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THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF $S_1S_2^{-1}$ UNDER VIOLATIONS[1]

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The paper deals with the density of the characteristic roots of $S_1S_2^{-1}$ where S_1 has a noncentral Wishart distribution, $W(p, n_1, \Sigma_1, \Omega)$, and S_2 has an independently distributed central Wishart distribution $W(p, n_2 \Sigma_2, 0)$, under a condition. This density is basic for an exact study of robustness of tests of at least two multivariate hypotheses.

1. Introduction. Consider the test of the following two hypotheses: 1) equality of covariance matrices in two p-variate normal populations and 2) equality of p-dimensional mean vectors in l p-variate normal populations having a common covariance matrix. In order to carry out some exact investigations of robustness of tests of 1) when the assumption of normality is violated and of 2) when that of a common covariance matrix is disturbed, the density of the characteristic roots of $S_1S_2^{-1}$ is studied, where S_1 has a noncentral Wishart distribution $W(p, n_1, \Sigma_1, \Omega)$, S_2 has an independently distributed central Wishart distribution $W(p, n_2, \Sigma_2, 0)$, and an assumption is made on Ω or $\Sigma_1\Sigma_2^{-1}$ (see Section 3 for definitions). The results of the robustness studies will be reported in a second report. The distribution is also considered when Ω is a random matrix.

2. Motivation. The distribution of the latent roots of $S_1S_2^{-1}$ is derived when $S_1(p \times p)$ has a noncentral Wishart distribution with n_1 d.f. and noncentrality parameter matrix Ω and covariance matrix Σ_1, $W(p, n_1, \Sigma_1, \Omega)$, and $S_2(p \times p)$ has an independently distributed central Wishart distribution with n_2 d.f. and covariance matrix Σ_2, $W(p, n_2, \Sigma_2, \mathbf{0})$, where $n_1, n_2 \geqq p$. Let $R = \operatorname{diag}(r_1, \cdots, r_p)$, where $0 < r_1 \leqq \cdots \leqq r_p < \infty$ are the latent roots of $S_1S_2^{-1}$. Then the distribution of $\mathbf{R}$ is obtained in two forms: the first when Ω is partially random (denoted "random" hereafter) and the second when $\Lambda = \Sigma_1\Sigma_2^{-1}$ is "random". Here "random" implies diagonalization by an orthogonal transformation $\mathbf{H}$ and integration over $\mathbf{H}$; in other words putting a Haar prior on $\mathbf{H}$ leaving the latent roots non-random. The method not only serves to separate variables but also, as for example in Theorem 1, leads to non-normality in the distribution, because with Ω "random" the Wishart distribution is no longer Wishart. Hence $W(p, n_1, \Sigma_1, \Omega)$ with Ω "random" is more analogical to the Edgeworth series type

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expansions in the univariate case. In Theorem 2, Λ is "random" and not Ω, as is appropriate for MANOVA. For canonical correlation Ω can be made totally random. It is also appropriate for non-normality in the sense that the noncentral Wishart distribution itself is analogical to Edgeworth series expansions for tests of $\Sigma_1 = \Sigma_2$, although the approach of Theorem 1 is more interesting in this case. However, from the discussion in the next section, convergence questions favor the results of Theorem 2 for both studies.

3. The distribution of the latent roots of $S_1S_2^{-1}$. The joint distribution of $r_1, \cdots, r_p$ is derived in this section, (a) when Ω is "random", (b) when Λ is "random", and (c) when Ω is completely random under (a) or (b).

