3,2,1}}}{b^{\tilde{u}_{1,2,3,4;3,1}+\cdots+\tilde{u}_{1,2,3,4;3,s}+\tilde{u}^{\ddagger}_{1,2,3,4}}}\vee 1\right)\left(\frac{b^m}{b^{\tilde{u}_{1,2,3,4;4,1}+\cdots+\tilde{u}_{1,2,3,4;4,s}}}\vee 1\right)$$
$$+ O(1)m^{-2(s-1)}b^{3m+\tilde{m}}\xi_{s_1,\dots,s_{14}}$$
$$\times\left\{\prod_{l=1}^{4}\left[\sum_{\tilde{u}_{4,3,2,1;l,1},\dots,\tilde{u}_{4,3,2,1;l,s}\geq\tilde{m}:\,\tilde{u}_{4,3,2,1;l,1}+\cdots+\tilde{u}_{4,3,2,1;l,s}\geq m+1}\right.\right.$$
$$\left.\left.\times b^{-(\tilde{u}_{4,3,2,1;l,1}+\cdots+\tilde{u}_{4,3,2,1;l,s})}\right]\right\} \tag{39}$$

$$\times \mathcal{I}\{\tilde{u}^{\dagger}_{4,3,2,1} \geq \tilde{u}^{\dagger}_{1,2,3,4}\}\left(\frac{b^m}{b^{\tilde{u}^{*}_{4,3,2,1}+\tilde{u}^{**}_{4,3,2,1}+\tilde{u}^{\dagger}_{4,3,2,1}+\tilde{u}^{\ddagger\ddagger}_{4,3,2,1}}} \vee 1\right)$$

$$\times \left(\frac{b^{m+\tilde{u}^{\dagger}_{1,2,3,4}}}{b^{\tilde{u}_{4,3,2,1;2,1}+\cdots+\tilde{u}_{4,3,2,1;2,s}+\tilde{u}^{\ddagger}_{4,3,2,1}}} \vee 1\right)\left(\frac{b^m}{b^{\tilde{u}_{4,3,2,1;1,1}+\cdots+\tilde{u}_{4,3,2,1;1,s}}} \vee 1\right)$$

$$= O\left(\frac{1}{m^2}\right)$$

as $m \to \infty$. On the other hand, if $s_6 = 1$, $s_1 + s_5 + s_7 + s_8 + s_{10} + \cdots + s_{13} = 0$ and $s_9 + s_{14} \geq 1$, the above argument can be applied with the order of the summations for i_1, i_2, i_3, i_4 suitably altered to obtain the bound in (39). Next suppose that $s_6 = 1$ and $s_3 + s_4 = s - 1$ (i.e., $s_2 = 0$). Then $\tilde{u}^{\dagger}_{4,3,2,1} = \tilde{u}^{\dagger}_{1,2,3,4} = 0$ and, on simplifying (39), we have

$$\begin{aligned}
&m^{-2(s-1)} b^{2m+\tilde{m}} \left\{ \prod_{l=1}^{4} \left[\sum_{u_{l,1},\ldots,u_{l,s} \geq \tilde{m}\,:\, u_{l,1}+\cdots+u_{l,s} \geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})} \right] \right\} \\
&\quad \times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{i_3=1}^{b^m} \sum_{i_4=1}^{b^m} \mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\} \xi_{s_1,\ldots,s_{14}} \\
&\quad = O(1) m^{-2(s-1)} b^{3m} \xi_{s_1,\ldots,s_{14}} \\
&\qquad \times \sum_{\tilde{u}_{4,3,2,1;4,1},\ldots,\tilde{u}_{4,3,2,1;4,s} \geq \tilde{m}\,:\, \tilde{u}_{4,3,2,1;4,1}+\cdots+\tilde{u}_{4,3,2,1;4,s} \geq m+1} \\
&\qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;4,1}+\cdots+\tilde{u}_{4,3,2,1;4,s})} \\
&\qquad \times \sum_{\tilde{u}_{4,3,2,1;3,j} \geq \tilde{m}\,:\, j \in \Xi_{4,3,2,1;3}, \tilde{u}_{4,3,2,1;3,1}+\cdots+\tilde{u}_{4,3,2,1;3,s} \geq m+1} \\
&\qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;3,1}+\cdots+\tilde{u}_{4,3,2,1;3,s})} \\
&\qquad \times \sum_{\tilde{u}_{4,3,2,1;2,j} \geq \tilde{m}\,:\, j \in \Xi_{4,3,2,1;2}, \tilde{u}_{4,3,2,1;2,1}+\cdots+\tilde{u}_{4,3,2,1;2,s} \geq m+1} \\
&\qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;2,1}+\cdots+\tilde{u}_{4,3,2,1;2,s})} \\
&\quad = O(1) m^{s-1+(s_3+s_4-1)+(s_6-1)-2(s-1)} \\
&\quad = O\left(\frac{1}{m}\right)
\end{aligned} \tag{40}$$

as $m \to \infty$. The second equality uses $\#\Xi_{4,3,2,1;3} = s_3 + s_4$ and $\#\Xi_{4,3,2,1;2} = s_2 + s_6$. Using symmetry, it remains to consider $s_6 = 1$, $s_2 + s_3 + s_4 = s - 1$ and

