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SEQUENTIAL ESTIMATION FOR THE AUTOCORRELATIONS OF LINEAR PROCESSES

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This paper considers sequential point estimation of the autocorrelations of stationary linear processes within the framework of the sequential procedure initiated by Robbins. The sequential estimator proposed here is based on the usual sample autocorrelations and is shown to be risk efficient in the sense of Starr as the cost per observation approaches zero. To achieve the asymptotic risk efficiency, we are led to study the uniform integrability and random central limit theorem of the sample autocorrelations. Some moment conditions are provided for the errors of the linear processes to establish the uniform integrability and random central limit theorem.

1. Introduction. Let $\{X_t;\ t \in Z\}$, $Z = \{0, \pm 1, \pm 2 \dots\}$, be a stationary linear process defined on a probability space $(\Omega, \mathscr{F}, P)$ of the form:

$$X_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}, \qquad t \in Z, \tag{1.1}$$

where the real sequence $\{a_i\}$ satisfies the absolute summability condition $\sum_{i=0}^{\infty} |a_i| < \infty$ and $\{\varepsilon_t; t \in Z\}$ are unobservable iid random variables with $E\varepsilon_1 = 0$ and $E\varepsilon_1^2 = \sigma^2 \in (0, \infty)$. The linear processes form a general class of stationary processes covering ARMA (autoregressive and moving average) and infinite-order autoregressive models. Applications to economics, engineering and the physical sciences are extremely broad, and a vast amount of literature is devoted to the study of linear processes under a variety of circumstances; for instance, see Rosenblatt (1985), page 26. Moreover, the model (1.1) gives easy access to asymptotic studies of parameter estimates such as the sample mean, autocovariance and autocorrelation. Most standard texts like Fuller (1976) and Brockwell and Davis (1990) put the linear process in the central position for asymptotic studies. See also Phillips and Solo (1993).

In time series, an accurate estimation of the autocorrelations is crucial, for example, in selecting an appropriate ARMA model. In this paper we consider the problem of estimating the autocorrelations within the framework of the sequential method initiated by Robbins (1959). Compared to iid cases, the literature on sequential estimation in time series emerged somewhat recently. See Sriram (1987, 1988), Fakhre-Zakeri and Lee (1992, 1993) and Lee (1994). For the history of iid cases, see the references cited in these papers.

As one can see in the literature on sequential estimation, the loss function is often the sum of quadratic loss for the discrepancy between target parameters

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and their estimates and sampling costs equal to the unit cost multiplied by the sample size. In order to develop the theory and thereby compute the risk, it is necessary to know a priori whether the expected value of the quadratic loss (after normalization) converges to a limit as the sample size increases. Thus it is natural to seek sufficient conditions under which moment convergence is guaranteed. For autocorrelations, Fuller [(1976), pages 240–242], and Lomnicki and Zaremba (1957) obtained the moment convergence results assuming the sixth and eighth moments of ε_1 in (1.1), respectively. In this paper we will focus on the uniform integrability rather than the moment convergence result itself, only assuming a fourth moment of ε_1.

In Section 2 we propose sequential procedures to deal with point estimation and then state the main results of this paper. Some preliminary lemmas are given in Section 3 and the proofs of the theorems are given in Section 4. Finally, in the Appendix we establish the asymptotic normality of the autocorrelation vector when the sample size itself is random. This may be of independent interest.

2. Main results. Let $X_1, \ldots, X_n$ be n consecutive observations following the model (1.1), and denote by $\gamma(k)$ and $\rho(k)$ the autocovariance and autocorrelation at lag k, respectively. As estimates of $\gamma(k)$ and $\rho(k)$, we use the sample autocovariances and autocorrelations

$$\hat{\gamma}_n(k) = n^{-1} \sum_{t=1}^{n-k} X_t X_{t+k}, \qquad 0 \le k \le n-1, \tag{2.1}$$

and

$$\hat{\rho}_n(k) = \hat{\gamma}_n(k)/\hat{\gamma}_n(0), \tag{2.2}$$

respectively. It is well known that these random sequences are strongly consistent estimates of the true parameters when the indices k are fixed. Also, it is known that if $\hat{\boldsymbol{\rho}}_n(r) = (\hat{\rho}_n(1), \ldots, \hat{\rho}_n(r))'$ and $\boldsymbol{\rho}(r) = (\rho(1), \ldots, \rho(r))'$, $r = 1, 2, \ldots$, then under the moment condition $E\varepsilon_1^4 < \infty$ we have, as $n \to \infty$,

$$n^{1/2}(\hat{\boldsymbol{\rho}}_n(r) - \boldsymbol{\rho}(r)) \to_{\mathscr{D}} (Y_1, \ldots, Y_r)', \tag{2.3}$$

where

$$Y_k = \sum_{i=1}^{\infty} \{\rho(k+i) + \rho(k-i) - 2\rho(k)\rho(i)\} Z_i, \tag{2.4}$$

with Z_i being iid $\mathscr{N}(0, 1)$ random variables [cf. Brockwell and Davis (1990) Theorem 7.2.1, page 221]. The following procedures will rely heavily on the fact (2.3).

