

SHARP ADAPTIVE ESTIMATION OF LINEAR FUNCTIONALS

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We consider estimation of a linear functional $T(f)$ where f is an unknown function observed in Gaussian white noise. We find asymptotically sharp adaptive estimators on various scales of smoothness classes in multidimensional situations. The results allow evaluating explicitly the effect of dimension and treating general scales of classes. Furthermore, we establish a connection between sharp adaptation and optimal recovery. Namely, we propose a scheme that reduces the construction of sharp adaptive estimators on a scale of functional classes to a solution of the corresponding optimization problem.

1. Introduction. Adaptation is now commonly considered a crucial element of curve estimation procedures. The literature on adaptive estimation suggests various methods, starting from the classical cross-validation or C_p criterion and ending with more recent techniques such as wavelet shrinkage or the method of Lepski. A more complete review of the existing approaches and further references can be found in Donoho, Johnstone, Kerkycharian and Picard (1995), Jones, Marron and Sheather (1996), Tsybakov (1998), Lepski and Spokoiny (1997), Härdle, Kerkycharian, Picard and Tsybakov (1998), Tsybakov (1998), Barron, Birgé and Massart (1999), Nemirovski (2000).

How to choose a method of adaptation which is optimal in a certain sense? Comparing the rates of convergence does not suffice for this purpose. In fact, it is proved in the literature that most of the adaptive estimates attain optimal rates (exactly or up to a logarithmic factor), and thus the rate criterion does not allow distinguishing between them. This suggests studying exact asymptotics of the estimation error.

Let $f: \mathbf{R}^d \rightarrow \mathbf{R}$ be the unknown function to be estimated. Intuitively, the aim of adaptation would be to select the estimator which has the smallest risk among all estimators for every f . Unfortunately, this is not possible. We have either to restrict the class of estimators, considering, for example, kernel, spline or orthogonal series estimators, and to mimic the best estimator in this class for fixed f , or to restrict the class of functions f (usually, assuming that f has some smoothness which is unknown) and to adapt among all estimators, but in a minimax sense. Restricting the class of estimators disagrees with our initial wish to seek optimality among all estimators. To satisfy it, the approach starting from smoothness classes of f and using a minimax seems more relevant. Note that, for curve estimation problems, it is often not a big

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limitation to assume that f belongs to some class of functions \mathcal{F}_ν , where ν is an *unknown* smoothness parameter.

There exist several results on exact asymptotics in minimax adaptation: Efroimovich and Pinsker (1984), Gobubev and Nussbaum (1990), Nemirovski (2000), Cavalier and Tsybakov (2000) (estimation of f in the L_2 -norm), Lepski (1992b), Tsybakov (1998) (estimation in sup-norm), Lepski and Spokoiny (1997), Tsybakov (1998), Lepski and Levit (1998) (estimation at a fixed point). These papers consider the one-dimensional case ($d = 1$). Recently some first results on multidimensional exact constants appeared: Lepski and Levit (1999) study the estimation of analytical functions in d dimensions and Efroimovich (2000) extends the L_2 -results of Efroimovich and Pinsker (1984) to the multivariate case.

We call the collection $\mathcal{F} = \{\mathcal{F}_\nu\}_{\nu \in B}$, where B is a given set of indices ν , the *scale of classes*. A typical form of \mathcal{F}_ν is

$$(1) \quad \mathcal{F}_\nu = \mathcal{F}_{s,L} = \{f: \mathbf{R}^d \rightarrow \mathbf{R} \mid \rho_s(f) \leq L\}$$

where $\nu = (s, L)$, $\rho_s(\cdot)$ is a given functional, usually a seminorm (for example, the Hölder or Sobolev seminorm), $s > 0$ is a smoothness parameter (for example, the number of derivatives) and $L > 0$ is the radius of the ball \mathcal{F}_ν .

In this paper we consider the estimation of f at a fixed point of \mathbf{R}^d , or, more generally, the estimation of some linear functional $T(f)$ with values in \mathbf{R} . Ibragimov and Hasminskii (1981, 1984), Stone (1980), Sacks and Ylvisaker (1981), Donoho and Liu (1991), Donoho and Low (1992), Donoho (1994b) obtained optimal rates of convergence and linear minimax estimates in this problem for various examples of seminorms $\rho_s(\cdot)$ and classes \mathcal{F}_ν . In particular, as noticed by Donoho and Low (1992), the optimal rates can be expressed in terms of renormalization exponents related to the functionals ρ_s and T . It is shown in these papers that the optimal rate has the form ε^κ [where $\kappa = \kappa(\rho_s, T) > 0$ is an exponent depending on ρ_s and T] if the observations $Y_\varepsilon(t)$ follow the *Gaussian white noise model*,

$$(2) \quad dY_\varepsilon(t) = f(t) dt + \varepsilon dW(t), \quad t \in \mathbf{R}^d,$$

where W is the standard Brownian sheet in \mathbf{R}^d and $0 < \varepsilon < 1$ is a small parameter. The use of the Gaussian white noise model has recently become standard in the literature on nonparametric estimation: it approximates asymptotically (in the sense of convergence of experiments) some common models with discrete observations, such as nonparametric regression or density estimation [Brown and Low (1996a), Nussbaum (1996)]. In such an approximation $\varepsilon \sim 1/\sqrt{n}$ where n is the number of discrete observations. To our knowledge, the equivalence results are now available only in dimension $d = 1$. Also, the equivalence is valid only for large enough smoothness s . Nevertheless, this does not restrict extension of our results to other nonparametric models, since this can be done directly, without equivalence considerations. For example, results for density estimation similar to ours and obtained by a direct method are actually available [see Butucea (2001a)].

For an estimator T_ε based on the observation $Y_\varepsilon(t)$ consider the maximal risk

$$\mathcal{R}_{\varepsilon, \nu}(T_\varepsilon) = \sup_{f \in \mathcal{F}_\nu} E_f(|T_\varepsilon - T(f)|^p),$$

where $p > 0$ and E_f denotes the expectation w.r.t. the distribution of the observations when the underlying function is f . For the adaptive setup, ν is unknown, and the minimax approach consists in looking for estimators T_ε^* such that the supremum of the normalized risk $\sup_{\nu \in B} \varphi_{\varepsilon, \nu}^{-p} \mathcal{R}_{\varepsilon, \nu}(T_\varepsilon^*)$ is as small as possible, where $\varphi_{\varepsilon, \nu}$ is the rate of convergence. As shown by Lepski (1990, 1992a), Efromovich and Low (1994), Brown and Low (1996b), the last expression does not tend to 0 asymptotically as $\varepsilon \rightarrow 0$ if $\varphi_{\varepsilon, \nu}$ equals the optimal rate ε^κ . The correct rate for adaptation deteriorates to $\varphi_{\varepsilon, \nu} = (\varepsilon \sqrt{\log(1/\varepsilon)})^\kappa$; that is the best we can guarantee in terms of the rate is

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\nu \in B} \varphi_{\varepsilon, \nu}^{-p} \mathcal{R}_{\varepsilon, \nu}(T_\varepsilon^*) < \infty,$$

except for the upper boundary of B where the normalization ε^κ can be maintained [see Lepski (1992a), Tsybakov (1998)]. The results of Lepski (1990, 1992a), Brown and Low (1996b) are proved for the case where \mathcal{F}_ν are Hölder classes of functions, $T(f) = f(0)$ and $d = 1$. Efromovich and Low (1994) considered more general linear functionals and Tsybakov (1998) proved the result for the Sobolev classes and $d = 1$. Following the scheme of Lepski (1992a) or of Tsybakov (1998), it is not difficult to show that in the general situation with $d \geq 1$ and Hölder or Sobolev classes of functions the correct asymptotic normalization $\varphi_{\varepsilon, \nu}$ in the risk $\sup_{\nu \in B} \varphi_{\varepsilon, \nu}^{-p} \mathcal{R}_{\varepsilon, \nu}(T_\varepsilon^*)$ is of the form $(\varepsilon \sqrt{\log(1/\varepsilon)})^\kappa$, up to a boundary effect.

Here we do not go into details of these results about the rates. For a more comprehensive discussion, the definition of adaptive rate and the lower bounds, see Tsybakov (1998). Below we assume as given the normalization $\varphi_{\varepsilon, \nu} = (\varepsilon \sqrt{\log(1/\varepsilon)})^\kappa$ for the risk, where κ is the exponent of the optimal rate that is calculated as in Ibragimov and Hasminskii (1984), Donoho and Liu (1991). Our aim is to find the exact asymptotical constant c_ν in the expression for the minimax “adaptive” risk and to construct the adaptive estimator T_ε^* that attains this constant.

Such an estimator T_ε^* will be called sharp adaptive on the scale of classes $\{\mathcal{F}_\nu\}$. Examples of sharp adaptive estimators of functionals are known for the case where $d = 1$, $T(f) = f(0)$. The first example has been given by Lepski and Spokoiny (1997) who considered the Hölder scale of classes with smoothness $0 < s \leq 2$. Tsybakov (1998) obtained sharp adaptive estimators for the Sobolev scale of classes where s takes discrete values without upper restriction on s and with fixed L . His set-up is somewhat different from the one considered here and his results cannot be formally deduced from ours. In the present paper we assume that s belongs to a bounded interval: this allows, in particular, a unified treatment of the Hölder and Sobolev cases (in the Hölder case, if s is large, we cannot guarantee the necessary assumptions and the optimal

solutions for the kernels are not explicitly known). Also, the Gaussian model here differs from those in Lepski and Spokoiny (1997) and Tsybakov (1998): we consider the observations on \mathbf{R}^d , while in those papers the observations are on $[0, 1]$. From the mathematical point of view, the difference is not significant between considering functions on \mathbf{R} and periodic functions on $[0, 1]$ [as in Tsybakov (1998)] or neglecting the boundary effects on $[0, 1]$ [as in Lepski and Spokoiny (1997)]. However, working with the infinite interval of observations leads to more transparent notation. In practice we always have finite intervals, but if they are large enough they can be approximately considered as infinite. This is commonly done in the literature on signal processing [(cf.) a discussion in Donoho and Low (1992)]. Lepski and Levit (1998, 1999) considered the Gaussian white noise model on the infinite interval and obtained sharp adaptation results for the case where \mathcal{F}_v are classes of analytic or super-smooth functions.

Here we find sharp adaptive estimators of linear functionals for the general problem of dimension $d \geq 1$, classes (1) with some general functional $\rho_s(\cdot)$ and both s and L unknown, and a functional T satisfying some assumptions that are relevant for the nonregular case where the “ \sqrt{n} -consistent” estimation is not possible. The main examples are $T(f) = f(0)$ or $T(f)$ being a partial derivative of f at a point.

We consider a general framework that makes transparent the connection between sharp adaptation and optimal estimation of linear functionals (optimal recovery). An explicit scheme is proposed that reduces the construction of sharp adaptive estimators to a solution of the corresponding optimal recovery (OR) problem. Donoho (1994a, b) was the first to point out a connection between OR and nonparametric statistics. He showed that the OR argument can be used to get exact asymptotics of linear minimax risks in estimation problems. More recently Lepski and Tsybakov (2000) proved that by means of OR one can construct asymptotically sharp minimax nonparametric tests. The present paper describes one more field of application of OR: construction of sharp adaptive estimators. Our conclusion can be formulated as follows: it is possible to construct sharp adaptive estimators of linear functionals by action of the Lepski-type selection procedure (with properly chosen thresholds) over families of linear estimates with optimal recovery kernels.

2. Assumptions and preliminaries. Let $s > 0$ and let ρ_s be a functional defined on a subset \mathcal{D} of the space of all functions $f: \mathbf{R}^d \rightarrow \mathbf{R}$. Suppose in the sequel that $\mathcal{D} = \{f: \rho_s(f) < \infty\}$. We assume the following conditions on the functional ρ_s .

ASSUMPTION 1.

- (i) *The functional ρ_s is convex, nonnegative and symmetric; that is, $\rho_s(f) = \rho_s(-f)$, and $\rho_s(f) \neq 0$,*
- (ii) *$\rho_s(af(b\cdot)) = ab^s \rho_s(f(\cdot))$ for any $a \geq 0$, $b > 0$, $f \in \mathcal{D}$.*

Furthermore, we assume that the functional T satisfies the following conditions.

ASSUMPTION 2.

- (i) T is a linear functional on \mathcal{D} .
- (ii) There exists $r \geq 0$ such that $T(af(b \cdot)) = ab^r T(f(\cdot))$ for $a \geq 0$, $b > 0$, $f \in \mathcal{D}$.
- (iii) The modulus of continuity is well defined:

$$\omega_{s,L}(\varepsilon) \stackrel{\text{def}}{=} \sup\{T(f): \|f\|_2 \leq \varepsilon, \rho_s(f) \leq L\} < \infty$$

for all $s > r$, $L > 0$, $\varepsilon > 0$ where $\|\cdot\|_2$ is the $L_2(\mathbf{R}^d)$ -norm.

Assumptions 1(ii) and 2(ii) are usual renormalization assumptions [see Donoho and Low (1992) for discussion and examples].

As described by Donoho (1994a, b), Donoho and Liu (1991), Donoho and Low (1992), the minimax estimation of functionals from random noisy data is closely related to the deterministic problem of minimax optimal recovery that considers estimation from observations in nonrandom noise. These papers show that, by calibrating the algorithms of optimal recovery one can construct linear minimax estimators for the statistical estimation of linear functionals and asymptotic minimax estimators for the statistical estimation of functions with supremum loss. We refer to these papers for a detailed discussion. Here we show that by calibrating the algorithms of optimal recovery, one can construct a family of linear estimators such that choosing one of these linear estimators with a certain data-based decision rule will result in a sharp adaptive estimation procedure. Next we give a brief summary of the results on optimal recovery that will be used below.