$s_i \geq 1$ whenever $2 \leq i \leq 4$. Then following essentially (39), we have

$$
\begin{aligned}
& m^{-2(s-1)} b^{2m+\tilde{m}} \left\{ \prod_{l=1}^{4} \left[\sum_{u_{l,1},\dots,u_{l,s} \geq \tilde{m} : u_{l,1}+\cdots+u_{l,s} \geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})} \right] \right\} \\
& \quad \times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{i_3=1}^{b^m} \sum_{i_4=1}^{b^m} \mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\} \xi_{s_1,\dots,s_{14}} \\
& \quad = O(1) m^{-2(s-1)} b^{3m} \xi_{s_1,\dots,s_{14}} \\
& \qquad \times \sum_{\tilde{u}_{1,2,3,4;1,1},\dots,\tilde{u}_{1,2,3,4;1,s} \geq \tilde{m} : \tilde{u}_{1,2,3,4;1,1}+\cdots+\tilde{u}_{1,2,3,4;1,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{1,2,3,4;1,1}+\cdots+\tilde{u}_{1,2,3,4;1,s})} \\
& \qquad \times \sum_{\tilde{u}_{1,2,3,4;2,j} \geq \tilde{m} : j \in \Xi_{1,2,3,4;2}, \tilde{u}_{1,2,3,4;2,1}+\cdots+\tilde{u}_{1,2,3,4;2,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{1,2,3,4;2,1}+\cdots+\tilde{u}_{1,2,3,4;2,s})} \\
& \qquad \times \sum_{\tilde{u}_{1,2,3,4;3,j} \geq \tilde{m} : j \in \Xi_{1,2,3,4;3}, \tilde{u}_{1,2,3,4;3,1}+\cdots+\tilde{u}_{1,2,3,4;3,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{1,2,3,4;3,1}+\cdots+\tilde{u}_{1,2,3,4;3,s})} \qquad (41) \\
& \qquad + O(1) m^{-2(s-1)} b^{3m} \xi_{s_1,\dots,s_{14}} \\
& \qquad \times \sum_{\tilde{u}_{4,3,2,1;4,1},\dots,\tilde{u}_{4,3,2,1;4,s} \geq \tilde{m} : \tilde{u}_{4,3,2,1;4,1}+\cdots+\tilde{u}_{4,3,2,1;4,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;4,1}+\cdots+\tilde{u}_{4,3,2,1;4,s})} \\
& \qquad \times \sum_{\tilde{u}_{4,3,2,1;3,j} \geq \tilde{m} : j \in \Xi_{4,3,2,1;3}, \tilde{u}_{4,3,2,1;3,1}+\cdots+\tilde{u}_{4,3,2,1;3,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;3,1}+\cdots+\tilde{u}_{4,3,2,1;3,s})} \\
& \qquad \times \sum_{\tilde{u}_{4,3,2,1;2,j} \geq \tilde{m} : j \in \Xi_{4,3,2,1;2}, \tilde{u}_{4,3,2,1;2,1}+\cdots+\tilde{u}_{4,3,2,1;2,s} \geq m+1} \\
& \qquad\qquad \times b^{-(\tilde{u}_{4,3,2,1;2,1}+\cdots+\tilde{u}_{4,3,2,1;2,s})} \\
& \quad = O(1) \big[m^{s-1+(s_3+s_4+s_6-1)+(s_2-1)-2(s-1)} \\
& \qquad\qquad + m^{s-1+(s_3+s_4-1)+(s_2+s_6-1)-2(s-1)} \big] \\
& \quad = O\left(\frac{1}{m}\right)
\end{aligned}
$$

as $m \to \infty$. The second equality uses $\#\Xi_{1,2,3,4;2} = s_3 + s_4 + s_6$ and $\#\Xi_{1,2,3,4;3} = s_2$.

