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NORMAL, GAMMA AND INVERSE-GAUSSIAN ARE THE ONLY NEFS WHERE THE BILATERAL UMPU AND GLR TESTS COINCIDE

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Consider an NEF F on the real line parametrized by $\theta \in \Theta$. Also let θ_0 be a specified value of θ. Consider the test of size α for a simple hypothesis $H_0 : \theta = \theta_0$ versus two sided alternative $H_1 : \theta \neq \theta_0$. A UMPU test of size α then exists for any given α. Suppose that F is continuous. Therefore the UMPU test is nonrandomized and then becomes comparable with the generalized likelihood ratio test (GLR). Under mild conditions we show that the two tests coincide iff F is either a normal or inverse Gaussian or gamma family. This provides a new global characterization of this set of NEFs. The proof involves a differential equation obtained by the cancelling of a determinant of order 6.

1. Introduction. Consider a natural exponential family (NEF) $F = F(\mu)$ generated by a positive measure μ on the real line R:

$$F(\mu) = \left\{P(\theta, \mu)(dx) = \exp\{\theta x - k_\mu(\theta)\}\mu(dx) : \theta \in \Theta(\mu)\right\} \tag{1}$$

where

$$\begin{aligned} L_\mu(\theta) &= \int_R \exp(\theta x)\mu(dx) = e^{k_\mu(\theta)}, \\ D(\mu) &= \{\theta \in R : L_\mu(\theta) < \infty\}, \\ \Theta(\mu) &\doteq \operatorname{int} D(\mu). \end{aligned} \tag{2}$$

Consider the testing of a simple hypothesis versus a two-sided alternative:

$$H_0 : \theta = \theta_0 \qquad \text{versus} \quad H_1 : \theta \neq \theta_0, \tag{3}$$

where θ_0 is fixed but arbitrary in $\Theta(\mu)$. A uniformly most powerful unbiased (UMPU) test of size α for testing (3) exists for any given α [see Lehmann (1986)]. The test function of such a UMPU test is given by

$$\varphi(x) = \begin{cases} 1, & \text{if } x < c_1 \text{ or } x > c_2, \\ \gamma_i, & \text{if } x = c_i \text{ for } i = 1, 2, \\ 0, & \text{if } c_1 < x < c_2. \end{cases} \tag{4}$$

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The critical points c_1 and c_2 in (4) are uniquely determined as the simultaneous solution of the equations

$$\alpha = E_{\theta_0}(\varphi(X)) \tag{5}$$

and

$$\alpha E_{\theta_0}(X) = E_{\theta_0}(X\varphi(X)). \tag{6}$$

Such a UMPU test exists for any size $\alpha \in (0,1)$. For discrete distributions (i.e., when the NEF F is generated by a counting measure) this UMPU test is randomized for almost all tests of size α (except perhaps on a countable set of α values depending on the support points of F). If F is generated by a continuous measure then the UMPU test is nonrandomized, in which case $\gamma_i = 0$ for $i = 1, 2$, and the UMPU test function attains the values 1 and 0 only.

On the other hand, the generalized likelihood ratio (GLR) test is a nonrandomized test and is given by a test function of the form:

$$\varphi_1(x) = \begin{cases} 1, & \text{if } g(x) \le \lambda_0, \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where $g(x)$ is the GLR statistic and λ_0 is determined by the equation

$$\alpha = E_{\theta_0}(\varphi_1(X)). \tag{8}$$

Accordingly, for delineating situations in which the UMPU and GLR tests coincide for all size α, the UMPU test must be nonrandomized. Evidently, such a situation occurs only when the NEF F is generated by a continuous measure. In such a case, the distribution of the likelihood ratio statistic $g(x)$ is also continuous: this comes from the fact that since we are dealing with an NEF model, g is analytic [see (13)]. Consequently, the two tests are then comparable for any size α.

The aim of the present paper is to prove under some regularity conditions, that the UMPU and GLR tests for testing (3) coincide if and only if up to an affine transformation, F is either a normal, inverse Gaussian or gamma family. This set of families has been characterized in the literature by Blaesild and Jensen (1985). Our result provides therefore an additional new characterization. The "only if" part of the statement is proved in Section 3. Section 2 gathers some required preliminaries and notation, a formal description of the problem of the coincidence of the two tests, and the proof of the "if" part of the characterization. Some of the painful computations of Section 3, like a derivative of order 7 and a determinant of order 6 whose entries are functions, have been performed with the computer software Mathematica.