By the generalized Weierstrass theorem, under the Assumptions 1 and 2 there exists a function $g_{s,L,\varepsilon}$ which attains the supremum of the modulus of continuity; that is,

$$(3) \quad T(g_{s,L,\varepsilon}) = \omega_{s,L}(\varepsilon)$$

[cf. Gabushin (1970), Micchelli and Rivlin (1977), Arestov (1989)]. These authors show that the extremal problem

$$(4) \quad \max T(f) \text{ subject to } \begin{cases} \|f\|_2 \leq 1, \\ \rho_s(f) \leq 1, \end{cases}$$

is related to the optimal recovery problem: find a function K_s such that

$$(5) \quad \sup_{\rho_s(f) \leq 1, \|f-g\|_2 \leq 1} \left| \int K_s g - T(f) \right| = \inf_K \sup_{\rho_s(f) \leq 1, \|f-g\|_2 \leq 1} \left| \int K g - T(f) \right| \stackrel{\text{def}}{=} E(s).$$

In particular, Theorems 6, 8 and 11 in Micchelli and Rivlin (1977) and Theorems 2.4 and 2.5 in Arestov (1989) show that under Assumptions 1 and 2 there

exists $K_s \in L_2(\mathbf{R}^d)$ that satisfies (5) and, moreover,

$$(6) \quad E(s) = T(g_{s,1,1}) = \sup_{\rho_s(f) \leq 1} \left| \int K_s f - T(f) \right| + \|K_s\|_2.$$

The property (6) plays crucial role in our argument. In the sequel K_s denotes the optimal recovery kernel, that is, the function in $L_2(\mathbf{R}^d)$ satisfying (6). Note that if $T(f) = f(0)$ and $0 < \int g_{s,1,1} < \infty$, the kernel K_s has a particular form $K_s = K_s^0$ where

$$K_s^0 = g_{s,1,1} / \int g_{s,1,1}.$$

This can be shown in a simple way [see, e.g., Lemma 1 of Lepski and Tsybakov (2000)]. For general functionals T , a similar condition usually holds:

$$(7) \quad K_s = C g_{s,1,1},$$

where the constant $C > 0$ depends only on s, r, d . In fact, Assumptions 1 and 2 and the renormalization argument entail that $\omega_{s,L}(\varepsilon)$ is of power law form: $\omega_{s,L}(\varepsilon) = \omega_{s,1}(1)L^{2(r+d)/(2s+d)}\varepsilon^\kappa$ where

$$\kappa = \kappa(s) = 2(s-r)/(2s+d).$$

Hence, as in Donoho (1994b), Donoho and Liu [(1991), Section 4.3] and Donoho and Low [(1992), Section 8], one gets (7).

We assume that the observations $Y_\varepsilon(t)$, $t \in \mathbf{R}^d$, are obtained from the Gaussian white noise model (2). As follows from Donoho and Liu (1991), Donoho and Low (1992), Donoho (1994b), the linear minimax estimator of $T(f)$ under the mean squared risk on the class of functions $\mathcal{F}_{s,L}$ is the kernel estimator with properly rescaled optimal recovery kernel K_s and the bandwidth

$$(8) \quad h_l(s, L, \varepsilon) = (\varepsilon/L)^{2/(2s+d)}.$$

The rate of convergence of the linear minimax estimator is respectively ε^κ .

Our definition of sharp adaptive estimator starts from the family of kernel estimators with optimal kernels K_s , though with the bandwidths different from those of the linear minimax case.

We suppose that the following continuity condition on the kernel holds.

ASSUMPTION 3. *The optimal recovery kernel K_s satisfies (7) and $\|K_{s'} - K_s\|_2 \rightarrow 0$, as $s' \rightarrow s$, for any $s > r$.*

Let the scale of classes $\{\mathcal{F}_{s,L}\}_{(s,L) \in B}$ be defined by (1) with

$$B = \{(s, L): s_* \leq s \leq s^*, L_* \leq L \leq L^*\},$$

where $r < s_* < s^* < \infty$, $0 < L_* < L^* < \infty$. This means that we are certain that $f \in \mathcal{F}_{s,L}$ for some $L \in [L_*, L^*]$ and $s \in [s_*, s^*]$. The values r, s_*, s^* are supposed to be known but L_* and L^* can be unknown: we do not need L_*, L^* for the construction of our sharp adaptive estimators.

Define the bias constant

$$b_{s,s'} = \sup_{\rho_s(f) \leq 1} \left| \int K_{s'} f - T(f) \right|.$$

The following boundedness and continuity condition on the bias constant will be assumed.

ASSUMPTION 4.

(i) For any $r < r' < s^*$ there exists a positive constant $b_{\max}(r', s^*)$ such that

$$b_{s,s'} \leq b_{\max}(r', s^*) \quad \forall r' \leq s' \leq s \leq s^*.$$

(ii)

$$\limsup_{\delta \rightarrow 0} \sup_{s, s' \in [s_*, s^*]: |s-s'| \leq \delta} \frac{b_{s,s'}}{b_{s,s}} \leq 1.$$

Clearly, the main interest of our construction is in the case where the solution $g_{s,L,\varepsilon}$ and the kernel K_s can be expressed explicitly. In this case Assumptions 3 and 4 can be checked directly; see the examples in Section 5.

3. Results. For any $h > 0$ denote $K_{s,h}(\cdot) = h^{-d-r} K_s(\cdot/h)$ where K_s is defined in Section 2. Consider kernel estimators of the form $\int K_{s,h}(t) dY_\varepsilon(t)$ where h is a suitably chosen bandwidth. Denote

$$(9) \quad h(s, \varepsilon) = \varepsilon^{2/(2s+d)},$$

and introduce the “effective noise level under adaptation”

$$\tilde{\varepsilon} = \tilde{\varepsilon}(s) = \varepsilon d_\varepsilon(s) = (\lambda(s) \varepsilon^2 \log \varepsilon^{-1})^{1/2},$$

where

$$d_\varepsilon(s) = (\lambda(s) \log \varepsilon^{-1})^{1/2},$$

$$\lambda(s) = 2p(2r+d) \left(\frac{1}{2s+d} - \frac{1}{2s^*+d} \right).$$

We use the bandwidth computed at the effective noise level,

$$h(s, \tilde{\varepsilon}(s)) = (\lambda(s) \varepsilon^2 \log \varepsilon^{-1})^{1/(2s+d)}.$$

This bandwidth is by a logarithmic factor larger in order than the bandwidth (8) of the linear minimax estimate.

We will introduce a sufficiently fine grid on $[s_*, s^*]$ and a statistic \hat{s} having values on this grid. To each point of the grid we assign a linear kernel estimator. The statistic \hat{s} will choose one of these estimators. Namely, we consider the grid

$$S = \{s_1, \dots, s_m\},$$

where

$$r' < s_1 < \dots < s_m < s^*$$

with a fixed r' satisfying $r < r' \leq s_*$, and we assume that there exist $k_2 > k_1 > 0$ and $\gamma_1 \geq \gamma > 1$ such that

$$(10) \quad k_1(\log \varepsilon^{-1})^{-\gamma_1} \leq s_{i+1} - s_i \leq k_2(\log \varepsilon^{-1})^{-\gamma}, \quad i = 0, \dots, m - 1,$$

where $s_0 = r'$, $s_m - s^* = o(1)$, as $\varepsilon \rightarrow 0$. Note that the same grid S can be used for different values s_* , provided $s_* > r'$. In this sense the exact knowledge of s_* is not required for the construction of the estimator.

For any $s \in S$ introduce the linear kernel estimator of the functional $T(f)$:

$$T_{s, \varepsilon} = \int K_{s, h(s, \tilde{\varepsilon}(s))}(t) dY_\varepsilon(t).$$

The sharp adaptive estimator has the form $T_{\hat{s}, \varepsilon}$ where \hat{s} is a suitably chosen statistic. To define \hat{s} we follow the approach used in different statistical models starting from the paper of Lepski (1990). That is, the statistic \hat{s} is defined as the largest of those s -values in the grid for which the estimator $T_{s, \varepsilon}$ does not differ significantly from the estimators corresponding to the smaller s -values. We choose

$$\hat{s} = \max\{s \in S: |T_{s, \varepsilon} - T_{s', \varepsilon}| \leq \eta(s') \text{ for all } s' \in S, s' \leq s\}$$

with the threshold

$$\eta(s) = d_\varepsilon(s) \sigma_s = \tilde{\varepsilon}(s)^{2(s-r)/(2s+d)} \|K_s\|_2,$$

where σ_s is the standard deviation of $T_{s, \varepsilon}$,

$$(11) \quad \sigma_s = \varepsilon \|K_{s, h(s, \tilde{\varepsilon}(s))}\|_2 = \varepsilon h^{-r-d/2}(s, \tilde{\varepsilon}(s)) \|K_s\|_2.$$

Finally, define the estimator of $T(f)$ as

$$(12) \quad T_\varepsilon^* = T_{\hat{s}, \varepsilon}.$$

The next theorem is the main result of this paper. It states that the estimator T_ε^* is sharp adaptive and that the exact asymptotical constant c_ν for the minimax adaptive risk is given by the expression

$$(13) \quad c_\nu = c_{\nu, s^*} = L^{(2r+d)/(2s+d)} T(g_{s, 1, 1}) \times \left[2p(2r+d) \left(\frac{1}{2s+d} - \frac{1}{2s^*+d} \right) \right]^{(s-r)/(2s+d)}.$$

To formulate the theorem we introduce, for any $\psi > 0$, $p > 0$, the normalized risk:

$$\mathcal{R}_{\varepsilon, \nu}(T_\varepsilon, \psi) = \sup_{f \in \mathcal{F}_\nu} E_f(\psi^{-p} |T_\varepsilon - T(f)|^p)$$

and denote

$$(14) \quad \psi_\nu = c_\nu \varphi_{\varepsilon, \nu} = c_\nu (\varepsilon^2 \log \varepsilon^{-1})^{(s-r)/(2s+d)}.$$

[Recall that we defined $\varphi_{\varepsilon, \nu} = (\varepsilon \sqrt{\log \varepsilon^{-1}})^\kappa$ and $\kappa = 2(s-r)/(2s+d)$.] The normalizing factor ψ_ν may be expressed as the value of the modulus of continuity at the effective noise level $\tilde{\varepsilon}$. Indeed, by standard renormalization argument, $g_{s, L, \tilde{\varepsilon}}(\cdot) = a g_{s, 1, 1}(b \cdot)$ where $a = L b^{-s}$ and $b = (L/\tilde{\varepsilon})^{2/(2s+d)}$. Thus,

$$\begin{aligned} \omega_{s, L}(\tilde{\varepsilon}) &= T(g_{s, L, \tilde{\varepsilon}}) = a b^r T(g_{s, 1, 1}) \\ (15) \quad &= \tilde{\varepsilon}^{2(s-r)/(2s+d)} L^{(2r+d)/(2s+d)} T(g_{s, 1, 1}) \\ &= c_\nu (\varepsilon^2 \log \varepsilon^{-1})^{(s-r)/(2s+d)} = \psi_\nu. \end{aligned}$$

THEOREM 1. *Let Assumptions 1–4 hold, let $p > 0$ and denote $B_q = [s_*, q] \times [L_*, L^*]$, where $s_* < q < s^*$. Then the estimator T_ε^* defined in (12) is sharp adaptive,*

$$(16) \quad \sup_{s_* < q < s^*} \limsup_{\varepsilon \rightarrow 0} \sup_{\nu \in B_q} \mathcal{R}_{\varepsilon, \nu}(T_\varepsilon^*, \psi_\nu) \leq 1$$

and

$$(17) \quad \sup_{s_* < q < s^*} \liminf_{\varepsilon \rightarrow 0} \inf_{T_\varepsilon} \sup_{\nu \in B_q} \mathcal{R}_{\varepsilon, \nu}(T_\varepsilon, \psi_\nu) \geq 1.$$

Here \inf_{T_ε} denotes the infimum over all estimators.

Proofs of Theorem 1 and of the further results are given in Section 7.

REMARK 1. Since s and L are not fixed, it is more precise to call $c_\nu = c_{\nu, s^*}$ the “optimal normalizing function” rather than the optimal constant. An insight on the structure of this function has been first given by Lepski (1992a), Theorem 8, where he shows that

$$c_{\nu, s^*} \asymp \left(\frac{1}{2s+1} - \frac{1}{2s^*+1} \right)^{(s-r)/(2s+1)}$$

(he considers the case $d = 1$ and the Hölder scale of classes, but it is not hard to extend his result to our multivariate setting). Here we specify the exact value of c_{ν, s^*} which contains, of course, the same factor, but also turns out to contain another factor expressed in terms of optimal recovery solutions [cf. (13)].