Case 4. Suppose that $k \in \{s_1 + \cdots + s_{12} + 1, \ldots, s_1 + \cdots + s_{13}\}$. Then if $s_1 + s_5 + \cdots + s_8 + s_{10} + \cdots + s_{13} \geq 2$, we have, as in (36),

$$
\begin{aligned}
&m^{-2(s-1)} b^{2m+\tilde{m}} \left\{ \prod_{l=1}^{4} \left[\sum_{u_{l,1},\ldots,u_{l,s} \geq \tilde{m} : u_{l,1}+\cdots+u_{l,s} \geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})} \right] \right\} \\
&\qquad \times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{i_3=1}^{b^m} \sum_{i_4=1}^{b^m} \mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\} \xi_{s_1,\ldots,s_{14}} = O\left(\frac{1}{m^2}\right)
\end{aligned} \tag{42}
$$

as $m \to \infty$. On the other hand, if $s_{13} = 1$, $s_1 + s_5 + \cdots + s_8 + s_{10} + \cdots + s_{12} = 0$ and $s_9 + s_{14} \geq 1$, the above argument can be applied with the order of the summations for i_1, i_2, i_3, i_4 suitably altered to obtain the bound in (42). Next suppose that $s_{13} = 1$ and $s_3 + s_4 = s - 1$. Arguing as in (40), we have

$$
\begin{aligned}
&m^{-2(s-1)} b^{2m+\tilde{m}} \left\{ \prod_{l=1}^{4} \left[\sum_{u_{l,1},\ldots,u_{l,s} \geq \tilde{m} : u_{l,1}+\cdots+u_{l,s} \geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})} \right] \right\} \\
&\qquad \times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{i_3=1}^{b^m} \sum_{i_4=1}^{b^m} \mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\} \xi_{s_1,\ldots,s_{14}} = O\left(\frac{1}{m}\right)
\end{aligned}
$$

as $m \to \infty$. Using symmetry, it remains to consider $s_{13} = 1$, $s_2 + s_3 + s_4 = s - 1$, $s_2 \geq 1$ and $s_3 + s_4 \geq 1$. Then arguing as in (41), we have

$$
\begin{aligned}
&m^{-2(s-1)} b^{2m+\tilde{m}} \left\{ \prod_{l=1}^{4} \left[\sum_{u_{l,1},\ldots,u_{l,s} \geq \tilde{m} : u_{l,1}+\cdots+u_{l,s} \geq m+1} b^{-(u_{l,1}+\cdots+u_{l,s})} \right] \right\} \\
&\qquad \times \sum_{i_1=1}^{b^m} \sum_{i_2=1}^{b^m} \sum_{i_3=1}^{b^m} \sum_{i_4=1}^{b^m} \mathcal{I}\{\Omega_{i_1,i_2,i_3,i_4;k;\tilde{m}}\} \xi_{s_1,\ldots,s_{14}} = O\left(\frac{1}{m}\right)
\end{aligned}
$$

as $m \to \infty$. Using symmetry, (35) and Cases 1–4 implies that $b^{2(\tilde{m}-1)} E(S^4) = O(1)$ as $m \to \infty$. This completes the proof of Lemma 4. □

PROPOSITION 5. *With the notation and assumptions of Theorem* 3,

$$
\begin{aligned}
&\left| E[g_z^{(1)}(\tilde{W})] E\left[\int K_{\tilde{W},\tilde{W}^*}(w)\,dw \right] - E\left[g_z^{(1)}(\tilde{W}) \int K_{\tilde{W},\tilde{W}^*}(w)\,dw \right] \right| \\
&\qquad = O\left(\left(\frac{\tilde{m}}{m} \right)^{1/2} \right)
\end{aligned}
$$

as $m \to \infty$ *uniformly over* $z \in \mathcal{R}$.

PROOF. Since $|g_z^{(1)}(w)| \le 1$ for all $w \in \mathcal{R}$, we observe from (17), (19) and Proposition 3 that

$$\begin{aligned} &\left| E[g_z^{(1)}(\tilde W)] E\left[\int K_{\tilde W, \tilde W^*}(w)\,dw\right] - E\left[g_z^{(1)}(\tilde W)\int K_{\tilde W, \tilde W^*}(w)\,dw\right]\right| \\ &\quad \le E\left|E^{\mathcal{W}}\left[\int K_{\tilde W, \tilde W^*}(w)\,dw - 1\right]\right| + O\left(\frac{\tilde m}{m}\right) \\ &\quad = E\left|E^{\mathcal{W}}\left[\frac{b^{\tilde m - 1}}{2}(\tilde S - S)^2 - 1\right]\right| + O\left(\frac{\tilde m}{m}\right) \end{aligned} \tag{43}$$