THEOREM 1. *Under the assumption that Ω is "random",*

(i) *the density function of* $\mathbf{R}$ *is given by*

$$C_0(p, n_1, n_2)e^{-tr\Omega}|\Lambda|^{\frac{1}{2}n_2}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}|\Lambda + \mathbf{R}|^{-\frac{1}{2}(n_1+n_2)} \times \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_\kappa}{(\frac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I})k!} C_\kappa((\Lambda + \mathbf{R})^{-1}\mathbf{R})C_\kappa(\Omega) , \tag{3.1}$$

and

(ii) *the joint density function of $r_1, \cdots, r_p$ is given by*

$$\begin{aligned} &C(p, n_1, n_2)e^{-tr\Omega}|\Lambda|^{-\frac{1}{2}n_1}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)} \prod_{i>j} (r_i - r_j) \\ &\quad \times \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\Omega)}{(\frac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I}_p)k!} \\ &\quad \times \sum_{n=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n g_{\kappa,\nu}^{\delta}(\frac{1}{2}(n_1+n_2))_\delta C_\delta(\Lambda^{-1})C_\delta(\mathbf{R})}{C_\delta(\mathbf{I}_p)n!} , \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} C_0(p, n_1, n_2) &= \Gamma_p(\tfrac{1}{2}(n_1+n_2))/[\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)] , \\ C(p, n_1, n_2) &= \pi^{\frac{1}{2}p^2}C_0(p, n_1, n_2)/\Gamma_p(\tfrac{1}{2}P) , \qquad \Lambda = \Sigma_1^{\frac{1}{2}}\Sigma_2^{-1}\Sigma_1^{\frac{1}{2}} , \end{aligned} \tag{3.3}$$

and $g_{\kappa,\nu}^{\delta}$ are constants (see Equation (27) *of* [2], *Lemma* 2 *of* [8] *and tabulations in* [6]) *with $\delta = (d_1 \geqq d_2 \geqq \cdots \geqq d_p \geqq 0)$ such that $\sum_{i=0}^{p} d_i = k + n$.*

PROOF. The joint density of S_1 and S_2 is given in [1], [4],

$$[\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)|2\Sigma_1|^{\frac{1}{2}n_1}|2\Sigma_2|^{\frac{1}{2}n_2}]^{-1}e^{-tr\Omega} \exp[-\tfrac{1}{2}tr\Sigma_1^{-1}S_1]|S_1|^{\frac{1}{2}(n_1-p-1)} \times \exp[-\tfrac{1}{2}tr\Sigma_2^{-1}S_2]|S_2|^{\frac{1}{2}(n_2-p-1)} {}_0F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}\Sigma_1^{-1}\Omega S_1) .$$

Let us transform $A_1 = \frac{1}{2}\Sigma_1^{-\frac{1}{2}}S_1\Sigma_1^{-\frac{1}{2}}$ and $A_2 = \frac{1}{2}\Sigma_1^{-\frac{1}{2}}S_2\Sigma_1^{-\frac{1}{2}}$, since the characteristic roots are invariant under the simultaneous transformations. ($\Sigma_1^{-\frac{1}{2}}(p \times p)$ is symmetric positive definite like other matrices of the form $A^{\frac{1}{2}}$ defined later.) We now have the joint density of A_1 and A_2 in the form

$$C_1(p, n_1, n_2)e^{-tr\Omega}|\Lambda|^{\frac{1}{2}n_2}e^{-trA_1}|A_1|^{\frac{1}{2}(n_1-p-1)} \times e^{-tr\Lambda A_2}|A_2|^{\frac{1}{2}(n_2-p-1)} {}_0F_1(\tfrac{1}{2}n_1; \Omega A_1) , \tag{3.4}$$

where

$$C_1(p, n_1, n_2) = [\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)]^{-1} .$$

Now transform $\mathbf{A}_1 = \mathbf{A}_2^{\frac{1}{2}}\mathbf{R}\mathbf{A}_2^{\frac{1}{2}}$ and $\mathbf{A}_2 = \mathbf{A}_2$. (The same notation is used to denote the matrix R both before and after diagonalization.) The Jacobian is $|\mathbf{A}_2|^{\frac{1}{2}(p+1)}$, and hence the joint density of $\mathbf{R}$ and $\mathbf{A}_2$ is given by

$$C_1(p, n_1, n_2)e^{-tr\mathbf{\Omega}}|\mathbf{\Lambda}|^{\frac{1}{2}n_2}e^{-tr\mathbf{R}\mathbf{A}_2}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}e^{-tr\mathbf{\Lambda}\mathbf{A}_2}|\mathbf{A}_2|^{\frac{1}{2}(n_1+n_2-p-1)}{}_0F_1(\tfrac{1}{2}n_1;\ \mathbf{\Omega}\mathbf{A}_2^{\frac{1}{2}}\mathbf{R}\mathbf{A}_2^{\frac{1}{2}}) .$$