We should remark that if the set of hypotheses is one-sided versus one-sided alternative, for example, $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$, then the two tests coincide for any continuous NEF F. In such a situation both tests are nonrandomized and there is only one critical value which is solved by the same unique equation. It also should be noted that our characterization does not seem to be extended to NEFs on R^2. Indeed, consider the NEF on R^2 which is associated to the normal family on R with two unknown parameters m and σ^2, where m is the mean and σ^2 the variance. (These parameters are not the natural parameters of the associated NEF.) Then, for testing $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$, where σ_0^2 is fixed but arbitrary and m is unknown, the GLR test does not coincide with the UMPU test. In fact the corresponding GLR test is biased in this case as shown in Stuart and Ord [(1991), Example 22.5, page 258]. Note, however, that H_0 in the latter example is a composite hypothesis whereas in our characterization H_0 is a simple hypothesis.

2. Mathematical description of the problem. We first introduce some notation and basic definitions and properties of NEFs [for a thorough description see Letac and Mora (1990)]. Let μ be a positive, non-Dirac measure on R such that $\Theta(\mu)$ as defined by (2) is not empty. The NEF $F = F(\mu)$ generated by μ is given by (1). Note that the cumulant function k_μ is strictly convex and real analytic on $\Theta(\mu)$ and that

$$k'_\mu(\theta) = \int_R x \exp\{\theta x - k_\mu(\theta)\}\mu(dx)$$

is the mean function of F. The open interval $\Omega_F = k'_\mu(\Theta(\mu))$ is called the mean domain of F. Since the map $\theta \mapsto k'_\mu(\theta)$ is one-to-one, its inverse function $\psi_\mu : \Omega_F \to \Theta(\mu)$ is well defined. Hence the map $m \mapsto P(m, F) = P(\psi_\mu(m), \mu)$ is one-to-one from Ω_F onto F and is called the mean domain parameterization of F.

The variance of the probability $P(m, F)$ is $V_F(m) = 1/\psi'_\mu(m) = k''_\mu(\theta)$. The map $m \to V_F(m)$ from Ω_F into $(0, \infty)$ is called the variance function (VF) of F. In fact, the VF of an NEF F is a pair (V_F, Ω_F). It uniquely determines an NEF within the class of NEFs. Morris (1982) characterized all NEFs having quadratic VFs and Letac and Mora (1990) characterized all NEFs having a cubic VFs.

The VFs of the form $(V_F, \Omega_F) = (\sigma^2, R)$, $(Am^2, (0, \infty))$ and $(Bm^3, (0, \infty))$ where A and B are positive constants correspond, respectively, to families of normal, gamma and inverse Gaussian. Let $f_{\alpha,\beta}(F)$ be the image of a NEF F under the affine transformation $x \mapsto \alpha x + \beta$. Then by applying such a transformation on the above three categories of NEFs, their VFs become $C_1\sigma^2$, $(C_2 m + C_3)^2$ and $(C_4 m + C_5)^3$, respectively, for suitable constants C_i and a suitable change of Ω_F (depending, of course, on α and β). Consequently, the latter three VFs represent affine transformations of the normal, gamma and inverse Gaussian NEFs. We shall use this fact in Theorem 3 in which we provide a characterization of this set of NEFs.

We also need one more property of NEFs, namely steepness. The measure μ or the NEF $F = F(\mu)$ are called steep if $\Omega_F = \operatorname{int} C(\mu)$, where $C(\mu)$ denotes the closed convex support of μ [Barndorff-Nielsen (1978) gives a characterization of the steepness of an NEF]. Moreover, if x is the sample mean, and if μ is continuous and steep, the maximum likehood estimate (MLE), $\hat{\theta} = \hat{\theta}(x)$, exists and is the unique solution of the maximum likelihood equation $k'(\hat{\theta}) = x$ exists, since $x \in \Omega_F$ a.s.

