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SOME NEW RESULTS FOR DIRICHLET PRIORS[1]

BY DONATO MICHELE CIFARELLI AND EUGENIO MELILLI

Universitá "L. Bocconi," Milano

Let p be a random probability measure chosen by a Dirichlet process whose parameter α is a finite measure with support contained in $[0, +\infty)$ and suppose that $V = \int x^2 p(dx) - [\int xp(dx)]^2$ is a (finite) random variable. This paper deals with the distribution of V, which is given in a rather general case. A simple application to Bayesian bootstrap is also illustrated.

1. Introduction. This paper deals with the probability distribution of the random variable (r.v.) $V(\omega) = \int x^2 p(dx, \omega) - [\int xp(dx, \omega)]^2 = [\int (x - y)^2 p(dx, \omega) p(dy, \omega)]/2$, when p is a Dirichlet process whose parameter α is a finite measure on the Borel σ-field of $\mathbb{R}$ having support contained in $[0, +\infty)$. Our results may be a useful complement to what is already known about the Dirichlet process [Diaconis and Kemperman (1996)]. The Dirichlet process was discovered by Freedman (1963) and developed by Fabius (1964); a careful history appears in Diaconis and Freedman (1986). For background, a general reference is Ferguson (1973), who discussed this process and its applications to nonparametric Bayesian inference. One of the reasons for the success of this prior is its well-known property of closure: when the prior distribution is a Dirichlet process with parameter α, then the posterior, given a sample $X_1, X_2, \ldots, X_n$, is also a Dirichlet process having parameter $\alpha + \sum \delta_{x_i}$, where δ_ξ denotes point mass at ξ.

In this framework, the assessment of the distribution of V (or at least of some characteristics of it) represents a useful tool in Bayesian inference. Because of the closure property, the expression for the distribution function of V can be employed both for prior and posterior Bayesian analyses.

In particular, the results of this paper can be easily applied to the so-called Bayesian bootstrap, introduced by Rubin (1981). Suppose that $\{X_n\}_{n\geq 1}$ is an exchangeable sequence of nonnegative r.v.'s. Then, by de Finetti's theorem, $X_1, X_2, \ldots$ are independent identically distributed according to p given the random probability measure p. A statistician may be interested in the variance V of p without the information needed to construct a prior for this random probability measure. Then the Bayesian bootstrap may help. For a fixed sample $x_1, x_2, \ldots, x_k$, the Bayesian bootstrap distribution of p and its functionals can be approximated as a weak limit of the posterior distribution under the Dirichlet prior when the total mass $\alpha(\mathbb{R}) = \alpha([0, +\infty)) = a$ tends to zero [Sethuraman and Tiwari (1982)].

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This program has been followed by Cifarelli and Regazzini (1990) for the bootstrap distribution of the mean functional and by Lo (1987), who used the Bayesian bootstrap distribution to obtain a 95% probability band for a univariate distribution function. Gasparini (1995) extended these results to the multivariate Bayesian bootstrap distribution of the moments. More recently, Choudhuri (1998) developed the Bayesian bootstrap for the multidimensional mean functional. Our formula (15) yields the Bayesian bootstrap distribution of the variance, when $0 \le x_1 < x_2 < \cdots < x_k$, with $k \ge 3$. (We limit ourselves to this case even though, by using (vi) of Section 2, we could actually consider the general case.) Note that V is not an a.s. weakly continuous functional, so its convergence is not a direct consequence of the result by Sethuraman and Tiwari, but follows from an application of Corollary 2.7 of Hannum, Hollander and Langberg (1981).

The setup of the paper is as follows. Section 2 recalls the definition of a Dirichlet process and some results about the distribution of general linear functionals of it. Section 3 gives the main results of the paper, that is, the expressions for the Laplace transform and the probability density of V. Section 4 gives the proofs of these results.

2. Preliminaries. Let us consider the space $\mathscr{P}$ of all probability measures on $(\mathbb{R}, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-field of $\mathbb{R}$. If $\mathscr{P}$ is endowed with the Prohorov metric ρ, then $(\mathscr{P}, \rho)$ is a Polish space, whose corresponding Borel σ-field we denote by $\mathscr{C}$. Given a probability space $(\Omega, \mathscr{F}, q)$, a random probability measure on $(\mathbb{R}, \mathscr{B})$ is a measurable function $p: \Omega \to \mathscr{P}$, that is, a function on $\mathscr{B} \times \Omega$ such that:

1. $A \to p(A, \omega)$ is a probability measure on $(\mathbb{R}, \mathscr{B})$ for any ω in Ω, and
2. $\omega \to p(A, \omega)$ is an $\mathscr{F}$-measurable function for any A in $\mathscr{B}$.

We now recall the definition of the Dirichlet process. Given a finite, positive measure α on $(\mathbb{R}, \mathscr{B})$, let us associate with each finite, measurable partition $\{A_1, A_2, \dots, A_k\}$ of $\mathbb{R}$ a random vector $(T_1, T_2, \dots, T_k)$ having a Dirichlet distribution with parameters $\alpha(A_1), \alpha(A_2), \dots, \alpha(A_k)$, with the proviso that $T_i = 0$ a.s. when $\alpha(A_i) = 0$. Denote by $D(x_1, \dots, x_{k-1}; \alpha(A_1), \alpha(A_2), \dots, \alpha(A_k))$ the Dirichlet distribution function with parameters $\alpha(A_1), \alpha(A_2), \dots, \alpha(A_k)$. Given any finite class $\{E_1, E_2, \dots, E_n\}$ of Borel subsets of $\mathbb{R}$, consider the family $\{A_1, A_2, \dots, A_k\}$ of its constituents (i.e., the family of all intersections of the E_i and their complements). Then, let us write the probability distribution function

$$H(y_1, y_2, \dots, y_n) = \int_{\Xi(y)} dD(x_1, \dots, x_{k-1}; \alpha(A_1), \dots \alpha(A_k)), \tag{1}$$

where $\Xi(y) = \{(x_1, \dots, x_{k-1}) \in [0, +\infty)^{k-1}\colon \sum_{i=1}^{k-1} x^i \le 1, \sum_{i \in \mathscr{C}(j)} x_i \le y_j,\ j = 1, \dots, n,\ x_k = 1 - \sum_{i=1}^{k-1} x_i\}$ and $C(j)$ denotes the set of all i's for which $A_j \subset E_j$. Ferguson (1973) showed that (1) defines a consistent family of distribution functions and that, according to Kolmogorov's theorem, there exists a unique probability measure q on the σ-algebra of cylinders of $[0,1]^{\mathscr{B}}$ such that the probability distribution function of the random vector $(p(E_1,\omega),\dots,p(E_n,\omega))$ is given by (1) for all $n\geq 1$ and all $E_1,\dots,E_n$ in $\mathscr{B}$; moreover, q selects almost surely probability measures. Basu and Tiwari (1982) and Berk and Savage (1979) solve some measure-theoretic difficulties existing in the Ferguson's construction. The family of r.v.'s $\{p(E,\omega), E\in\mathscr{B}\}$ is said to be a Dirichlet process with parameter α and defines, as already observed, a random probability measure on $(\mathbb{R},\mathscr{B})$. From now on we shall consider Dirichlet processes with parameter α having support contained in $[0,+\infty)$.

Let ψ be a nonnegative and measurable function defined in $[0,+\infty)$. The random functional

$$Y(\omega)=\int_{[0,+\infty)}\psi(x)p(dx,\omega) \tag{2}$$

has been studied by Cifarelli and Regazzini (1979, 1990, 1993), Hannum, Hollander and Langberg (1981), Yamato (1984), Diaconis and Kemperman (1996). The last two authors gave a number of applications. Resorting now to a condition for the existence of the distribution of Y given by Feigin and Tweedie (1989) and by Cifarelli and Regazzini (1995), and using some results in Cifarelli and Regazzini (1990, 1993), we can state the following.

THEOREM 2.1. *Let p be a Dirichlet process on $(\mathbb{R},\mathscr{B})$ with parameter α having support contained in $[0,+\infty)$ and consider a measurable function ψ: $[0,+\infty)\to[0,+\infty)$ such that the measure $\alpha_\psi:=\alpha\psi^{-1}$ is nondegenerate. Call A_ψ the distribution function of α_ψ and put $a=\alpha([0,+\infty))>0$. Then*

$$q\left(\omega\in\Omega:\int_{[0,+\infty)}\psi(x)p(dx,\omega)<+\infty\right)=1$$

if and only if

$$\int_{[0,+\infty)}\log[1+\psi(x)]\alpha(dx)<+\infty. \tag{3}$$

In this case the distribution function M_ψ of Y is absolutely continuous and satisfies the equation

$$\int_0^{+\infty}\frac{1}{(s+x)^a}M'_\psi(x)\,dx=\exp\left\{-\int_{[0,+\infty)}\log(s+x)\,dA_\psi(x)\right\},\quad s\in\mathbb{C}\setminus(-\infty,0]. \tag{4}$$

Moreover, if $a=1$, then

$$M'_\psi(x)=\frac{1}{\pi}\sin\{\pi A_\psi(x)\}\exp\left\{-\int_{[0,+\infty)}\log|t-x|dA_\psi(t)\right\} \tag{5}$$

for almost all x *in* $[0, +\infty)$, *while, if* $a > 1$ *and* α *has no concentrated mass greater than or equal to* 1, *then*

$$M'_\psi(x) = \tfrac{a-1}{\pi} \int_0^x (x-u)^{a-2} \sin\{\pi A_\psi(u)\} \times \exp\left\{-\int_{[0,+\infty)} \log|t-u|\, dA_\psi(t)\right\} du \tag{6}$$

for almost all x *in* $[0, +\infty)$.