If we transform $\mathbf{\Omega} \to \mathbf{H}\mathbf{\Omega}\mathbf{H}'$ where $\mathbf{H} \in O(p)$, and integrate over $O(p)$ (see [1]), all the factors in the above expression remain the same except the hypergeometric function which becomes ${}_0F_1(\frac{1}{2}n_1;\ \mathbf{\Omega}, \mathbf{R}\mathbf{A}_2)$. Now expand the latter hypergeometric function in terms of zonal polynomials (see [4]), integrate out $\mathbf{A}_2$ using Eq. (1) of [1] and obtain (i) of Theorem 1. (The integration above and subsequently is with respect to Lebesgue measure on the sets of variables involved.)

Further, in the expression before integrating out $\mathbf{A}_2$, expand $\exp(-tr\mathbf{R}\mathbf{A}_2) = {}_0F_0(-\mathbf{R}\mathbf{A}_2)$ in zonal polynomials (see [4]) and apply Equation (27) of [2] (see also [8]). The joint density of $\mathbf{R}$ and $\mathbf{A}_2$ becomes

$$C_1(p, n_1, n_2)e^{-tr\mathbf{\Omega}}|\mathbf{\Lambda}|^{\frac{1}{2}n_2}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}e^{-tr\mathbf{\Lambda}\mathbf{A}_2}|\mathbf{A}_2|^{\frac{1}{2}(n_1+n_2-p-1)}$$
$$\times \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{\Omega})}{(\frac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I}_p)k!}\sum_{n=0}^{\infty}\sum_{\nu,\delta}\frac{(-1)^n g_{\kappa,\nu}^{\delta}C_\delta(\mathbf{R}\mathbf{A}_2)}{n!} .$$

Now integrate out $\mathbf{A}_2$ using (1) of [1] as before and obtain the density of $\mathbf{R}$ in the alternate form

$$C_1(p, n_1, n_2)e^{-tr\mathbf{\Omega}}|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}$$
$$\times \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{\Omega})}{(\frac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I}_p)k!}\sum_{n=0}^{\infty}\sum_{\nu,\delta}\frac{(-1)^n g_{\kappa,\nu}^{\delta}\Gamma_p\left(\frac{n_1+n_2}{2},\ \delta\right)C_\delta(\mathbf{R}\mathbf{\Lambda}^{-1})}{n!} .$$

$\mathbf{R}$ can be diagonalized by an orthogonal transformation $\mathbf{H}$ such that $\mathbf{H}\mathbf{R}\mathbf{H}' = \operatorname{diag}(r_1, \cdots, r_p)$ where $r_1, \cdots, r_p$ are the characteristic roots of $\mathbf{R}$. For uniqueness, we assume that the elements in the first row of $\mathbf{H}$ are positive and the roots are arranged in the order $0 < r_1 \leqq r_2 < \cdots \leqq r_p < \infty$. The volume element $d\mathbf{R}$ becomes (see [1], [4])

$$d\mathbf{R} = \prod_{i>j}(r_i - r_j)\prod_{i=1}^{p} dr_i(d\mathbf{H}) . \tag{3.5}$$