If F is steep and and if $\mu(dx) = h(x)\,dx$ the test function of the UMPU test in (4) becomes

$$\varphi(x) = \begin{cases} 1, & \text{if } x \le c_1 \text{ or } x \ge c_2, \\ 0, & \text{if } c_1 < x < c_2. \end{cases} \tag{9}$$

Whereas the two critical points c_1 and c_2 in (5) and (6) are determined by the two equations as

$$1 - \alpha = P_{\theta_0}(c_1 < X < c_2) \tag{10}$$

and

$$\int_{c_1}^{c_2} x e^{\theta_0 x - k_\mu(\theta_0)} h(x)\,dx = m_0 \int_{c_1}^{c_2} e^{\theta_0 x - k_\mu(\theta_0)} h(x)\,dx, \tag{11}$$

where $m_0 = k'_\mu(\theta_0) = E_{\theta_0}(X)$ and where we have used (10) in (6).

The test function $\varphi_1(x)$ of the GLR test is given in (7). Let us give an expression for the likelihood ratio test statistics $g(x)$. This will allow us then to compare the two tests. For testing the hypothesis in (4), $g(x)$ has the following form:

$$g(x) = \frac{h(x)\exp(\theta_0 x - k_\mu(\theta_0))}{\sup_{\theta\in\Theta(\mu)}\{h(x)\exp(\theta x - k_\mu(\theta))\}} = \frac{\exp\{(\theta_0 x - k_\mu(\theta_0))\}}{\exp(\hat{\theta}x - k_\mu(\hat{\theta}x))}. \tag{12}$$

Since $x \in \Omega_F$ μ-a.e. we can write $\hat{\theta} = \psi_\mu(x)$ so that

$$\sup_{\theta\in\Theta(\mu)} \{\theta x - k_\mu(\theta)\} = x\psi_\mu(x) - k_\mu(\psi_\mu(x)) \doteq k^*_\mu(x),$$

where $k^*_\mu : \Omega_F \to R$ is called the Legendre transform of k_μ. Hence, (11) becomes

$$g(x) = \exp\bigl(\theta_0 x - k_\mu(\theta_0) - k^*_\mu(x)\bigr), \tag{13}$$

where $g : \Omega_F \to (0, 1)$. The function $\log g$ is strictly concave on Ω_F and attains a unique maximum at $m_0 = k'_\mu(\theta_0)$ with $g(m_0) = 1$. Fix $\lambda_0 = (0, 1)$, then

$$g(x) > \lambda_0 \iff c_1 < x < c_2, \qquad \text{where } g(c_1) = g(c_2) = \lambda_0. \tag{14}$$

Hence the GLR test function (7) becomes

$$\varphi_1(x) = \begin{cases} 1, & \text{if } x \le c_1 \text{ or } x \ge c_2, \\ 0, & \text{if } c_1 < x < c_2, \end{cases} \tag{15}$$

where, by using (8) and (14), the numbers c_1 and c_2 are determined by the two equations

$$1 - \alpha = P_{\theta_0}(c_1 < X < c_2) \tag{16}$$

with

$$g(c_1) = g(c_2) = \lambda_0. \tag{17}$$

Accordingly, by comparing the two equations (10) and (11) relating to the UMPU test with the respective equations (16) and (17) relating to the GLR test, it follows that the two tests coincide iff the two relations (11) and (17) hold for the same numbers c_1 and c_2. More specifically, fix $\theta \in \Theta(\mu)$ and consider the two functions on the mean domain Ω_F of the NEF $F = F(\mu)$,

$$\begin{aligned} G_\theta(y) &= \int_y^\infty \big(x - k'_\mu(\theta)\big) P(\theta, \mu)(dx), \\ g_\theta(x) &= \exp\big(\theta x - k_\mu(\theta) - k^*_\mu(x)\big) \end{aligned} \tag{18}$$

and assume that F is steep and that μ is continuous. We want to find F such that for all θ the two curves

$$\begin{aligned} &\big\{(c_1, c_2) \in \Omega_F \times \Omega_F;\ c_1 \le c_2;\ G_\theta(c_1) - G_\theta(c_2) = 0\big\} \\ &\quad = \big\{(c_1, c_2) \in \Omega_F \times \Omega_F;\ c_1 \le c_2;\ g_\theta(c_1) - g_\theta(c_2) = 0\big\} \end{aligned} \tag{19}$$

coincide. Examples of such curves are presented in the proof of Theorem 1.