Let us consider the problem of estimating the unknown correlations $\rho(1), \ldots, \rho(r)$ by the sample autocorrelations $\hat{\rho}_n(1), \ldots, \hat{\rho}_n(r)$ defined in (2.2), subject to the loss function

$$L_n = \sum_{k=1}^{r} A_k (\hat{\rho}_n(k) - \rho(k))^2 + cn, \tag{2.5}$$

where the preassigned A_k reflects the importance of quadratic error at lag k and c denotes the cost per observation. Note that the A_k are not necessarily equal. One may wish to put large values of A_k for the first few terms and minimize the others.

Provided that $E\varepsilon_1^{4\alpha} < \infty$ for some $\alpha \geq 1$, it follows from Theorem 1 below that

$$E(\hat{\rho}_n(k) - \rho(k))^2 = w^2(k)/n + o(n^{-1}) \quad \text{as } n \to \infty, \tag{2.6}$$

where

$$w^2(k) = \sum_{i=1}^{\infty}\{\rho(k+i) + \rho(k-i) - 2\rho(k)\rho(i)\}^2. \tag{2.7}$$

Then, denoting $\tau^2 = \sum_{k=1}^{r} A_k w^2(k)$, the associated risk is

$$R_n = EL_n = n^{-1}\tau^2 + cn + o(n^{-1}), \tag{2.8}$$

which is minimized by

$$n_0 \simeq c^{-1/2}\tau, \tag{2.9}$$

with corresponding risk

$$R_{n_0} \simeq 2cn_0, \tag{2.10}$$

where, for any real sequences $\{u_n\}, \{v_n\}$, the notation $u_n \simeq v_n$ indicates that $u_n/v_n \to 1$ as $n \to \infty$. However, since τ^2 is unknown, there is no fixed sample size procedure that achieves the risk (2.10). Thus we follow the sequential procedure that Robbins (1959) has proposed.

From now on, assume $\alpha > 1$ and let $\{h_n;\ n = 1, 2, \ldots\}$ be a sequence of positive integers such that, as $n \to \infty$,

$$h_n \to \infty \quad \text{and} \quad h_n = O(n^{\beta}) \quad \text{for some } \beta \in (0, (\alpha-1)/2\alpha). \tag{2.11}$$

The expression (2.7) suggests that as an estimate of $w^2(k)$ it is reasonable to employ

$$\hat{w}_n^2(k) = \sum_{i=1}^{h_n}\{\hat{\rho}_n(k+i) + \hat{\rho}_n(k-i) - 2\hat{\rho}_n(k)\hat{\rho}_n(i)\}^2. \tag{2.12}$$

Setting

$$\hat{\tau}_n^2 = \sum_{k=1}^{r} A_k \hat{w}_n^2(k), \tag{2.13}$$

define the stopping rule, in analogy of n_0, by

$$N_c = \inf\{n;\ n \geq c^{-1/2}(\hat{\tau}_n + n^{-\lambda})\}, \tag{2.14}$$

where $n^{-\lambda}$, $\lambda > 0$, is the delay factor which has to be chosen later [cf. Chow and Yu (1981)]. It will be shown later that under certain conditions the proposed stopping rule is asymptotically risk efficient in the sense of Starr (1966), that is to say, $R_{N_c}/R_{n_0} \to 1$ as $c \to 0$.

The following are the main theorems asserted by the performance of the sequential methods described above.

THEOREM 1 (Uniform integrability). *Assume that* $E\varepsilon_1^{4\alpha} < \infty$ *for some* $\alpha \geq 1$. *Then, for fixed* $k = 1, 2, \ldots,$ *we have that* $\{(n^{1/2}(\hat{\rho}_n(k) - \rho(k)))^{2\alpha};\ n \geq 1\}$ *is uniformly integrable. Hence, in particular, we have as,* $n \to \infty$,

$$En(\hat{\rho}_n(k) - \rho(k))^2 \to w^2(k), \tag{2.15}$$

where $w^2(k)$ *is the number in* (2.7).