as $m \to \infty$. Hence it follows from (16) that

$$\begin{aligned} &E\left|E^{\mathcal{W}}\left[\frac{b^{\tilde m-1}}{2}(\tilde S - S)^2 - 1\right]\right| \\ &\quad = \frac{1}{2}E|E^{\mathcal{W}}[b^{\tilde m-1}(\tilde S^2 - 2\tilde S S + S^2) - 2]| \\ &\quad \le \frac{1}{2}E|b^{\tilde m-1}E^{\mathcal{W}}(S^2) - 1| + b^{\tilde m-1}E|E^{\mathcal{W}}(\tilde S S)| + \frac{1}{2}E|b^{\tilde m-1}E^{\mathcal{W}}(\tilde S^2) - 1| \\ &\quad \le \frac{1}{2}\{E[b^{\tilde m-1}E^{\mathcal{W}}(S^2) - 1]^2\}^{1/2} + b^{\tilde m-1}E|E^{\mathcal{W}}[SE^{\mathcal{W},I,J}(\tilde S)]| \\ &\qquad + \frac{1}{2}\{E[b^{\tilde m-1}E^{\mathcal{W}}(\tilde S^2) - 1]^2\}^{1/2} \\ &\quad \le \{E[b^{\tilde m-1}E^{\mathcal{W}}(S^2)]^2 - 2b^{\tilde m-1}E(S^2) + 1\}^{1/2}. \end{aligned} \tag{44}$$

The last inequality uses (25) and that

$$E[b^{\tilde m-1}E^{\mathcal{W}}(\tilde S^2)]^2 \le E[b^{\tilde m-1}E^{\mathcal{W}}(S^2)]^2$$

via Jensen's inequality and symmetry. We observe from (32) that

$$\begin{aligned} &E\{[b^{\tilde m-1}E^{\mathcal{W}}(S^2)]^2\} \\ &\quad = \frac{1}{s^2 b^{4m}\sigma_{0,m,s}^4} \\ &\qquad \times E\Bigg\{\Bigg[\sum_{k_1=1}^{s}\sum_{i_1=1}^{b^m}\sum_{u_{1,1},\dots,u_{1,s}\ge\tilde m:\, u_{1,1}+\dots+u_{1,s}\ge m+1} \\ &\qquad\qquad \times \nu_{u_{1,1},\dots,u_{1,s}}[\pi_j(a_{i_1,j,1}),\dots, \\ &\qquad\qquad\qquad \pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_{1,j}-1}}(a_{i_1,j,u_{1,j}}) : 1 \le j \le s] \end{aligned} \tag{45}$$

$$\times \sum_{i_2=1}^{b^m} \sum_{u_{2,1},\dots,u_{2,s}\geq \tilde{m}\,:\,u_{2,1}+\cdots+u_{2,s}\geq m+1} \mathscr{I}\{\Omega_{i_1,i_2;k_1;\tilde{m}}\}$$

$$\times \nu_{u_{2,1},\dots,u_{2,s}}\bigl[\pi_j(a_{i_2,j,1}),\dots,$$

$$\pi_{j;a_{i_2,j,1},\dots,a_{i_2,j,u_{2,j}-1}}(a_{i_2,j,u_{2,j}}) : 1\leq j\leq s\bigr]\Bigr]^2\Biggr\}$$

$$= R_1 + R_2,$$

where

$$R_1 = \frac{1}{s^2 b^{4m}\sigma_{0,m,s}^4}$$

$$\times E\Biggl\{\sum_{k_1=1}^{s}\sum_{i_1=1}^{b^m}$$

$$\times \sum_{u_{1,1},\dots,u_{1,s}\geq \tilde{m}\,:\,u_{1,1}+\cdots+u_{1,s}\geq m+1}$$

$$\times \nu_{u_{1,1},\dots,u_{1,s}}\bigl[\pi_j(a_{i_1,j,1}),\dots,$$

$$\pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_{1,j}-1}}(a_{i_1,j,u_{1,j}}) : 1\leq j\leq s\bigr]$$

$$\times \sum_{i_2=1}^{b^m} \sum_{u_{2,1},\dots,u_{2,s}\geq \tilde{m}\,:\,u_{2,1}+\cdots+u_{2,s}\geq m+1} \mathscr{I}\{\Omega_{i_1,i_2;k_1;\tilde{m}}\}$$

$$\times \nu_{u_{2,1},\dots,u_{2,s}}\bigl[\pi_j(a_{i_2,j,1}),\dots,$$

$$\pi_{j;a_{i_2,j,1},\dots,a_{i_2,j,u_{2,j}-1}}(a_{i_2,j,u_{2,j}}) : 1\leq j\leq s\bigr]$$

$$\times \sum_{k_3=1}^{s}\sum_{i_3=1}^{b^m} \sum_{u_{3,1},\dots,u_{3,s}\geq \tilde{m}\,:\,u_{3,1}+\cdots+u_{3,s}\geq m+1}$$

$$\times \nu_{u_{3,1},\dots,u_{3,s}}\bigl[\pi_j(a_{i_3,j,1}),\dots,$$

$$\pi_{j;a_{i_3,j,1},\dots,a_{i_3,j,u_{3,j}-1}}(a_{i_3,j,u_{3,j}}) : 1\leq j\leq s\bigr]$$