Under the assumptions that F is steep, that μ is absolutely continuous and that its density is C^2 we show in Section 3 that (19) is statisfied for all θ only for the normal, gamma and inverse Gaussian families. Note that the characterization of this set of NEFs given by Blaesild and Jensen (1985) is using the same kind of regularity assumptions. To remove them seems to be a hard problem. The next theorem shows the much easier direct result: these families and their affine transformations fulfill (19).

THEOREM 1. *Suppose that* $\mu(dx)$ *is either*:

(i) $\exp(-x^2/2)\, dx/\sqrt{2\pi}$, *or*
(ii) $x^{a-1}\mathbf{1}_{(0,\infty)}(x)\, dx/\Gamma(a)$ *for some* $a > 0$, *or*
(iii) $ax^{-3/2}\exp(-a^2/(2x))\mathbf{1}_{(0,\infty)}(x))\, dx/\sqrt{2\pi}$ *for some* $a > 0$.

For $\theta \in \Theta(\mu)$ *consider the two functions* G_θ *and* g_θ *defined by* (18) *on the mean domain* Ω_F *of the NEF* $F = F(\mu)$. *Then there exists a strictly increasing continuous function* $z \mapsto H_\theta(z)$ *on* $(0, 1)$ *such that*

$$G_\theta(y) = H_\theta(g_\theta(y))$$

for all y *in* Ω_F. *In particular*, (19) *holds and the two tests GLR and UMPU coincide for these NEFs as well as for their affine transformations*.

PROOF. Note that the end of the statement is obvious. We prove the first part by inspection.

The normal case. Here $\Theta(\mu) = \Omega_F = R$, $k_\mu(\theta) = \theta^2/2$ and

$$g_\theta(x) = e^{-(x-\theta)^2/2},$$
$$G_\theta(y) = \int_y^\infty (x-\theta)e^{-(x-\theta)^2/2}\frac{dx}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}e^{-(y-\theta)^2/2}.$$

Thus $H_\theta(z) = \frac{z}{\sqrt{2\pi}}$. The curves (19) are the lines $c_1 + c_2 = \text{constant}$.

The gamma case. Here $\Theta(\mu) = (-\infty, 0)$, $\Omega_F = (0,\infty)$, $k_\mu(\theta) = -a\log(-\theta)$ and

$$g_\theta(x) = \left(\frac{-\theta e}{a}\right)^a x^a e^{\theta x},$$
$$G_\theta(y) = \int_y^\infty \left(x + \frac{a}{\theta}\right)e^{\theta x + a\log(-\theta)}\frac{x^{a-1}}{\Gamma(a)}\,dx = \frac{1}{-\theta}e^{\theta y + a\log(-\theta)}\frac{y^a}{\Gamma(a)}.$$

Thus $H_\theta(z) = C_\theta z$, where C_θ is a positive constant. The curves (19) are $\frac{\log c_2 - \log c_1}{c_2 - c_1} = \text{constant}$.

The inverse Gaussian case. This case is somewhat harder. Here $\Theta(\mu) = (-\infty, 0)$, $\Omega_F = (0,\infty)$, $k_\mu(\theta) = -a\sqrt{-2\theta}$. For simplification we denote $s = \sqrt{-2\theta}$. We get

$$g_\theta(x) = e^{-(xs-a)^2/(2x)}, \qquad G_\theta(y) = \frac{a}{s^2\sqrt{2\pi}}\int_y^\infty (xs-a)x^{-3/2}g_\theta(x)\,dx.$$

We now show that the function

$$H_\theta(z) = \frac{2a}{s^2\sqrt{2\pi}}\int_0^z (4as - 2\log t)^{-1/2}\,dt$$

satisfies $G_\theta(y) = H_\theta(g_\theta(y))$. To prove this we observe that the derivative of the $y \mapsto G_\theta(y) - H_\theta(g_\theta(y))$ is 0 (we skip this standard computation). Thus the function is constant. We also observe that $\lim_{y\to 0} G_\theta(y) = 0$, $\lim_{y\to 0} g_\theta(y) = 0$, which proves that the constant is 0. The curves (19) are the hyperbolas $c_2 c_1 = \text{constant}$. $\square$

3. Main result. In this section we show the hard part of the characterization. We work with a fixed NEF $F = F(\mu)$. Henceworth for simplicity we suppress the dependence on μ and F and write k, ψ, ΘV and Ω instead of k_μ, ψ_μ, $\Theta(\mu)$, V_F and Ω_F, respectively. In what follows we also use $f^{(k)}$, $k = 1, 2, \ldots$, to denote the kth derivative of a mapping f. Since the proof is long and delicate we split it into three theorems.