Usually it is not easy to obtain manageable analytical expressions for M'_ψ even when α has a simple form, and much work has been done to derive suitable methods for numerical approximation [Guglielmi (1998)]. We give now the density function $M' := M'_x$ of $W_1(\omega) = W_1 := \int_{[0,+\infty)} x p(dx, \omega)$ obtained from (4), (5) or (6) with $\psi(x) = x$, for a few simple α's.

1. Let $\alpha(\{0\}) = a_0, \alpha(\{1\}) = a_1$, $a = a_0 + a_1 > 0$. Then
$$M'(x) = \frac{x^{a_1-1}(1-x)^{a_0-1}}{B(a_0, a_1)}, \qquad 0 < x < 1,$$
where $B(a_0, a_1)$ denotes the Beta function with parameters a_0 and a_1.
2. Let $\alpha(dx) = \{x(1-x)[\pi^2 + \log^2 x/(1-x)]\}^{-1} dx$, $0 < x < 1$. Here $a = 1$ and
$$M'(x) = 1, \qquad 0 < x < 1.$$
3. Let $\alpha(dx) = a\pi^{-1}[x(1-x)]^{-1/2} dx$, $0 < x < 1$, $a > 0$. Then
$$M'(x) = \frac{x^{a-1/2}(1-x)^{a-1/2}}{B(a+1/2, a+1/2)}, \qquad 0 < x < 1.$$
4. Let $\alpha(\{0\}) = 1/2$, $\alpha(dx) = (a-1/2)\pi^{-1}[x(1-x)]^{-1/2} dx$, $0 < x < 1$, $a > 1/2$. Then
$$M'(x) = \frac{x^{a-3/2}(1-x)^{a-1/2}}{B(a+1/2, a+1/2)}, \qquad 0 < x < 1.$$
5. Let $\alpha(\{0\}) = \alpha(\{1\}) = 1/2$, $\alpha(dx) = (a-1)\pi^{-1}[x(1-x)]^{-1/2} dx$, $0 < x < 1$, $a \geq 1$. Then
$$M'(x) = \frac{x^{a-3/2}(1-x)^{a-3/2}}{B(a-1/2, a-1/2)}, \qquad 0 < x < 1.$$
6. Let $\alpha(\{x_i\}) = n_i \geq 1$, $i = 1, 2, \dots, k$, $k \geq 2$, $a = \sum_i n_i = n$, $0 \leq x_1 < x_2 < \cdots < x_k$; then, for $x \geq 0$,
$$M'(x) = (n-1) \sum_{h=1}^{k} \sum_{q=0}^{n_h-1} (-1)^{n_{h-q-1}} \binom{n-2}{q} \times (x-x_h)_+^{n-2-q} \sum_{(*)} \prod_{j\neq h}^{k} \frac{\binom{n_j+i_j-1}{i_j}}{(x_j-x_h)^{n_j+i_j}},$$

where the sum extends to all nonnegative integers $i_1, i_2, \dots, i_{h-1}, i_{h+1}, \dots, i_k$ such that $i_1+i_2+\cdots+i_{h-1}+i_{h+1}+\cdots+i_k = n_h-q-1$ and $x_+ = \max\{x, 0\}$.

7. Let $\alpha(dx) = a\pi^{-1}x^{-1/2}(1+x)^{-1}\,dx$, $x > 0$, $a \ge 1$. Then

$$M'(x) = \frac{\Gamma(a+1)}{\Gamma(a-1/2)\sqrt{\pi}}\frac{x^{a-1/2}}{(1+x)^{a+1}}, \qquad x > 0.$$

Note, in the last example, that α has no mean and yet W_1 exists almost surely.

REMARK 2.1. Suppose $\xi_j := \int_{[0,+\infty)} x^j \alpha(dx) < +\infty$ for each $j > 0$ and consider the r.v. W_1. Then, from (4) we can write, for each $k \ge 0$,

$$\mathrm{E}(W_1^k) = \frac{\Gamma(a)}{\Gamma(a+k)} B_k, \tag{7}$$

where B_k is defined recursively by

$$B_k = \Gamma(k) \sum_{i=0}^{k-1} \frac{\xi_{k-i} B_i}{i!}, \tag{8}$$

with $B_0 = 1$.

Let p be, as above, a Dirichlet process whose parameter α has support contained in $[0, +\infty)$ and denote by A its distribution function. For any positive C, let α^C be the measure with distribution function

$$A^C(x) = \begin{cases} 0, & x < 0, \\ A(x), & 0 \le x < C, \\ a, & x \ge C, \end{cases} \tag{9}$$

and denote by p^C a Dirichlet process with parameter α^C.

PROPOSITION 2.1. *Suppose that* $\int_{[0,+\infty)} \log(1+x)\alpha(dx) < +\infty$ *and consider the r.v.*:

$$W_1^C(\omega) = \int_{[0,+\infty)} x p^C(dx, \omega),$$

whose distribution function is denoted by M^C. *Then, for every* x *in* $\mathbb{R}$,

$$\lim_{C \to +\infty} M^C(x) = M(x).$$

PROOF. Let F and F^C be the random distribution functions corresponding to p and p^C, respectively. Of course, $F^C(x)$ and $F(x)$ are equally distributed for each x in $(0, C)$, so that, letting $\overset{\mathrm{st}}{=}$ denote equality in distribution,

$$\begin{aligned} W_1(\omega) = \int_0^{+\infty} [1 - F(x, \omega)\,dx &= \int_0^C [1 - F(x, \omega)]\,dx + \int_C^{+\infty} [1 - F(x, \omega)]\,dx \\ &\overset{\mathrm{st}}{=} \int_0^C [1 - F^C(x, \omega)]\,dx + \int_C^{+\infty} [1 - F(x, \omega)]\,dx \\ &= W_1^C(\omega) + \int_C^{+\infty} [1 - F(x, \omega)]\,dx. \end{aligned}$$

Since W_1 is finite with probability 1, then $\int_C^{+\infty}[1-F(x,\omega)]\,dx \to 0$ a.s. for $C \to +\infty$; hence $W_1^C \to^D W_1$ for $C \to +\infty$ and the result follows from the continuity of M. □

3. The main result. We consider a Dirichlet process p whose parameter α is a finite measure, with support contained in $[0,+\infty)$, such that $\int_{[0,+\infty)} \log \times (1+x^2)\alpha(dx) < +\infty$; by Theorem 2.1, W_1, $W_2(\omega) = W_2 := \int_{[0,+\infty)} x^2\, p(dx,\omega)$ and $V = W_2 - W_1^2$ are (finite) r.v.'s. The results are given for simplicity when $\alpha([0,+\infty]) = a$ is a positive integer, although this condition could actually be dropped. The following proposition, whose proof will be given in the next section, yields the Laplace transform of the density of V.

PROPOSITION 3.1. *Let p be a Dirichlet process with parameter α, where α, is a finite measure with support contained in $[0,+\infty)$, having as total mass an integer $a \ge 1$ and such that (3) is satisfied with $\psi(x) = x^2$. Let X and Y be independent r.v.'s distributed as W_1 and denote by φ any version of the probability density of $U = (X-Y)^2$. Then, for all $z > 0$,*

$$\mathrm{E}(e^{-zV}) = \int_0^{+\infty} {}_1F_1(a; 1/2; -zu/4)\varphi(u,a)\,du, \tag{10}$$

where ${}_1F_1$ denotes the confluent hypergeometric function.

We shall prove this proposition in several steps. First, we give the proof when a is an integer and α has no concentrated mass greater than or equal to 1, then we prove the proposition when α has also integer masses concentrated on some points $x_1, x_2, \dots, x_k$.

REMARK 3.1. Of course, when the parameter of the Dirichlet process is given by $\alpha + \sum_{i=1}^k \delta_{x_i}$, with the x_is in the support of α, then (10) gives the Laplace transform of the posterior distribution of V.

REMARK 3.2. If W_1 has all moments, then (10) is true if and only if, for all $m \ge 0$,

$$\mathrm{E}(V^m) = \frac{\Gamma(a+m)\Gamma(1/2)}{4^m\Gamma(m+1/2)\Gamma(a)}\mathrm{E}(U^m). \tag{11}$$

Moreover, if α has all its moments, then (7), (11) and the definition of U give the moments of V in terms of those of α; that is, for $k \ge 0$,

$$\mathrm{E}(V^k) = \sum_{j=0}^{2k} \frac{(-1)^j\Gamma(a+k)\Gamma(1/2)\Gamma(a)(2k)!}{4^k\Gamma(k+1/2)\Gamma(a+j)\Gamma(a+2k-j)j!(2k-j)!}B_jB_{2k-j}, \tag{12}$$

where B_k is given by (8). Equation (12) can be useful to approximate the distribution of V using a finite sequence of its moments.

Now we are in a position to give the main result of the paper, that is, the expression for the probability density function of the variance V of p.