THEOREM 2. *Suppose that* $E\varepsilon_1^{4\alpha} < \infty$ *for some* $\alpha > 1$ *and* $\lambda \in (0, \alpha(\alpha - 2\alpha\beta - 1)/(\alpha - 1))$. *Then, as* $c \to 0$,

$$N_c/n_0 \ \to \ 1 \quad a.s., \tag{2.16}$$

$$E|N_c/n_0 - 1| \ \to \ 0, \tag{2.17}$$

$$N_c^{1/2}(\hat{\boldsymbol{\rho}}_{N_c}(r) - \boldsymbol{\rho}(r)) \ \to_{\mathscr{D}} \mathscr{N}(\mathbf{0}, \Gamma) \qquad \textit{asymptotic normality}, \tag{2.18}$$

$$R_{N_c}/R_{n_0} \ \to \ 1 \qquad \textit{asymptotic risk efficiency}, \tag{2.19}$$

where Γ *is the* $r \times r$ *matrix whose* (k, l)*th entry equals*

$$\begin{aligned} w(k, l) = \sum_{i=1}^{\infty} &\{\rho(k + i) + \rho(k - i) - 2\rho(k)\rho(i)\} \\ &\times \{\rho(l + i) + \rho(l - i) - 2\rho(l)\rho(i)\}. \end{aligned} \tag{2.20}$$

3. Preliminary lemmas. Throughout the sequel, $||\cdot||_p$ denotes the norm in $L^p(\Omega, \mathscr{F}, P)$.

DEFINITION 3.1. The sequence of random variables $\{z_t;\ t \in Z\}$ is said to be m-dependent if, for any $r < s$ with $s - r > m$, $(\ldots, z_{r-1}, z_r)$ and $(z_s, z_{s+1}, \ldots)$ are independent.

LEMMA 3.1. *Suppose that* $\xi_1, \xi_2, \ldots$ *are* m*-dependent,* $m \geq 1$, *and strictly stationary random variables with mean* 0 *and* $E|\xi_1|^p < \infty$, $p \geq 2$. *In addition, assume that* $\xi_1, \ldots, \xi_m$ *are independent random variables. Then, for all* $n \geq 1$,

$$\left|\left|\max_{1 \leq j \leq n} |\xi_1 + \cdots + \xi_j|\right|\right|_p \leq C_p ||\xi_1||_p n^{1/2},$$

where the positive constant C_p *depends only on* p (*regardless of* m).

REMARK. An example of $\{\xi_j\}$ satisfying the assumptions of Lemma 3.1 is $\{\varepsilon_j \varepsilon_{j+m}\}$, where ε_j are iid with $E\varepsilon_1 = 0$ and $E|\varepsilon_1|^p < \infty$.

PROOF. For simplicity, we only consider the case $n = 2mk + \nu_0$, where $k \in \{0, 1, \ldots\}$ and $\nu_0 \in \{1, \ldots, m-1\}$. The other cases can be treated in a similar way. Define, for each $\nu \in \{1, \ldots, m\}$, $x_1(\nu) = \xi_1 + \cdots + \xi_\nu$, $y_1(\nu) = \xi_{m+1} + \cdots + \xi_{m+\nu}, \ldots,$ $x_k(\nu) = \xi_{2(k-1)m+1} + \cdots + \xi_{2(k-1)m+\nu}$, $y_k(\nu) = \xi_{(2k-1)m+1} + \cdots + \xi_{(2k-1)m+\nu}$ and $x_{k+1}(\nu) = \xi_{2mk+1} + \cdots + \xi_{2mk+\nu}$. Set $x = \max_{1\le j\le k} |x_1(m) + \cdots + x_j(m)|$, $y = \max_{1\le j\le k} |y_1(m) + \cdots + y_j(m)|$, $z_j = \max_{1\le\nu\le m-1} |x_j(\nu)|$, $z = \max_{1\le j\le k+1} z_j$, $w_j = \max_{1\le\nu\le m-1} |y_j(\nu)|$ and $w = \max_{1\le j\le k} w_j$.