THEOREM 2. *Suppose that* (19) *holds for all* $\theta \in \Theta$. *Assume that* F *is steep, that* $\mu(dx) = h(x)\,dx$ *and that* h *is twice continuously differentiable on* Ω. *Then* $h(t)$ *satisfies*

$$2h'(t)\psi'(t) - h(t)\psi''(t) + 2h(t)\psi'(t)\psi(t) \equiv 0 \tag{20}$$

for all t *in* Ω. *Furthermore*, h *is real analytic on the open interval* Ω.

PROOF. We fix $\theta_0 \in \Theta$. For simplicity set $g_{\theta_0}(t) = \exp(g_1(t))$. Then

$$g_1(t) = \theta_0 t - k(\theta_0) - t\psi(t) + k(\psi(t)). \tag{21}$$

Let $c_1 \le m_0 \le c_2$, then $g_{\theta_0}(c_1) = g_{\theta_0}(c_2) = \lambda_0 \in [0, 1]$ iff $g_1(c_1) = g_2(c_2) = \log\lambda_0 \in (-\infty, 0]$.

The basic idea of the proof is to solve the equation

$$g_1(c_1) = g_1(c_2) = -\varepsilon^2/2, \qquad \varepsilon > 0, \tag{22}$$

for c_1 and c_2. Note that g_1 is increasing on the left of m_0 and decreasing on the right of m_0, and $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are well defined. In these two intervals g_1 is not invertible in a closed form, but as we are going to see, for small values of ε, $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$ are real analytic in ε so that they can be traced by a power series expansion. To see this we observe that the function g_1, which is analytic at m_0 satisfies

$$g_1(m_0) = g_1'(m_0) = 0. \tag{23}$$

This comes from the fact that $g_1'(t) = \theta_0 - \psi(t) - t\psi'(t) + k'(\psi(t))\psi'(t) = \theta_0 - \psi(t)$. Since $g_1''(t) = -\psi'(t)$ we may write

$$g_1(t) = -\frac{(t-m_0)^2}{2!}\psi'(m_0) - \frac{(t-m_0)^3}{3!}\psi''(m_0) + O\big((t-m_0)^4\big). \tag{24}$$

This last equality and the implicit function theorem for analytic functions show that the two functions c_1 and c_2 are analytic functions of ε at 0.

We set

$$c(\varepsilon) = \sum_{i=0}^{\infty} \alpha_i \varepsilon^i = \alpha_0 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + O(\varepsilon^3),$$

where c is either c_1 or c_2. We have $c_1(0) = c_2(0) = \alpha_0 = m_0$.

Consequently, from (24),

$$\begin{aligned} g_1(c(\varepsilon)) = {}& -\frac{(\alpha_1\varepsilon + \alpha_2\varepsilon^2 + O(\varepsilon^3))^2}{2!}\psi'(m_0) \\ & - \frac{(\alpha_1\varepsilon + \alpha_2\varepsilon^2 + O(\varepsilon^3))^3}{3!}\psi''(m_0) + O(\varepsilon^4) = -\varepsilon^2/2 \end{aligned}$$

yields

$$\left(\frac{1-\alpha_1^2\psi'(m_0)}{2}\right)\varepsilon^2 - \left(\alpha_1\alpha_2\psi'(m_0) + \frac{\alpha_1^3\psi''(m_0)}{6}\right)\varepsilon^3 = O(\varepsilon^4).$$

By comparing coefficients we conclude that

$$\alpha_1^2 = \frac{1}{\psi'(m_0)} \quad \text{and} \quad \alpha_2 = -\frac{\alpha_1^2\psi''(m_0)}{6\psi'(m_0)} = -\frac{\psi''(m_0)}{6(\psi'(m_0))^2}.$$

Hence,

$$c_1(\varepsilon) = m_0 - \varepsilon(\psi'(m_0))^{-1/2} - \varepsilon^2\frac{\psi''(m_0)}{6(\psi'(m_0))^2} + O(\varepsilon^3)$$

and

$$c_2(\varepsilon) = m_0 + \varepsilon(\psi'(m_0))^{-1/2} - \varepsilon^2\frac{\psi''(m_0)}{6(\psi'(m_0))^2} + O(\varepsilon^3).$$

