The Annals of Statistics
1975, Vol. 3, No. 1, 257-265

TESTING HYPOTHESES IN UNBALANCED VARIANCE COMPONENTS MODELS FOR TWO-WAY LAYOUTS

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Consider the model equation $y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$ ($i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$; $k = 1, 2, \cdots, n_{ij}$), where μ is a constant and α_i, β_j, γ_{ij}, e_{ijk} are distributed independently and normally with zero means and variances σ_A^2, σ_B^2, σ_{AB}^2, σ^2, respectively. In this paper procedures for testing hypotheses on σ_A^2/σ_B^2, σ^2/σ^2, and σ_{AB}^2/σ^2 are given. The test procedure for σ_{AB}^2/σ^2 is compared with the corresponding test procedures when α_i, β_j, and γ_{ij} are fixed effects instead of being random.

1. Introduction. The analysis of the variance method of estimating variance components from balanced data is based on equating mean squares of analyses of variance to their expected values. Futhermore, expected values of mean squares will suggest which mean squares are the appropriate denominators for testing hypotheses concerning the variance components (Searle (1971), pages 411–15). However, with unbalanced data no unique set of sums of squares or quadratic forms in the observations can be optimally used for estimating variance components.

In this paper we shall find some exact tests concerning the variance components in an unbalanced, random two-way layout by modifying an approach suggested by Graybill and Hultquist (1961), who describe a variance components model as follows:

A $(n \times 1)$ vector of observations $\mathbf{Y}$ is assumed to be a linear sum of $k + 2$ quantities,

$$\mathbf{Y} = \mathbf{J}_n \beta_0 + \sum_{i=1}^{k} \mathbf{B}_i \boldsymbol{\beta}_i + \boldsymbol{\beta}_{k+1} \, . \tag{1.1}$$

Here β_0 is a fixed unknown constant. $\boldsymbol{\beta}_i$ is a $(p_i \times 1)$ vector of multinormally distributed random variables with mean $\mathbf{0}$ and covariance matrix $\sigma_i^2 \mathbf{I}_{p_i}$. ($\mathbf{I}_k$ denotes a k-dimensional identity matrix and $\mathbf{0}$ a null matrix.) The vectors $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_{k+1}$ are stochastically independent. $\mathbf{J}_k$ is a $(k \times 1)$ vector with all elements equal to 1. $\mathbf{B}_i$ $(i = 1, 2, \cdots, k)$ a $(n \times p_i)$ matrix of known constants.

Some general theorems concerning this model have been derived by Graybill and Hultquist (1961) under one or both of the following assumptions:

(i) $\mathbf{A}_i$ and $\mathbf{A}_j$ commute, where $\mathbf{A}_i = \mathbf{B}_i \mathbf{B}_i'$ $(i, j = 1, 2, \cdots, k)$,

(ii) The matrix $\mathbf{B}_i$ is such that $\mathbf{J}_n' \mathbf{B}_i = r_i \mathbf{J}_{r_i}'$ and $\mathbf{B}_i \mathbf{J}_{p_i} = \mathbf{J}_n$, where r_i is a positive integer.

Received January 1972; revised February 1974.

Key words and phrases. Testing hypotheses, unbalanced variance components model, two-way layouts.

The assumptions (i) are not satisfied in most unbalanced models.

In this paper we will consider a special case of model (1.1) without assumptions (i), viz. the common variance components model for a complete two-way layout. Spjøtvoll (1968) has treated the same model in a different manner. Bush and Anderson (1963) suggest a similar procedure as proposed in this paper, but they are primarily concerned with estimation.

In Section 2 we shall transform our model to a "semi-canonical" form and find a method for obtaining confidence intervals and testing hypotheses concerning the σ_i^2. In Section 3 these tests are compared with the corresponding tests in a fixed effects model. In Section 4 the test statistics are expressed in terms of the original observations. In Sections 2–4 we assume that there is at least one observation in each cell. This assumption is removed in Section 5.

2. Modification of the model of Graybill and Hultquist. We consider the following model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} ; \tag{2.1}$$

$i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$, and $k = 1, 2, \cdots, n_{ij}$. Here μ is a constant, while α_i, β_j, γ_{ij}, and e_{ijk} are independent normally distributed random variables with means 0 and variances σ_A^2, σ_B^2, σ_{AB}^2, and σ^2, respectively.

Define $\bar{y}_{ij} = (1/n_{ij}) \sum_{k=1}^{n_{ij}} y_{ijk}$; $i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$. Then

$$\bar{y}_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \bar{e}_{ij} , \tag{2.2}$$

with $\bar{e}_{ij} = (1/n_{ij}) \sum_{k=1}^{n_{ij}} e_{ijk}$.