$$\times \sum_{i_4=1}^{b^m} \sum_{u_{4,1},\dots,u_{4,s}\geq \tilde{m}\,:\,u_{4,1}+\cdots+u_{4,s}\geq m+1} \mathscr{I}\{\Omega_{i_3,i_4;k_3;\tilde{m}}\}$$

$$\times \nu_{u_{4,1},\dots,u_{4,s}}\bigl[\pi_j(a_{i_4,j,1}),\dots,$$

$$\pi_{j;a_{i_4,j,1},\dots,a_{i_4,j,u_{4,j}-1}}(a_{i_4,j,u_{4,j}}) : 1\leq j\leq s\bigr]\Biggr\}$$

$$\times \mathscr{I}\{\#S_2[\{1,2\},\{3,4\}] = s\},$$

$S_2[\{1,2\},\{3,4\}]$ is as in (34) and $R_2 = E\{[b^{\tilde{m}-1}E^{\mathcal{W}}(S^2)]^2\} - R_1$. We observe from (22) that

$$
\begin{aligned}
R_1 = {} & \frac{1}{s^2 b^{4m}\sigma_{0,m,s}^4} \\
& \times E\Biggl\{\sum_{k_1=1}^{s}\sum_{i_1=1}^{b^m} \\
& \qquad \times \sum_{u_{1,1},\dots,u_{1,s}\ge\tilde{m}\,:\,u_{1,1}+\cdots+u_{1,s}\ge m+1}\Biggl(\sum_{i_2=1}^{b^m}\prod_{j=1}^{s}\Upsilon_{i_1,i_2;j,u_{1,j}}\Biggr) \\
& \qquad\qquad \times \nu_{u_{1,1},\dots,u_{1,s}}\bigl[\pi_j\bigl(a_{i_1,j,1}\bigr),\dots, \\
& \qquad\qquad\qquad \pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_{1,j}-1}}\bigl(a_{i_1,j,u_{1,j}}\bigr):1\le j\le s\bigr]^2 \\
& \qquad \times \sum_{k_3=1}^{s}\sum_{i_3\ne i_1} \\
& \qquad \times \sum_{u_{3,1},\dots,u_{3,s}\ge\tilde{m}\,:\,u_{3,1}+\cdots+u_{3,s}\ge m+1}\Biggl(\sum_{i_4=1}^{b^m}\prod_{j=1}^{s}\Upsilon_{i_3,i_4;j,u_{3,j}}\Biggr) \\
& \qquad\qquad \times \nu_{u_{3,1},\dots,u_{3,s}}\bigl[\pi_j\bigl(a_{i_3,j,1}\bigr),\dots, \\
& \qquad\qquad\qquad \pi_{j;a_{i_3,j,1},\dots,a_{i_3,j,u_{3,j}-1}}\bigl(a_{i_3,j,u_{3,j}}\bigr):1\le j\le s\bigr]^2\Biggr\} \\
& \qquad\qquad\qquad \times \mathcal{I}\bigl\{\#S_2[\{1,2\},\{3,4\}]=s\bigr\}.
\end{aligned}
$$

For each $j = 1,\dots,s$, define

$$M_{i_1,i_3;j} = \max\bigl\{k\ge 1:\bigl(a_{i_1,j,1},\dots,a_{i_1,j,k-1}\bigr) = \bigl(a_{i_3,j,1},\dots,a_{i_3,j,k-1}\bigr)\bigr\}.$$

Since $\#S_2[\{1,2\},\{3,4\}] = s$ implies $M_{i_1,i_3;j}\le u_{1,j}\wedge u_{3,j}-1$, we have