Note that $\max_{1\le j\le n} |\xi_1 + \cdots + \xi_j| \le x + y + z + w$, and thus $\|\max_{1\le j\le n} |\xi_1 + \cdots + \xi_j|\,\|_p \le 2(\|x\|_p + \|z\|_p)$ because $x = y$ in distribution and $\|z\|_p \ge \|w\|_p$. Since $x_1(m), \ldots, x_k(m)$ are iid with mean 0 by assumption, it follows from Doob's maximal inequality and the Marcinkiewicz-Zygmund inequality [cf. Theorem 2 and Corollary 2 of Chow and Teicher (1988), pages 367–368] that

$$\|x\|_p \le B_p \|x_1(m)\|_p k^{1/2} \le B_p^2 \|\xi_1\|_p (mk)^{1/2}, \tag{3.1}$$

where B_p is a positive constant. On the other hand, using the same inequalities, we can show that $\|z\|_p \le (E \max_{1\le j\le k+1} |z_j|^p)^{1/p} \le 2B_p \|\xi_1\|_p (km)^{1/2}$. This together with (3.1) yields $\|x\|_p + \|z\|_p \le \max\{B_p^2, 2B_p\} \|\xi_1\|_p n^{1/2}$, which completes the proof. □

LEMMA 3.2. *Assume that the ξ_j satisfy the conditions of Lemma* 3.1. *Then, for all* $M = 1, 2, \ldots,$

$$\left\| \sup_{n\ge M} \left| n^{-1} \sum_{j=1}^n \xi_j \right| \right\|_p \le D_p \|\xi_1\|_p M^{-1/2},$$

where D_p is the positive constant that depends only on p.

PROOF. We first consider the case where $m < M$. Define, for $t \in \{0, 1, \ldots\}$ and $\nu \in \{1, \ldots, M\}$, $x_t(\nu) = \xi_{2tM+1} + \cdots + \xi_{2tM+\nu}$ and $y_t(\nu) = \xi_{(2t+1)M+1} + \cdots + \xi_{(2t+1)M+\nu}$. Set $x_t(0) = y_t(0) = 0$. For convenience, assume that $n = 2kM + \nu_0$, where $k \ge 1$ and $\nu_0 \in \{0, \ldots, M-1\}$. Then we have

$$\begin{aligned} \left| n^{-1} \sum_{j=1}^n \xi_j \right| &= n^{-1} \left| \sum_{j=0}^{k-1} \{x_j(M) + y_j(M)\} + x_k(\nu_0) \right| \\ &\le M^{-1/2}(Q_1 + Q_2) + R, \end{aligned} \tag{3.2}$$

where

$$Q_1 = \sup_{u\ge1} \left| u^{-1} \sum_{j=0}^{u-1} \{M^{-1/2} x_j(M)\} \right|,$$

$$Q_2 = \sup_{u\ge1} \left| u^{-1} \sum_{j=0}^{u-1} \{M^{-1/2} y_j(M)\} \right|$$

and

$$R = \sup_{k\ge1} (2kM)^{-1} \max_{0\le\nu\le M-1} |x_k(\nu)|.$$

Note that, due to Lemma 3.1, $x_j(M)$ are iid random variables with mean 0, such that $E|M^{-1/2}x_0(M)|^p \le C_p^p||\xi_1||_p^p < \infty$. Thus, it follows from the maximal inequality of reverse martingales [cf. Theorem 3 of Chow and Teicher (1988), page 369] that $||Q_1||_p \le C_p||\xi||_p$. Similarly, $||Q_2||_p \le C_p||\xi_1||_p$. Since

$$ER^p \le (2M)^{-p}\left(\sum_{k=1}^{\infty} k^{-p}\right)E \max_{1\le \nu \le M-1} |x_1(\nu)|^p \le (C_p')^p||\xi_1||_p^p M^{-p/2}, \qquad C_p' > 0,$$

where the last inequality follows from Lemma 3.1, we have that

$$\left|\left|\sup_{n\ge M}\left|n^{-1}\sum_{j=1}^{n}\xi_j\right|\right|\right|_p \le \max\{C_p, C_p'\}||\xi_1||_p M^{-1/2}. \tag{3.3}$$

The result for the case where $M \le m$ can be established by Lemma 3.1, the argument in (3.3) and the Marcinkiewicz–Zygmund inequality. □

The following lemma is a direct result of Lee (1994), Lemma 2.

LEMMA 3.3. *Suppose that $\xi_1, \xi_2, \dots$ are m-dependent strictly stationary random variables such that $\xi_1 \ge 0$ a.s. and $\mu = E\xi_1 \in (0, \infty)$. Then, for $\theta < \mu$, there exists $B > 0$ such that*

$$P\left(n^{-1}\sum_{j=1}^{n}\xi_j \le \theta\right) \le e^{-Bn} \quad \text{for all } n$$

and

$$P\left(n^{-1}\sum_{j=1}^{n}\xi_j \le \theta \text{ for some } n \ge M\right) \le Ce^{-BM} \quad \text{for all } M \ge 1,$$

where $C = (1 - e^{-B})^{-1}$.