Substituting $\mathbf{R}$ in the above expression and integrating out $\mathbf{H}$ (where (dH) is not normalized), we have the joint density of $r_1, \cdots, r_p$ in the form

$$C_1(p, n_1, n_2)e^{-tr\mathbf{\Omega}}|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}\prod_{i>j}(r_i - r_j)$$
$$\times \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{\Omega})}{(\frac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I}_p)k!}\sum_{k=0}^{\infty}\sum_{\nu,\delta}\frac{(-1)^n g_{\kappa,\nu}^{\delta}\Gamma_p\left(\frac{n_1+n_2}{2},\ \delta\right)}{n!}$$
$$\times 2^{-p}\int_{O(p)} C_\delta(\mathbf{H}\mathbf{R}\mathbf{H}'\mathbf{\Lambda}^{-1})(d\mathbf{H}) .$$

The factor 2^{-p} multiplying the integral arises from the restriction that the elements in the first row of $\mathbf{H}$ are positive. Finally, by using the property of the zonal polynomials given by James [4] for normalized measure,

$$\int_{O(p)} C_\kappa(\mathbf{HSH'T})(d\mathbf{H}) = [C_\kappa(\mathbf{S})C_\kappa(\mathbf{T})]/C_\kappa(\mathbf{I}_p)\,, \tag{3.6}$$

and (15) and (44) of [1], we have the result as stated in (ii) of Theorem 1.

For $\mathbf{\Omega} = \mathbf{0}$, since $g_{0,\nu}^{\delta} = 1$ and $\delta = \nu$, the expression (3.2) gives the result stated in (65) of [4].

The expression (3.2) may not converge for all values of $\mathbf{R}$, $\mathbf{\Lambda}$, and $\mathbf{\Omega}$ since the remarks on page 480 of [4] concerning (65) also apply here. Hence we prove the following theorem.

THEOREM 2. *Let $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{R}$ and r_i's be as stated in Theorem 1. Then the joint density function of $r_1, \cdots, r_p$ is given by*

$$\begin{aligned} &C(p, n_1, n_2)e^{-tr\mathbf{\Omega}}|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}|\mathbf{I} + \lambda\mathbf{R}|^{-\frac{1}{2}(n_1+n_2)} \textstyle\prod_{i>j}(r_i - r_j) \\ &\quad \times \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\left(\frac{n_1+n_2}{2}\right)_\kappa \dfrac{C_\kappa(\lambda\mathbf{R}(\mathbf{I}+\lambda\mathbf{R})^{-1})}{k!} \\ &\quad \times \textstyle\sum_{d=0}^{k}\sum_{\delta} \dfrac{a_{\kappa,\delta}C_\delta(-\lambda^{-1}\mathbf{\Lambda}^{-1})L_\delta^{\frac{1}{2}(n_1-p-1)}(\mathbf{\Omega})}{\left(\frac{n_1}{2}\right)_\delta C_\delta(\mathbf{I})C_\delta(\mathbf{I})}\,, \end{aligned} \tag{3.7}$$

where $\mathbf{\Lambda}$ is "random", λ is a positive real number, and $C(p, n_1, n_2)$ is as defined in (3.3). *The generalized Laguerre polynomial $L_\delta^\gamma(S)$ is defined in Equation* (14) *of* [2] *and $a_{\kappa,\delta}$ are constants* (*see Equation* (20) *of* [2], *tabulations in* [7]).

PROOF. We start from the expression (3.4). Now apply (29) of [4] to ${}_0F_1(\frac{1}{2}n_1;\ \mathbf{\Omega}^{\frac{1}{2}}\mathbf{A}_1\mathbf{\Omega}^{\frac{1}{2}})$ to get the joint density of $\mathbf{A}_1$ and $\mathbf{A}_2$ as

$$\begin{aligned} &C_2(p, n_2)e^{-tr\mathbf{\Omega}}|\mathbf{\Lambda}|^{\frac{1}{2}n_2}e^{-tr\mathbf{\Lambda}\mathbf{A}_2}|\mathbf{A}_2|^{\frac{1}{2}(n_2-p-1)}|\mathbf{A}_1|^{\frac{1}{2}(n_1-p-1)} \\ &\qquad \times \int_{\mathrm{Re}(\mathbf{T})=\mathbf{X}_0>0} e^{trT}|\mathbf{T}|^{-\frac{1}{2}n_1}e^{-tr(\mathbf{I}-\mathbf{W})\mathbf{A}_1}(d\mathbf{T})\,, \end{aligned}$$