For any set of variables a_{ij} ($i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$), let $\mathbf{a}$ be the vector $(a_{11}, a_{12}, \cdots, a_{1s}, a_{21}, \cdots, a_{rs})'$. Then with this ordering $\bar{\mathbf{e}}$ is multivariate normally distributed with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}(\bar{\mathbf{e}}) = \mathbf{K}\sigma^2$, where

$$\mathbf{K} = \text{Diag}\,(n_{11}^{-1}, n_{12}^{-1}, \cdots, n_{rs}^{-1}) . \tag{2.3}$$

Formula (2.2) may be written in matrix forms as

$$\bar{\mathbf{y}} = \mathbf{J}_{rs}\mu + \mathbf{B}_1\boldsymbol{\alpha} + \mathbf{B}_2\boldsymbol{\beta} + \mathbf{B}_3\boldsymbol{\gamma} + \bar{\mathbf{e}} \tag{2.4}$$

with

$$\mathbf{B}_1' = \text{Diag}\,(\mathbf{J}_s, \cdots, \mathbf{J}_s) , \qquad \mathbf{B}_2' = [\mathbf{I}_s, \cdots, \mathbf{I}_s]$$

and $\mathbf{B}_3 = \mathbf{I}_{rs}$, which is of the same form as (1.1). The covariance matrix for $\bar{\mathbf{y}}$ turns out as

$$\mathbf{\Sigma}(\bar{\mathbf{y}}) = \mathbf{B}_1\mathbf{B}_1'\sigma_A^2 + \mathbf{B}_2\mathbf{B}_2'\sigma_B^2 + \mathbf{I}_{rs}\sigma_{AB}^2 + \mathbf{K}\sigma^2 .$$

As $\mathbf{B}_1\mathbf{B}_1'$ and $\mathbf{B}_2\mathbf{B}_2'$ commute, it follows that there exists an orthogonal matrix $\mathbf{P}$ with the property that $\mathbf{PA}_1\mathbf{P}'$ and $\mathbf{PA}_2\mathbf{P}'$ are diagonal matrices with the eigenvalues on the diagonal (Herbach, 1959). $\mathbf{P}$ may be chosen so that the first row in $\mathbf{P}$ is $(rs)^{-\frac{1}{2}}\,(1, 1, \cdots, 1)$. ($\mathbf{A}_1 = \mathbf{B}_1\mathbf{B}_1'$; $\mathbf{A}_2 = \mathbf{B}_2\mathbf{B}_2'$).

If $\mathbf{Z} = \mathbf{P}\bar{\mathbf{y}}$, the covariance matrix for $\mathbf{Z}$ is

$$\mathbf{\Sigma}(\mathbf{Z}) = \mathbf{PA}_1\mathbf{P}'\sigma_A^2 + \mathbf{PA}_2\mathbf{P}'\sigma_B^2 + \mathbf{I}_{rs}\sigma_{AB}^2 + \mathbf{PKP}'\sigma^2 .$$

LEMMA 1.

(i) $\operatorname{rank}(\mathbf{B}_1) = r$;
(ii) $\operatorname{rank}(\mathbf{B}_2) = s$;
(iii) $\operatorname{rank}(\mathbf{B}_1 \vdots \mathbf{B}_2) = r + s - 1$;
(iv) $\operatorname{rank}(\mathbf{A}_1 + \mathbf{A}_2) = \operatorname{rank}(\mathbf{B}_1 \vdots \mathbf{B}_2)$.

PROOF. (i), (ii), and (iii) are seen from (2.4). (iv) follows from the proof of Graybill and Hultquist's (1961) Theorem 1. □

From the fact that $\operatorname{rank}(\mathbf{A}_1) = \operatorname{rank}(\mathbf{B}_1) = r$ and because $\mathbf{A}_1$ has the eigenvalues s of multiplicity r and 0 of multiplicity $(rs - r) = r(s-1)$, it follows that $\mathbf{PA'P'}$ has r diagonal elements all equal to s and the rest equal to 0. In the same way it is seen that $\mathbf{PA}_2\mathbf{P'}$ has s diagonal elements all equal to r and the other elements equal to 0.

From (iii) and (iv) it is seen that the matrix $(\mathbf{PA}_1\mathbf{P'} + \mathbf{PA}_2\mathbf{P'})$ has $(r + s - 1)$ diagonal elements different from zero. Thus when the diagonal element in $\mathbf{PA}_1\mathbf{P'}$ is different from zero, the corresponding element in $\mathbf{PA}_2\mathbf{P'}$ is equal to zero except in one place (in the first row).