4. Proofs of theorems. Throughout the sequel, we denote, for $k = 0, 1, \dots,$

$$\gamma_n^*(k) = n^{-1}\sum_{t=1}^{n} X_t X_{t+k} \tag{4.1}$$

and

$$\rho_n^* = \gamma_n^*(k)/\gamma_n^*(0). \tag{4.2}$$

LEMMA 4.1. *Assume that the random variables ε_t in (1.1) satisfy the moment condition $E\varepsilon_1^{4\alpha} < \infty$ for some $\alpha \ge 1$. Then we have, as $M \to \infty$,*

$$\sup_k \left|\left|\sup_{n\ge M}|\gamma_n^*(k) - \gamma(k)|\right|\right|_{2\alpha} = O(M^{-1/2}). \tag{4.3}$$

PROOF. As with (2.12) of Fakhre-Zakeri and Lee (1992), without additional assumptions on $\{a_j\}$, we can write

$$\begin{aligned}\gamma_n^*(k) - \gamma(k) &= \sum_{u=0}^{\infty} a_u a_{u+k} \left\{ n^{-1} \sum_{j=1}^{n} (\varepsilon_{j-u}^2 - \sigma^2) \right\} \\ &\quad + \sum_{\nu \neq u+k} a_u a_\nu \left\{ n^{-1} \sum_{j=1}^{n} \varepsilon_{j-u} \varepsilon_{j+k-\nu} \right\}. \end{aligned} \tag{4.4}$$

Thus, by Minkowski's inequality and the stationary property, we have

$$\begin{aligned}\left|\left| \sup_{n\geq M} |\gamma_n^*(k) - \gamma(k)| \right|\right|_{2\alpha} &\leq \sum_{u=0}^{\infty} |a_u a_{u+k}| \left|\left| \sup_{n\geq M} \left| n^{-1} \sum_{j=1}^{n} (\varepsilon_j^2 - \sigma^2) \right| \right|\right|_{2\alpha} \\ &\quad + \sum_{\nu \neq u+k} |a_u a_\nu| \sup_{l\neq 0} \left|\left| \sup_{n\geq M} \left| n^{-1} \sum_{j=1}^{n} \varepsilon_j \varepsilon_{j+l} \right| \right|\right|_{2\alpha}. \end{aligned} \tag{4.5}$$

First, note that

$$\left|\left| \sup_{n\geq M} \left| n^{-1} \sum_{j=1}^{n} (\varepsilon_j^2 - \sigma^2) \right| \right|\right|_{2\alpha} = O(M^{-1/2}) \tag{4.6}$$

by Theorem 3 of Chow and Teicher (1988), page 369. Second, notice that $\{\varepsilon_1 \varepsilon_{j+l};\ j = 1, 2, \ldots\}$ are $|l|$-dependent random variables and $\{\varepsilon_1\varepsilon_{1+l}, \ldots, \varepsilon_l \varepsilon_{2l}\}$ are independent. Then it follows from Lemma 3.2 that

$$\sup_{l\neq 1} \left|\left| \sup_{n\geq M} \left| n^{-1} \sum_{j=1}^{n} \varepsilon_1 \varepsilon_{j+l} \right| \right|\right|_{2\alpha} \leq D_p ||\varepsilon_1||_{2\alpha}^2 M^{-1/2}. \tag{4.7}$$

Combining (4.6) and (4.7), we can see that the right-hand side of (4.5) is $O(M^{-1/2})$ uniformly in k. This establishes (4.3). □

LEMMA 4.2. *Suppose that* $E\varepsilon_1^{4\alpha} < \infty$ *for some* $\alpha \geq 1$. *Then the following hold*:

(i) *For each* $k = 0, \ldots, r$,

$$\left|\left| \sup_{n\geq M} |\hat{\gamma}_n(k) - \gamma(k)| \right|\right|_{2\alpha} = O(M^{-1/2}) \quad \textit{as } M \to \infty.$$

(ii) *If* $\{h_n\}$ *is a sequence of real numbers satisfying the property in* (2.11), *then*

$$\max_{0\leq k\leq 2h_n} ||\hat{\gamma}_n(k) - \gamma(k)||_{2\alpha} = O(n^{-1/2}) \quad \textit{as } n \to \infty.$$

PROOF. The lemma follows from Lemma 4.1 immediately due to the fact that $\hat{\gamma}_n(k) = ((n-k)/n)\gamma_{n-k}^*(k)$. □

The following two lemmas are concerned with uniform integrability. We only state them without proof.