where

$$C_2(p, n_2) = 2^{\frac{1}{2}p(p-1)}\Big/\left[(2\pi i)^{\frac{1}{2}p(p+1)}\Gamma_p\left(\frac{n_2}{2}\right)\right], \qquad \mathbf{W} = \mathbf{\Omega}^{\frac{1}{2}}\mathbf{T}^{-1}\mathbf{\Omega}^{\frac{1}{2}}$$

and $\mathbf{T} = \mathbf{X}_0 + i\mathbf{Y}$ with $\mathbf{X}_0$ a positive definite symmetric matrix and $\mathbf{Y}$, real symmetric. The characteristic roots are invariant under simultaneous transformations

$$\mathbf{B}_1 = \mathbf{\Lambda}^{\frac{1}{2}}\mathbf{A}_1\mathbf{\Lambda}^{\frac{1}{2}} \qquad \text{and} \qquad \mathbf{B}_2 = \mathbf{\Lambda}^{\frac{1}{2}}\mathbf{A}_2\mathbf{\Lambda}^{\frac{1}{2}}\,.$$

Apply the transformation $\mathbf{B}_2 = \mathbf{B}_1^{\frac{1}{2}}\mathbf{R}_1\mathbf{B}_1^{\frac{1}{2}}$ and $\mathbf{B}_1 = \mathbf{B}_1$ and integrate out $\mathbf{B}_1$ using (1) of [1] with $\kappa = (0)$, and also (15) of [1]. Then we have the density of $\mathbf{R}_1$ in the form

$$\begin{aligned} &C_3(p, n_1, n_2)e^{-tr\mathbf{\Omega}}|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{R}_1|^{\frac{1}{2}(n_2-p-1)} \\ &\qquad \times \int_{\mathrm{Re}(\mathbf{T})=\mathbf{X}_0>0} e^{tr\mathbf{T}}|\mathbf{T}|^{-\frac{1}{2}n_1}|\mathbf{R}_1 + \mathbf{\Lambda}^{-\frac{1}{2}}(\mathbf{I}-\mathbf{W})\mathbf{\Lambda}^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}(d\mathbf{T})\,, \end{aligned} \tag{3.8}$$

where

$$C_3(p, n_1, n_2) = C_2(p, n_2)\Gamma_p\left(\frac{n_1+n_2}{2}\right).$$

Upon transforming $\mathbf{R} = \mathbf{R}_1^{-1}$, (the Jacobian of this transformation is $|\mathbf{R}|^{-(p+1)}$), (3.8) becomes

$$(3.9) \qquad C_3(p, n_1, n_2)e^{-tr\Omega}|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)} \times \int_{\mathrm{Re}(\mathbf{T})=\mathbf{X}_0>0} e^{tr\mathbf{T}}|\mathbf{T}|^{-\frac{1}{2}n_1}|\mathbf{I} + \mathbf{R}\mathbf{\Lambda}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{W})\mathbf{\Lambda}^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}(d\mathbf{T}) .$$

Since $\mathbf{R}$ is symmetric we can diagonalize by an orthogonal transformation $\mathbf{H}$ and use the same technique as before. After making necessary substitutions and integrating out $\mathbf{H}$ using (3.5), we get the joint density of $r_1, \cdots, r_p$ in the form

$$(3.10) \qquad C_3(p, n_1, n_2)e^{-tr\Omega}|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)} \prod_{i>j}(r_i - r_j) \times \int_{\mathrm{Re}(\mathbf{T})=\mathbf{X}_0>0} e^{tr\mathbf{T}}|\mathbf{T}|^{-\frac{1}{2}n_1} \times \int_{O(p)} 2^{-p}|\mathbf{I} + \mathbf{H}\mathbf{R}\mathbf{H}'\mathbf{\Lambda}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{W})\mathbf{\Lambda}^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}(d\mathbf{H})(d\mathbf{T}) .$$