For $s = s^*$ we have $c_{\nu, s^*} = 0$. Thus $\mathcal{R}_{\varepsilon, \nu}(T_\varepsilon^*, \psi_\nu)$ is not defined for the single point $s = s^*$. This explains why the set B_q appears in place of B in Theorem 1: indeed, the difference between (16) and (17) and analogous expressions with $\lim_{\varepsilon \rightarrow 0} \sup_{\nu \in B}$ is really minor but we have to use the form (16) and (17) in order to exclude the point $s = s^*$. It is possible to construct an adaptive estimator that has the property given in Theorem 1 and attains for $s = s^*$ a faster rate, without the logarithmic factor: $\varepsilon^{2(s^*-r)/(2s^*+d)}$ [cf. Lepski (1992a), Theorem 8]. Such an estimator is defined similarly to T_ε^* , but with the enlarged grid

$\{s_1, \dots, s_m, s^*\}$ and with $T_{s, \varepsilon}$ replaced by

$$\tilde{T}_{s, \varepsilon} = \begin{cases} \int K_{s, h(s, \tilde{\varepsilon}(s))}(t) dY_\varepsilon(t), & \text{for } s = s_1, \dots, s_m, \\ \int K_{s, h(s, \varepsilon)}(t) dY_\varepsilon(t), & \text{for } s = s^*. \end{cases}$$

Also one should impose an assumption on the rate of approximation of s^* by s_m as $\varepsilon \rightarrow 0$. The effect of improving the rate at a single boundary point s^* is discussed for example by Lepski (1992a) and Tsybakov (1998). This issue is of minor importance, although involving more technical details, and we do not pursue it here.

The estimator T_ε^* depends on the value s^* which is not always available. One can propose a suboptimal modification of T_ε^* that does not depend on s^* . It is obtained by putting formally $s^* = \infty$ in the definition for T_ε^* . In other words, we replace in all the formulas $\lambda(s)$ by

$$\tilde{\lambda}(s) = \frac{2p(2r + d)}{2s + d},$$

and we set

$$\tilde{c}_\nu = L^{(2r+d)/(2s+d)} T(g_{s, 1, 1}) \left(\frac{2p(r + d)}{2s + d} \right)^{(s-r)/(2s+d)},$$

$\tilde{\psi}_\nu = \tilde{c}_\nu (\varepsilon^2 \log \varepsilon^{-1})^{(s-r)/(2s+d)}$. We also assume that the grid $S = \{s_1, \dots, s_m\}$ is extended to the right beyond s^* :

$$r' < s_1 < \dots < s_m = s_{\max},$$

where $s_{\max} > s^*$ and s_i satisfy (10). Here s_{\max} is an arbitrary large fixed number.

Let \tilde{T}_ε be the estimate defined as T_ε^* with the change of $\lambda(s)$ to $\tilde{\lambda}(s)$ and with the extended grid S as above. Note that \tilde{T}_ε is completely data-driven: the dependence on s^* disappears.

THEOREM 2. *Let Assumptions 1–4 hold and let $p > 0$. Then*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\nu \in B} \mathcal{R}_{\varepsilon, \nu}(\tilde{T}_\varepsilon, \tilde{\psi}_\nu) \leq 1.$$

Proof of Theorem 2 is omitted: it follows the same lines as that of Theorem 1, with a minor modification concerning the extension of the grid beyond s^* (this is done as in the proof of Theorem 3 below). Comparing with that proof, we observe that Theorem 2 remains valid if one takes $s_{\max} \rightarrow \infty$ as $\varepsilon \rightarrow 0$, but not faster than $\log(1/\varepsilon)$.

Note that in view of Theorems 1 and 2 the asymptotical risk of the estimator \tilde{T}_ε can be larger than that of T_ε^* on any set \mathcal{F}_s (with $s_* \leq s < s^*$) at most by a factor of

$$\frac{\tilde{c}_\nu}{c_\nu} = \left(\frac{2s^* + d}{2(s^* - s)} \right)^{(s-r)/(2s+d)}.$$

This factor is particularly close to 1 if s is fixed and s^* gets large. Thus, for s^* large enough and fixed s the behavior of \tilde{T}_ε and T_ε^* is similar, but \tilde{T}_ε has an advantage since it does not depend on s^* . Although, for s close to s^* the estimator \tilde{T}_ε is much less efficient than T_ε^* .

A useful modification of Theorem 1 consists in constructing a grid on the values of smoothing parameter h and not on the s -values as above. To get correspondence with the s -grid satisfying (10), the h -grid should have a geometrical character. This means that an “economic” choice of h (among a logarithmic number of possible candidates) is in fact sufficient to attain sharp asymptotic adaptivity: increasing the cardinality of the grid or passing to the choice of h in a continuum of values complicates the procedure but does not improve the result. We set $h_0 = h(r', \tilde{\varepsilon}(r'))$ and define the sequence $\{h_i\}$ by the recursion

$$(18) \quad h_{i+1} = h_i(1 + \alpha(h_i))$$

where $\alpha(h_i)$ is a slowly varying function of h_i . It will be sufficient to consider

$$(19) \quad \alpha(h) = (\log(1/h))^{-\gamma_0},$$

where $\gamma_0 > 0$ is a constant. Given the grid

$$\mathcal{H} = \{h_1, h_2, \dots, h_m\},$$

with $m = \max\{i: h_i < h_{\max}\}$, $h_{\max} \geq \varepsilon^{2/(2s^*+d)}$, consider the bandwidth

$$\hat{h} = \max\{h \in \mathcal{H}: |T_\varepsilon(h) - T_\varepsilon(h')| \leq \eta_{h'}, \text{ for all } h' \leq h, h' \in \mathcal{H}\},$$

where $T_\varepsilon(h) = \int K_{s(h), h}(t) dY_\varepsilon(t)$, and

$$\eta_h = \varepsilon h^{-r-d/2} \sqrt{p(2r+d) \log \frac{h_{\max}}{h}} \|K_{s(h)}\|_2$$

with

$$s(h) = \left(\frac{\log \varepsilon}{\log h} - \frac{d}{2} \right).$$

Define

$$(20) \quad T_\varepsilon^* = T_\varepsilon(\hat{h}).$$

We now state an analogue of Theorems 1 and 2 for the estimator (20).

THEOREM 3. *Let Assumptions 1–4 be satisfied, $p > 0$, and let T_ε^* be defined in (20).*

(i) *If $h_{\max} = \varepsilon^{2/(2s^*+d)}$ then (16) holds; that is, the estimator (20) is sharp adaptive.*

(ii) *If $h_{\max} = 1$ then the estimator (20) satisfies*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\nu \in B} \mathcal{R}_{\varepsilon, \nu}(T_\varepsilon^*, \tilde{\psi}_\nu) \leq 1.$$

We observe that the result of Theorem 3(ii) is robust to the choice of h_{\max} . Inspection of the proof shows that $h_{\max} = 1$ can be replaced by $h_{\max} = h^*$ for any positive h^* , and other choices of $h_{\max} > \varepsilon^{2/(2s^*+d)}$ are acceptable as well. We also note that a special case of the above construction is given by Lepski and Spokoiny (1997). They considered a grid on h -values defined in (18), (19) with $\gamma_0 = 1/4$ and for the particular case of Hölder scale of classes with $0 < s \leq 2$, $r = 0$, $d = 1$. Their Theorem 3.3 in the proper form follows from Theorem 3 modulo the fact that their procedure is slightly different: it includes an additional factor $(1 + \alpha(h))$ in the threshold η_h .

REMARK 2 (Pointwise and spatial adaptivity). Application of our adaptive procedure to the particular functionals $T(f) = f(x)$ for each point x of an interval where f is defined gives an estimator of f on this interval. Therefore, a special case of our results is sharp pointwise adaptivity for estimation of a whole function f on an interval. Arguing as in Lepski, Mammen, and Spokoiny (1997), Goldenshluger and Nemirovskii (1997) one can deduce spatial adaptivity of such an estimator of a function from its pointwise adaptivity.

REMARK 3 (Data-dependent kernels). One of the most studied topics in nonparametric curve estimation is the choice of the smoothing parameter in kernel estimation. We do not restrict ourselves to kernel estimators but find asymptotically sharp adaptive estimator among all estimators. Observe that nevertheless this estimator turns out to be a kernel one, with smoothing parameter selected in a data-dependent way. What is more, our results suggest that, to attain optimality, not only the smoothing parameter but also the kernel function of this estimator should be chosen in a data-dependent way.

4. Other statistical models. It is possible to modify the proposed estimator for other types of observations than the Gaussian white noise model. Consider some examples.

Density estimation. Let X_1, \dots, X_n be i.i.d. observations with density f on \mathbf{R}^d and consider estimation of $T(f) = f(x)$ for some $x \in \mathbf{R}^d$. Construct a preliminary estimator $\hat{f}_n(x) > 0$ for $f(x)$ and consider the family of kernel estimators

$$T_{s,n} = \frac{1}{n} \sum_{i=1}^n K_{s, h(s, \tilde{\varepsilon}(s))}(X_i - x),$$

where K_s , h and $\tilde{\varepsilon}(s)$ are defined as above where one substitutes $\varepsilon = (\hat{f}_n(x)/n)^{1/2}$. The adaptive procedure is defined as in Section 3; it is a data-driven selection of an appropriate member of this family. Again, in the definition of the threshold $\eta(s)$ of the adaptive procedure one should set $\varepsilon = (\hat{f}_n(x)/n)^{1/2}$. Sharp adaptation properties of this adaptive density estimator on the Sobolev scale of classes are proved by Butucea (2001a). Butucea (2001b) presents a large simulation study showing a successful behavior of the adaptive procedure for different densities f . In particular, the procedure is quite robust to the choice of the preliminary estimator \hat{f}_n .

Nonparametric regression. Consider the nonparametric regression model

$$Y_i = f(X_i) + \xi_i, \quad i = 1, \dots, n,$$

where $X_i = i/n - 1/2$ are equispaced regressors on the interval $[-1/2, 1/2]$, f is an unknown regression function and ξ_i are i.i.d. random variables such that $E(\xi_i) = 0$, $E(\xi_i^2) = \sigma^2 > 0$, satisfying some additional moment conditions. Let again $T(f) = f(x)$. An adaptive procedure analogous to ours can be suggested similar to the density case. Construct a preliminary estimator $\hat{\sigma}$ of σ , and consider the family of linear kernel estimators

$$T_{s,n} = \frac{1}{n} \sum_{i=1}^n Y_i K_{s,h(s,\tilde{\varepsilon}(s))}(X_i - x),$$

where K_s , h and $\tilde{\varepsilon}$ are defined as in Section 3, with $\varepsilon = \hat{\sigma}/\sqrt{n}$. Finally, apply the thresholding procedures of Section 3 to get \hat{s} (or \hat{h}), with the above definition of ε used to compute the threshold $\eta(s)$. The adaptive estimator is then $T_{\hat{s},n}$. We conjecture that $T_{\hat{s},n}$ has sharp optimality properties, as in Theorems 1–3.

5. Examples. In this section we give examples of classes \mathcal{F}_ν (respectively, functionals ρ_s) that satisfy the assumptions of Section 2 and allow explicit construction of sharp adaptive estimators.

EXAMPLE 1 (Sobolev classes). Let $\beta > d/2$ and denote $s = \beta - d/2$. Define the Sobolev seminorm ρ_s by

$$\rho_s^2(f) = (2\pi)^d \int_{\mathbf{R}^d} \|\omega\|^{2\beta} |\hat{f}(\omega)|^2 d\omega,$$

where the Fourier transform of a function $f \in L_1(\mathbf{R}^d)$ is

$$\hat{f}(\omega) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} f(x) \exp(-ix^T \omega) dx$$

and $\|\cdot\|$ denotes the Euclidean norm in \mathbf{R}^d . Note that if β is an integer,

$$\rho_s^2(f) = \sum_{|\alpha|=\beta} \int_{\mathbf{R}^d} |f^{(\alpha)}|^2,$$

where

$$f^{(\alpha)}(x) = i^{|\alpha|} \int_{\mathbf{R}^d} \omega^\alpha \hat{f}(\omega) \exp(ix^T \omega) d\omega$$

and $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_d$, $\omega^\alpha = \omega_1^{\alpha_1} \dots \omega_d^{\alpha_d}$, for $\omega = (\omega_1, \dots, \omega_d)$.

Consider the estimation of the functional $T(f) = f^{(\alpha_0)}(0)$ where α_0 is a multi-index, $|\alpha_0| = r$ and $r \geq 0$ is an integer, $s > r$. The kernel K_s is then obtained as a renormalized version of the basic kernel

$$(21) \quad \tilde{K}_s(x) = (2\pi)^{-d} i^r \int_{\mathbf{R}^d} \omega^{\alpha_0} (1 + \|\omega\|^{2\beta})^{-1} \exp(ix^T \omega) d\omega.$$

Note that \tilde{K}_s is always real-valued: it is the directional derivative corresponding to the multi-index α_0 of the function whose Fourier transform is $(2\pi)^{-d} (1 + \|\omega\|^{2\beta})^{-1}$. Introduce the constant

$$C_* = \left[\frac{1}{2\beta} B\left(1 + \frac{2r + d}{2\beta}, 1 - \frac{2r + d}{2\beta}\right) (2\pi)^{-d} \int_{S_d} \xi^{2\alpha_0} d\mu(\xi) \right]^{1/2},$$

where $S_d = \{x \in \mathbf{R}^d: \|x\| = 1\}$ for $d = 2, 3, \dots$, $S_1 = [-1, 1]$, μ is the Lebesgue measure of S_d so that $\mu(S_d) = 2\pi^{d/2}/\Gamma(d/2)$, $d = 1, 2, \dots$, and $B(\cdot, \cdot)$ denotes the beta-function.