LEMMA 4.3(a). *Let $\zeta > 0$ and let $\{E_n\}$ be a family of Borel sets in $\mathscr{F}$. Assume that $\{W_{nj};\ n \geq 1\}$, $j = 1, \ldots, J$, are the families of random variables such that $\{W_{nj}^{\zeta};\ n \geq 1\}$ is uniformly integrable. Then if, for all $A \in \mathscr{F}$,*

$$n^{\zeta} P(E_n \cap A) \leq K \sum_{j=1}^{J} ||W_{nj} I(A)||_{\zeta}^{\zeta} \quad \textit{for some } K > 0,$$

where $I(\cdot)$ denotes the indicator function, it holds that $\{n^{\zeta} I(E_n);\ n \geq 1\}$ is uniformly integrable.

LEMMA 4.3(b). *Let W be a random variable with $E|W|^{\zeta} < \infty$, $\zeta > 0$. Then, for $A \in \mathscr{F}$ and $\delta > 0$, we have*

$$P(|W| > \delta, A) \leq \delta^{-\zeta} E|W|^{\zeta} I(A).$$

LEMMA 4.4. *Suppose that $E\varepsilon_1^{4\alpha} < \infty$ for some $\alpha \geq 1$. Then, for each $k = 0, \ldots, r$, $\{n^{\alpha}(\hat{\gamma}_n(k) - \gamma(k))^{2\alpha};\ n \geq 1\}$ is uniformly integrable.*

PROOF. We first show that $\{n^{\alpha}(\gamma_n^*(k) - \gamma(k))^{2\alpha};\ n \geq 1\}$ is uniformly integrable. In view of (4.4), we split $n^{1/2}(\gamma_n^*(k) - \gamma(k))$ into I_n and II_n, where

$$I_n = \sum_{u=0}^{\infty} a_u a_{u+k} \left\{ n^{-1/2} \sum_{j=1}^{n} (\varepsilon_j^2 - \sigma^2) \right\}$$

and

$$II_n = \sum_{\nu \neq u+k} a_u a_{\nu} \left\{ n^{-1/2} \sum_{j=1}^{n} \varepsilon_{j-u} \varepsilon_{j+k+\nu} \right\}.$$

Note first that $\{|I_n|^{2\alpha};\ n \geq 1\}$ is uniformly integrable, since, for each u, $\{(n^{1/2} \sum_{j=1}^{n} (\varepsilon_{j-u}^2 - \sigma^2))^{2\alpha};\ n \geq 1\}$ is uniformly integrable [cf. (4.8) of Gut (1988), page 18]. Next, to show the uniform integrability of $\{|II_n|^{2\alpha};\ n \geq 1\}$, use Minkowski's inequality and Lemma 3.2 to obtain

$$||II_n||_{4\alpha} \leq \left(\sum_{u=0}^{\infty} |a_u| \right)^2 C_{\alpha} ||\varepsilon_1||_{4\alpha}^2 \quad \text{for some } C_{\alpha} > 0,$$

which in turn implies that $\{|II_n|^{2\alpha};\ n \geq 1\}$ is uniformly integrable. Since $\{n^{\alpha}(\gamma_n^*(k) - \hat{\gamma}_n(k))^{2\alpha};\ n \geq 1\}$ is uniformly integrable, the lemma is established. □

PROOF OF THEOREM 1. Tentatively fix $\delta > 0$; δ will be chosen properly later. Decompose $n^{\alpha}(\hat{\rho}_n(k) - \rho(k))^{2\alpha}$ into I_n and II_n, where

$$I_n = n^{\alpha}(\hat{\rho}_n(k) - \rho(k))^{2\alpha} I(\hat{\gamma}_n(0) \geq \delta)$$

and

$$II_n = n^{\alpha}(\hat{\rho}_n(k) - \rho(k))^{2\alpha} I(\hat{\gamma}_n(0) < \delta).$$

Note that

$$\hat{\rho}_n(k) - \rho(k) = \hat{\gamma}_n^{-1}(0)\{(\hat{\gamma}_n(k) - \gamma(k)) + (\gamma(0) - \hat{\gamma}_n(0))\rho(k)\}, \tag{4.8}$$

and, accordingly,

$$I_n \leq 4^{\alpha}\delta^{-2\alpha}\{n^{\alpha}(\hat{\gamma}_n(k) - \gamma(k))^{2\alpha} + n^{\alpha}(\hat{\gamma}_n(0) - \gamma(0))^{2\alpha}\}.$$

Thus it follows from Lemma 4.4 that $\{I_n;\ n \geq 1\}$ is uniformly integrable for each $\delta > 0$.