We now partition $\mathbf{Z}$ in the following way:

(i) $Z_1 = (rs)^{\frac{1}{2}}y\cdots$, which is the first element in $\mathbf{Z}$.

(ii) $\mathbf{Z}_A$ consists of the $(r-1)$ elements in $\mathbf{Z}$ whose covariance matrix is independent of σ_B^2.

(iii) $\mathbf{Z}_B$ consists of the $(s-1)$ elements in $\mathbf{Z}$ whose covariance matrix is independent of σ_A^2.

(iv) $\mathbf{Z}_{AB}$ consists of the $(r-1)(s-1)$ elements in $\mathbf{Z}$ whose covariance matrix is independent of σ_A^2 and σ_B^2.

LEMMA 2. $E\mathbf{Z}_A = E\mathbf{Z}_B = E\mathbf{Z}_{AB} = 0$.

PROOF. This follows from the fact that $\mathbf{P}$ is orthogonal with a first row which is $(rs)^{-\frac{1}{2}}(1, \cdots, 1)$. □

We have

$$\Sigma(\mathbf{Z}_A) = s\mathbf{I}_{r-1}\sigma_A^2 + \mathbf{I}_{r-1}\sigma_{AB}^2 + \mathbf{K}_1\sigma^2\,, \tag{2.5}$$

$$\Sigma(\mathbf{Z}_B) = r\mathbf{I}_{s-1}\sigma_B^2 + \mathbf{I}_{s-1}\sigma_{AB}^2 + \mathbf{K}_2\sigma^2\,,$$

and

$$\Sigma(\mathbf{Z}_{AB}) = \mathbf{I}_{(r-1)(s-1)}\sigma_{AB}^2 + \mathbf{K}_3\sigma^2\,.$$

Here $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ are the corresponding submatrices of $\mathbf{PKP'}$.

In what follows, $\mathbf{Z}_A$, $\mathbf{Z}_B$ and $\mathbf{Z}_{AB}$ will be used for testing hypotheses concerning σ_A^2/σ^2, σ_B^2/σ^2, and σ_{AB}^2/σ^2.

2.a. *Test for* σ_{AB}^2/σ^2. $\Sigma(\mathbf{Z}_{AB})$ may be written as $(\mathbf{I}_{(r-1)(s-1)}\Delta_{AB} + \mathbf{K}_3)\sigma^2$, where $\Delta_{AB} = \sigma_{AB}^2/\sigma^2$. Then

$$Q_{AB}/\sigma^2 = \mathbf{Z}'_{AB}(\mathbf{I}_{(r-1)(s-1)}\Delta_{AB} + \mathbf{K}_3)^{-1}\mathbf{Z}_{AB}/\sigma^2 \tag{2.6}$$

has a χ^2-distribution with $(r-1)(s-1)$ degrees of freedom. There exists an orthogonal matrix $\mathbf{A}$ such that $\mathbf{AK}_3\mathbf{A}' = \mathbf{D}_1$ is a diagonal matrix. Introduce $\mathbf{Z}_{AB}^* = \mathbf{AZ}_{AB}$. The covariance matrix for $\mathbf{Z}_{AB}^*$ is $(\mathbf{I}_{(r-1)(s-1)}\Delta_{AB} + \mathbf{D}_1)$ and

$$\begin{aligned}\mathbf{Z}'_{AB}(\mathbf{I}_{(r-1)(s-1)}\Delta_{AB} + \mathbf{K}_3)^{-1}\mathbf{Z}_{AB} &= \mathbf{Z}^{*\prime}_{AB}(\mathbf{I}_{(r-1)(s-1)}\Delta_{AB} + \mathbf{D}_1)^{-1}\mathbf{Z}^*_{AB} \\ &= \textstyle\sum_{j=1}^{(r-1)(s-1)} (Z^*_{jAB})^2/(\Delta_{AB} + d_j)\,.\end{aligned}$$

Here $d_1, \cdots, d_{(r-1)(s-1)}$ are the diagonal elements of $\mathbf{D}_1$. We see that Q_{AB}/σ^2 is a decreasing function of Δ_{AB}.

Define $Q = \sum_{i,j,k}(y_{ijk} - \bar{y}_{ij\cdot})^2$. Then Q/σ^2 has a χ^2-distribution with $(n - rs)$ degrees of freedom. Q is stochastically independent of Q_{AB}. Thus $F(\Delta_{AB}) = (n - rs)Q_{AB}/(r-1)(s-1)\,Q$ has an F-distribution. Since Q_{AB}/σ^2 decreases with Δ_{AB}, $F(\Delta_{AB})$ decreases with Δ_{AB}. Hence a confidence interval can be obtained in the usual way.