PROPOSITION 1. *Let $r \geq 0$ be an integer and $s > r$. Then for the Sobolev seminorm ρ_s and $T(f) = f^{(\alpha_0)}(0)$, $|\alpha_0| = r$, the extremal function $g_{s,1,1}$ is given by $g_{s,1,1}(x) = (-1)^r a \tilde{K}_s(bx)$ where \tilde{K}_s is defined in (21), $a = C_*^{-1} b^{-s}$ and $b = \left(\frac{2(s-r)}{2r+d}\right)^{1/(2s+d)}$. Furthermore,*

$$\begin{aligned} K_s(x) &= b^{r+d} \tilde{K}_s(bx), \\ \|K_s\|_2 &= C_* \left(\frac{2(s-r)}{2r+d}\right)^{(s+r+d)/(2s+d)}, \\ T(g_{s,1,1}) &= C_* \left(\frac{2r+d}{2(s-r)}\right)^{(s-r)/(2s+d)} \frac{2s+d}{2r+d}. \end{aligned}$$

For the one-dimensional case ($d = 1$) the extremal function $g_{s,1,1}$ was found by Taikov (1968).

The kernel \tilde{K}_s can be expressed in terms of the Bessel functions. Thus, if $r = 0$, we have

$$\tilde{K}_s(x) = (2\pi)^{-d/2} \|x\|^{1-d/2} \int_0^\infty \frac{t^{d/2}}{1 + t^{2\beta}} J_{(d-1)/2}(t\|x\|) dt,$$

where J_n is the ordinary Bessel function of order n . For $r \neq 0$ the kernel \tilde{K}_s is the corresponding directional derivative of the right-hand side of the last equality.

PROPOSITION 2. *Let ρ_s be the Sobolev seminorm, and let $T(f)$ be as in Proposition 1. Then Assumptions 1–4 are satisfied with any s_*, s^*, r' such that $r < r' \leq s_* < s^* < \infty$, $r' > (s^* - r - d)/2$.*

EXAMPLE 2 (Taylor and Hölder classes). The Taylor classes are defined by (1) with

$$(22) \quad \rho_s(f) = \sup_{x \neq 0} \|x\|^{-s} \left| f(x) - \sum_{i=0}^{\lceil s-1 \rceil} \sum_{|\alpha|=i} \frac{x^\alpha}{\alpha!} f^{(\alpha)}(0) \right|,$$

where $s > 0$, $f^{(\alpha)}(0)$ is the partial derivative corresponding to the multi-index α and $\alpha! = \alpha_1! \cdots \alpha_d!$. Consider the estimation of the functional $T(f) = f(0)$ when f is in a Taylor class. This problem was studied in a nonadaptive setting of linear minimax estimation by Legostaeva and Shiryaev (1971), Sacks and Ylvisaker (1981). The extremal function $g_{s,1,1}$ of the maximization problem (4) is given by the following proposition which is a multivariate generalization of their results. This function is a renormalized version of the basic kernel

$$(23) \quad \tilde{K}_s(x) = (1 - \|x\|^s) I(\|x\| \leq 1).$$

PROPOSITION 3. Let $r = 0$, $0 < s \leq 2$. Then, for ρ_s defined in (22) and $T(f) = f(0)$ the extremal function $g_{s,1,1}$ is given by $g_{s,1,1}(x) = a \tilde{K}_s(bx)$ where \tilde{K}_s is defined in (23), $a = b^{-s}$, and

$$b = \left(\frac{2s^2 \mu(S_d)}{(2s+d)(s+d)d} \right)^{1/(2s+d)}.$$

Furthermore,

$$\begin{aligned} K_s(x) &= \left(\frac{\mu(S_d)s}{(s+d)d} \right)^{-2s/(2s+d)} \left(\frac{2s}{2s+d} \right)^{d/(2s+d)} \tilde{K}_s(bx), \\ \|K_s\|_2 &= \frac{(s+d)d}{\mu(S_d)s} \left(\frac{2s^2 \mu(S_d)}{(2s+d)(s+d)d} \right)^{(s+d)/(2s+d)}, \\ T(g_{s,1,1}) &= \left(\frac{(2s+d)(s+d)d}{2s^2 \mu(S_d)} \right)^{s/(2s+d)}. \end{aligned}$$

For $0 < s \leq 1$ the Hölder seminorm is defined by

$$(24) \quad \rho_s(f) = \sup_{x, y \in \mathbf{R}^d, x \neq y} \|x\|^{-s} |f(x) - f(y)|$$

and the Hölder classes are defined by (1) with ρ_s as in (24).

PROPOSITION 4. The result of Proposition 3 remains valid if ρ_s is the Hölder seminorm and $0 < s \leq 1$.

PROPOSITION 5. Let $T(f) = f(0)$. Then Assumptions 1–4 are satisfied with any $0 < s_* < s^* \leq 2$ for the Taylor classes and with any $0 < s_* < s^* \leq 1$ for the Hölder classes.

The Hölder seminorm can be also defined for $s > 1$. However, explicit expressions for the solution $g_{s,1,1}$ and the kernel K_s are not generally known for $s > 1$, even in the dimension $d = 1$. This does not allow constructing sharp adaptive estimators on the Hölder scale with $s^* > 1$. Thus, for $d = 1, s > 1$ the Hölder seminorm is

$$\rho_s(f) = \sup_{x, y \in \mathbf{R}, x \neq y} \frac{|f^{(l)}(x) - f^{(l)}(y)|}{|x - y|^{s-l}},$$

where $l = \lfloor s \rfloor$. Explicit solutions of the extremal problem (4) with this seminorm are available only for $0 < s \leq 1$ and $s = 2$ [see Fuller (1961), Gabushin (1968), Korostelev (1993), Leonov (1997, 1999), Zhao (1997)].

6. Lemmas. The following lemma gives a bound for the bias of a kernel estimator.

LEMMA 1. *Let $h > 0$ and $s', s > r$. Then under Assumptions 1 and 2,*

$$\sup_{f \in \mathcal{F}_{s,L}} \left| E_f T_{s', \varepsilon} - T(f) \right| \leq L h^{s-r} b_{s, s'}.$$

PROOF. Using Assumptions 1 and 2, we have

$$\begin{aligned} \sup_{f \in \mathcal{F}_{s,L}} \left| E_f \int K_{s', h} dY_\varepsilon - T(f) \right| &= \sup_{\rho_s(f) \leq L} \left| \int K_{s', h} f - T(f) \right| \\ &= \sup_{\rho_s(f) \leq L} \left| \int K_{s'}(x) h^{-r} f(hx) dx - T(f) \right| \\ &= L h^{s-r} \sup_{\rho_s(f) \leq L} \left| \int K_{s'}(x) \frac{f(hx)}{L h^s} dx - T\left(\frac{f(h \cdot)}{L h^s}\right) \right| \\ &= L h^{s-r} \sup_{\rho_s(f) \leq 1} \left| \int K_{s'} f - T(f) \right| \end{aligned}$$

since

$$\left\{ \frac{f(h \cdot)}{L h^s} : \rho_s(f) \leq L \right\} = \{f : \rho_s(f) \leq 1\}. \quad \square$$

Recall that $h_l(s, L, \varepsilon) = (\varepsilon/L)^{2/(2s+d)}$. Define the bias term

$$B(s, L, \varepsilon) = L h_l^{s-r}(s, L, \varepsilon) b_{s, s}$$

and the standard deviation term

$$R(s, L, \varepsilon) = \varepsilon h_l^{-r-d/2}(s, L, \varepsilon) \|K_s\|_2.$$

LEMMA 2. *Suppose that Assumptions 1 and 2 hold and let $g_{s, L, \varepsilon}$ be defined by (3), $\varepsilon > 0, L > 0, s > r$. Then:*

(i) $\|g_{s, L, \varepsilon}\|_2 = \varepsilon.$

- (ii) If $\rho_s(g) \leq L$ and $T(g) = T(g_{s,L,\varepsilon})$, then $\int g g_{s,L,\varepsilon} \geq \|g_{s,L,\varepsilon}\|_2^2$.
- (iii) $\psi_\nu = B(s, L, \tilde{\varepsilon}(s)) + R(s, L, \tilde{\varepsilon}(s))$, where ψ_ν is defined by (14).

PROOF. Assertion (i) means that the solution $g_{s,L,\varepsilon}$ is attained at the boundary of the set of restrictions, while (ii) follows from the usual duality argument [see Gabushin (1970), Micchelli and Rivlin (1977), Arestov (1989)]. An elementary proof of (i) and (ii) for the case $T(f) = f(0)$ is given in Lepski and Tsybakov (2000). It can be easily extended to our general case. Let us prove (iii). In view of (6),

$$(25) \quad T(g_{s,1,1}) = b_{s,s} + \|K_s\|_2.$$

Now, by standard renormalization argument, $g_{s,L,\varepsilon}(x) = a g_{s,1,1}(bx)$ where

$$a = Lb^{-s} = \varepsilon^{2s/(2s+d)} L^{d/(2s+d)}, \quad b = (L/\varepsilon)^{2/(2s+d)} = h_l^{-1}(s, L, \varepsilon).$$

Thus,

$$ab^r = Lh_l^{s-r}(s, L, \varepsilon) = \varepsilon h_l^{-r-d/2}(s, L, \varepsilon),$$

and, by (25),

$$T(g_{s,L,\varepsilon}) = ab^r T(g_{s,1,1}) = B(s, L, \varepsilon) + R(s, L, \varepsilon).$$

This and (15) yield (iii). \square

For $s > s' > 0$ define

$$(26) \quad \begin{aligned} d_\varepsilon(s', s) &= \left[2p(\kappa(s) - \kappa(s')) \log \varepsilon^{-1} \right]^{1/2} \\ &= \left[2p(2r + d) \left(\frac{1}{2s' + d} - \frac{1}{2s + d} \right) \log \varepsilon^{-1} \right]^{1/2}. \end{aligned}$$

LEMMA 3. *Let Assumptions 1–4 hold. Let $s, s' \in [r', q]$, $r < r' < q$, $s' < s$, $L \in [L_*, L^*]$, and denote $\nu = (s, L)$. Then there exist positive constants D_1, \dots, D_5 (that can depend only on $s_*, s^*, L_*, L^*, r, q, d, p$) such that*

$$(27) \quad \frac{\psi_{s',L}}{\psi_\nu} \leq D_1 \exp \left\{ \frac{1}{2p} d_\varepsilon^2(s', s) \right\},$$

$$(28) \quad \frac{\psi_{s',L}}{\psi_\nu} \geq D_2 \left(\varepsilon^2 \log \frac{1}{\varepsilon} \right)^{(\kappa(s') - \kappa(s))/2},$$

$$(29) \quad \frac{h(s, \tilde{\varepsilon}(s))}{h(s', \tilde{\varepsilon}(s'))} \geq D_3 \exp \left\{ \frac{2}{(2s^* + d)^2} (s - s') \log \varepsilon^{-1} \right\}$$

and

$$(30) \quad D_4 \leq \frac{\psi_{s,L}}{\eta(s)} \leq D_5.$$

PROOF. From (15) we have $\psi_{s,L} = \tilde{\varepsilon}^{2(s-r)/(2s+d)} L^{(2r+d)/(2s+d)} T(g_{s,1,1})$ where $\tilde{\varepsilon} = \varepsilon d_\varepsilon(s, s^*)$. Thus,

$$\frac{\psi_{s',L}}{\psi_\nu} = \exp\left\{\frac{1}{2p} d_\varepsilon^2(s', s)\right\} \frac{d_\varepsilon^{2(s'-r)/(2s'+d)}(s', s^*) T(g_{s',1,1})}{d_\varepsilon^{2(s-r)/(2s+d)}(s, s^*) T(g_{s,1,1})} L^{\frac{2r+d}{2s'+d} - \frac{2r+d}{2s+d}}.$$

Note that $\|K_s\|_2 > 0$ for every s , since otherwise $\|g_{s,1,1}\|_2 = 0$ [by (7)] which contradicts Lemma 2(i). Also, by Assumption 3, $\|K_s\|_2$ is a continuous function of s on the interval $[r', s^*]$ for $r' > r$. Hence

$$(31) \quad \inf_{s \in [r', s^*]} \|K_s\|_2 > 0, \quad \sup_{s \in [r', s^*]} \|K_s\|_2 < \infty.$$

Combining (31) with (25) and taking into account Assumption 4(i) we get

$$(32) \quad \inf_{s \in [r', s^*]} T(g_{s,1,1}) > 0, \quad \sup_{s \in [r', s^*]} T(g_{s,1,1}) < \infty.$$

Now, for $s_0 < s_1$,

$$(33) \quad d_\varepsilon^2(s_0, s_1) = \frac{4p(2r+d)}{(2s_0+d)(2s_1+d)} (s_1 - s_0) \log \varepsilon^{-1}$$

and thus, for $s \in [r', q]$,

$$\frac{4p(2r+d)}{(2s^*+d)^2} (s^* - q) \log \varepsilon^{-1} \leq d_\varepsilon^2(s, s^*) \leq \frac{4p}{2r+d} (s^* - r) \log \varepsilon^{-1}.$$

Therefore,

$$D_6 (\log \varepsilon^{-1})^{(\kappa(s') - \kappa(s))/2} \leq \frac{d_\varepsilon^{2(s'-r)/(2s'+d)}(s', s^*)}{d_\varepsilon^{2(s-r)/(2s+d)}(s, s^*)} \leq D_7 (\log \varepsilon^{-1})^{(\kappa(s') - \kappa(s))/2}$$

for $D_6, D_7 > 0$. Observing that $(\kappa(s') - \kappa(s))/2 < 0$ and using (32) we obtain (27) and (28). To prove the bound (29) it is enough to note that

$$(34) \quad \frac{h(s, \tilde{\varepsilon}(s))}{h(s', \tilde{\varepsilon}(s'))} = \exp\left\{\frac{4(s-s')}{(2s+d)(2s'+d)} \left(\log \varepsilon^{-1} - \frac{1}{2} \log \log \varepsilon^{-1}\right)\right\} \frac{\lambda(s)^{1/(2s+d)}}{\lambda(s')^{1/(2s'+d)}}.$$

The bounds (30) follow from the equality

$$\frac{\psi_{s,L}}{\eta(s)} = L^{(2r+d)/(2s+d)} \frac{T(g_{s,1,1})}{\|K_s\|_2}$$

in view of (31) and (32). \square

We need some exponential bounds for the stochastic part of the estimator. Define

$$(35) \quad Z_s = \varepsilon \int K_{s, h(s, \tilde{\varepsilon}(s))}(t) dW(t).$$

LEMMA 4. Let σ_s be as defined in (11). Then for $u > 0, p \geq 0$,

$$E[|Z_s|^p I(|Z_s| \geq u)] \leq D(p)(\sigma_s^p + u^p) \exp\left\{-\frac{u^2}{2\sigma_s^2}\right\},$$

where $D(p) > 0$ is a constant depending only on p .