THEOREM 3.1. *Under the hypothesis of Proposition* 3.1, *the probability density function of the variance V is given by*

$$
\begin{aligned}
fv(v) &= \frac{(-1)^a}{\Gamma(a)} v^{a-1} \frac{d^a}{dv^a}\left[\int_{4v}^{+\infty} u^{1/2}(u-4v)^{-1/2}\varphi(u,a)\,du\right] I_{(0,+\infty)}(v) \\
&= \frac{(-1)^a}{\Gamma(a)} v^{a-1} 4^a \left[\int_{4v}^{+\infty} (u-4v)^{-1/2}\left\{\frac{d^a}{du^a} u^{1/2}\varphi(u,a)\right\} du\right] I_{(0,+\infty)}(v).
\end{aligned}
\tag{13}
$$

PROOF. Consider the function ${}_1F_1(a;1/2;-zu/4)$ in (10) and write, using well-known properties of this hypergeometric function [Gradshteyn and Ryzhik (1980), page 1086],

$$
\begin{aligned}
{}_1F_1(a;1/2;-zu/4) &= 1 + \sum_{j=1}^{a} (-1)^j \frac{\binom{a}{j}}{2^j(2j-1)!!}(zu)^j\, {}_1F_1(j;(2j+1)/2;-zu/4) \\
&= 1 + \sum_{j=1}^{a} (-1)^j \frac{\binom{a}{j}}{2^j(2j-1)!!}(zu)^j \frac{\Gamma(j+1/2)2^{2j}}{\Gamma(j)\Gamma(1/2)u^{j-1/2}} \\
&\qquad \times \int_0^{u/4} e^{-zv} v^{j-1}(u-4v)^{-1/2}\,dv.
\end{aligned}
$$

Hence, from (10),

$$
\begin{aligned}
\mathrm{E}(e^{-zV}) &= 1 + \sum_{j=1}^{a} (-1)^j \frac{\binom{a}{j} 2^j z^j \Gamma(j+1/2)}{(2j-1)!!\Gamma(j)\Gamma(1/2)} \\
&\qquad \times \int_0^{+\infty} u^{1/2}\left(\int_0^{u/4} e^{-zv} v^{j-1}(u-4v)^{-1/2}\,dv\right)\varphi(u,a)\,du \\
&= 1 + \sum_{j=1}^{a} (-1)^j \frac{\binom{a}{j} 2^j z^j \Gamma(j+1/2)}{(2j-1)!!\Gamma(j)\Gamma(1/2)} \int_0^{+\infty} e^{-zv} v^{j-1} \\
&\qquad \times \left[\int_{4v}^{+\infty} u^{1/2}(u-4v)^{-1/2}\varphi(u,a)\,du\right] dv \\
&= 1 + \sum_{j=1}^{a} (-1)^j \binom{a}{j} z^j \int_0^{+\infty} e^{-zv} \frac{v^{j-1}}{(j-1)!} \\
&\qquad \times \left[\int_{4v}^{+\infty} u^{1/2}(u-4v)^{-1/2}\varphi(u,a)\,du\right] dv
\end{aligned}
\tag{14}
$$

and then [see Erdelyi (1954), page 130, formula (13)]

$$
\mathrm{E}(e^{-zV}) = \int_0^{+\infty} e^{-zV} \frac{(-1)^a}{\Gamma(a)} v^{a-1} \frac{d^a}{dv^a}\left[\int_{4v}^{+\infty} u^{1/2}(u-4v)^{-1/2}\varphi(u,a)\,du\right] dv
$$

for every $z > 0$, so that f_V in (13) is a version of the probability density of V. The second expression in (13) is easily derived by means of integration by parts and differentiations. □

REMARK 3.3. When α is a probability measure, that is, $a = 1$, then the distribution function of V is

$$F_V(t) = \begin{cases} 0, & t < 0, \\ 1 - \int_{4t}^{+\infty} u^{1/2}(u-4t)^{-1/2}\varphi(u,1)\,du, & t \geq 0. \end{cases}$$

REMARK 3.4. As in the bootstrap application given in the introduction, let us suppose that $\{X_n\}_{n\geq 1}$ is a sequence of nonnegative exchangeable r.v.'s with de Finetti's measure given by a Dirichlet process. Then, when α satisfies (3) with $\psi(x) = x^2$, it is easy to show that, for $n \to +\infty$,

$$q\left(\left\{\omega : \frac{1}{n(n-1)} \sum_{1\leq i<j\leq n} (X_i(\omega) - X_j(\omega))^2 \leq v\right\}\right) \to F_v(v)$$

for each v in $\mathbb{R}$, where F_V is the probability distribution function whose density is given in (13).

The remaining part of the section is devoted to the illustration of some examples in which the distribution of the variance V is explicitly given.

EXAMPLE 3.1. Take as parameter α of the Dirichlet process the measure giving mass 1 to each of the points $0, 1, \dots, k-1$; in this case, $a = \alpha([0, +\infty)) = k \geq 2$ and (6) of Section 2 gives, for $x \in (0, k-1)$,

$$M'(x) = (k-1) \sum_{j=0}^{k-1} (-1)^j \frac{(x-j)_+^{k-2}}{j!(k-j-1)!}.$$

Then, for the r.v. U we have, for $u \in (0, (k-1)^2)$,

$$\varphi(u,k) = \frac{1}{\Gamma(2k-2)\sqrt{u}} \sum_{q=0}^{2k-2} (-1)^q (\sqrt{u} + k - 1 - q)_+^{2k-3} \binom{2k-2}{q}.$$

For instance, if $k = 3$, so that $\alpha = \delta_0 + \delta_1 + \delta_2$, then

$$M'(x) = \begin{cases} x, & \text{if } x \in (0,1), \\ 2 - x, & \text{if } x \in (1,2), \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varphi(u,3) = \begin{cases} u/2 - \sqrt{u} + 2u^{-1/2}/3, & \text{if } u \in (0,1), \\ (2-\sqrt{u})^3 u^{-1/2}/6, & \text{if } u \in (1,4), \\ 0, & \text{otherwise.} \end{cases}$$

Then, substitution in (13) gives

$$f_V(v) = \begin{cases} 2\sqrt{1-v} - 2\sqrt{1-4v}, & \text{if } v \in (0,1/4), \\ 2\sqrt{1-v}, & \text{if } v \in (1/4,1), \\ 0, & \text{otherwise.} \end{cases}$$

More generally, if α has k masses$=1$ concentrated on the points $0 \le x_1 < x_2 < \cdots < x_k$, with $k \ge 3$, it is possible to show that, for $v \in (0, (x_k - x_1)^2/4)$,

$$\begin{aligned} f_V(v) = (k-1)(k-2) \sum_{1 \le q < j \le k} & \frac{(-1)^{q+j-1}(x_j - x_q)}{\prod_{i \ne q} |x_i - x_q| \prod_{i \ne j} |x_i - x_j|} \\ & \times \int_0^v (v-u)^{k-3} \left(\frac{(x_j - x_q)^2}{4} - u \right)_+^{-1/2} du. \end{aligned} \tag{15}$$

EXAMPLE 3.2. Take $\alpha = \delta_0 + 2\delta_1 + \delta_2$, so that $a = 4$. Then, (6) of Section 2 gives

$$M'(x) = \begin{cases} 3x^2/2, & \text{if } x \in (0,1), \\ 3(x-2)^2/2, & \text{if } x \in (1,2), \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varphi(u,4) = \begin{cases} -3u^{1/2} + (9/10)u^{-1/2} - (3/40)u^2 - (3/4)u^{3/2} + 3u, & \text{if } 0 < u < 1, \\ (3/40)(2 - \sqrt{u})^5/\sqrt{u}, & \text{if } 1 < u < 4, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the probability density of V is given by

$$f_V(v) = \begin{cases} 4(1-v)(\sqrt{1-v} - \sqrt{1-4v}), & \text{if, } v \in (0, 1/4), \\ 4(1-v)^{3/2}, & \text{if } v \in (1/4, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Now we give a proposition due to Lu and Richards (1993) which, together with (11), turns out to be useful in order to explicitly obtain the distribution of the variance in some other simple cases.

PROPOSITION 3.2. *If X and Y are independent and identically distributed according to a Beta distribution with parameters c and d, then $U = (X - Y)^2 =^{\mathrm{st}} Y_1 Y_2 Y_3$, where $Y_1 \sim \mathrm{Beta}(c, d)$, $Y_2 \sim \mathrm{Beta}(1/2, (c+d)/2)$,*

$$Y_3 \sim \begin{cases} \mathrm{Beta}(\min(c,d), |c-d|/2), & \textit{if } c \ne d, \\ \delta_1, & \textit{if } c = d \end{cases}$$

and Y_1, Y_2, Y_3 are mutually independent.

EXAMPLE 3.3. (The easiest example) Take as p a Dirichlet process with parameter $\alpha = a_1\delta_1 + a_0\delta_0$, where $a = a_0 + a_1$ is an arbitrary integer and $a_0 < a_1$. From case (1) of Section 2, the distribution of W_1 is Beta with parameters a_1 and a_0; if X and Y are independent and both have this distribution, then, by the previous proposition, $U = (X - Y)^2 =^{\mathrm{st}} Y_1 Y_2 Y_3$, where $Y_1 \sim \mathrm{Beta}(a_1, a_0)$, $Y_2 \sim \mathrm{Beta}(1/2, a/2)$, $Y_3 \sim \mathrm{Beta}(a_0, (a_1 - a_0)/2)$ and Y_1, Y_2, Y_3 are independent. Hence we can write, for each $m \ge 1$,

$$\begin{aligned} \mathrm{E}(U^m) &= \prod_{i=1}^{3} \mathrm{E}(Y_i^m) \\ &= \frac{\Gamma(a_1 + m)\Gamma(a_0 + m)\Gamma((a+1)/2)\Gamma(a/2)\Gamma(a)\Gamma(m+1/2)}{\Gamma(a_1)\Gamma(a_0)\Gamma(m + (a+1)/2)\Gamma(m + a/2)\Gamma(a+m)\Gamma(1/2)}, \end{aligned}$$

so that, by (11),

$$
\begin{aligned}
\mathrm{E}(V^m) &= \frac{\Gamma(m+a)\Gamma(1/2)}{4^m\Gamma(m+1/2)\Gamma(a)}\mathrm{E}(U^m)\\
&= \frac{B(m+a_1, m+a_0)}{B(a_1, a_0)}\\
&= \frac{1}{B(a_1, a_0)}\int_0^1 [x(1-x)]^m x^{a_1-1}(1-x)^{a_0-1}\,dx\\
&= \frac{1}{2^{a-2}B(a_1, a_0)}\\
&\quad\times\int_0^{1/4} t^m \frac{(1-\sqrt{1-4t})^{a_1-1}(1+\sqrt{1-4t})^{a_0-1}+(1+\sqrt{1-4t})^{a_1-1}(1-\sqrt{1-4t})^{a_0-1}}{\sqrt{1-4t}}\,dt.
\end{aligned}
$$

Then, we can conclude that the variance V of p has probability density

$$
\begin{aligned}
f_V(t) = {}& \frac{1}{2^{a-2}B(a_1, a_0)\sqrt{1-4t}}\\
&\times\{(1-\sqrt{1-4t})^{a_1-1}(1+\sqrt{1-4t})^{a_0-1}+(1+\sqrt{1-4t})^{a_1-1}\\
&\times(1-\sqrt{1-4t})^{a_0-1}\},
\end{aligned}
$$

for $t \in (0, 1/4)$. Note that in this case, of course, $V =^{\mathrm{st}} W_1(1-W_1)$.