Now we can write

$$|\mathbf{I} + \mathbf{H}\mathbf{R}\mathbf{H}'\mathbf{A}| = |\mathbf{I} + \lambda\mathbf{R}|\,|\mathbf{I} - (\mathbf{I} - \lambda^{-1}\mathbf{A})\mathbf{H}(\lambda\mathbf{R})(\mathbf{I} + \lambda\mathbf{R})^{-1}\mathbf{H}'|$$

where λ is a positive real number and in our case $\mathbf{A} = \mathbf{\Lambda}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{W})\mathbf{\Lambda}^{-\frac{1}{2}}$. After making use of (James [4])

$$\int_{O(p)} |\mathbf{I} - \mathbf{H}\mathbf{R}\mathbf{H}'\mathbf{A}|^{-a}(d\mathbf{H}) = {}_1F_0(a; \mathbf{A}, \mathbf{R}) ,$$

and (44) of [1], the expression (3.10) becomes

$$(3.11) \qquad C_4(p, n_1, n_2)e^{-tr\Omega}|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}|\mathbf{I} + \lambda\mathbf{R}|^{-\frac{1}{2}(n_1+n_2)} \prod_{i>j}(r_i - r_j) \times \int_{\mathrm{Re}(\mathbf{T})=\mathbf{X}_0>0} e^{tr\mathbf{T}}|\mathbf{T}|^{-\frac{1}{2}n_1} \times {}_1F_0\left(\frac{n_1 + n_2}{2}; \mathbf{I} - \lambda^{-1}\mathbf{\Lambda}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{W})\mathbf{\Lambda}^{-\frac{1}{2}}, \lambda\mathbf{R}(\mathbf{I} + \lambda\mathbf{R})^{-1}\right) d\mathbf{T} ,$$

where

$$C_4(p, n_1, n_2) = [C_3(p, n_1, n_2)\pi^{\frac{1}{2}p^2}]/\Gamma_p(\tfrac{1}{2}p) .$$

Now expand ${}_1F_0$ and integrate term by term. The expression (3.11) involves the integral of the form

$$(3.12) \qquad \int_{\mathrm{Re}(\mathbf{T})=\mathbf{X}_0>0} e^{tr\mathbf{T}}|\mathbf{T}|^{-\frac{1}{2}n_1} \frac{C_\kappa(\mathbf{I} - \lambda^{-1}\mathbf{\Lambda}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{W})\mathbf{\Lambda}^{-\frac{1}{2}})}{C_\kappa(\mathbf{I})} d\mathbf{T} .$$

Let us use the relation (Constantine [2])

$$(3.13) \qquad \frac{C_\kappa(\mathbf{I} - \mathbf{S})}{C_\kappa(\mathbf{I})} = \sum_{n=0}^{k} \sum_\nu (-1)^n a_{\kappa,\nu} \frac{C_\nu(\mathbf{S})}{C_\nu(\mathbf{I})} ,$$

where $a_{\kappa,\nu}$ are constants discussed earlier. Then (3.12) becomes

$$(3.14) \qquad \sum_{n=0}^{k} \sum_\nu \frac{(-\lambda^{-1})^n a_{\kappa,\nu}}{C_\nu(\mathbf{I})} \int_{\mathrm{Re}(\mathbf{T})=\mathbf{X}_0>0} e^{tr\mathbf{T}}|\mathbf{T}|^{-\frac{1}{2}n_1}C_\nu(\mathbf{\Lambda}^{-1}(\mathbf{I} - \mathbf{W}))\, d\mathbf{T} .$$