Proof of this lemma is straightforward since $Z_s \sim \mathcal{N}(0, \sigma_s^2)$.

7. Proofs of the results. In the following we denote C, C', C_1, C_2, \dots positive constants that can depend only on $s_*, s^*, L_*, L^*, r, q, d, p$. These constants may be different in different occasions.

PROOF OF THE UPPER BOUND IN THEOREM 1. Here we prove the bound (16). Consider $\nu = (s, L) \in B_q$ and define $s^- = s^-(s)$ by

$$s^- = s - \frac{\log \log \log(1/\varepsilon)}{\log(1/\varepsilon)}.$$

We have

$$\sup_{f \in \mathcal{F}_\nu} E_f(\psi_\nu^{-p} |T_\varepsilon^* - T(f)|^p) = R_{\varepsilon, \nu}^- + R_{\varepsilon, \nu}^+,$$

where

$$R_{\varepsilon, \nu}^- = \sup_{f \in \mathcal{F}_\nu} E_f(\psi_\nu^{-p} |T_\varepsilon^* - T(f)|^p I(\hat{s} < s^-)),$$

$$R_{\varepsilon, \nu}^+ = \sup_{f \in \mathcal{F}_\nu} E_f(\psi_\nu^{-p} |T_\varepsilon^* - T(f)|^p I(\hat{s} \geq s^-)).$$

To show (16), we will prove that

$$(36) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{\nu \in B_q} R_{\varepsilon, \nu}^- = 0$$

and

$$(37) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{\nu \in B_q} R_{\varepsilon, \nu}^+ \leq 1.$$

PROOF OF (36). Let $s \in [s_*, q], s' \in S, s' < s^-, L \in [L_*, L^*]$. Let $f \in \mathcal{F}_\nu, \nu = (s, L)$. Then for sufficiently small ε , using Lemma 1, (29) and the fact that $s' < s$, we get

$$(38) \quad |E_f T_{s', \varepsilon} - T(f)| \leq L h^{s-r}(s', \tilde{\varepsilon}(s')) b_{s, s'} \leq L h^{s-r}(s, \tilde{\varepsilon}(s)) b_{s, s'}.$$

By definition,

$$(39) \quad h^{s-r}(s, \tilde{\varepsilon}(s)) = \tilde{\varepsilon}^{\kappa(s)}(s, \tilde{\varepsilon}(s)),$$

where $\kappa(s) = 2(s-r)/(2s+d)$. Comparing this to (15) and using the inequality $b_{s, s'} \leq b_{\max}(r', s^*)$ [Assumption 4(i)], we get

$$(40) \quad L h^{s-r}(s, \tilde{\varepsilon}(s)) b_{s, s'} \leq C_1 \psi_\nu.$$

Then

$$|T_{s', \varepsilon} - T(f)| \leq |E_f T_{s', \varepsilon} - T(f)| + |Z_{s'}| \leq C_1 \psi_\nu + |Z_{s'}|,$$

where $Z_{s'}$ is defined in (35) as a stochastic error of the kernel estimator. Using this we find

$$R_{\varepsilon, \nu}^- \leq \sum_{s' \in S, s' < s^-} \sup_{f \in \mathcal{F}_\nu} E_f (\psi_\nu^{-p} |T_{s', \varepsilon} - T(f)|^p I(\hat{s} = s')) \leq g_1(\nu) + g_2(\nu),$$

where

$$(41) \quad g_1(\nu) = C \sum_{s' \in S, s' < s^-} \sup_{f \in \mathcal{F}_\nu} P_f(\hat{s} = s')(1 + \psi_\nu^{-1} \tau(s'))^p$$

and

$$(42) \quad g_2(\nu) = C \sum_{s' \in S, s' < s^-} E \left[(1 + \psi_\nu^{-1} |Z_{s'}|)^p I(|Z_{s'}| \geq \tau(s')) \right]$$

with

$$\tau(s') = \sigma_{s'} \left[d_\varepsilon(s', s) + (\log \varepsilon^{-1})^{1/4} \right],$$

where $\sigma_{s'}$ is defined in (11) and $d_\varepsilon(s', s)$ is defined in (26).

Let us prove that the probability of greatly underestimating largely the value of s by the statistic \hat{s} is small, uniformly over $f \in \mathcal{F}_\nu$.

LEMMA 5. *Let $s \in [s_*, q]$, $s' \in S$, $s' < s^-$, $L \in [L_*, L^*]$ and $\nu = (s, L)$. Then for every $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ independent of s, s', L and such that for all $0 < \varepsilon < \varepsilon_0$ we have*

$$\sup_{f \in \mathcal{F}_\nu} P_f(\hat{s} = s') \leq Cm \exp\left\{-\frac{1}{2} d_\varepsilon^2(s', s^*)(1 - \delta)\right\}.$$

PROOF. Since $\text{Card}(S) = m$, we have

$$(43) \quad \begin{aligned} \sup_{f \in \mathcal{F}_\nu} P_f(\hat{s} = s') &\leq \sum_{s'' \in S, s'' \leq s'} \sup_{f \in \mathcal{F}_\nu} P_f(|T_{\bar{s}', \varepsilon} - T_{s'', \varepsilon}| > \eta(s'')) \\ &\leq m \max_{s'' \in S, s'' \leq s'} \sup_{f \in \mathcal{F}_\nu} P_f(|T_{\bar{s}', \varepsilon} - T_{s'', \varepsilon}| > \eta(s'')), \end{aligned}$$

where $\bar{s}' = \bar{s}'(s')$ is the smallest element of S greater than s' . Let $f \in \mathcal{F}_\nu$. Arguing as in (38) and (40), and using (28), for sufficiently small ε we get

$$|E_f T_{\bar{s}', \varepsilon} - T(f)| + |E_f T_{s'', \varepsilon} - T(f)| \leq C_1 \psi_\nu \leq C_1 \gamma_\varepsilon \psi_{s'', L},$$

where

$$\begin{aligned} \gamma_\varepsilon = \gamma_\varepsilon(s, s'') &= D_2^{-1} \left(\varepsilon^2 \log \frac{1}{\varepsilon} \right)^{(\kappa(s) - \kappa(s''))/2} \leq D_2^{-1} \exp\{-C(\kappa(s) - \kappa(s'')) \log(1/\varepsilon)\} \\ &\leq D_2^{-1} \exp\{-C(\kappa(s) - \kappa(s^-)) \log(1/\varepsilon)\} \leq D_2^{-1} \exp\{-C' \log \log \log(1/\varepsilon)\}. \end{aligned}$$

Note that $\gamma_\varepsilon(s, s'') \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly in $s, s'' \in [r', q]$.

Using (30) we obtain $\psi_{s'', L} \leq D_5 \eta(s'')$ and thus

$$\begin{aligned} |T_{\bar{s}', \varepsilon} - T_{s'', \varepsilon}| &\leq |E_f T_{\bar{s}', \varepsilon} - T(f)| + |E_f T_{s'', \varepsilon} - T(f)| + |Z_{\bar{s}'} - Z_{s''}| \\ &\leq C_3 \gamma_\varepsilon \eta(s'') + |Z_{\bar{s}'} - Z_{s''}|. \end{aligned}$$

Hence, for sufficiently small ε ,

$$(44) \quad P_f(|T_{\bar{s}', \varepsilon} - T_{s'', \varepsilon}| > \eta(s'')) \leq P_f(|Z_{\bar{s}'} - Z_{s''}| > \eta(s'')(1 - C_3 \gamma_\varepsilon)).$$

Denote $h_0 = h(s'', \tilde{\varepsilon}(s''))$, $h_1 = h(\bar{s}', \tilde{\varepsilon}(\bar{s}'))$, $\bar{K}_0 = K_{s''}$, $\bar{K}_1 = K_{\bar{s}'}$, and, as usual, $\bar{K}_{i, h_i} = h_i^{-r-d} \bar{K}_i(\cdot/h_i)$. Now $Z_{\bar{s}'} - Z_{s''} \sim \mathcal{N}(0, \varepsilon^2 A^2)$ where

$$A = \|\bar{K}_{0, h_0} - \bar{K}_{1, h_1}\|_2 \leq \|\bar{K}_{0, h_0} - \bar{K}_{0, h_1}\|_2 + \|\bar{K}_{0, h_1} - \bar{K}_{1, h_1}\|_2 \stackrel{\text{def}}{=} A_1 + A_2.$$

Now

$$\begin{aligned} A_1^2 &= h_0^{-2r-d} \int \left[\bar{K}_0(x) - \left(\frac{h_0}{h_1}\right)^{r+d} \bar{K}_0\left(\frac{h_0}{h_1}x\right) \right]^2 dx \\ &= h_0^{-2r-d} \left[\int \bar{K}_0^2 + \left(\frac{h_0}{h_1}\right)^{2r+d} \int \bar{K}_0^2 - 2 \left(\frac{h_0}{h_1}\right)^{r+d} \int \bar{K}_0(x) \bar{K}_0\left(\frac{h_0}{h_1}x\right) dx \right]. \end{aligned}$$

Denote

$$\tilde{g}(x) = \left(\frac{h_0}{h_1}\right)^{-r} g_{s'', 1, 1}\left(\frac{h_0}{h_1}x\right).$$

Then, because $h_0 < h_1$,

$$\rho_{s''}(\tilde{g}) = \left(\frac{h_0}{h_1}\right)^{s''-r} \rho_{s''}(g_{s'', 1, 1}) \leq 1$$

and $T(\tilde{g}) = T(g_{s'', 1, 1})$. Lemma 2(ii) gives

$$\int g_{s'', 1, 1} \tilde{g} \geq \int g_{s'', 1, 1}^2$$

and therefore, by (7),

$$\left(\frac{h_0}{h_1}\right)^{-r} \int \bar{K}_0(x) \bar{K}_0\left(\frac{h_0}{h_1}x\right) dx \geq \int \bar{K}_0^2.$$

Thus

$$(45) \quad A_1^2 \leq h_0^{-2r-d} \left[1 - \left(\frac{h_0}{h_1}\right)^{2r+d} \right] \int \bar{K}_0^2 \leq h_0^{-2r-d} \int \bar{K}_0^2.$$

Also,

$$(46) \quad A_2^2 = h_1^{-2r-d} \|\bar{K}_0 - \bar{K}_1\|_2^2 = h_0^{-2r-d} \|\bar{K}_0\|_2^2 \gamma'_\varepsilon,$$

where

$$\gamma'_\varepsilon = \gamma'_\varepsilon(\bar{s}', s'') = \left(\frac{h_0}{h_1}\right)^{2r+d} \frac{\|\bar{K}_0 - \bar{K}_1\|_2^2}{\|\bar{K}_0\|_2^2} \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Indeed, to prove this consider the two cases: (i) $\bar{s}' - s'' \geq (\log \varepsilon^{-1})^{-1/2}$ and (ii) $0 < \bar{s}' - s'' < (\log \varepsilon^{-1})^{-1/2}$. If (i) holds then, by (29), $h_0/h_1 \leq D_3^{-1} \exp(-C\sqrt{\log \varepsilon^{-1}})$, and, using (31), we get $\gamma'_\varepsilon \leq C \exp(-C\sqrt{\log \varepsilon^{-1}})$. If (ii) holds, then (29) and the inequality $\bar{s}' > s''$ entail $h_0/h_1 \leq C$, while $\|\bar{K}_0 - \bar{K}_1\|_2 \leq \Omega(\bar{s}' - s'') \leq \Omega((\log \varepsilon^{-1})^{-1/2})$ where $\Omega(\delta) = \sup\{\|K_s - K_{s'}\|_2: |s - s'| \leq \delta, s, s' \in [r', s^*]\}$. Note that $\Omega(\delta) \rightarrow 0$, as $\delta \rightarrow 0$, by the uniform continuity of the function $F(s, s') = \|K_s - K_{s'}\|_2$ on $[r', s^*] \times [r', s^*]$ which follows from Assumption 3. Thus, in both cases (i) and (ii) we have $\gamma'_\varepsilon(\bar{s}', s'') \rightarrow 0$ as $\varepsilon \rightarrow 0$, uniformly in $\bar{s}', s'' \in [r', s^*]$.