EXAMPLE 3.4. Consider a Dirichlet process with parameter α as given in case (2) of Section 2, so that W_1 has a uniform distribution on $(0, 1)$. By Proposition 2.3 $\mathrm{E}(U^n) = 1/[2(n+1)(n+1/2)]$ and hence, by (11),

$$
\begin{aligned}
\mathrm{E}(V^n) &= \frac{\Gamma(1/2)\Gamma(n+1)}{4^n\Gamma(n+1/2)}\frac{1/2}{(n+1)(n+1/2)} = \frac{B(1/2, n+1)}{4^n 2(n+1)}\\
&= \frac{1}{4^n 2}\int_0^1 x^n\,dx\int_0^1 y^{-1/2}(1-y)^n dy\\
&= 2\int_0^1 y^{-1/2}(1-y)^{-1}\left\{\int_0^{(1-y)/4} t^n\,dt\right\}dy\\
&= \int_0^{1/4} t^n\left\{2\int_0^{1-4t} y^{-1/2}(1-y)^{-1}dy\right\}dt.
\end{aligned}
$$

Hence, the density of V is

$$
\begin{aligned}
f_V(t) &= 2\int_0^{1-4t} y^{-1/2}(1-y)^{-1}dy\\
&= 4\ \mathrm{Arth}(\sqrt{1-4t}, \quad t \in (0, 1/4).
\end{aligned}
$$

EXAMPLE 3.5. We now consider some examples in which α has, at least as one component, a Beta distribution with parameter $(1/2, 1/2)$.

(α) First suppose α to be such that $\alpha(dx) = a\pi^{-1}x^{-1/2}(1-x)^{-1/2}I_{(0,1)}(x)\,dx$, with a arbitrary positive integer. Case (3) of Section 2 shows that W_1 has density $M'(x) = B(a+1/2, a+1/2)^{-1}x^{a-1/2}(1-x)^{a-1/2}I_{(0,1)}(x)$. Then, by

Proposition 2.3, U is distributed as $Y_1 \cdot Y_2$, where Y_1 and Y_2 have Beta distributions with parameters $a+1/2$, $a+1/2$ and, respectively, $1/2, a+1/2$. Hence by (11), we have, for $n \geq 1$,

$$\begin{aligned}
\mathrm{E}(V^n) &= \frac{\Gamma(a+n)\Gamma(1/2)}{4^n\Gamma(n+1/2)\Gamma(a)}\mathrm{E}(U^n)\\
&= \frac{a}{4^n(a+n)}\frac{\Gamma(2a+1)\Gamma(n+a+1/2)}{\Gamma(a+1/2)\Gamma(2a+n+1)}\\
&= \frac{a4^a}{(a+n)4^{a+n}}\frac{B(n+a+1/2, a+1/2)}{B(a+1/2, a+1/2)}\\
&= \frac{a4^a}{B(a+1/2, a+1/2)}\frac{1}{(a+n)4^{a+n}}\int_0^1 x^{a+n-1/2}(1-x)^{a-1/2}\,dx\\
&= \frac{a4^a}{B(a+1/2, a+1/2)}\int_0^{1/4} v^{a+n-1}\left\{\int_{4v}^1 x^{-1/2}(1-x)^{a-1/2}\,dx\right\}dv\\
&= \int_0^{1/4} v^n\left\{\frac{4^a a v^{a-1}}{B(a+1/2, a+1/2)}\int_{4v}^1 x^{-1/2}(1-x)^{a-1/2}\,dx\right\}dv.
\end{aligned}$$

Then, the density of V is

$$f_V(v) = \frac{4^a a v^{a-1}}{B(a+1/2, a+1/2)}\int_{4v}^1 x^{-1/2}(1-x)^{a-1/2}\,dx, \qquad 0 < v < 1/4.$$

(β) Take α as in case (4) of Section 2. Then, $M'(x) = B(a-1/2, a+1/2)^{-1}x^{a-3/2}\cdot(1-x)^{a-1/2}$. The same procedure followed above yields

$$f_V(v) = \frac{\Gamma(2a)4^{a-1/2}}{\Gamma(a-1/2)^2}v^{a-3/2}\int_{4v}^1 x^{-1}(1-x)^{a-1/2}\,dx, \qquad 0 < v < 1/4.$$

(γ) Now suppose α is as in case (5) of Section 2, so that

$$M'(x) = \frac{x^{a-3/2}(1-x)^{a-3/2}}{B(a-1/2, a-1/2)}, \qquad 0 < x < 1;$$

then

$$f_V(v) = \frac{\Gamma(a)4^{2a-3/2}}{\Gamma(1/2)\Gamma(a-1/2)}[v(1-4v)]^{a-3/2}, \qquad v \in (0, 1/4).$$

4. The proof of Proposition 3.1. In this section we prove Proposition 3.1. The section is divided into two parts: Section 4.1 includes the proof when the base measure α is diffuse or has, at most, concentrated masses less than 1; Section 4.2 deals with the case in which α also has concentrated masses ≥ 1. We introduce, for brevity, the following notations:

$$h_a^{\psi}(x) = \exp\left\{-\int_{[0,+\infty)}\log|t-x|\,dA_{\psi}(t)\right\}, \qquad a = A(+\infty)$$

and

$$g_a(x) = \frac{1}{\pi}\sin\{\pi A(x)\}\exp\left\{-\int_{[0,+\infty)}\log|t-x|\,dA(t)\right\} = \frac{1}{\pi}\sin\{\pi A(x)\}h_a^x(x).$$

4.1. *The base measure α has no concentrated masses greater than or equal to* 1. The proof of Proposition 3.1 in this first case will be given, for better readability, in more steps.

STEP 1. As a first step, we prove that, if α has support contained in a compact interval $[0, C]$ for some $C > 0$, then, for positive z_1 and z_2,

$$\begin{aligned}
&\mathrm{E}\big(\exp(-z_1 W_1 - z_2 W_2)\big)\\
&\quad= \frac{\Gamma(a)}{z_2^{a-1}}\delta_{1,a}\int_0^{+\infty}\exp\Big(-z_1 x - z_2 x^2\Big) g_a(x)\,dx\\
&\qquad+\frac{\Gamma(a)}{z_2^{a-2}}\int_0^{\infty}\exp(-z_1 t)\bigg\{\int_0^t \exp(-z_2 x^2) g_a(x)\\
&\qquad\qquad\times\bigg[\int_0^{\infty}\exp(-z_2(t-x)(x+v))\\
&\qquad\qquad\qquad\times(x-v)g_a(v)dv\bigg]dx\bigg\}\,dt.
\end{aligned}\tag{16}$$

To prove (16), first suppose $a = 1$. By Theorem 2.1, a version of the density of W_1 is $g_1(x)$. Taking in (5), for positive z_1 and z_2, $\psi(x) = z_1 x + z_2 x^2$, we have

$$\begin{aligned}
&\mathrm{E}\big(\exp(-z_1 W_1 - z_2 W_2)\big)\\
&\quad=\int_0^{+\infty}\exp(-x)\frac{1}{\pi}\sin\{\pi A_\psi(x)\} h_a^\psi(x)\,dx\\
&\quad=\frac{1}{\pi}\int_0^{+\infty}\exp\Big(-z_1 u - z_2 u^2\Big)(z_1 + 2z_2 u)\sin[\pi A(u)]\\
&\qquad\qquad\times\exp\bigg\{-\int_{[0,+\infty)}[\log|t-u| + \log|z_1 + z_2 u + z_2 t|]\,dA(t)\bigg\}\,du\\
&\quad=\int_0^{+\infty}\exp\Big(-z_1 u - z_2 u^2\Big)(z_1 + 2z_2 u) g_1(u)\\
&\qquad\times\exp\bigg\{-\int_{[0,+\infty)}\log|z_1 + z_2 u + z_2 t|\,dA(t)\bigg\}\,du\\
&\quad=\int_0^{+\infty}\exp(-z_1 u - z_2 u^2)\bigg(\int_0^{+\infty}\frac{z_1 + 2z_2 u}{z_1 + z_2 u + z_2 v}\\
&\qquad\qquad\qquad\times g_1(v)\,dv\bigg) g_1(u)\,du\\
&\quad=\int_{[0,+\infty)^2}\exp\Big(-z_1 u - z_2 u^2\Big)\frac{z_1 + 2z_2 u}{z_1 + z_2 u + z_2 v}\\
&\qquad\times g_1(v) g_1(u)\,dv\,du.
\end{aligned}\tag{17}$$

Since

$$\begin{aligned}
&\exp(-z_1 u)\frac{z_1 + 2z_2 u}{z_1 + z_2 u + z_2 v}\\
&\qquad= z_2(u-v)\int_0^{+\infty}\exp(-z_1 t - z_2)(t-u)(u+v)\,\mathrm{dt} + \exp(-z_1 u),
\end{aligned}\tag{18}$$

from (17) we have

$$\begin{aligned}
&\mathrm{E}\big(\exp(-z_1 W_1 - z_2 W_2)\big)\\
&\quad= \int_0^{+\infty} \exp\Big(-z_1 u - z_2 u^2\Big) g_1(u)\, du\\
&\qquad + z_2 \int_0^{+\infty}\int_0^{+\infty} \exp\Big(-z_2 U^2\Big)(u-v) g_1(u) g_1(v)\\
&\qquad\qquad \times \left[\int_u^{+\infty} \exp\big(-z_1 t - z_2 (t-u)(u+v)\big)\, dt\right] dv\, du\\
&\quad= \int_0^{+\infty} \exp\Big(-z_1 u - z_2 u^2\Big) g_1(u)\, du\\
&\qquad + z_2 \int_0^{+\infty} \exp(-z_1 t)\bigg(\int_0^t \bigg\{\int_0^{+\infty} \exp\Big(-z_2 u^2 - z_2 (t-u)\\
&\qquad\qquad \times (u+v)\Big)(u-v) g_1(u) g_1(v)\, dv\bigg\}\, du\bigg)\, dt,
\end{aligned}$$

which is the thesis for $a = 1$.