$$
\begin{aligned}
R_1 = {} & \frac{1}{s^2 b^{4m}\sigma_{0,m,s}^4} \\
& \times E\Biggl\{\sum_{k_1=1}^{s}\sum_{i_1=1}^{b^m} \\
& \qquad \times \sum_{u_{1,1},\dots,u_{1,s}\ge\tilde{m}\,:\,u_{1,1}+\cdots+u_{1,s}\ge m+1}\Biggl(\sum_{i_2=1}^{b^m}\prod_{j=1}^{s}\Upsilon_{i_1,i_2;j,u_{1,j}}\Biggr) \\
& \qquad\qquad \times \nu_{u_{1,1},\dots,u_{1,s}}\bigl[\pi_j\bigl(a_{i_1,j,1}\bigr),\dots, \\
& \qquad\qquad\qquad \pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_{1,j}-1}}\bigl(a_{i_1,j,u_{1,j}}\bigr):1\le j\le s\bigr]^2
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&\qquad \times \sum_{k_3=1}^{s} \sum_{i_3 \neq i_1} \\
&\qquad\quad \times \sum_{u_{3,1},\dots,u_{3,s} \geq \tilde{m} \,:\, u_{3,1}+\cdots+u_{3,s} \geq m+1} \left(\sum_{i_4=1}^{b^m} \prod_{j=1}^{s} \Upsilon_{i_3,i_4;\,j,u_{3,j}} \right) \\
&\qquad\qquad\qquad \times \nu_{u_{3,1},\dots,u_{3,s}} \big[\pi_j(a_{i_3,j,1}),\dots, \\
&\qquad\qquad\qquad\qquad \pi_{j;a_{i_3,j,1},\dots,a_{i_3,j,u_{3,j}-1}}(a_{i_3,j,u_{3,j}}) : 1 \leq j \leq s\big]^2 \Bigg\} \\
&\quad \times \mathbb{1}\{\#S_2[\{1,2\},\{3,4\}] = s\} \\
&\quad \times \left(\sum_{m_1=1}^{u_{1,1}\wedge u_{3,1}-1} \cdots \sum_{m_s=1}^{u_{1,s}\wedge u_{3,s}-1} \mathbb{1}\{M_{i_1,i_3;\,j} = m_j : 1 \leq j \leq s\} \right) \\
= {}& \frac{1}{b^{4m}\sigma_{0,m,s}^4} \\
&\quad \times \sum_{u_{1,1},\dots,u_{1,s} \geq \tilde{m} \,:\, u_{1,1}+\cdots+u_{1,s} \geq m+1} \left(\sum_{i_2=1}^{b^m} \prod_{j=1}^{s} \Upsilon_{1,i_2;\,j,u_{1,j}} \right) \\
&\quad \times \frac{1}{b^{u_{1,1}+\cdots+u_{1,s}}} \\
&\quad \times \sum_{0 \leq \tilde{c}_{1,j,k} \leq b-1 \,:\, 1 \leq j \leq s,\, 1 \leq k \leq u_{1,j}} \nu_{u_{1,1},\dots,u_{1,s}}[\tilde{c}_{1,j,1},\dots,\tilde{c}_{1,j,u_{1,j}} : 1 \leq j \leq s]^2 \\
&\quad \times \sum_{u_{3,1},\dots,u_{3,s} \geq \tilde{m} \,:\, u_{3,1}+\cdots+u_{3,s} \geq m+1} \left(\sum_{i_4=1}^{b^m} \prod_{j=1}^{s} \Upsilon_{1,i_4;\,j,u_{3,j}} \right) \\
&\quad \times \left(\sum_{m_1=1}^{u_{1,1}\wedge u_{3,1}-1} \cdots \sum_{m_s=1}^{u_{1,s}\wedge u_{3,s}-1} \sum_{i_1=1}^{b^m} \sum_{i_3 \neq i_1} \mathbb{1}\{M_{i_1,i_3;\,j} = m_j : 1 \leq j \leq s\} \right) \\
&\quad \times \frac{1}{(b-1)^s b^{u_{3,1}+\cdots+u_{3,s}-m_1-\cdots-m_s}} \\
&\quad \times \sum_{0 \leq \tilde{c}_{3,j,m_j} \leq b-1 \,:\, \tilde{c}_{3,j,m_j} \neq \tilde{c}_{1,j,m_j},\, 1 \leq j \leq s} \\
&\quad \times \sum_{0 \leq \tilde{c}_{3,j,k} \leq b-1 \,:\, 1 \leq j \leq s,\, m_j+1 \leq k \leq u_{3,j}} \\
&\qquad\qquad\qquad \times \nu_{u_{3,1},\dots,u_{3,s}}[\tilde{c}_{1,j,1},\dots, \\
&\qquad\qquad\qquad\qquad \tilde{c}_{1,j,m_j-1}, \tilde{c}_{3,j,m_j},\dots,\tilde{c}_{3,j,u_{3,j}} : 1 \leq j \leq s]^2.
\end{aligned}
$$

From the definition of a $(0, m, s)$ net that every elementary interval in base b of s-dimensional Lebesgue measure b^{-m} contains exactly one point of the net, we have

$$\sum_{i_3 \neq i_1} \mathcal{I}\{M_{i_1,i_3;j} = m_j : 1 \leq j \leq s\} = (b-1)^s b^{m-m_1-\cdots-m_s} \qquad \forall m_1 + \cdots + m_s \leq m$$

and

$$\sum_{i_3 \neq i_1} \mathcal{I}\Big\{M_{i_1,i_3;j} = m_j : 1 \leq j \leq s\Big\} \leq b^{m-m_1-\cdots-m_s+s} \qquad \forall m_1 + \cdots + m_s \geq m+1.$$