When testing the hypothesis

$$\Delta_{AB} \leqq \Delta_0 \quad \text{against} \quad \Delta_{AB} > \Delta_0\,,$$

we reject when $F(\Delta_0)$ is larger than the upper α-quantile, $f_{1-\alpha}$, of the corresponding F-distribution. The power function is

$$\begin{aligned}\beta(\Delta_{AB}) &= P\{(n-rs)[\textstyle\sum_{i=1}^{n} Z_{iAB}^2/(\Delta_0 + d_i)]/[(r-1)(s-1)Q)] > f_{1-\alpha}\} \\ &= P\{(n-rs)[\textstyle\sum_{i=1}^{n}(\Delta_{AB} + d_i)R_i/(\Delta_0 + d_i)]/[(r-1)(s-1)] > f_{1-\alpha}\}\,,\end{aligned}$$

where $R_1, \cdots, R_{(r-1)(s-1)}$ are independent χ^2-distributed random variables with 1 degree of freedom. $\beta(\Delta_{AB})$ increases with Δ_{AB}. The test is unbiased, size α, but with no established optimality properties.

2.b. *Test for $\sigma_A{}^2/\sigma^2$ assuming $\sigma_{AB} = 0$.* When $\sigma_{AB} = 0$ the covariance matrix for $\{{\mathbf{Z}_A \atop \mathbf{Z}_{AB}}\}$ is equal to

$$\Sigma\begin{Bmatrix}\mathbf{Z}_A \\ \mathbf{Z}_{AB}\end{Bmatrix} = \begin{Bmatrix}s\mathbf{I}_{(r-1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{Bmatrix}\sigma_A{}^2 + \begin{Bmatrix}\mathbf{K}_1 & \mathbf{K}_4 \\ \mathbf{K}_4 & \mathbf{K}_3\end{Bmatrix}\sigma^2\,,$$

where $E\{\mathbf{Z}_A \;\; \mathbf{Z}'_{AB}\} = \mathbf{K}_4$. $\{{\mathbf{I}_{(r-1)} \atop \mathbf{0}}\;{\mathbf{0} \atop \mathbf{0}}\}$ is positive semi-definite, and $\{{\mathbf{K}_1 \atop \mathbf{K}_4}\;{\mathbf{K}_4 \atop \mathbf{K}_3}\}$ is positive definite, so we can find a non-singular matrix H such that $\mathbf{H}\{{\mathbf{K}_1 \atop \mathbf{K}_4}\;{\mathbf{K}_4 \atop \mathbf{K}_3}\}\mathbf{H}' = \mathbf{I}$, and $\mathbf{H}\{{s\mathbf{I}_{(r-1)} \atop \mathbf{0}}\;{\mathbf{0} \atop \mathbf{0}}\}\mathbf{H}' = \lambda = \operatorname{diag}\{\lambda_1, \cdots, \lambda_{r-1}, 0, \cdots, 0\}$.

Define $\mathbf{U} = \{{\mathbf{U}_A \atop \mathbf{U}_{AB}}\} = \mathbf{H}\{{\mathbf{Z}_A \atop \mathbf{Z}_{AB}}\}$. If $\Delta_A = \sigma_A{}^2/\sigma^2$, $Q_A/\sigma^2 = \mathbf{U}_A'(\lambda\Delta_A + \mathbf{I}_{(r-1)})^{-1}\mathbf{U}_A/\sigma^2$ has a χ^2-distribution with $(r-1)$ degrees of freedom, and $Q^*_{AB} = \mathbf{U}'_{AB}\mathbf{I}_{(r-1)(s-1)}\mathbf{U}_{AB}/\sigma^2$ has a χ^2-distribution with $(r-1)(s-1)$ degrees of freedom. Q_A, Q^*_{AB} and Q are stochastically independent.

To test the hypothesis $\Delta_A \leqq \Delta_0$ against $\Delta_A > \Delta_0$, we reject when

$$G(\Delta_A) = Q_A\{(n-rs) + (r-1)(s-1)\}/(Q + Q^*_{AB})(r-1) \tag{2.7}$$

is larger than the upper α-quantile, $f_{1-\alpha}$, of the corresponding F-distribution. This test is not the same as the test given by Spjøtvoll (1968).

In the same way as above it may be proved that the test is unbiased.

A corresponding test exists concerning $\sigma_B{}^2/\sigma^2$.