Now suppose a to be an integer greater than or equal to 2. In this case, a version of the density of the r.v. $Y(\omega) = z_1 W_1(\omega) + z_2 W_2(\omega)$ is, for every $x \geq 0$, $M'_\psi(x) = (a - 1/\pi) \int_0^x (x-u)^{a-2} \sin\{\pi A_\psi(u)\} h_a^\psi(u)\, du$. Hence

$$\begin{aligned}
&\mathrm{E}\big(\exp(-z_1 W_1 - z_2 W_2)\big) = \mathrm{E}(\exp(-Y)) = \int_0^{+\infty} \exp(-x) M'_\psi(x)\, dx\\
&\quad = \int_0^{+\infty} \exp(-x)\left[\frac{a-1}{\pi} \int_0^x (x-u)^{a-2} \sin\{\pi A_\psi(u)\} h_a^\psi(u)\, du\right] dx,
\end{aligned} \tag{19}$$

which is finite, since the function

$$\rho(x,u) = \frac{a-1}{\pi} \exp(-x)(x-u)^{a-2} \sin\{\pi A_\psi(u)\} h_a^\psi(u)$$

is integrable with respect to the Lebesgue measure on the region $S = \{(x,u) \in (0,+\infty)^2 \colon x > u\}$. Indeed, if ρ, a measurable function, were not integrable, that is, $\int_S |\rho(x,u)|\, dx\, du = +\infty$, then

$$\begin{aligned}
&\int_0^{+\infty} \frac{a-1}{\pi}\left\{\int_0^{+\infty} y^{a-2} \exp(-y-u) dy\right\} |\sin\{\pi A_\psi(u)\}| h_a^\psi(u)\, du\\
&\quad = \Gamma(a) \int_0^{+\infty} e^{-u} \frac{1}{\pi} |\sin\{\pi A_\psi(u)\}| h_a^\psi(u)\, du = +\infty.
\end{aligned}$$

But this is not possible, since, for every $u > 0$,

$$\begin{aligned}
&e^{-u} \frac{1}{\pi} |\sin\{\pi A_\psi(u)\}| h_a^\psi(u)\\
&\quad < \frac{1}{\pi} |\sin\{\pi A_\psi(u)\}| h_a^\psi(u) = |\sigma_a(u)|
\end{aligned}$$

and σ_a is integrable on $(0, +\infty)$. Then, we can apply the Fubini theorem and change the order of integration in (19), obtaining

$$\begin{aligned}
&\mathrm{E}\big(\exp(-z_1 W_1 - z_2 W_2)\big)\\
&\quad = \Gamma(a) \int_0^{+\infty} e^{-u} \frac{1}{\pi} \sin\{\pi A_\psi(u)\} h_a^\psi(u)\,du\\
&\quad = \Gamma(a) \frac{1}{\pi} \int_0^{+\infty} \exp\Big(-z_1 x - z_2 x^2\Big)(z_1 + 2z_2 x)\\
&\qquad\qquad \times \sin(\pi A(x)) h_a^1(x) \exp\bigg\{-\int_{[0,\infty)} \log|z_1 + z_2 x + z_2 v|\,dA(v)\bigg\}\,dx \qquad (20)\\
&\quad = \Gamma(a) \int_0^{+\infty} \exp\Big(-z_1 x - z_2 x^2\Big)(z_1 + 2z_2 x) g_a(x)\\
&\qquad\qquad \times \exp\bigg\{-\int_{[0,+\infty)} \log|z_1 + z_2 x + z_2 v|\,dA(v)\bigg\}\,dx.
\end{aligned}$$

But, from (4),

$$\begin{aligned}
&\exp\bigg\{-\int_{[0,+\infty)} \log|z_1 + z_2 x + z_2 v|\,dA(v)\bigg\}\\
&\quad = \frac{1}{z_2^a} \int_0^{+\infty} \bigg(\frac{z_1 + z_2 x}{z_2} + v\bigg)^{-a} \bigg[(a-1) \int_0^v (v-u)^{a-2} g_a(u)\,du\bigg]\,dv\\
&\quad = \frac{a-1}{z_2^a} \int_0^{+\infty} g_a(u) \bigg[\int_0^{+\infty} y^{a-2} \bigg(\frac{z_1 + z_2 x}{z_2} + y + u\bigg)^{-a} dy\bigg]\,du\\
&\quad = \frac{1}{z_2^{a-1}} \int_0^{+\infty} \frac{1}{z_1 + z_2 x + z_2 u} g_a(u)\,du,
\end{aligned}$$

so that

$$\begin{aligned}
&\Gamma(a) \int_0^{+\infty} \exp(-z_1 x - z_2 x^2)(z_1 + 2z_2 x) g_a(x)\\
&\qquad \times \exp\bigg\{-\int_{[0,+\infty)} \log|z_1 + z_2 x + z_2 v|\,dA(v)\bigg\}\,dx \qquad (21)\\
&\quad = \frac{\Gamma(a)}{z_2^{a-1}} \int_0^{+\infty} \bigg[\int_0^{+\infty} \exp(-z_1 x - z_2 x^2) \frac{z_1 + 2z_2 x}{z_1 + z_2 x + z_2 v} g_a(x) g_a(v)\,dx\bigg]\,dv,
\end{aligned}$$

where the change in the order of integration is justified as in the case before. Then, from (20) and (21) there follows

$$
\begin{aligned}
&\mathrm{E}\exp(-z_1W_1-z_2W_2)\\
&\quad=\frac{\Gamma(a)}{z_2^{a-1}}\int_0^{+\infty}\int_0^{+\infty}\exp(z_1x-z_2x^2)g_a(x)g_a(v)\,dx\,dv\\
&\qquad+\frac{\Gamma(a)}{z_2^{a-2}}\int_0^{+\infty}\int_0^{+\infty}\exp(-z_2x^2)(x-v)g_a(x)g_a(v)\\
&\qquad\qquad\times\left(\int_0^{+\infty}\exp(-z_1t)\exp(-z_2(t-x)(x+v))\,dt\right)dx\,dv\\
&\quad=\frac{\Gamma(a)}{z_2^{a-1}}\int_0^{+\infty}\exp(-z_1x-z_2x^2)g_a(x)\left\{\int_0^{+\infty}g_a(v)\,dv\right\}dx\\
&\qquad+\frac{\Gamma(a)}{z_2^{a-2}}\int_0^{+\infty}\exp(-z_1t)\left\{\int_0^t\left[\int_0^{+\infty}\exp\big(-z_2x^2-z_2(t-x)\right.\right.\\
&\qquad\qquad\left.\left.\times(x+v)\big)(x-v)g_a(v)\,dv\right]g_a(x)\,dx\right\}dt.
\end{aligned}
\tag{22}
$$

Now observe that, by Theorem 2.1, the density of W_1 is

$$
M'(x)=(a-1)\int_0^x(x-u)^{a-2}g_a(u)\,du, \tag{23}
$$

from which we obtain the equality $M^{(a)}(x)=\Gamma(a)g_a(x)$, true for almost all x in $[0,+\infty)$. Hence

$$
\int_0^{+\infty}g_a(v)\,dv=\Gamma(a)^{-1}\int_0^{+\infty}M^{(a)}(v)\,dv=\Gamma(a)^{-1}\lim_{x\to+\infty}M^{(a-1)}(x)=0,
$$

so that (22) becomes

$$
\begin{aligned}
&\frac{\Gamma(a)}{z_2^{a-2}}\int_0^{+\infty}\exp(-z_1t)\left\{\int_0^t\exp(-z_2x^2)g_a(x)\right.\\
&\qquad\left.\times\left[\int_0^{+\infty}\exp\big(-z_2(t-x)(x+v)\big)(x-v)g_a(v)\,dv\right]dx\right\}dt,
\end{aligned}
\tag{24}
$$

which is the thesis for $a\geq 2$; this completes the proof of (16). □

STEP 2. Now we prove that, if α has support contained in $[0,C]$, then

$$
E(e^{-zV})=\int_0^{+\infty}{}_1F_1(a;1/2;-zu/4)\varphi(u,a)\,du;
$$

that is, the thesis of Proposition 3.1 is true.