From (45) and (46) it follows that for sufficiently small ε ,

$$\varepsilon^2 A^2 \leq \varepsilon^2 h^{-2r-d}(s'', \tilde{\varepsilon}(s'')) \|K_{s''}\|_2^2 (1 + \gamma'_\varepsilon) = \sigma_{s''}^2 (1 + \gamma'_\varepsilon).$$

Now, for sufficiently small ε ,

$$\frac{\eta^2(s'')}{\varepsilon^2 A^2} \geq \frac{\eta^2(s'')}{\sigma_{s''}^2 (1 + \gamma'_\varepsilon)} = \frac{d_\varepsilon^2(s'', s^*)}{1 + \gamma'_\varepsilon} \geq \frac{d_\varepsilon^2(s', s^*)}{1 + \gamma'_\varepsilon}.$$

Thus, using Lemma 4 we get

$$\begin{aligned} P_f(|Z_{\bar{s}'} - Z_{s''}| > \eta(s'')(1 - C_3 \gamma_\varepsilon)) &\leq C_4 \exp\left\{-\frac{1}{2\varepsilon^2 A^2} \eta^2(s'')(1 - C_3 \gamma_\varepsilon)^2\right\} \\ &\leq C_4 \exp\left\{-\frac{1}{2} d_\varepsilon^2(s', s^*) (1 - C_3 \gamma_\varepsilon)^2 (1 + \gamma'_\varepsilon)^{-1}\right\}. \end{aligned}$$

Comparing this to (43) and (44) we get the lemma. \square

LEMMA 6. *Let g_1 be defined in (41). Then,*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\nu \in B_q} g_1(\nu) = 0.$$

PROOF. Let $s \in [s_*, q]$, $s' \in S$, $s' < s^-$, $L \in [L_*, L^*]$, $\nu = (s, L)$. By definitions [see also (33)],

$$\frac{\tau(s')}{\eta(s')} = \frac{d_\varepsilon(s', s) + (\log \varepsilon^{-1})^{1/4}}{d_\varepsilon(s', s^*)} \leq C_1.$$

Next, by (30),

$$\frac{\eta(s')}{\psi_\nu} \leq C_2 \frac{\psi_{s', L}}{\psi_\nu}.$$

Combining the two previous inequalities and using (27), we get

$$(47) \quad \frac{\tau(s')}{\psi_\nu} \leq C_3 \frac{\psi_{s', L}}{\psi_\nu} \leq C_4 \exp\left\{\frac{1}{2p} d_\varepsilon^2(s', s)\right\}.$$

Using this and Lemma 5, we find

$$\begin{aligned} g_1(\nu) &\leq C_5 \sum_{s' \in S, s' < s^-} \sup_{f \in \mathcal{F}_\nu} P_f(\hat{s} = s') \left(\frac{\tau(s')}{\psi_\nu} \right)^p \\ &\leq C_6 m \sum_{s' \in S, s' < s^-} \left[\exp \left\{ \frac{1}{2} d_\varepsilon^2(s', s) \right\} \exp \left\{ -\frac{1}{2} d_\varepsilon^2(s', s^*)(1 - \delta) \right\} \right] \\ &\leq C_7 m^2 \exp \{ -c(\delta) \log \varepsilon^{-1} \}, \end{aligned}$$

where

$$c(\delta) = p(2r + d) \left(\frac{1}{2s + d} - \frac{\delta}{2r + d} - \frac{1 - \delta}{2s^* + d} \right).$$

Choose $\delta = \frac{(s^* - q)(2r + d)}{2(s^* - r)(2q + d)}$. Then $c(\delta) \geq C_8 > 0$. Moreover, in view of (10),

$$(48) \quad m = O([\log(1/\varepsilon)]^{\gamma_1}), \quad \varepsilon \rightarrow 0,$$

and the lemma follows. \square

LEMMA 7. Let g_2 be defined in (42). Then,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\nu \in B_q} g_2(\nu) = 0.$$

PROOF. Let $s \in [s_*, q]$, $s' \in S$, $s' < s^-$, $L \in [L_*, L^*]$, $\nu = (s, L)$. Now,

$$\frac{\tau^2(s')}{\sigma_{s'}^2} = [d_\varepsilon(s', s) + (\log \varepsilon^{-1})^{1/4}]^2 \geq (\log \varepsilon^{-1})^{1/2}.$$

This, together with Lemma 4 and (47), (48) yields

$$\begin{aligned} g_2(\nu) &\leq C_1 \sum_{s' \in S, s' < s^-} [1 + \psi_\nu^{-p}(\sigma_{s'}^p + \tau^p(s'))] \exp \left\{ -\frac{\tau^2(s')}{2\sigma_{s'}^2} \right\} \\ &\leq C_2 m \exp \left\{ -\frac{1}{2} (\log \varepsilon^{-1})^{1/2} \right\} \rightarrow 0 \end{aligned}$$

uniformly in $\nu \in B_q$, as $\varepsilon \rightarrow 0$. \square

Lemmas 6 and 7 imply (36).

PROOF OF (37). Let $s \in [s_*, q]$, $L \in [L_*, L^*]$, $\nu = (s, L)$. Let $\bar{s} = \bar{s}(s)$ be defined by

$$\bar{s} = s - \frac{(2s + d) \log L}{2 \log(1/\varepsilon) - \log \log(1/\varepsilon) + 2 \log L}.$$

In other words, \bar{s} is chosen so that

$$(49) \quad (L^{-2} \varepsilon^2 \log \varepsilon^{-1})^{1/(2s+d)} = (\varepsilon^2 \log \varepsilon^{-1})^{1/(2\bar{s}+d)}.$$

Let $s^+ \in S$ be the largest grid point $\leq \bar{s}$. Denote $\mathcal{S}_1 = \mathcal{S}_1(s) = \{s' \in S: s^- \leq s' \leq s^+\}$ and $\mathcal{S}_2 = \mathcal{S}_2(s) = \{s' \in S: s^+ < s' \leq q\}$. Assume that ε is small enough, so that $s^- < s^+$. We have

$$R_{\varepsilon, \nu}^+ = \sup_{f \in \mathcal{F}_\nu} E_f(\psi_\nu^{-p} |T_\varepsilon^* - T(f)|^p I(\hat{s} \in \mathcal{S}_1 \cup \mathcal{S}_2)).$$

Let $s' \in \mathcal{S}_1$ and $f \in \mathcal{F}_\nu$. Using successively Lemma 1, the fact that $s' \leq \bar{s}$ and (49) we get

$$\begin{aligned} |E_f T_{s', \varepsilon} - T(f)| &\leq L h^{s-r}(s', \tilde{\varepsilon}(s')) b_{s, s'} \\ &= L \lambda(s')^{(s-r)/(2s'+d)} (\varepsilon^2 \log \varepsilon^{-1})^{(s-r)/(2s'+d)} b_{s, s'} \\ (50) \quad &\leq L \lambda(s')^{(s-r)/(2s'+d)} (\varepsilon^2 \log \varepsilon^{-1})^{(s-r)/(2\bar{s}+d)} b_{s, s'} \\ &= L \lambda(s')^{(s-r)/(2s'+d)} (L^{-2} \varepsilon^2 \log \varepsilon^{-1})^{(s-r)/(2s+d)} b_{s, s'} \\ &= \Lambda(s, s') L h_l^{s-r}(s, L, \tilde{\varepsilon}(s)) b_{s, s'}, \end{aligned}$$

where $\Lambda(s, s') = \lambda(s')^{(s-r)/(2s'+d)} \lambda(s)^{-(s-r)/(2s+d)}$. Note that

$$(51) \quad |s - s'| \leq \frac{C \log \log \log(1/\varepsilon)}{\log(1/\varepsilon)} \quad \forall s' \in \mathcal{S}_1.$$

This and the uniform continuity of $\Lambda(s, s')$ in $s, s' \in [s_*, q]$ yields that $\Lambda(s, s') \leq 1 + \gamma_{\varepsilon 1}$. (Here and later we denote $\gamma_{\varepsilon i}, i = 1, 2, \dots$ the functions of ε that can depend only on $s_*, s^*, L_*, L^*, r, q, d, p$ and such that $\lim_{\varepsilon \rightarrow 0} \gamma_{\varepsilon i} = 0$.) Next, using Assumption 4(ii) and (51), for every $s' \in \mathcal{S}_1, s \in [s_*, s^*]$, we get $b_{s, s'} \leq b_{s, s}(1 + \gamma_{\varepsilon 2})$. These remarks, (50) and Lemma 2(iii), yield

$$(52) \quad \begin{aligned} |E_f T_{s', \varepsilon} - T(f)| &\leq L h_l^{s-r}(s, L, \tilde{\varepsilon}(s)) b_{s, s} (1 + \gamma_{\varepsilon 3}) \\ &= B(s, L, \tilde{\varepsilon}(s)) (1 + \gamma_{\varepsilon 3}) \leq \psi_\nu (1 + \gamma_{\varepsilon 3}) \quad \forall s' \in \mathcal{S}_1. \end{aligned}$$

From (27) and (51) we have $\psi_{s', L} / \psi_\nu \leq (\log \log(1/\varepsilon))^{C_1} \forall s' \in \mathcal{S}_1$. This and (30) entail

$$\eta(s') \leq D_4^{-1} \psi_{s', L} \leq C_2 (\log \log(1/\varepsilon))^{C_1} \psi_\nu \quad \forall s' \in \mathcal{S}_1$$

and

$$(53) \quad \begin{aligned} \frac{\sigma_{s'}}{\psi_\nu} &\leq C_3 (\log \log(1/\varepsilon))^{C_1} \frac{\sigma_{s'}}{\eta(s')} = \frac{C_3 (\log \log(1/\varepsilon))^{C_1}}{d_\varepsilon(s', s^*)} \\ &\leq \frac{C_3 (\log \log(1/\varepsilon))^{C_1}}{d_\varepsilon(q, s^*)} \leq \frac{C_4}{\log^{1/4}(1/\varepsilon)} \quad \forall s' \in \mathcal{S}_1. \end{aligned}$$

Note also that, since $s^+ \leq \bar{s}$, the argument similar to (50), (52) and Assumption 3 yield

$$\begin{aligned}
\eta(s^+) &= (\lambda(s^+) \varepsilon^2 \log \varepsilon^{-1})^{(s^+-r)/(2s^++d)} \|K_{s^+}\|_2 \\
&\leq \lambda(s^+)^{(s^+-r)/(2s^++d)} (\varepsilon^2 \log \varepsilon^{-1})^{(\bar{s}-r)/(2\bar{s}+d)} \|K_{s^+}\|_2 (1 + \gamma_{\varepsilon 4}) \\
(54) \quad &= \lambda(s^+)^{(s^+-r)/(2s^++d)} \sqrt{\varepsilon^2 \log \varepsilon^{-1}} \left(\frac{\varepsilon^2 \log \varepsilon^{-1}}{L^2} \right)^{-(r+d/2)/(2s^++d)} \\
&\quad \times \|K_{s^+}\|_2 (1 + \gamma_{\varepsilon 4}) \\
&\leq \tilde{\varepsilon}(s) h_l^{-r-d/2}(s, L, \tilde{\varepsilon}(s)) \|K_s\|_2 (1 + \gamma_{\varepsilon 5}) \\
&= R(s, L, \tilde{\varepsilon}(s)) (1 + \gamma_{\varepsilon 5}).
\end{aligned}$$

[For the first inequality in this display we used the fact that, by (10), $|\bar{s} - s^+| \leq k_2 [\log(1/\varepsilon)]^{-\gamma}$, $\gamma > 1$, and thus $(\varepsilon^2 \log \varepsilon^{-1})^{(s^+-r)/(2s^++d) - (\bar{s}-r)/(2\bar{s}+d)} \leq (1 + \gamma_{\varepsilon 4})$.]

Now we are ready for the main argument of the proof. Let first $\hat{s} = s' \in \mathcal{S}_1$. Then, in view of (52),

$$(55) \quad |T_\varepsilon^* - T(f)| = |T_{s', \varepsilon} - T(f)| \leq |E_f T_{s', \varepsilon} - T(f)| + |Z_{s'}| \leq \psi_\nu (1 + \gamma_{\varepsilon 3}) + |Z_{s'}|.$$

Next, let $\hat{s} = s' \in \mathcal{S}_2$. Then, using successively the definition of \hat{s} , (52), (54) and Lemma 2(iii), we get

$$\begin{aligned}
|T_\varepsilon^* - T(f)| &\leq |T_{s', \varepsilon} - T_{s^+, \varepsilon}| + |T_{s^+, \varepsilon} - T(f)| \\
&\leq \eta(s^+) + |E_f T_{s^+, \varepsilon} - T(f)| + |Z_{s^+}| \\
&\leq R(s, L, \tilde{\varepsilon}(s)) (1 + \gamma_{\varepsilon 5}) + B(s, L, \tilde{\varepsilon}(s)) (1 + \gamma_{\varepsilon 3}) + |Z_{s^+}| \\
&\leq \psi_\nu (1 + \gamma_{\varepsilon 6}) + |Z_{s^+}|.
\end{aligned}$$

This and (55) entail

$$\begin{aligned}
(56) \quad &E_f (\psi_\nu^{-p} |T_\varepsilon^* - T(f)|^p I(\hat{s} \in \mathcal{S}_1 \cup \mathcal{S}_2)) \\
&\leq \sum_{s' \in \mathcal{S}_1} E_f ((1 + \gamma_{\varepsilon 3} + \psi_\nu^{-1} |Z_{s'}|)^p I(\hat{s} = s')) \\
&\quad + \sum_{s' \in \mathcal{S}_2} E_f ((1 + \gamma_{\varepsilon 6} + \psi_\nu^{-1} |Z_{s^+}|)^p I(\hat{s} = s')).
\end{aligned}$$