Now assume $\mathbf{\Lambda}$ "random" and transform $\mathbf{\Lambda} \to \mathbf{H}\mathbf{\Lambda}\mathbf{H}'$ by an orthogonal transformation $\mathbf{H}$ and integrate over $\mathbf{H}$ using (3.6). Then (3.14) becomes

$$(3.15) \qquad \sum_{n=0}^{k} \sum_\nu \frac{(-\lambda^{-1})^n a_{\kappa,\nu} C_\nu(\mathbf{\Lambda}^{-1})}{C_\nu(\mathbf{I})C_\nu(\mathbf{I})} \int_{\mathrm{Re}(\mathbf{T})=\mathbf{X}_0>0} e^{tr\mathbf{T}}|\mathbf{T}|^{-\frac{1}{2}n_1}C_\nu(\mathbf{I} - \mathbf{T}^{-1}\Omega)\, d\mathbf{T} .$$

Applying (17) of [2] to (3.15) gives

$$\frac{(2\pi i)^{\frac{1}{2}p(p+1)}}{2^{\frac{1}{2}p(p-1)}} \sum_{n=0}^{k} \sum_{\nu} \frac{a_{\kappa,\nu} C_\nu(-\lambda^{-1}\mathbf{\Lambda}^{-1}) L_\nu^{\frac{1}{2}(n_1-p-1)}(\mathbf{\Omega})}{\Gamma_p\left(\frac{n_1}{2}, \nu\right) C_\nu(\mathbf{I}) C_\nu(\mathbf{I})} . \tag{3.16}$$

Combining (3.11)—(3.16) and making use of (15) of [1] we have the result as stated in the theorem. □

Formula (3.7) yields the following special cases:

(a) For $\mathbf{\Omega} = \mathbf{0}$, it is seen from (21) of [2] (see also Herz [3], page 487), $L_\kappa^\gamma(\mathbf{0}) = (\gamma + \frac{1}{2}(p+1))_\kappa C_\kappa(\mathbf{I})$. Substituting $L_\delta^{\frac{1}{2}(n_1-p-1)}(\mathbf{0})$ into (3.7) and making use of (3.13), we have the result of Khatri [5].

(b) By letting $\mathbf{\Lambda} = \mathbf{I}$ and $\lambda = 1$ and by using the relation (Constantine [2])

$$\frac{L_\delta^\gamma(\mathbf{S})}{(\gamma + m)_\kappa C_\kappa(\mathbf{I})} = \sum_{n=0}^{k} \sum_{\nu} \frac{(-1)^n a_{\kappa,\nu} C_\nu(\mathbf{S})}{(\gamma + m)_\nu C_\nu(\mathbf{I})} , \tag{3.17}$$

where $m = \frac{1}{2}(p+1)$, whenever $\mathbf{S}$ is a $(p \times p)$ matrix and $\nu = \delta = \kappa$ we have the result of Constantine [1], James [4].

In order to discuss the convergence of the series (3.7) let us note that series (3.7) (excluding the factors outside the summation) is dominated termwise by the series

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_\kappa}{k!} \frac{C_\kappa(\lambda\mathbf{R}(I + \lambda\mathbf{R})^{-1})}{C_\kappa(\mathbf{I})} \sum_{d=0}^{k} \sum_{\delta} a_{\kappa,\delta} C_\delta(-\lambda^{-1}\mathbf{\Lambda}^{-1}) \frac{e^{tr\mathbf{\Omega}}}{C_\delta(\mathbf{I})}$$

in view of Theorem 3 of [2]. Further

$$\sum_{d=0}^{k} \sum_{\delta} a_{\kappa,\delta} C_\delta(-\lambda^{-1}\mathbf{\Lambda}^{-1})/C_\delta(\mathbf{I}) = C_\kappa(\mathbf{I} - \lambda^{-1}\mathbf{\Lambda}^{-1})/C_\kappa(\mathbf{I})$$

by (19) of [2]. Hence (3.7) is dominated termwise by

$$C(p, n_1, n_2)|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}|\mathbf{I} + \lambda\mathbf{R}|^{-\frac{1}{2}(n_1+n_2)} \prod_{i>j}(r_i - r_j)$$
$$\times \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_\kappa}{k!\, C_\kappa(\mathbf{I})} C_\kappa(\mathbf{I} - \lambda^{-1}\mathbf{\Lambda}^{-1}) C_\kappa(\lambda R(\mathbf{I} + \lambda\mathbf{R})^{-1}) ,$$

which is independent of $\mathbf{\Omega}$ and $a_{\kappa,\delta}$ coefficients, and is in fact the joint density function of the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ given by Khatri [5].