Indeed, consider the transform (16) and invert it with respect to z_1, so that

$$
\begin{aligned}
&E\big(\exp(-z_2 W_2) I_{(w,\, w+dw)}(W_1)\big)\\
&\quad = \delta_{1,\,a} \exp\big(-z_2 w^2\big) g_a(w)\, dw\\
&\qquad + \frac{\Gamma(a)}{z_2^{a-2}} \int_0^w \exp\big(-z_2 x^2\big) g_a(x)\\
&\qquad \times \bigg[\int_0^{+\infty} \exp\big(-z_2 (w-x)(x+v)\big)\\
&\qquad\qquad \times (x-v) g_a(v)\, dv\bigg]\, dx\, dw.
\end{aligned}
\tag{25}
$$

Then when multiplying both sides by $\exp(z_2 w^2)$ and integrating with respect to w we have (with $z_2 = z > 0$)

$$
\begin{aligned}
&\mathrm{E}(e^{-zV})\\
&\quad = \delta_{1,\,a} + \frac{\Gamma(a)}{z^{a-2}} \int_0^{+\infty} \bigg\{\int_0^w g_a(x) \bigg[\int_0^{+\infty} \exp\big(-z(w-x)(v-w)\big)\\
&\qquad\qquad \times (x-v) g_a(v)\, dv\bigg] dx\bigg\}\, dw\\
&\quad = \delta_{1,\,a} - \frac{\Gamma(a)}{z^{a-2}} \int_0^{+\infty} \bigg\{\int_0^w g_a(x) \bigg[\int_w^{+\infty} \exp\big(-z(w-x)(v-w)\big)\\
&\qquad\qquad \times (v-x) g_a(v)\, dv\bigg]\, dx\bigg\}\, dw.
\end{aligned}
$$

Since $M^{(a)}(x) = \Gamma(a) g_a(x)$ a.e., we can write

$$
\begin{aligned}
&\mathrm{E}(e^{-zV})\\
&\quad = \delta_{1,\,a} - \frac{1}{\Gamma(a) z^{a-2}} \int_0^{+\infty} \bigg\{\int_0^w M^{(a)}(x) \bigg[\int_0^{+\infty} \exp\big(-z(w-x)(v-w)\big)\\
&\qquad\qquad \times (v-x) M^{(a)}(v)\, dv\bigg]\, dx\bigg\}\, dw\\
&\quad = \delta_{1,\,a} - \frac{1}{\Gamma(a) z^{a-2}} \int_0^{+\infty} M^{(a)}(x) \bigg\{\int_x^{+\infty} (v-x) M^{(a)}(v)\\
&\qquad\qquad \times \bigg[\int_x^v \exp\big(-z(w-x)(v-w)\big)\, dw\bigg]\, dv\bigg\}\, dx.
\end{aligned}
$$

Writing $\int_x^v \exp\{-z(w-x)(v-w)\}\, dw = \int_x^{(x+v)/2} \exp\{-z(w-x)(v-w)\, dw + \int_{(x+v)/2}^v \exp\{-z(w-x)(v-w)\}\, dw$ and making in each of these integrals the change of variable $t = 4(w-x)(v-w)(v-x)^{-2}$, we have

$$\begin{aligned}
&\mathrm{E}(e^{-zV})\\
&\quad = \delta_{1,a} - \frac{1}{\Gamma(a)z^{a-2}} \int_0^{+\infty} M^{(a)}(x) \biggl\{ \int_x^{+\infty} (v-x)^2 M^{(a)}(v)\\
&\qquad\qquad \times \biggl[\frac{1}{2} \int_0^1 \exp\Bigl(-\frac{1}{4} z t (v-x)^2\Bigr)(1-t) - 1/2 dt \biggr] dv \biggr\} dx \\
&\quad = \delta_{1,a} - \frac{1}{\Gamma(a)z^{a-2}} \int_0^{+\infty} M^{(a)}(x)\\
&\qquad\qquad \times \biggl\{ \int_x^{+\infty} (v-x)^2 M^{(a)}(v) {}_1F_1\Bigl(1; \frac{3}{2}; \frac{-z(v-x)^2}{4}\Bigr) dv \biggr\} dx.
\end{aligned} \tag{26}$$

Integrating by parts, first with respect to x and then with respect to v, and iterating this process $a-1$ times, from (26) we obtain

$$\begin{aligned}
&\mathrm{E}(e^{-zV})\\
&\quad = \delta_{1,a} - \frac{1}{\Gamma(a)z^{a-2}} \int_0^{+\infty} M'(x) \biggl\{ \int_0^{+\infty} M'(v) \frac{d^{2a-2}}{dv^{a-1}dx^{a-1}}\\
&\qquad\qquad \times \biggl[(v-x)^2 {}_1F_1\Bigl(1; \frac{3}{2}; \frac{-z(v-x)^2}{4}\Bigr) \biggr] dv \biggr\} dx.
\end{aligned}$$

Since $(v-x)^2 {}_1F_1(1; 3/2; -z(v-x)^2/4) = 2z^{-1}(1 - {}_1F_1(1; 1/2; -z(v-x)^2/4)$, then

$$\begin{aligned}
\mathrm{E}(e^{-zV}) &= \delta_{1,a} - \frac{2}{\Gamma(a)z^{a-1}} \int_0^{+\infty} M'(x)\\
&\qquad \times \biggl\{ \int_x^{+\infty} M'(v) \biggl[\delta_{1,a} - \frac{d^{2a-2}}{dv^{a-1}dx^{a-1}} {}_1F_1\Bigl(1; \frac{1}{2}; \frac{-z(v-x)^2}{4}\Bigr) \biggr] dv \biggr\} dx\\
&= \delta_{1,a} - \frac{2}{\Gamma(a)z^{a-1}} \delta_{1,a} \int_0^{+\infty} M'(x) \biggl\{ \int_x^{+\infty} M'(v) dv \biggr\} dx + \frac{2}{\Gamma(a)z^{a-1}}\\
&\qquad \times \int_0^{+\infty} M'(x) \biggl\{ \int_x^{+\infty} M'(v) \biggl[\frac{d^{2a-2}}{dv^{a-1}dx^{a-1}} {}_1F_1\Bigl(1; \frac{1}{2}; \frac{-z(v-x)^2}{4}\Bigr) \biggr] dv \biggr\} dx.
\end{aligned} \tag{27}$$

Using well-known properties of the confluent hypergeometric function we can write

$$\begin{aligned}
&\frac{d^{2a-2}}{dv^{a-1}dx^{a-1}} {}_1F_1\Bigl(1; \frac{1}{2}; \frac{-z(v-x)^2}{4}\Bigr)\\
&\qquad = (-1)^{a-1} \frac{d^{2a-2}}{dv^{2a-2}} {}_1F_1\Bigl(1; \frac{1}{2}; \frac{-z(v-x)^2}{4}\Bigr)\\
&\qquad = \Gamma(a) z^{a-1} {}_1F_1\Bigl(a; \frac{1}{2}; \frac{-z(v-x)^2}{4}\Bigr).
\end{aligned} \tag{28}$$

Finally, by substituting (28) in (27) and observing that

$$\int_0^{+\infty} M'(x)\left\{\int_x^{+\infty} M'(v)\,dv\right\}dx = 1/2$$

we have

$$\begin{aligned}
\mathrm{E}\,(e^{-zV}) &= \delta_{1,a} - \frac{\delta_{1,a}}{\Gamma(a)z^{a-1}} \\
&\quad + 2\int_0^{+\infty} M'(x)\left\{\int_x^{+\infty} M'(v)\,{}_1F_1\left(a;\frac{1}{2};\frac{-z(v-x)^2}{4}\right)dv\right\}dx \qquad (29)\\
&= \int_0^{+\infty} {}_1F_1\left(a;\frac{1}{2};\frac{-zu}{4}\right)\varphi(u,a)\,du. \qquad \square
\end{aligned}$$

STEP 3. Now we complete the proof of Proposition 3.1 in the first case, showing that its thesis is true also when α does not have bounded support. To this aim, consider the measure α^C whose distribution function is defined in (9), the Dirichlet process p^C whose parameter is α^c and the r.v. $V^C(\omega) = \int (x - W_1^C(\omega))^2 p^C(dx,\omega)$. Denote by Φ and Φ^C, respectively, the distribution functions of $(X-Y)^2$ and $(X^C - Y^C)^2$, where X^C and Y^C are independent r.v.'s distributed as W_1^C. Then, since ${}_1F_1(a;1/2;t)$ is, for $t<0$, a bounded and continuous function, from (29) and Proposition 2.1 there follows

$$\begin{aligned}
\mathrm{E}\,(e^{-zV}) &= \lim_{C\to+\infty} \mathrm{E}\big(\exp(-zV^C)\big) \\
&= \lim_{C\to+\infty} \int_0^{+\infty} {}_1F_1\left(a;\frac{1}{2};-\frac{zu}{4}\right)d\Phi^C(u,a) \\
&= \int_0^{+\infty} {}_1F_1\left(a;\frac{1}{2};-\frac{zu}{4}\right)d\Phi(u,a) \\
&= \int_0^{+\infty} {}_1F_1\left(a;\frac{1}{2};-\frac{zu}{4}\right)\varphi(u,a)\,du
\end{aligned}$$

and the proof of Proposition 3.1 is then complete when α has no concentrated mass greater than or equal to 1. $\square$

4.2. *The base measure α also has k integer massess ≥ 1 concentrated on the points $x_1, x_2, \ldots, x_k$.* It is enough to consider the case of a single mass equal to 1 on a point $\xi \geq 0$. Hence, consider a Dirichlet process with parameter $\beta = \alpha + \delta_\xi$, where α has no concentrated mass greater than or equal to 1 and ξ is a nonnegative number; for the sake of simplicity, we will give the proof when $a = \alpha([0,+\infty)) = 1$. Let us call p_β this process, while p_α will denote a Dirichlet process with parameter α; moreover, let $W_{k,\beta}$ and V_β be the kth moment and the variance of p_β and $W_{k,\alpha}$ and V_α the same quantities relative to p_α. When the moments of $W_{1,\alpha}$ exist, to prove Proposition 3.1 we will show that condition (11) holds. We can observe that, by (4), the following relationships between the moments of p_β and those of p_α hold:

$$W_{1,\,\beta|y} \overset{\text{st}}{=} (1-y)\xi + yW_{1,\,\alpha}; \tag{30}$$

$$W_{2,\,\beta|y} \overset{\text{st}}{=} (1-y)\xi^2 + yW_{2,\,\alpha}; \tag{31}$$

$$V_{\beta|y} \overset{\text{st}}{=} y(1-y)\overline{W}_{1,\,\alpha}^2 + yV_\alpha, \tag{32}$$

where $\overline{W}_{1,\,\alpha} = W_{1,\,\alpha} - \xi$ and y is a r.v. with uniform distribution on $(0,1)$.

Also for this second case, we split the proof in three steps.