It remains now to substitute $c_1(\varepsilon)$ and $c_2(\varepsilon)$ in (10), to calculate the integral

$$l(\varepsilon) = \int_{c_1(\varepsilon)}^{c_2(\varepsilon)} (x-m_0)h(x)\exp\bigl(\theta_0 x - k(\theta_0)\bigr)\,dx, \tag{25}$$

and then to equate it to 0. Now, by changing the variable of integration, we obtain

$$\begin{aligned}
l(\varepsilon) &= e^{-k(\theta_0)+m_0\theta_0}\int_{c_1(\varepsilon)}^{c_2(\varepsilon)} (x-m_0)h(x)e^{\theta_0(x-m_0)}\,dx \\
&= e^{-k(\theta_0)+m_0\theta_0}\int_{c_1(\varepsilon)-m_0}^{c_2(\varepsilon)-m_0} th(t+m_0)e^{\theta_0 t}\,dt \\
&= e^{-k(\theta_0)+m_0\theta_0}\int_{c_1(\varepsilon)-m_0}^{c_2(\varepsilon)-m_0} \bigl[t\bigl(h(m_0)+th'(m_0)+O(t^2)\bigr)\bigl(1+\theta_0 t+O(t^2)\bigr)\bigr]\,dt \\
&= e^{-k(\theta_0)+m_0\theta_0}\left[\frac{t^2}{2}h(m_0) + \frac{t^3}{3}\bigl(h'(m_0)+\theta_0 h(m_0)\bigr) + O(t^4)\right]_{c_1(\varepsilon)-m_0}^{c_2(\varepsilon)-m_0} \\
&= \varepsilon^3 e^{-k(\theta_0)+m_0\theta_0}\left[\frac{-\psi''(m_0)h(m_0)}{3(\psi'(m_0))^{5/2}} + \frac{2}{3}\left(\frac{h'(m_0)+\theta_0 h(m_0)}{(\psi'(m_0))^{3/2}}\right)\right] + O(\varepsilon^4) \equiv 0.
\end{aligned}$$

Hence

$$-\psi''(m_0)h(m_0) + 2\bigl(h'(m_0)+\theta_0 h(m_0)\bigr)\psi'(m_0) = 0.$$

By noting that $\theta_0 = \psi(m_0)$, the result (20) follows.

Finally, we observe that in (20), a division by $\psi'(t) = 1/V(t)$ on Ω_F is possible and $h' = ah$ where $a = \psi''/(2\psi) - \psi$ is real analytic on Ω_F. Thus $h(t) = C\exp\int a(t)\,dt$ is real analytic on Ω_F and the proof is complete. $\square$

Since $l(\varepsilon)$ as defined by (25) is 0, we obtain that $l^{(5)}(0)=0$ and $l^{(7)}(0)=0$. We are going to use these high order variations to reduce (20) into equations that involve ψ and its derivatives only.

THEOREM 3. *Under the hypothesis and notation of Theorem* 2 *we have*

$$12\psi''^3 - 14\psi'\psi''\psi^{(3)} + 3\psi'^2\psi^{(4)} = 0, \tag{26}$$

$$12\psi''^5 - 17\psi'\psi''^3\psi^{(3)} + 6\psi'^2\psi''(\psi^{(3)})^2 = 0. \tag{27}$$

PROOF. We shall use (27) for the characterization and not (26). The principle of the proof of (27) will be explained by showing (26) only. For simplicity set $h(t)=\sum_{i=0}^{3} h_i(t-m_0)^i + O((t-m_0)^4)$ and $\psi(t)=\sum_{i=0}^{\infty} p_i(t-m_0)^i$. Consider the function l defined by (25). Then elementary calculations show that $l^{(5)}(0)=0$ reduces to

$$\begin{aligned} &24h_3p_1^3 + 72h_1p_0^2p_1^3 + 24h_0p_0^3p_1^3 + 12h_2p_1^2(6p_0p_1 - 5p_2)\\ &\quad - 120h_1p_0p_1^2p_2 - 60h_0p_0^2p_1^2p_2 + 70h_1p_1p_2^2 + 70h_0p_0p_1p_2^2\\ &\quad - 35h_0p_2^3 - 30h_1p_1^2p_3 - 30h_0p_0p_1^2p_3 + 35h_0p_1p_2p_3 - 6h_0p_1^2p_4 = 0. \end{aligned} \tag{28}$$