To deal with II_n, note that

$$II_n \leq 4^{\alpha} n^{\alpha} I(\hat{\gamma}_n(0) < \delta). \tag{4.9}$$

Let δ be a positive number less than $\gamma(0) = \sum_{j=0}^{\infty} a_j^2 \sigma^2$ and let m be a positive integer with $\sum_{j=0}^{m} a_j^2\sigma^2 > \delta$. Define $X_t(m) = \sum_{j=0}^{m} a_j \varepsilon_{t-j}$, $\hat{\gamma}_{n,m}(0) = n^{-1}\sum_{t=1}^{n} X_t^2(m)$ and $\gamma_m(0) = \sum_{j=0}^{m} a_j^2\sigma^2$. Let $\eta > 0$ be such that $3\eta < \min\{\gamma(0) - \gamma_m(0), \gamma(0) - \delta\}$ and $\delta_0 = \delta + \eta$. Denoting $E_n = \{|\hat{\gamma}_n(0) - \hat{\gamma}_{n,m}(0)| > \eta\}$, write $n^{\alpha} I(\hat{\gamma}_n(0) < \delta) = III_n + IV_n$, where $III_n = n^{\alpha} I(\hat{\gamma}_n(0) < \delta, E_n)$ and $IV_n = n^{\alpha} I(\hat{\gamma}_n(0) < \delta, E_n^c)$. To obtain the uniform integrability of III_n, in view of Lemma 4.3(a), consider

$$n^{\alpha} P(\hat{\gamma}_n(0) < \delta,\ |\hat{\gamma}_n(0) - \hat{\gamma}_{n,m}(0)| > \eta, A), \tag{4.10}$$

where A is a Borel set in $\mathscr{F}$. By using Lemma 4.3(b), we can show that (4.10) is bounded by

$$(3/\eta)^{2\alpha}(||n^{1/2}(\hat{\gamma}_n(0) - \gamma(0))I(A)||_{2\alpha}^{2\alpha} + ||n^{1/2}(\hat{\gamma}_{n,m}(0) - \gamma_m(0))I(A)||_{2\alpha}^{2\alpha}),$$

whence, in view of Lemmas 4.4 and 4.3(a), $\{III_n;\ n \geq 1\}$ is uniformly integrable. On the other hand, $\{IV_n;\ n \geq 1\}$ is uniformly integrable since applying Lemma 3.4 to $\{X_t^2(m)\}$, we obtain $EIV_n \leq n^{\alpha} P(\hat{\gamma}_{n,m}(0) < \delta_0) \leq e^{-Bn}$, $B > 0$. Hence $\{n^{\alpha}(\hat{\rho}_n(k) - \rho(k))^{2\alpha};\ n \geq 1\}$ is uniformly integrable. This together with (2.3) and (2.4) implies (2.15). □

The following lemmas are aimed at proving Theorem 2.

LEMMA 4.5. *Suppose that* $E\varepsilon_1^{4\alpha} < \infty$ *for some* $\alpha \geq 1$. *Then we have the following*:

(i) *For each* $k = 0, \ldots, r$,

$$\left|\left|\sup_{n \geq M} |\hat{\rho}_n(k) - \rho(k)|\ \right|\right|_{2\alpha} = O(M^{-1/2}) \quad \text{as } M \to \infty. \tag{4.11}$$

(ii) *As* $n \to \infty$,

$$\max_{1 \le k \le 2h_n} ||\hat{\rho}_n(k) - \rho(k)||_{2\alpha} = O(n^{-1/2}). \tag{4.12}$$

PROOF. Fix $\delta > 0$ and let E denote the event on which $\hat{\gamma}_n(0) < \delta$ for some $n \ge M$. From (4.8), we can write

$$\left|\left|\sup_{n \ge M} |\hat{\rho}_n(k) - \rho(k)| \right|\right|_{2\alpha} \le B(M, \delta) + 2\{P(E)\}^{1/2\alpha},$$

where

$$B(M, \delta) = \delta^{-2\alpha}\left(\left|\left|\sup_{n \ge M} |\hat{\gamma}_n(k) - \gamma(k)| \right|\right|_{2\alpha} + \left|\left|\sup_{n \ge M} |\hat{\gamma}_n(0) - \gamma(0)| \right|\right|_{2\alpha}\right).$$

Since $B(M, \delta) = O(M^{-1/2})$ for each $\delta > 0$ by (i) of Lemma 4.2, (4.11) will be true if there exists $\delta > 0$, such that

$$\{P(E)\}^{1/2\alpha} = \{P(\hat{\gamma}_n(0) < \delta \text{ for some } n \ge M)\}^{1/2\alpha} = O(M^{-1/2}). \tag{4.13}$$