STEP 1. Here we prove the following formula, which yields the moments of V_β in terms of those of $\overline{W}_{1,\,\alpha}$, when α and then β have compact support; for each $m \ge 0$,

$$\begin{aligned} \mathrm{E}\Big(V_\beta^m\Big) &= \frac{m!m!}{(2m+1)!}\mathrm{E}\Big(\overline{W}_{1,\,\alpha}^{2m}\Big) + \sum_{j=0}^{m-1} \frac{m!m!}{2j!(2m-j)!} \sum_{k=0}^{2j+2} (-1)^k \binom{2j+2}{k} \\ &\quad \times \sum_{q=0}^{2m-2-2j} \frac{(2m-2-2j)!(2m-j-q-2)!(q+j)!}{q!(2m-2-2j-q)!(2m-1)!} \\ &\quad \times \mathrm{E}\Big(\overline{W}_{1,\,\alpha}^{2m-2j-q+k-2}\Big)\mathrm{E}\Big(\overline{W}_{1,\,\alpha}^{2j+q-k+2}\Big). \end{aligned} \tag{33}$$

Take (25) with $a = 1$, multiply both members of it by $\exp\{z_2 w^2 - z_1(w-\xi)^2\}$ and then integrate with respect to w, obtaining

$$\begin{aligned} &\mathrm{E}\Big(\exp(-z_2 V_\alpha - z_1)\overline{W}_{1,\,\alpha}^2\Big) \\ &\quad = \mathrm{E}\Big(\exp\big(-z_1\overline{W}_{1,\,\alpha}^2\big)\Big) + Z_2 \int_0^{+\infty} \exp\big(z_2\omega^2 - z_t(\omega-\xi)\big)^2 \\ &\qquad \times \bigg\{\int_0^w \exp\big(-z_2x^2\big)g_1(x)\bigg[\int_0^{+\infty} \exp\big(-z_2(w-x)(x+v)\big) \\ &\qquad\qquad \times (x-v)g_1(v)\,dv\bigg]\,dx\bigg\}\,dw. \end{aligned} \tag{34}$$

Putting $z_2 = yz$ and $z_1 = y(1-y)z$ in (34) we have

$$\begin{aligned} &\mathrm{E}\Big(\exp\big(yzV_\alpha - y(1-y)z\overline{W}_{1,\,\alpha}^2\big)\Big) \\ &= \mathrm{E}\Big(\exp\big(-zy(1-y)\overline{W}_{1,\,\alpha}^2\big)\Big) - zy\int_0^{+\infty}\bigg\{\int_0^w g_1(x)\bigg[\int_w^{+\infty}(v-x)g_1(v) \\ &\qquad \times \exp\big(-z[y(v-w)(w-x) + y(1-y)(w-\xi)^2]\,dv\bigg]\,dx\bigg\}\,dw. \end{aligned} \tag{35}$$

Then, from (32) and (35) there follows, for $z > 0$,

$$\begin{aligned} \mathrm{E}\big(\exp(-zV_\beta)|y\big) &= \mathrm{E}\Big(\exp\big(-zy(1-y)\overline{W}^2_{1,\alpha}\big)\Big) \\ &\quad -zy\int_0^{+\infty}\bigg\{\int_0^w g_1(x)\bigg[\int_w^{+\infty}(v-x)g_1(v) \\ &\qquad \times \exp\big(-z[y(v-w)(w-x)+y(1-y)(w-\xi)^2]\,dv\bigg]\,dx\bigg\}\,dw, \end{aligned} \tag{36}$$

so that, for each $m \geq 0$,

$$\begin{aligned} \mathrm{E}\big(V^m_\beta|y\big) &= y^m(1-y)^m\mathrm{E}\Big(\overline{W}^{2m}_{1,\alpha}\Big) \\ &\quad +my^m\int_0^{+\infty}\bigg\{\int_0^w g_1(x)\bigg[\int_w^{+\infty}(v-x)g_1(v) \\ &\quad \times \sum_{j=0}^{m-1}\binom{m-1}{j}(v-w)^j(w-x)^j(1-y)^{m-j-1}(w-\xi)^{2m-2j-2}dv\bigg]dx\bigg\}dw. \end{aligned} \tag{37}$$

Integrating with respect to y we have, for $m \geq 0$,

$$\begin{aligned} \mathrm{E}\big(V^m_\beta\big) &= \frac{m!m!}{(2m+1)!}\mathrm{E}\Big(\overline{W}^{2m}_{1,\alpha}\Big) \\ &\quad + m\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{m!(m-j-1)!}{(2m-j)!}\int_0^{+\infty}\bigg\{\int_0^w g_1(x) \\ &\quad \times\bigg[\int_w^{+\infty}(w-x)^j(v-w)^j(w-\xi)^{2m-2j-2} \\ &\qquad \times(v-x)g_1(v)dv\bigg]dx\bigg\}\,dw \\ &= \frac{m!m!}{(2m+1)!}\mathrm{E}\Big(\overline{W}^{2m}_{1,\alpha}\Big) + \sum_{j=0}^{m-1}\frac{m!m!}{j!(2m-j)!}\int_0^{+\infty}g_1(x) \\ &\quad \times\bigg\{\int_0^{+\infty}(v-x)_+g_1(v)\bigg[\int_x^v(w-x)^j(v-w)^j \\ &\qquad \times(w-\xi)^{2m-2j-2}dw\bigg]\,dv\bigg\}\,dx \\ &= \frac{m!m!}{(2m+1)!}\mathrm{E}\Big(\overline{W}^{2m}_{1,\alpha}\Big) + \sum_{j=0}^{m-1}\frac{m!m!}{j!(2m-j)!}\int_0^{+\infty} \\ &\quad \times g_1(x)\bigg\{\int_0^{+\infty}g_1(v)\bigg[\int_0^1(v-x)_+^{2j+2}t^j(1-t)^j \\ &\qquad \times(t(v-x)+x-\xi)^{2m-2-2j}dt\bigg]\,dv\bigg\}\,dx \end{aligned} \tag{38}$$

$$
\begin{aligned}
&= \frac{m!m!}{(2m+1)!}\mathrm{E}\Big(\overline{W}_{1,\alpha}^{2m}\Big) + \sum_{j=0}^{m-1} \frac{m!m!}{j!(2m-j)!}\int_0^{+\infty} g_1(x)\bigg\{\int_0^{+\infty} g_1(v) \\
&\quad \times \bigg[\int_0^1 \sum_{k=0}^{2m-2-2j} \binom{2m-2-2j}{k} t^{j+k}(1-t)^j\,dt\bigg] \\
&\qquad\qquad \times (v-x)_+^{2j+2+k}(x-\xi)^{2m-2-2j-k}\,dv\bigg\}\,dx \\
&= \frac{m!m!}{(2m+1)!}\mathrm{E}\Big(\overline{W}_{1,\alpha}^{2m}\Big) + \sum_{j=0}^{m-1}\sum_{k=0}^{2m-2-2j} \\
&\quad \times \frac{m!m!(2m-2-2j)!(j+k)!}{(2m-j)!k!(2m-2-2j-k)!(2j+k+1)!}\int_0^{+\infty} g_1(x) \\
&\quad \times (x-\xi)^{2m-2-2j-k}\bigg\{\int_o^{+\infty} I_{(x,+\infty)}(v) g_1(v)(v-x)^{2j+2+k}\,dv\bigg\}\,dx
\end{aligned} \tag{39}
$$

Let us now consider a r.v. $W_{1,\alpha}^*$ independent of $W_{1,\alpha}$ and with the very same distribution and denote by $\overline{W}_{1,\alpha}^*$ the r.v. $W_{1,\alpha}^* - \xi$; then (39) gives

$$
\begin{aligned}
\mathrm{E}\big(V_\beta^m\big) &= \frac{m!m!}{(2m+1)!}\mathrm{E}\Big(\overline{W}_{1,\alpha}^{2m}\Big) + \sum_{j=0}^{m-1}\sum_{k=0}^{2m-2-2j} \\
&\quad \times \frac{m!m!(2m-2-2j)!(j+k)!}{(2m-j)!k!(2m-2-2j-k)!(2j+k+1)!} \\
&\quad \times \mathrm{E}\Big(\Big[W_{1,\alpha}-\xi\Big]^{2m-2j-2-k}\Big[W_{1,\alpha}^* - W_{1,\alpha}\Big]^{2j+2+k} I_{\left(W_{1,\alpha},+\infty\right)}\Big(W_{1,\alpha}^*\Big)\Big) \\
&= \frac{m!m!}{(2m+1)!}\mathrm{E}\Big(\overline{W}_{1,\alpha}^{2m}\Big) + \sum_{j=0}^{m-1}\sum_{k=0}^{2m-2-2j} \\
&\quad \times \frac{m!m!(2m-2-2j)!(j+k)!}{(2m-j)!k!(2m-2-2j-k)!(2j+k+1)!} \\
&\quad \times \frac{1}{2}\sum_{l=0}^{2j+2+k}(-1)^{k-l}\binom{2j+2+k}{l}\mathrm{E}\Big(\overline{W}_{1,\alpha}^{2m-l}\Big)\mathrm{E}\Big(\overline{W}_{1,\alpha}^{l}\Big)
\end{aligned} \tag{40}
$$

which, after some simple manipulations, yields (33). □

STEP 2. Here we prove that, if α has compact support, then (11) holds. Take a r.v. $W_{1,\beta}^*$ independent of $W_{1,\beta}$ and with the very same distribution; by (30) we can write

$$W_{1,\beta}^*|z \overset{\text{st}}{=} (1-z)\xi + zW_{1,\alpha}^*,$$

where y and z are independent uniform $(0, 1)$ r.v.'s. Hence, if $U_\beta := [W_{1,\beta} - W^*_{1,\beta}]^2$, we have, for $m \geq 0$,

$$U_\beta^m | y,\, z \overset{\text{st}}{=} \Big[y\overline{W}_{1,\alpha} - z\overline{W}^*_{1,\alpha} \Big]^{2m}$$

and then

$$\text{(41)} \quad \mathrm{E}\big(U_\beta^m\big) = \sum_{j=0}^{2m} \binom{2m}{j} \frac{1}{(j+1)(2m-j+1)} (-1)^j \mathrm{E}\Big(\overline{W}^j_{1,\alpha}\Big) \mathrm{E}\Big(\overline{W}^{2m-j}_{1,\alpha}\Big).$$