3. Comparison with corresponding tests in fixed effects models. A two-way layout in fixed effects models may be described as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} ;$$

$i = 1, 2, \cdots, r; j = 1, 2, \cdots, s; k = 1, 2, \cdots, n_{ij}$, where μ, α_i, β_j, and γ_{ij} are unknown constants such that

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0 , \tag{3.1}$$

and the e_{ijk} have a joint normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}_n \sigma^2$.

The null hypothesis $\gamma_{ij} = 0$ ($i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$) is tested by minimizing the sum of squares $Q = \sum_{i,j,k} (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2$ under the null hypothesis and under the a priori specifications. Let the two minima of Q be Q_ω and Q_Ω, respectively. The null hypothesis is rejected when

$$(Q_\omega - Q_\Omega)(n - rs)/Q_\Omega(r-1)(s-1) \tag{3.2}$$

is larger than the upper α-quantile $f_{1-\alpha}$ of the corresponding F-distribution. The reader is referred to Scheffé (1959).

We will prove that the quantity in (3.2) is equal to the test-statistic $F(0)$ in Section 2a.

If, as in Section 2, we introduce $\bar{\mathbf{y}}$, we have that

$$\bar{\mathbf{y}} = \mathbf{J}_{rs}\mu + \mathbf{B}_1 \boldsymbol{\alpha} + \mathbf{B}_2 \boldsymbol{\beta} + \mathbf{I}_{rs} \boldsymbol{\gamma} + \bar{\mathbf{e}} . \tag{3.3}$$

The only difference from the random effects model (2.4) is that α_i, β_j, and γ_{ij} here are fixed constants with the side conditions (3.1).

(3.3) may be written in the form

$$\bar{\mathbf{y}} = (\mathbf{J}, \mathbf{B}_1\mathbf{A}, \mathbf{B}_2\mathbf{B}, \mathbf{C})(\mu, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)' + \bar{\mathbf{e}} , \tag{3.4}$$

where $\boldsymbol{\alpha}^* = (\alpha_1, \alpha_2, \cdots, \alpha_{r-1})'$; $\boldsymbol{\beta}^* = (\beta_1, \beta_2, \cdots, \beta_{s-1})'$, $\boldsymbol{\gamma}^* = (\gamma_{11}, \cdots, \gamma_{(r-1)(s-1)})'$, and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are defined such that

$$(\alpha_1, \alpha_2, \cdots, \alpha_r)' = \mathbf{A}^{(r\times(r-1))}\boldsymbol{\alpha}^*$$
$$(\beta_1, \beta_2, \cdots, \beta_s)' = \mathbf{B}^{(s\times(s-1))}\boldsymbol{\beta}^* , \qquad \text{and}$$
$$(\gamma_{11}, \cdots, \gamma_{rs})' = \mathbf{C}^{(rs\times(r-1)(s-1))}\boldsymbol{\gamma}^* .$$

(It is possible to write (3.3) in several other ways. This will lead to formally different $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ matrices, and formally different $\boldsymbol{\alpha}^*$, $\boldsymbol{\beta}^*$ and $\boldsymbol{\gamma}^*$ in (3.4) and (3.5).)

Denote $\mathbf{B}_1\mathbf{A} = \mathbf{W}_2$, $\mathbf{B}_2\mathbf{B} = \mathbf{W}_3$, $\mathbf{C} = \mathbf{W}_4$, and $(\mathbf{J}, \mathbf{B}_1\mathbf{A}, \mathbf{B}_2\mathbf{B}, \mathbf{C}) = \mathbf{W}$. Then

$$\bar{\mathbf{y}} = \mathbf{W}(\mu, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)' + \bar{\mathbf{e}} . \tag{3.5}$$

Define $\mathbf{V} = \mathbf{K}^{-\frac{1}{2}}\bar{\mathbf{y}}$, then

$$\mathbf{V} = \mathbf{K}^{-\frac{1}{2}}\mathbf{W}(\boldsymbol{\mu}, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)' + \mathbf{e}^* , \tag{3.6}$$

where $\mathbf{e}^*$ is normally distributed with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}_{rs}\sigma^2$. We have that

$$Q = \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij})^2 + (\mathbf{V} - E\mathbf{V})'(\mathbf{V} - E\mathbf{V}) . \tag{3.7}$$

Define $Q_p = (\mathbf{V} - E\mathbf{V})'(\mathbf{V} - E\mathbf{V})$, and let $Q_{p\omega}$ and $Q_{p\Omega}$ denote the minima of Q_p under the null hypothesis and under the a priori specifications, respectively. Then it follows that $Q_\omega - Q_\Omega = Q_{p\omega} - Q_{p\Omega}$.