Applying Lemma 4 and (53) we get, for any $s' \in \mathcal{S}_1$,

$$\begin{aligned}
&E_f ((1 + \gamma_{\varepsilon 3} + \psi_\nu^{-1} |Z_{s'}|)^p I(\hat{s} = s')) \\
&\leq (1 + \gamma_{\varepsilon 3} + \sqrt{\sigma_{s'} \psi_\nu^{-1}})^p P_f(\hat{s} = s') \\
&\quad + E_f ((1 + \gamma_{\varepsilon 3} + \psi_\nu^{-1} |Z_{s'}|)^p I(|Z_{s'}| \geq \sqrt{\sigma_{s'} \psi_\nu})) \\
&\leq (1 + \gamma_{\varepsilon 3} + \sqrt{\sigma_{s'} \psi_\nu^{-1}})^p P_f(\hat{s} = s') + C_5 [P_f(|Z_{s'}| \geq \sqrt{\sigma_{s'} \psi_\nu}) \\
&\quad + \psi_\nu^{-p} E_f(|Z_{s'}|^p I(|Z_{s'}| \geq \sqrt{\sigma_{s'} \psi_\nu}))]
\end{aligned}$$

$$\begin{aligned} &\leq (1 + \gamma_{\varepsilon 3} + \sqrt{\sigma_{s'} \psi_\nu^{-1}})^p P_f(\hat{s} = s') \\ &\quad + C_6 \left[1 + \psi_\nu^{-p} \left((\sigma_{s'} \psi_\nu)^{p/2} + \sigma_{s'}^p \right) \right] \exp \left\{ -\frac{\psi_\nu}{2\sigma_{s'}} \right\} \\ &\leq \left(1 + \gamma_{\varepsilon 3} + C_4^{1/2} (\log(1/\varepsilon))^{-1/8} \right)^p P_f(\hat{s} = s') + C_7 \exp \left\{ -\frac{\log^{1/4}(1/\varepsilon)}{2C_4} \right\}. \end{aligned}$$

Since $s^+ \in \mathcal{S}_1$, an analogous bound holds for $E_f((1 + \gamma_{\varepsilon 6} + \psi_\nu^{-1} |Z_{s^+}|)^p I(\hat{s} = s'))$. We conclude therefore that

$$\begin{aligned} (57) \quad R_{\varepsilon, \nu}^+ &= \sup_{f \in \mathcal{F}_\nu} E_f(\psi_\nu^{-p} |T_\varepsilon^* - T(f)|^p I(\hat{s} \in \mathcal{S}_1 \cup \mathcal{S}_2)) \\ &\leq \left(1 + \gamma_{\varepsilon 7} + \frac{C_4^{1/2}}{\log^{1/8}(1/\varepsilon)} \right)^p \sup_{f \in \mathcal{F}_\nu} P_f(\hat{s} \in \mathcal{S}_1 \cup \mathcal{S}_2) \\ &\quad + 2C_7 m \exp \left\{ -\frac{\log^{1/4}(1/\varepsilon)}{2C_4} \right\} \end{aligned}$$

where $\gamma_{\varepsilon 7} = \max\{\gamma_{\varepsilon 3}, \gamma_{\varepsilon 6}\}$. It remains to note that (37) follows from (57) and (48). \square

PROOF OF THE LOWER BOUND IN THEOREM 1. Here we prove the bound (17). The proof consists in reducing the problem to getting a lower bound on the risk of two hypotheses $f = f_0$ and $f = f_1$, which are chosen to be distant enough. In fact, f_1 will be chosen on the “boundary” of our scale of classes. Let $L \in [L_*, L^*]$, $\nu' = (s_*, L)$ and $\nu'' = (q, L)$. Consider the functions

$$f_0 \equiv 0, \quad f_1 = (1 - \delta)g_{s_*, L, \tilde{\varepsilon}(s_*)}$$

where $0 < \delta < 1/2$ and $\tilde{\varepsilon}(s_*) = \varepsilon d_\varepsilon(s_*, s^*)$. Obviously, $f_0 \in \mathcal{F}_{q, L}$. Furthermore, $\rho_{s_*}(f_1) = (1 - \delta)\rho_{s_*}(g_{s_*, L, \tilde{\varepsilon}(s_*)}) \leq (1 - \delta)L$ and thus $f_1 \in \mathcal{F}_{s_*, L}$. From (15), $T(f_1) = (1 - \delta)\psi_{\nu'}$. Also, $T(f_0) = 0$. Thus, for any estimator T_ε ,

$$|T_\varepsilon - T(f_i)| = \psi_{\nu'} D((1 - \delta)^{-1} \psi_{\nu'}^{-1} T_\varepsilon, i), \quad i = 0, 1,$$

where $D(u, v) = (1 - \delta)|u - v|$, $u, v \in \mathbf{R}$. Denoting $Q = \psi_{\nu'} / \psi_{\nu''}$ and $E_i = E_{f_i}$, we get

$$\begin{aligned} &\inf_{T_\varepsilon} \sup_{\nu \in B_q} \sup_{f \in \mathcal{F}_\nu} E_f(\psi_\nu^{-p} |T_\varepsilon - T(f)|^p) \\ &\geq \inf_{T_\varepsilon} \max \{ E_0(\psi_{\nu'}^{-p} |T_\varepsilon - T(f_0)|^p), E_1(\psi_{\nu'}^{-p} |T_\varepsilon - T(f_1)|^p) \} \\ &= \inf_{T_\varepsilon} \max \{ Q^p E_0 D^p(T_\varepsilon, 0), E_1 D^p(T_\varepsilon, 1) \}. \end{aligned}$$

Denote for brevity $P_i = P_{f_i}$, $i = 0, 1$. We apply now the following lemma, which is a special case of Theorem 6(i) in Tsybakov (1998) adapted to the present notation.

LEMMA 8. Let $Q > 0$, $\tau > 0$, $0 < \delta < 1/2$, $0 < \alpha < 1$ be fixed and let $D: \mathbf{R} \times \mathbf{R} \rightarrow [0, \infty)$ be a distance such that

$$D(0, 1) \geq 1 - \delta.$$

Suppose that

$$P_1\left(\frac{dP_0}{dP_1} \geq \tau\right) \geq 1 - \alpha.$$

Then, for $p > 0$,

$$\inf_{T_\varepsilon} \max\{Q^p E_0 D^p(T_\varepsilon, 0), E_1 D^p(T_\varepsilon, 1)\} \geq \frac{(1 - \alpha)(1 - 2\delta)^p \tau (Q\delta)^p}{(1 - 2\delta)^p + \tau(Q\delta)^p},$$

where the infimum is taken over all estimators.

Let us check the assumptions of Lemma 8. Clearly, the assumption $D(0, 1) \geq 1 - \delta$ is satisfied for our definition of $D(\cdot, \cdot)$. Next, by Lemma 2(i), $\|f_1\|_2^2 = (1 - \delta)^2 \varepsilon^2(s_*)$. Put

$$\tau = \exp\left\{-\frac{1 - \delta}{2} d_\varepsilon^2(s_*, s^*)\right\}.$$

Then

$$P_1\left(\frac{dP_0}{dP_1} \geq \tau\right) = P(\exp\{\varepsilon^{-1}\|f_1\|_2 \xi - \varepsilon^{-2}\|f_1\|_2^2/2\} \geq \tau) = 1 - \Phi(l_\varepsilon),$$

where $\xi \sim \mathcal{N}(0, 1)$, $\Phi(\cdot)$ is a standard normal c.d.f. and

$$l_\varepsilon = \frac{\varepsilon}{\|f_1\|_2} (\log \tau + \varepsilon^{-2}\|f_1\|_2^2/2) = -\frac{\delta}{2} d_\varepsilon(s_*, s^*) \rightarrow -\infty, \quad \varepsilon \rightarrow 0.$$

Hence, we can use Lemma 8 with the choice $\alpha = \Phi(l_\varepsilon)$ which results in

$$(58) \quad \inf_{T_\varepsilon} \max\{Q^p E_0 D^p(T_\varepsilon, 0), E_1 D^p(T_\varepsilon, 1)\} \geq \frac{(1 - \Phi(l_\varepsilon))(1 - 2\delta)^p \tau (Q\delta)^p}{(1 - 2\delta)^p + \tau(Q\delta)^p}.$$

Here, in view of (14) and (33),

$$\begin{aligned} \tau Q^p &= \exp\left\{-\frac{1 - \delta}{2} d_\varepsilon^2(s_*, s^*)\right\} \left(\frac{\psi_{\nu'}}{\psi_{\nu''}}\right)^p \\ &\geq \exp\left\{\frac{2p(2r + d)}{(2s_* + d)(2s^* + d)} [q - s^* + \delta(s^* - s_*)] \log \varepsilon^{-1}\right\} \\ &\geq \exp\left\{-\frac{p(2r + d)}{(2s_* + d)(2q + d)} \log \log \varepsilon^{-1}\right\} \left(\frac{c_{\nu'}}{c_{\nu''}}\right)^p. \end{aligned}$$

Choose $\delta = (1 - \delta_1) \frac{s^* - q}{s^* - s_*} + \delta_1$, where $0 < \delta_1 < 1/2$, and consider only q that is close enough to s^* , so that $\delta < 1/2$. Then $q - s^* + \delta(s^* - s_*) > 0$, and $\tau Q^p \rightarrow \infty$, as $\varepsilon \rightarrow 0$. Since also $l_\varepsilon \rightarrow -\infty$, we conclude that the RHS of (58) tends to

$(1 - 2\delta)^p$, as $\varepsilon \rightarrow 0$. Taking the limit of $(1 - 2\delta)^p$, as $q \rightarrow s^*$, and using the fact that δ_1 can be chosen arbitrarily small, we get

$$\lim_{q \rightarrow s^*} \liminf_{\varepsilon \rightarrow 0} \inf_{T_\varepsilon} \sup_{\nu \in B_q} \mathcal{R}_{\varepsilon, \nu}(T_\varepsilon, \psi_\nu) \geq 1. \quad \square$$

PROOF OF THEOREM 3. We start with the proof of Theorem 3(i). Note that the function

$$F(s) \stackrel{\text{def}}{=} \left(\max \left(\lambda(s), \lambda \left(\frac{s^* + q}{2} \right) \right) \varepsilon^2 \log \varepsilon^{-1} \right)^{1/(2s+d)}$$

is a continuous function of s on the interval $[r', s^*]$, and $F(r') = h(r', \tilde{\varepsilon}(r')) \leq h_i < h_{\max} = \varepsilon^{2/(2s^*+d)} < F(s^*)$ for any $i \in \{1, \dots, m\}$ and ε small enough. Hence for every $i \in \{1, \dots, m\}$ there exists at least one $s_i \in [r', s^*]$ such that

$$(59) \quad h_i = \left(\max \left(\lambda(s_i), \lambda \left(\frac{s^* + q}{2} \right) \right) \varepsilon^2 \log \varepsilon^{-1} \right)^{1/(2s_i+d)}.$$

Fix a sequence $S = \{s_1, s_2, \dots, s_m\}$ where s_i is a solution of (59). Using (18), (19) it is easy to check that s_i defined by (59) satisfies (10) with $\gamma_1 = \gamma = 1 + \gamma_0$, and

$$h_i = h(s_i, \tilde{\varepsilon}(s_i)) \quad \forall s_i \leq (s^* + q)/2.$$

Therefore, we can apply the argument as in the proof of Theorem 1 for this particular grid S . Some modifications of the proof are needed here since, unlike the case of Theorem 1, the kernel and the threshold are defined with $s(h_i)$ in place of s_i . These modifications are easy to establish if one notes that

$$(60) \quad |s_i - s(h_i)| \leq \frac{C \log \log(1/\varepsilon)}{\sqrt{\log(1/\varepsilon)}} \quad \forall s_i \leq (s^* + q)/2.$$

In fact, Assumption 3 and (60) entail

$$(61) \quad \left| \frac{\eta_{h_i}}{\eta(s_i)} - 1 \right| \leq \gamma_{\varepsilon 8} \quad \forall s_i \leq (s^* + q)/2,$$

and, in view of (60) and Assumption 4,

$$(62) \quad b_{s, s(h_i)} \leq b_{s, s_i}(1 + \gamma_{\varepsilon 9}) \leq b_{\max}(r', s^*)(1 + \gamma_{\varepsilon 9}) \quad \forall s, s_i \in [r', q], s_i \leq s.$$

Using (60)–(62), the proof of (36) is almost the same as in Theorem 1. For the proof of (37) we mention only the modifications in the key relations (50) and (54). Instead of (50) we now obtain (with the notation $s' = s_i$)

$$|E_f T_\varepsilon(h_i) - T(f)| \leq L h_i^{s-r} b_{s, s(h_i)} = L h^{s-r}(s_i, \tilde{\varepsilon}(s_i)) b_{s, s(h_i)},$$

where, by (51), $|s_i - s| \leq C \log \log \log(1/\varepsilon) / \log(1/\varepsilon)$. This and (60), together with Assumption 4(ii), yield $b_{s, s(h_i)} \leq b_{s, s}(1 + \gamma_{\varepsilon 10})$. Other elements of (50) remain as in the proof of Theorem 1. Turning to (54), we have to evaluate now η_{h^+} in place of $\eta(s^+)$, where $h^+ = h(s^+, \tilde{\varepsilon}(s^+))$. By virtue of (61), the only

difference from the case of Theorem 1 appears in the inclusion of the extra factor $(1 + \gamma_{\varepsilon 8})$.