Proposition 3.1 is proved if we show that $\mathrm{E}\big(V_\beta^m\big)$, as computed in (33), equals

$$\frac{\Gamma(m+2)\Gamma(1/2)}{4^m \Gamma(m+1/2)\Gamma(2)} \mathrm{E}\big(U_\beta^m\big).$$

By a reordering of (33), after some simple manipulations, we obtain

$$\text{(42)} \quad \begin{aligned} \mathrm{E}\big(V_\beta^m\big) = {} & \frac{2(m+1)!m!}{(2m+1)!} \mathrm{E}\Big(\overline{W}^{2m}_{1,\alpha}\Big) + \frac{m!m!}{2(2m-1)!} \Big[\mathrm{E}\Big(\overline{W}^m_{1,\alpha}\Big)\Big]^2 \sum_{j=0}^{m-1} \frac{1}{j!(2m-j)!} \\ & \times \sum_{t+q=m} (-1)^t \binom{2j+2}{t} \binom{2m-2j-2}{q} (2m-j-q-2)!(j+q)! \\ & \qquad + \frac{m!m!}{(2m-1)!} \sum_{s=1}^{m-1} \mathrm{E}\Big(\overline{W}^{2m-s}_{1,\alpha}\Big) \mathrm{E}\Big(\overline{W}^s_{1,\alpha}\Big) \sum_{j=0}^{m-1} \frac{1}{j!(2m-j)!} \\ & \times \sum_{t+q=s} (-1)^t \binom{2j+2}{t} \binom{2m-2j-2}{q} (2m-j-q-2)!(j+q)!, \end{aligned}$$

while

$$\text{(43)} \quad \begin{aligned} & \frac{\Gamma(m+2)\Gamma(1/2)}{4^m \Gamma(m+1/2)\Gamma(2)} \mathrm{E}\big(U_\beta^m\big) \\ & \quad = \frac{2m!(m+1)!}{(2m+1)!} \mathrm{E}\Big(\overline{W}^{2m}_{1,\alpha}\Big) + \frac{(-1)^m}{m+1} \Big[E\Big(\overline{W}^m_{1,\alpha}\Big)\Big]^2 \\ & \quad\quad + 2m!(m+1)! \sum_{s=1}^{m-1} \frac{(-1)^{2m-s}}{(s+1)!(2m-s+1)!} \mathrm{E}\Big(\overline{W}^{2m-s}_{1,\alpha}\Big) \\ & \qquad\qquad \times \mathrm{E}\Big(\overline{W}^s_{1,\alpha}\Big). \end{aligned}$$

It is simple to verify that (42) and (43) coincide, showing that the coefficients of $\mathrm{E}\Big(\overline{W}^j_{1,\alpha}\Big)\mathrm{E}\Big(\overline{W}^{2m-j}_{1,\alpha}\Big)$ are the same in the two expressions, for $j = 0, 1, \ldots, m$.

For instance, consider the coefficients of $\mathrm{E}\Big(\overline{W}_{1,\alpha}^{2m-1}\Big)\mathrm{E}\Big(\overline{W}_{1,\alpha}\Big)$ in (42); we have

$$\begin{aligned}
&\sum_{j=0}^{m-1} \frac{m!m!}{(2m-1)!\,j!(2m-j)!} \sum_{t+q=1} (-1)^t \\
&\quad\times \frac{(2j+2)!(2m-2j-2)!(2m-j-q-2)!(j+q)!}{t!(2j+2-t)!q!(2m-2j-2-q)!} \\
&\quad= \frac{m!m!}{(2m-1)!}\left\{\sum_{j=0}^{m-2} \frac{2(j+1)![(m-j-1)(2m-j-3)!-(2m-j-2)!]}{j!(2m-j)!}\right. \\
&\qquad\qquad\qquad\qquad\left. - \frac{2m!(m-1)!}{(m-1)!(m+1)!}\right\} \\
&\quad= -2\frac{m!m!}{(2m-1)!}\left\{(m-1)\sum_{j=0}^{m-2} \frac{j+1}{(2m-j-2)(2m-j-1)(2m-j)} + \frac{1}{m+1}\right\} \\
&\quad= -2\frac{m!m!}{(2m-1)!}\left\{(m-1)\sum_{p=0}^{m-2} \frac{m-1-p}{(m+p)(m+p+1)(m+p+2)} + \frac{1}{m+1}\right\} \\
&\quad= -2\frac{m!m!}{(2m-1)!}\left\{(m-1)\sum_{p=0}^{m-2} \frac{\sum_{k=1}^{m-1-p} 1}{(m+p)(m+p+1)(m+p+2)} + \frac{1}{m+1}\right\} \\
&\quad= -2\frac{m!m!}{(2m-1)!(m+1)}\left\{\frac{(m-1)^2}{4m} + 1\right\} = -\frac{m!(m+1)!}{(2m)!},
\end{aligned}$$

which coincides with the corresponding coefficient in (43). □

STEP 3. Now we can conclude the proof of Proposition 3.1. Indeed, by repeating the reasoning made in the third step of Section 4.1, we observe that (11) is true even if α has noncompact support, but all the moments of $W_{1,\alpha}$ are finite. □

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REFERENCES

BASU, D. and TIWARI, R. C. (1982). A note on the Dirichlet process. *In Statistics and Probability: Essays in Honor of C. R. Rao* (G. Kallinpur, P. R. Krishnaiah and J. K. Ghosh, eds.) 89–103.

BERK, R. H. and SAVAGE, I. R. (1979). Dirichlet processes produce discrete measures: an elementary proof. In *Contributions to Statistics. Jaroslav Hajek Memorial Volume* (J. Jureckova, ed.) 25–31. North-Holland, Prague.

CHOUDHURI, N. (1998). Bayesian bootstrap credible sets for multidimensional mean functional. *Ann. Statist.* **26** 2104–2127.

CIFARELLI, D. M. and REGAZZINI, E. (1979). Considerazioni generali sull'impostazione bayesiana di problemi non parametrici. Le medie associative nel contesto del processo aleatorio di Dirichlet I, II. *Riv. Mat. Sci. Econom. Social* **2** 39–52.

CIFARELLI, D. M. and REGAZZINI, E. (1990). Distribution functions of means of a Dirichlet process. *Ann. Statist.* **18** 429–442 (correction: *Ann. Statist.* **22** 1633–1634).

CIFARELLI, D. M. and REGAZZINI, E. (1993). Some remarks on the distribution functions of means of a Dirichlet process. *Quaderno IAMI 93.4,* CNR-IAMI, Milano.

CIFARELLI, D. M. and REGAZZINI, E. (1995). Further remarks on the tails of a Ferguson–Dirichlet random probability distribution function. *Quaderno IAMI 95.19,* CNR-IAMI, Milano.

DIACONIS, P. and FREEDMAN, D. A. (1986). On the consistency of Bayes estimates. *Ann. Statist.* **14** 1–26.

DIACONIS, P. and KEMPERMAŃ, J. (1996). Some new tools for Dirichlet priors In *Fifth International Meeting on Bayesian Statistics,* Alicante.

ERDELYI, A. (1954). *Tables of Integral Transforms.* McGraw-Hill, New York.

FABIUS, J. (1964). Asymptotic behaviour of Bayes estimate. *Ann. Math. Statist.* **35** 846–856.

FEIGIN, P. D. and TWEEDIE, R. L. (1989). Linear functionals and Markov chains associated with Dirichlet processes. *Math. Proc. Philos. Soc.* **105** 579–585.

FERGUSON, T. (1973). A Bayesian analysis of some nonparametric problems. *Ann. Statist.* **1** 209–230.

FREEDMAN, D. A. (1963). On the asymptotic behaviour of Bayes' estimate in the discrete case. *Ann. Math. Statist.* **34** 1386–1403.

GASPARINI, F. (1995). Exact multivariate Bayesian bootstrap distributions of moments. *Ann. Statist.* **23** 762–768.

GRADSHTEYN, I. S. and RYZHIK, I. M. (1980). *Table of Integrals, Series, and Products.* Academic Press, Boston.

GUGLIELMI, A. (1998). Numerical analysis for the distribution function of the mean of a Dirichlet process. *Quaderno IAMI 98.01*, CNR-IAMI, Milano.

HANNUM, R. C., HOLLANDER, M. and LANGBERG, N. A. (1981). Distributional results for random functionals of a Dirichlet process, *Ann. Probab.* **9** 665–670.

LO, A. Y. (1987). A larger sample study of the Bayesian bootstrap. *Ann. Statist.* **15** 360–375.

LU, I. and RICHARDS, D. (1993). Random discriminants. *Ann. Statist.* **21** 1982–2000.

RUBIN, D. B. (1981). The Bayesian bootstrap. *Ann. Statist.* **9** 130–134.

SETHURAMAN, J. and TIWARI, R. (1982). Convergence of Dirichlet measures and the interpretation of their parameter. In *Statistical Decision Theory and Related Topics* III, **2** 305–315.

YAMATO, H. (1984). Characteristic functions of means of distributions chosen from a Dirichlet process. *Ann. Probab.* **12** 262–267.

ISTITUTO DI METODI QUANTITATIVI
UNIVERSITÁ "L. BOCCONI"
VIA SARFATTI, 25
20136 MILANO
ITALY
E-MAIL: michele.cifarelli@uni-bocconi.it
eugenio.melilli@uni.bocconi.it