From the general theory for linear models it is known that

$$Q_{p\omega} - Q_{p\Omega} = \hat{\boldsymbol{\gamma}}^{*\prime}(\boldsymbol{\Sigma}_4)^{-1}\hat{\boldsymbol{\gamma}}^* , \tag{3.8}$$

where $\hat{\boldsymbol{\gamma}}^*$ is the least squares estimate of $\boldsymbol{\gamma}^*$, and $\boldsymbol{\Sigma}_4$ is the covariance matrix for $\hat{\boldsymbol{\gamma}}^*$. The least squares estimate of $(\mu, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)'$ is

$$(\mu, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)' = (\mathbf{W}'\mathbf{K}^{-\frac{1}{2}}\mathbf{K}^{-\frac{1}{2}}\mathbf{W})^{-1}\mathbf{W}\mathbf{K}^{-\frac{1}{2}}\mathbf{V} = \mathbf{W}^{-1}\bar{\mathbf{y}} .$$

The reader is referred to Searle (1971, page 120).

To prove that $\sigma^{-2}(Q_{p\omega} - Q_{p\Omega}) = Q_{AB}$ when $\Delta_{AB} = 0$, where Q_{AB} is defined as in Section 2, we introduce the transformation $\mathbf{P}$, where $\mathbf{P}$ is the orthogonal matrix with which the cell mean values were transformed in the random effects model. The following lemma is useful.

LEMMA 3. *Partition* $\mathbf{P}$ *into submatrices corresponding to the partitioning of* W,

$$\mathbf{P} = [\mathbf{P}_1^{(1\times rs)\prime}, \mathbf{P}_2^{((r-1)\times rs)\prime}, \mathbf{P}_3^{((s-1)\times rs)\prime}, \mathbf{P}_4^{((r-1)(s-1)\times rs)\prime}] .$$

For any choice of $\mathbf{W}$ *we have that*

(i) *The rows of* $\mathbf{P}_2$ *are orthogonal to the columns of* $\mathbf{W}_3$.
(ii) *The rows of* $\mathbf{P}_3$ *are orthogonal to the columns of* $\mathbf{W}_2$.
(iii) *The rows of* $\mathbf{P}_4$ *are orthogonal to the columns of* $\mathbf{W}_2$ *and* $\mathbf{W}_3$.

PROOF. From the results in Section 2 we have that $\mathbf{P}_2\mathbf{B}_2\mathbf{B}_2'\mathbf{P}_2' = \mathbf{0}$, then $\mathbf{P}_2\mathbf{B}_2 = \mathbf{0}$, and thus $\mathbf{P}_2\mathbf{W}_3 = \mathbf{0}$ because $\mathbf{W}_3 = \mathbf{B}_2\mathbf{B}$. The rest of the lemma now follows by treating $\mathbf{P}_3$ and $\mathbf{P}_4$ in a similar way. □

From Lemma 3 and from the facts that $\mathbf{P}_1\mathbf{W}_2 = \mathbf{P}_1\mathbf{W}_3 = \mathbf{P}_1\mathbf{W}_4 = \mathbf{0}$ it follows that $\mathbf{PW}$ has the form

$$PW = \begin{Bmatrix} \mathbf{P}_1\mathbf{J}_{rs} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2\mathbf{W}_2 & \mathbf{0} & \mathbf{P}_2\mathbf{W}_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_3\mathbf{W}_3 & \mathbf{P}_3\mathbf{W}_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_4\mathbf{W}_4 \end{Bmatrix} . \tag{3.9}$$

Now $\hat{\boldsymbol{\gamma}}^*$ is the $(r-1)(s-1)$ lower element of $\mathbf{W}^{-1}\bar{\mathbf{y}} = (\mathbf{PW})^{-1}\mathbf{P}\bar{\mathbf{y}}$. From (3.9) it follows that $(\mathbf{PW})^{-1}$ is a triangular matrix with zeroes to the left of the diagonal, from which follows that

$$\hat{\boldsymbol{\gamma}}^* = (\mathbf{P}_4\mathbf{W}_4)^{-1}\mathbf{P}_4\bar{\mathbf{y}} .$$