Hence it follows from (15), (30) and (46) that

$$\begin{aligned}
R_1 = \Bigg\{ & \frac{1}{b^m \sigma_{0,m,s}^2} \\
& \times \sum_{u_{1,1},\dots,u_{1,s} \geq \tilde{m} : u_{1,1}+\cdots+u_{1,s} \geq m+1} \left(\sum_{i_2=1}^{b^m} \prod_{j=1}^{s} \Upsilon_{1,i_2;j,u_{1,j}} \right) \frac{1}{b^{u_{1,1}+\cdots+u_{1,s}}} \\
& \times \sum_{0 \leq \tilde{c}_{1,j,k} \leq b-1 : 1 \leq j \leq s, 1 \leq k \leq u_{1,j}} \nu_{u_{1,1},\dots,u_{1,s}} \big[\tilde{c}_{1,j,1}, \dots, \tilde{c}_{1,j,u_{1,j}} : 1 \leq j \leq s\big]^2 \Bigg\}^2 \qquad (47) \\
& + R_{1,1} + R_{1,2} \\
= {} & [E(\tilde{W}^2)]^2 + R_{1,1} + R_{1,2} \\
= {} & 1 + O\left(\frac{\tilde{m}}{m}\right) + R_{1,1} + R_{1,2},
\end{aligned}$$

where we observe from (22), Theorem 1 and Lemma 3 that

$$\begin{aligned}
|R_{1,1}| \leq {} & \frac{O(1)}{b^{2m} \sigma_{0,m,s}^4} \\
& \times \sum_{u_{1,1},\dots,u_{1,s} \geq \tilde{m} : u_{1,1}+\cdots+u_{1,s} \geq m+1} \left(\sum_{i_2=1}^{b^m} \prod_{j=1}^{s} \Upsilon_{1,i_2;j,u_{1,j}} \right) \frac{1}{b^{u_{1,1}+\cdots+u_{1,s}}} \\
& \times \sum_{0 \leq \tilde{c}_{1,j,k} \leq b-1 : 1 \leq j \leq s, 1 \leq k \leq u_{1,j}} \nu_{u_{1,1},\dots,u_{1,s}} \big[\tilde{c}_{1,j,1}, \dots, \tilde{c}_{1,j,u_{1,j}} : 1 \leq j \leq s\big]^2 \\
& \times \sum_{u_{3,1},\dots,u_{3,s} \geq \tilde{m} : u_{3,1}+\cdots+u_{3,s} \geq m+1} \left(\sum_{i_4=1}^{b^m} \prod_{j=1}^{s} \Upsilon_{1,i_4;j,u_{3,j}} \right) \sum_{l=1}^{s} \frac{1}{b^{u_{3,1}+\cdots+u_{3,s}}} \qquad (48)
\end{aligned}$$

$$\times \sum_{\tilde{c}_{3,l,k}\,:\,\tilde{c}_{3,l,k}=\tilde{c}_{1,l,k},1\le k\le (u_{1,l}\wedge u_{3,l})-1} \sum_{0\le \tilde{c}_{3,l,k}\le b-1\,:\,u_{1,l}\wedge u_{3,l}\le k\le u_{3,l}}$$
$$\times \sum_{0\le \tilde{c}_{3,j,k}\le b-1\,:\,j\ne l,1\le k\le u_{3,j}} \nu_{u_{3,1},\dots,u_{3,s}}[\tilde{c}_{3,j,1},\dots,\tilde{c}_{3,j,u_{3,j}} : 1\le j\le s]^2$$
$$= O(b^{-\tilde{m}})$$