Now (20) and two successive derivatives thereof read

$$2h_1p_1 + 2h_0p_0p_1 - h_0p_2 = 0, \tag{29}$$

$$2h_2p_1 + 2h_1p_0p_1 + 2h_0p_1^2 + h_1p_2 + 2h_0p_0p_2 - h_0p_3 = 0, \tag{30}$$

$$\begin{aligned} &2h_3p_1 + 3h_2p_2 + 2h_0p_1p_2\\ &\quad + 4p_1(h_1p_1 + h_0p_2)\\ &\quad + 2p_0(h_2p_1 + 2h_1p_2 + h_0p_3) - h_0p_4 = 0. \end{aligned} \tag{31}$$

By noting that (28), (29), (30) and (31) form a homogeneous linear system in $h_i,\ i=0,1,2,3$, it follows that either $h_0=h_1=h_2=h_3=0$, or the determinant of the coefficient matrix must vanish, which is equivalent to

$$12p_2^3 - 14p_1p_2p_3 + 3p_1^2p_4 = 0. \tag{32}$$

Recall that from Theorem 2 that the density h is analytic on the open interval Ω. If $Z\subset\Omega$ is the set of zeroes of h, then (32) occurs for $m_0\notin Z$. Thus (26) holds for $t=m_0\notin Z$. In other terms, the first member of (26) is an analytic function which is 0 on $\Omega\setminus Z$. Now, since h is not identically 0, the principle of isolated zeroes applied to h and Z shows that (26) holds on the whole interval Ω. (We thank the referee for pointing out that Z has to be taken into account.) The proof of (26) is complete.

The same procedure applied to $l^{(7)}(0)=0$, to (29), (30), (31) and to two equations obtained from (20) by two more derivations provides a homogeneous linear system in $(h_i)_{i=0}^5$. Equating the determinant of the system to zero yields (27). We skip the details, which can be checked by Mathematica. □

We now use (27) to conclude.

THEOREM 4. *Under the hypothesis and notations of Theorem* 2, *up to affinity*, F *is an NEF corresponding to one of the normal*, *gamma or inverse Gaussian families*.

PROOF. We first translate the differential equation in ψ (27) into a differential equation for the variance function $V = 1/\psi'$ of the NEF F,

$$\frac{1}{V^{10}}(-2V'^5 + 7VV'^3V'' - 6V^2V'V''^2) = 0. \tag{33}$$

Now, the miracle is that (33) factors into the three factors related to the three groups of NEF,

$$V'(3VV'' - 2V'^2)(2VV'' - V'^2) = 0. \tag{34}$$

Since the function V is real analytic on the open interval Ω the principle of isolated zeros implies that one of the three factors in (34) is identically 0 on Ω. Therefore

$$\text{either } V' = 0 \quad \text{or} \quad \frac{V''}{V'} = \frac{2}{3}\frac{V'}{V} \quad \text{or} \quad \frac{V''}{V'} = \frac{1}{2}\frac{V'}{V}.$$

Straightforward integration yields

$$\text{either } V = A_1 \quad \text{or} \quad V' = A_2V^{2/3} \quad \text{or} \quad V' = A_3V^{1/2},$$

where A_1, A_2, A_3 are constants.

Actually, in the two latter cases, we have to use the fact that V is a nonconstant analytic function on Ω, thus the zeroes of V' are isolated ones. Furthermore, V never vanishes on Ω. Therefore the two equalities $V' = A_2V^{2/3}$ or $V' = A_3V^{1/2}$ which hold outside of this set of zeroes also show that this set is empty (we thank the referee for this point).

Again in the two later cases, another integration gives

$$V(t) = (C_3t + C_4)^3 \quad \text{or} \quad V(t) = (C_1t + C_2)^2$$

and the desired result follows. Note that the order of the differential equation (27) is three while (26) has order four. Thus the translation of (26) into an equation for V gives $2V'^3 - 4VV'V'' + 3V^2V''' = 0$, a differential equation of order three which is more complicated than (34). $\square$

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