Let us now consider $\mathbf{\Omega}$ as a random matrix $\frac{1}{2}\mathbf{\Sigma}_1^{-\frac{1}{2}}\mathbf{MYY'M'}\mathbf{\Sigma}_1^{-\frac{1}{2}}$ where $\mathbf{YY'}$ has a central Wishart distribution $W(q, n_3, \mathbf{\Sigma}_3, \mathbf{0})$, i.e.,

$$\{\Gamma_q(\tfrac{1}{2}n_3)|2\mathbf{\Sigma}_3|^{\frac{1}{2}n_3}\}^{-1}|\mathbf{YY'}|^{\frac{1}{2}(n_3-q-1)} \exp[\operatorname{tr}(-\tfrac{1}{2}\mathbf{\Sigma}_3^{-1}\mathbf{YY'})] . \tag{3.18}$$

Multiplying (3.2) by (3.18), integrating term by term with respect to $\mathbf{YY'}$, and using (1) of [1], we get the joint density of $r_1, \cdots, r_p$ in the form:

$$C(p, n_1, n_2)|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{I} + \mathbf{M'}\mathbf{\Sigma}_1^{-1}\mathbf{M}\mathbf{\Sigma}_3|^{-\frac{1}{2}n_3}|\mathbf{R}|^{\frac{1}{2}(n_1-p-1)}$$
$$\times \prod_{i>j}(r_i - r_j) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n_3)_\kappa C_\kappa[(\mathbf{I} + \mathbf{\Omega}_1)^{-1}\mathbf{\Omega}_1]}{(\frac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I}) k!} \tag{3.19}$$
$$\times \sum_{n=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n g_{\kappa,\nu}^{\delta}(\frac{1}{2}(n_1+n_2))_\delta C_\delta(\mathbf{\Lambda}^{-1}) C_\delta(\mathbf{R})}{C_\delta(\mathbf{I}) n!} ,$$

where $\Omega_1 = \Sigma_3^{\frac{1}{2}}M'\Sigma_1^{-1}M\Sigma_3^{\frac{1}{2}}$. Now, alternately, consider (3.7). Expand the generalized Laguerre polynomial in (3.7) using the expansion in (3.17). After multiplying this new form of (3.7) by (3.18), integrating YY' and using (1) of [1], we have the joint density of $r_1, \cdots, r_p$ in the form

$$\begin{aligned} & C(p, n_1, n_2)|\Lambda|^{-\frac{1}{2}n_1}|I + M'\Sigma_1^{-1}M\Sigma_3|^{-\frac{1}{2}n_3} \\ & \quad \times |R|^{\frac{1}{2}(n_1-p-1)}|I + \lambda R|^{-\frac{1}{2}(n_1+n_2)} \prod_{i>j}(r_i - r_j) \\ & \quad \times \sum_{k=0}^{\infty} \sum_{\kappa} (\tfrac{1}{2}(n_1+n_2))_\kappa \frac{C_\kappa(\lambda R(I+\lambda R)^{-1})}{k!} \\ & \quad \times \sum_{d=0}^{k} \sum_{\delta} a_{\kappa,\delta} \frac{C_\delta(-\lambda^{-1}\Lambda^{-1})}{C_\delta(I)} \\ & \quad \times \sum_{n=0}^{d} \sum_{\nu} \frac{(-1)^n a_{\delta,\nu}(\frac{1}{2}n_3)_\nu C_\nu[(I+\Omega_1)^{-1}\Omega_1]}{(\frac{1}{2}n_1)_\nu C_\nu(I)}. \end{aligned} \tag{3.20}$$

It is easy to see that the distribution of the characteristic roots for the canonical correlation problem is a special case of (3.20).

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