Consider now the proof of Theorem 3(ii). We have $F(r') = h(r', \tilde{\varepsilon}(r')) \leq h_i < h_{\max} = 1 = \lim_{s \rightarrow \infty} F(s)$. Hence, the solutions s_i exist, as above, but we get an additional set of gridpoints that extends to the right beyond s^* ,

$$\mathcal{S}_3 = \{s_i: s_i > (s^* + q)/2\}.$$

We can apply the argument as in the proof of Theorem 3(i) with a modification so as to address the set \mathcal{S}_3 and the choice $h_{\max} = 1$. The latter is equivalent to putting $s^* = \infty$, and all the calculations in the proof of Theorems 1 and 3(i) remain valid with this modification if ψ_ν is replaced by $\tilde{\psi}_\nu$. Inclusion of the set \mathcal{S}_3 leads to a modification only in the proof of (37). In fact,

$$R_{\varepsilon, \nu}^+ = \sup_{f \in \mathcal{F}_\nu} E_f(\psi_\nu^{-p} |T_\varepsilon^* - T(f)|^p I(\hat{s} \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3))$$

and the inclusion of \mathcal{S}_3 results in the consideration of the third component of $R_{\varepsilon, \nu}^+$, namely,

$$R_{\varepsilon, \nu}^{++} = \sup_{f \in \mathcal{F}_\nu} E_f(\psi_\nu^{-p} |T_\varepsilon^* - T(f)|^p I(\hat{s} \in \mathcal{S}_3)).$$

We treat this component similarly to (56) (we have the same expression with \mathcal{S}_3 instead of \mathcal{S}_2). Hence

$$R_{\varepsilon, \nu}^{++} \leq \sup_{f \in \mathcal{F}_\nu} \sum_{s' \in \mathcal{S}_3} E_f((1 + \gamma_{\varepsilon 8} + \psi_\nu^{-1} |Z_{s^+}|)^p I(\hat{s} = s'))$$

and acting as in the calculation following the formula (56), we get instead of (57)

$$R_{\varepsilon, \nu}^+ \leq \left(1 + \gamma_{\varepsilon 7} + \frac{C_4^{1/2}}{\log^{1/8}(1/\varepsilon)}\right)^p \sup_{f \in \mathcal{F}_\nu} P_f(\hat{s} \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3) + 2C_6 m \exp\left\{-\frac{\log^{1/4}(1/\varepsilon)}{2C_4}\right\}$$

and we conclude the proof by noting that $m = \text{Card}(S) \leq C(\log(1/\varepsilon))^{1+\gamma_0}$. \square

PROOF OF PROPOSITION 1. Note that

$$\begin{aligned} \|\tilde{K}_s\|_2^2 &= (2\pi)^d \|\widehat{\tilde{K}}_s\|_2^2 = (2\pi)^{-d} \int_{\mathbf{R}^d} \frac{\omega^{2\alpha_0}}{(1 + \|\omega\|^{2\beta})^2} d\omega \\ &= \frac{1}{2\beta} B\left(\frac{2r+d}{2\beta}, 2 - \frac{2r+d}{2\beta}\right) (2\pi)^{-d} \int_{S_d} \xi^{2\alpha_0} d\mu(\xi) \end{aligned}$$

and

$$\begin{aligned} \rho_s^2(\tilde{K}_s) &= (2\pi)^d \int_{\mathbf{R}^d} \|\omega\|^{2\beta} \left|\widehat{\tilde{K}}_s(\omega)\right|^2 d\omega = (2\pi)^{-d} \int_{\mathbf{R}^d} \frac{\omega^{2\alpha_0} \|\omega\|^{2\beta}}{(1 + \|\omega\|^{2\beta})^2} d\omega \\ &= \frac{1}{2\beta} B\left(1 + \frac{2r+d}{2\beta}, 1 - \frac{2r+d}{2\beta}\right) (2\pi)^{-d} \int_{S_d} \xi^{2\alpha_0} d\mu(\xi) = C_*^2. \end{aligned}$$

Since the beta- function satisfies $B(c, d) = B(c - 1, d + 1)(c - 1)/d, \forall c > 1, d > 0$, it follows that

$$\|\tilde{K}_s\|_2^2 = \frac{2(s-r)}{2r+d} \rho_s^2(\tilde{K}_s),$$

and thus

$$(63) \quad b = \left(\frac{\|\tilde{K}_s\|_2}{\rho_s(\tilde{K}_s)} \right)^{2/(2s+d)}.$$

Now,

$$\begin{aligned} (-1)^r T(\tilde{K}_s) &= (-i)^r \int \omega^{\alpha_0} \tilde{K}_s(\omega) d\omega \\ &= (2\pi)^{-d} \int_{\mathbf{R}^d} \frac{\omega^{2\alpha_0}}{1 + \|\omega\|^{2\beta}} d\omega = \|\tilde{K}_s\|_2^2 + \rho_s^2(\tilde{K}_s) \end{aligned}$$

and hence, using (63), we get

$$\begin{aligned} T(g_{s,1,1}(\cdot)) &= (-1)^r a T(\tilde{K}_s(b \cdot)) = (-1)^r a b^r T(\tilde{K}_s) \\ (64) \quad &= \left(\frac{\rho_s(\tilde{K}_s)}{\|\tilde{K}_s\|_2} \right)^{2(s-r)/(2s+d)} \rho_s(\tilde{K}_s)^{-1} (\|\tilde{K}_s\|_2^2 + \rho_s^2(\tilde{K}_s)) \\ &= C_* \left(\frac{2r+d}{2(s-r)} \right)^{(s-r)/(2s+d)} \frac{2s+d}{2r+d}. \end{aligned}$$

Now,

$$(65) \quad \|K_s\|_2 = b^{r+d/2} \|\tilde{K}_s\|_2.$$

To evaluate the bias constant we use the following lemma.

LEMMA 9. *Let the assumptions of Proposition 1 be satisfied, and let $r < r' < s^*, r' \leq s, s' \leq s^*, \beta = s + d/2, \beta' = s' + d/2, b' = \left(\frac{2(s'-r)}{2r+d}\right)^{1/(2s'+d)}$. Then*

$$b_{s,s'} \leq (b')^{r-s} \left[(2\pi)^{-d} \int_{\mathbf{R}^d} \frac{\omega^{2\alpha_0} \|\omega\|^{4\beta'-2\beta}}{(1 + \|\omega\|^{2\beta'})^2} d\omega \right]^{1/2}$$

provided the last integral is finite.

PROOF. We have $\widehat{K}_{s'}(\omega) = (2\pi)^{-d} i^r \omega^{\alpha_0} (1 + \|\omega/b'\|^{2\beta'})^{-1}$. Hence, by the Cauchy inequality,

$$\begin{aligned} b_{s,s'} &= \sup_{\rho_s(f) \leq 1} \left| \int K_{s'} f - T(f) \right| = \sup_{\rho_s(f) \leq 1} \left| \int_{\mathbf{R}^d} \hat{f}(\omega) ((2\pi)^d \widehat{K}_{s'}(\omega) - i^r \omega^{\alpha_0}) d\omega \right| \\ &= \sup_{\rho_s(f) \leq 1} \left| \int_{\mathbf{R}^d} \hat{f}(\omega) i^r \omega^{\alpha_0} \left(\frac{\|\omega/b'\|^{2\beta'}}{1 + \|\omega/b'\|^{2\beta'}} \right) d\omega \right| \end{aligned}$$

$$\begin{aligned} &\leq (b')^{r-s} \sup_{\rho_s(f) \leq 1} \left[(2\pi)^d \int_{\mathbf{R}^d} \|\omega\|^{2\beta} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} \\ &\quad \times \left[(2\pi)^{-d} \int_{\mathbf{R}^d} \frac{\omega^{2\alpha_0} \|\omega\|^{4\beta'-2\beta}}{(1 + \|\omega\|^{2\beta'})^2} d\omega \right]^{1/2}. \end{aligned}$$

Using the definition of ρ_s for the Sobolev classes, we get the lemma. \square

It follows from Lemma 9 with $s = s'$ that $b_{s,s} \leq \rho_s(\tilde{K}_s) b^{r-s}$. Moreover,

$$(66) \quad b_{s,s} = \rho_s(\tilde{K}_s) b^{r-s}.$$

Indeed, the function f_* with the Fourier transform

$$\hat{f}_*(\omega) = (2\pi)^{-d} \rho_s(\tilde{K}_s)^{-1} b^{-r-s-d} \frac{i^r \omega^{\alpha_0}}{1 + \|\omega/b\|^{2\beta}}$$

satisfies $\rho_s(f_*) = 1$ and

$$\left| \int K_s f_* - T(f_*) \right| = \left| \int_{\mathbf{R}^d} \hat{f}_*(\omega) i^r \omega^{\alpha_0} \left(\frac{\|\omega/b\|^{2\beta}}{1 + \|\omega/b\|^{2\beta}} \right) d\omega \right| = \rho_s(\tilde{K}_s) b^{r-s}.$$

Combining (64)–(66), and using (63), we get (6). This proves the proposition. \square

PROOF OF PROPOSITION 2. Assumptions 1–3 are straightforward to verify. We check only Assumption 4. For $s, s' \in [r', s^*]$ and $\beta = s + d/2, \beta' = s' + d/2$ using the inequality $r' > (s^* - r - d)/2$ assumed in Proposition 2, we obtain $\beta' \geq r' + d/2 > (s^* - r)/2 \geq (s - r)/2 = (\beta - r)/2 - d/4$. Thus

$$\begin{aligned} \int_{\mathbf{R}^d} \frac{\omega^{2\alpha_0} \|\omega\|^{4\beta'-2\beta}}{(1 + \|\omega\|^{2\beta'})^2} d\omega &\leq \int_{\mathbf{R}^d} \frac{\|\omega\|^{4\beta'-2\beta+2r}}{(1 + \|\omega\|^{2\beta'})^2} d\omega = \mu(S_d) \int_0^\infty \frac{t^{4\beta'-2\beta+2r+d-1}}{(1 + t^{2\beta'})^2} dt \\ &\leq \mu(S_d) \left(\int_0^1 t^{4\beta'-2\beta+2r+d-1} dt + \int_1^\infty t^{d-1-2\beta+2r} dt \right) \\ &\leq \mu(S_d) \left[\frac{1}{4r' - 2(s^* - r - d)} + \frac{1}{2(r' - r)} \right], \end{aligned}$$

where for the last inequality we used that $\beta, \beta' > r' + d/2$. This and Lemma 9 yield Assumption 4(i). Next, Assumption 4(ii) follows from (66), the continuity of the beta function and the fact that (by Lemma 9)

$$\begin{aligned} b_{s,s'} &\leq (b')^{r-s} \left[\frac{1}{2\beta'} B \left(1 + \frac{2r + d + 2(\beta' - \beta)}{2\beta'}, 1 - \frac{2r + d + 2(\beta' - \beta)}{2\beta'} \right) \right]^{1/2} \\ &\quad \times \left[(2\pi)^{-d} \int_{S_d} \xi^{2\alpha_0} d\mu(\xi) \right]^{1/2}. \quad \square \end{aligned}$$

PROOF OF PROPOSITIONS 3 AND 4. Note that, for \tilde{K}_s defined in (23),

$$\int \|x\|^s \tilde{K}_s(x) dx + \|\tilde{K}_s\|_2^2 = \int \tilde{K}_s.$$

Now, $g_{s,1,1}(\cdot) = a\tilde{K}_s(b\cdot)$, where $b = \|\tilde{K}_s\|_2^{2/(2s+d)}$. Let

$$K_s(x) = \left[\int g_{s,1,1} \right]^{-1} g_{s,1,1}(x) = \left[\int \tilde{K}_s \right]^{-1} b^d \tilde{K}_s(bx).$$

Then

$$\begin{aligned} \int \|x\|^s K_s(x) dx + \|K_s\|_2 &= \left(b^{-s} \int \|x\|^s \tilde{K}_s(x) dx + b^{d/2} \|\tilde{K}_s\|_2 \right) / \int \tilde{K}_s \\ &= \|\tilde{K}_s\|_2^{-2s/(2s+d)} \left(\int \|x\|^s \tilde{K}_s(x) dx + \|\tilde{K}_s\|_2^2 \right) / \int \tilde{K}_s \\ &= \|\tilde{K}_s\|_2^{-2s/(2s+d)}. \end{aligned}$$

For $f \in \{g: \|g\|_2 \leq 1, \rho_s(g) \leq 1\}$ we get

$$\begin{aligned} |f(0)| &\leq \left| \int K_s f - f(0) \right| + \left| \int K_s f \right| \leq \left| \int (f(x) - f(0)) K_s(x) dx \right| + \|f\|_2 \|K_s\|_2 \\ &\leq \int \|x\|^s K_s(x) dx + \|K_s\|_2 = \|\tilde{K}_s\|_2^{-2s/(2s+d)}. \end{aligned}$$

On the other hand, this upper bound is achieved by $g_{s,1,1}$ because

$$g_{s,1,1}(0) = a\tilde{K}_s(0) = a = \|\tilde{K}_s\|_2^{-2s/(2s+d)}.$$

Propositions 3 and 4 follow from these remarks and the equations

$$\int \tilde{K}_s = \frac{\mu(S_d)s}{(s+d)d}, \quad \|\tilde{K}_s\|_2^2 = \frac{2s^2\mu(S_d)}{(2s+d)(s+d)d}. \quad \square$$

PROOF OF PROPOSITION 5. Assumptions 1–3 are straightforward. To check Assumption 4, it suffices to remark that in this case $b_{s,s'} \leq \int \|x\|^s K_{s'}(x) dx$ and $b_{s,s} = \int \|x\|^s K_s(x) dx$. \square

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