From (3.9) it also follows that the covariance matrix for $\hat{\boldsymbol{\gamma}}^*$, $\boldsymbol{\Sigma}_4$, is

$$\boldsymbol{\Sigma}_4 = (\mathbf{P}_4\mathbf{W}_4)^{-1}(\mathbf{PKP}')_4(\mathbf{P}_4\mathbf{W}_4)^{-1} ,$$

where $(\mathbf{PKP}')_4$ is the $(r-1)(s-1) \times (r-1)(s-1)$ submatrix in the lower right-hand corner of $\mathbf{PKP}'$ in Section 2.
(3.8) may then be written in the form

$$\bar{\mathbf{y}}'\mathbf{P}_4'(\mathbf{P}_4\mathbf{W}_4)_4'^{-1}(\mathbf{P}_4\mathbf{W}_4)'(\mathbf{PKP}')_4^{-1}(\mathbf{P}_4\mathbf{W}_4)(\mathbf{P}_4\mathbf{W}_4)^{-1}\mathbf{P}_4\bar{\mathbf{y}}\sigma^2 = \bar{\mathbf{y}}'\mathbf{P}_4'(\mathbf{PKP}')_4^{-1}\mathbf{P}_4\bar{\mathbf{y}}\sigma^2 .$$

This quadratic form is independent of $\mathbf{W}$, $\boldsymbol{\alpha}^*$, $\boldsymbol{\beta}^*$, and $\boldsymbol{\gamma}^*$, and is equal to Q_{AB} in (2.6) when $\Delta_{AB} = 0$, because $\mathbf{Z}_{AB} = \mathbf{P}_4\bar{\mathbf{y}}$ and $\mathbf{K}_3 = (\mathbf{PKP}')_4$.

4. The test statistics expressed by the original observations.

LEMMA 4. *With the choice of* $\mathbf{W}$ *made in Section* 3, *the least squares estimates for* $(\mu, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)'$ *are* $\hat{\mu} = y_{\cdots}$, $\{\hat{\alpha}_i^*\} = \{y_{i\cdot\cdot} - y_{\cdots}\}$, $\{\hat{\beta}^*\} = (y_{\cdot j\cdot} - y_{\cdots}\}$, *and* $\{\hat{\gamma}_{ij}^*\} = \{y_{ij\cdot} - y_{i\cdot\cdot} - y_{\cdot j\cdot} + y_{\cdots}\}$. $(i = 1, 2, \cdots, r-1; j = 1, 2, \cdots, s-1)$.

PROOF. If we insert $\hat{\mu}$, $\{\hat{\alpha}_i^*\}$, $\{\hat{\beta}_j^*\}$ and $\{\hat{\gamma}_{ij}^*\}$ for μ, $\{\alpha_i\}$, $\{\beta_j\}$ and $\{\gamma_{ij}\}$ in (3.7), Q reduces to $\sum_{i,j,k}(y_{ijk} - y_{ij\cdot})^2$. □

When testing the null hypothesis $\Delta_{AB} \leqq 0$ against $\Delta_{AB} > 0$, we reject when

$$(4.1) \qquad (n - rs)\hat{\boldsymbol{\gamma}}^{*\prime}(\boldsymbol{\Sigma}_4)^{-1}\hat{\boldsymbol{\gamma}}^*/\textstyle\sum_{i,j,k}(y_{ijk} - y_{ij\cdot})^2(r-1)(s-1)$$

is larger than the upper α-quantile of the corresponding F-distribution. This test is the same as the one suggested by Spjøtvoll (1968).

It should be noted that the test statistic reduces to the usual one when the model is balanced.

5. Empty cells. In Sections 1–4 we have assumed that there is at least one observation per cell. In this section we shall remove this assumption. We shall show that the results in Sections 2a and 3 are not affected by empty cells (except that the number of degrees of freedom has to be adjusted), while the test given in 2b has to be modified.

As in Section 2 we define $\bar{y}_{ij} = (1/n_{ij}) \sum y_{ijk}$ for all cells with $n_{ij} > 0$. Then we have that

$$(5.1) \qquad \bar{\mathbf{y}} = \mathbf{J}_{(rs-p)}\mu + \mathbf{C}_1\boldsymbol{\alpha} + \mathbf{C}_2\boldsymbol{\beta} + \mathbf{C}_3\boldsymbol{\gamma} + \bar{\mathbf{e}} ,$$

where p is the number of empty cells. (5.1) is of the same form as (2.4), but $\mathbf{C}_i\mathbf{C}_i'$ $(i = 1, 2)$ do not commute as did $\mathbf{B}_i\mathbf{B}_i'$ in Section 2. We still have that

$$(5.2) \qquad \begin{aligned} &\text{(i)} && \operatorname{rank}(\mathbf{C}_1) = r \\ &\text{(ii)} && \operatorname{rank}(\mathbf{C}_2) = s \\ &\text{(iii)} && \operatorname{rank}(\mathbf{C}_1\mathbf{C}_2) = r + s - 1 \\ &\text{(iv)} && \operatorname{rank}(\mathbf{D}_1 + \mathbf{D}_2) = \operatorname{rank}(\mathbf{C}_1 \mid \mathbf{C}_2) , \end{aligned}$$

where $\mathbf{D}_i = \mathbf{C}_i\mathbf{C}_i'$ $(i = 1, 2)$.