and

$$|R_{1,2}| \le \frac{2e^2 b^s}{b^{4m}\sigma_{0,m,s}^4 (b-1)^s}$$
$$\times \sum_{u_{1,1},\dots,u_{1,s}\ge \tilde{m}\,:\,u_{1,1}+\cdots+u_{1,s}\ge m+1}$$
$$\times \Biggl(\frac{1}{b^{u_{1,1}+\cdots+u_{1,s}}}$$
$$\times \sum_{0\le \tilde{c}_{1,j,k}\le b-1\,:\,1\le j\le s,1\le k\le u_{1,j}}$$
$$\times \nu_{u_{1,1},\dots,u_{1,s}}[\tilde{c}_{1,j,1},\dots,\tilde{c}_{1,j,u_{1,j}} : 1\le j\le s]^4\Biggr)^{1/2}$$
$$\times \sum_{u_{3,1},\dots,u_{3,s}\ge \tilde{m}\,:\,u_{3,1}+\cdots+u_{3,s}\ge m+1}$$
$$\times \Biggl(\frac{1}{b^{u_{3,1}+\cdots+u_{3,s}}} \tag{49}$$
$$\times \sum_{0\le \tilde{c}_{3,j,k}\le b-1\,:\,1\le j\le s,1\le k\le u_{3,j}}$$
$$\times \nu_{u_{3,1},\dots,u_{3,s}}[\tilde{c}_{3,j,1},\dots,\tilde{c}_{3,j,u_{3,j}} : 1\le j\le s]^4\Biggr)^{1/2}$$
$$\times \sum_{1\le m_j\le u_{1,j}\wedge u_{3,j}-1\,:\,1\le j\le s,m_1+\cdots+m_s\ge m+1} b^{2m-m_1-\cdots-m_s}$$
$$= \frac{O(1)b^{4m}}{m^{2(s-1)}} \sum_{u_{1,1},\dots,u_{1,s}\ge \tilde{m}\,:\,u_{1,1}+\cdots+u_{1,s}\ge m+1} b^{-2(u_{1,1}+\cdots+u_{1,s})}$$
$$\times \sum_{u_{3,1},\dots,u_{3,s}\ge \tilde{m}\,:\,u_{3,1}+\cdots+u_{3,s}\ge m+1} b^{-2(u_{3,1}+\cdots+u_{3,s})}$$
$$\times \sum_{m_1,\dots,m_s\ge 1\,:\,m_1+\cdots+m_s\ge m+1} b^{-m_1-\cdots-m_s}$$
$$= O(m^{s-1}b^{-m})$$

as $m \to \infty$. Finally we observe that as $m \to \infty$,

$$
\begin{aligned}
R_2 = {} & \frac{1}{s^2 b^{4m} \sigma_{0,m,s}^4} \\
& \times E\Biggl\{ \sum_{k_1=1}^{s} \sum_{i_1=1}^{b^m} \sum_{u_{1,1},\dots,u_{1,s} \ge \tilde{m} : u_{1,1}+\cdots+u_{1,s} \ge m+1} \\
& \qquad \times \nu_{u_{1,1},\dots,u_{1,s}}\bigl[\pi_j(a_{i_1,j,1}),\dots, \\
& \qquad\qquad \pi_{j;a_{i_1,j,1},\dots,a_{i_1,j,u_{1,j}-1}}(a_{i_1,j,u_{1,j}}) : 1 \le j \le s\bigr] \\
& \qquad \times \sum_{i_2=1}^{b^m} \sum_{u_{2,1},\dots,u_{2,s} \ge \tilde{m} : u_{2,1}+\cdots+u_{2,s} \ge m+1} \mathbb{1}\{\Omega_{i_1,i_2;k_1;\tilde{m}}\} \\
& \qquad\qquad \times \nu_{u_{2,1},\dots,u_{2,s}}\bigl[\pi_j(a_{i_2,j,1}),\dots, \\
& \qquad\qquad\qquad \pi_{j;a_{i_2,j,1},\dots,a_{i_2,j,u_{2,j}-1}}(a_{i_2,j,u_{2,j}}) : 1 \le j \le s\bigr] \\
& \qquad \times \sum_{k_3=1}^{s} \sum_{i_3=1}^{b^m} \sum_{u_{3,1},\dots,u_{3,s} \ge \tilde{m} : u_{3,1}+\cdots+u_{3,s} \ge m+1} \\
& \qquad\qquad \times \nu_{u_{3,1},\dots,u_{3,s}}\bigl[\pi_j(a_{i_3,j,1}),\dots, \\
& \qquad\qquad\qquad \pi_{j;a_{i_3,j,1},\dots,a_{i_3,j,u_{3,j}-1}}(a_{i_3,j,u_{3,j}}) : 1 \le j \le s\bigr] \\
& \qquad \times \sum_{i_4=1}^{b^m} \sum_{u_{4,1},\dots,u_{4,s} \ge \tilde{m} : u_{4,1}+\cdots+u_{4,s} \ge m+1} \mathbb{1}\{\Omega_{i_3,i_4;k_3;\tilde{m}}\} \\
& \qquad\qquad \times \nu_{u_{4,1},\dots,u_{4,s}}\bigl[\pi_j(a_{i_4,j,1}),\dots, \\
& \qquad\qquad\qquad \pi_{j;a_{i_4,j,1},\dots,a_{i_4,j,u_{4,j}-1}}(a_{i_4,j,u_{4,j}}) : 1 \le j \le s\bigr] \Biggr\} \\
& \times \mathbb{1}\{\#S_2[\{1,2\},\{3,4\}] \le s-1\} \\
= {} & O\left(\frac{1}{m}\right).
\end{aligned} \tag{50}
$$

The proof of the last equality follows in a similar manner as the proof of Lemma 4. Proposition 5 is now a direct consequence of (43)–(45), (47)–(50) and Lemma 4. □

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