Instead of applying the transformation $\mathbf{P}$ as in Section 2, we now apply the matrix of contrast vectors, $\mathbf{C}$, suggested by Bush and Anderson (1963).

Define $\mathbf{Z} = \mathbf{C}\bar{\mathbf{y}}$. Then

(5.3) $$\Sigma\mathbf{Z} = \mathbf{CD}_1\mathbf{C}'\sigma_A^2 + \mathbf{CD}_2\mathbf{C}'\sigma_B^2 + \mathbf{CC}'\sigma_{AB}^2 + \mathbf{CKC}'\sigma^2 .$$

As in Section 2, $\mathbf{Z}$ may be partitioned such that

(i) Z_1, has a variance dependent of σ_A^2, σ_B^2, σ_{AB}^2 and σ^2.

(ii) $\mathbf{Z}_A$ consists of the $(r-1)$ elements whose covariance matrix is independent of σ_B^2.

(iii) $\mathbf{Z}_B$ consists of the $(s-1)$ elements whose covariance matrix is independent of σ_A^2.

(iv) $\mathbf{Z}_{AB}$ consists of the $((r-1)(s-1)-p)$ elements whose covariance matrix is independent of σ_A^2 and σ_B^2.

The only difference from Section 2 is that $\mathbf{CD}_i\mathbf{C}'$ $(i = 1, 2)$ is not diagonal as in Section 2.

The covariance matrix of $\mathbf{Z}_{AB}$, Σ, $\mathbf{Z}_{AB}$ is of the form

$$\Sigma\mathbf{Z}_{AB} = \mathbf{D}\sigma_{AB}^2 + E\sigma^2 = (\mathbf{D}\Delta_{AB} + E)\sigma^2 ,$$

where $\mathbf{D}$ and $\mathbf{E}$ are matrices of known constants.

In the same way as in Section 2 it is seen that

$$F(\Delta_{AB}) = \mathbf{Z}_{AB}'(\mathbf{D}\Delta_{AB} + E)^{-1}\mathbf{Z}_{AB}(n - (rs - p))/Q((r-1)(s-1) - p)$$

has an F-distribution. When testing the hypothesis $\Delta_{AB} \leqq \Delta_0$ against $\Delta_{AB} > \Delta_0$ we reject when $F(\Delta_0)$ is larger than the upper α-quantile, $f_{1-\alpha}$, of the corresponding f-distribution.

For $\Delta_0 = 0$ this test is the same as the corresponding test in a fixed effects model, which is seen by applying $\mathbf{C}$ instead of $\mathbf{P}$ in the discussion in Section 3.

Assuming $\sigma_{AB} = 0$, the covariance matrix of $\mathbf{Z}_A$ can be written

$$\Sigma\mathbf{Z}_A = [\mathbf{L}\sigma_A^2 + \mathbf{F}\sigma^2] = [\mathbf{L}\Delta_A^2 + \mathbf{F}]\sigma^2 ,$$

where $\mathbf{L}$ and $\mathbf{F}$ are matrices of known constants. Then $\mathbf{Z}_A'(\mathbf{L}\Delta_A^2 + \mathbf{F})^{-1}\mathbf{Z}_A/\sigma^2$ has a χ^2-distribution and is independent of Q.

When testing the hypothesis $\Delta_A \leqq \Delta_0$ against $\Delta_A > \Delta_0$ we reject when $K(\Delta_0)$ is larger than the upper α-quantile, $f_{1-\alpha}$, of the corresponding F-distribution, where

$$K(\Delta_0) = \mathbf{Z}_A'(\mathbf{L}\Delta_0 + \mathbf{F})^{-1}\mathbf{Z}_A(n - (rs - p))/Q(r-1) .$$

It should be noted that this test is not the same as the test given in Section 2b.

If $n_{ij} = m$ for all nonempty cells it is possible to test hypotheses concerning σ_A^2/σ^2 and σ_B^2/σ^2 without assuming $\sigma_{AB}^2 = 0$ because the factors of σ_{AB}^2 and σ^2 are proportional matrices in (5.3).

The tests suggested in this section are the same as the tests suggested by Spjøtvoll (1968).

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