Let δ, δ_0, $\hat{\gamma}_{n,m}(0)$ and $\gamma_m(0)$ be the same as in the arguments between (4.9) and (4.10). Denoting $K_M = \{\sup_{n \ge M} |\hat{\gamma}_n(0) - \hat{\gamma}_{n,m}(0)| > \eta\}$, write $P(\hat{\gamma}_n(0) < \delta \text{ for some } n \ge M) \le Q_{n1} + Q_{n2}$, where $Q_{n1} = P(K_M)$ and $Q_{n2} = P(\hat{\gamma}_n(0) < \delta \text{ for some } n \ge M,\ K_M^c)$. Note that, from Markov's inequality and (i) of Lemma 4.2,

$$\begin{aligned} Q_{n1} &\le (3/\eta)^{2\alpha}\left(\left|\left|\sup_{n \ge M} |\hat{\gamma}_n(0) - \gamma(0)| \right|\right|_{2\alpha}^{2\alpha} + \left|\left|\sup_{n \ge M} |\hat{\gamma}_{n,m}(0) - \gamma_m(0)|\right|\right|_{2\alpha}^{2\alpha}\right) \\ &= O(M^{-\alpha}) \end{aligned} \tag{4.14}$$

[(i) of Lemma 4.2 is also true for $\hat{\gamma}_{n,m}(0)$]. Meanwhile, Q_{n2} is bounded by

$$\begin{aligned} &P(\hat{\gamma}_{n,m}(0) < \delta_0 \text{ for some } n \ge M) \\ &\quad \le \sum_{n=M}^{\infty} P(\hat{\gamma}_{n,m}(0) < \delta_0) = O(e^{-BM}), \qquad B > 0, \end{aligned}$$

where the last equality follows from Lemma 4.5. Combining this and (4.14), we obtain (4.13), and therefore (4.11) is established.

Now, we are going to verify (4.12). Observe from (4.8) that, for all $\delta > 0$,

$$\begin{aligned} &\max_{1 \le k \le 2h_n} ||\hat{\rho}_n(k) - \rho(k)||_{2\alpha} \\ &\quad \le 2\delta^{-1} \max_{0 \le h \le 2h_n} ||\hat{\gamma}_n(k) - \gamma(k)||_{2\alpha} + 2\{P(\hat{\gamma}_n(0) < \delta)\}^{1/2\alpha}. \end{aligned} \tag{4.15}$$

In view of (4.13), there exists $\delta > 0$, such that $\{P(\hat{\gamma}_n(0) < \delta)\}^{1/2\alpha} = O(n^{-1/2})$. Hence, the right hand side of (4.15) is $O(n^{-1/2})$ by (ii) of Lemma 4.2. This completes the proof. □

LEMMA 4.6. *Let $w^2(k)$ and $\hat{w}_n^2(k)$ be as defined in* (2.7) *and* (2.12), *respectively. If $E|\varepsilon_1|^{4\alpha} < \infty$ for some $\alpha \geq 1$, then, for $k = 1, \ldots, r$,*

$$P(|\hat{w}_n^2(k) - w^2(k)| > \delta) = O(n^{-\alpha} h_n^{2\alpha}) \quad \textit{for all } \delta > 0, \tag{4.16}$$

and, consequently,

$$P(|\hat{\tau}_n^2 - \tau^2| > \delta) = O(n^{-\alpha} h_n^{2\alpha}) \quad \textit{for all } \delta > 0. \tag{4.17}$$

PROOF. Split $\hat{w}_n^2(k) - w^2(k)$ into I_n and II_n, where

$$I_n = \sum_{i=1}^{h_n} \big(\{\hat{\rho}_n(k+i) + \hat{\rho}_n(k-i) - 2\hat{\rho}_n(k)\hat{\rho}_n(i)\}^2 - \{\rho(k+i) + \rho(k-i) - 2\rho(k)\rho(i)\}^2\big),$$

$$II_n = \sum_{i=h_n+1}^{\infty} \{\rho(k+i) + \rho(k-i) - 2\rho(k)\rho(i)\}^2.$$

Note that $II_n \to 0$ as $n \to \infty$, and thus $P(|\hat{w}_n^2(k) - w^2(k)| > \delta)$ is no more than $P(|I_n| > \delta/2)$ for all sufficiently large n. By simple algebra, we can see that, for all sufficiently large n, $\|I_n\|_{2\alpha} \leq 48 h_n \max_{1 \leq j \leq 2h_n} \|\hat{\rho}_n(j) - \rho(j)\|_{2\alpha}$. Therefore, $P(|I_n| > \delta/2) = O(n^{-\alpha} h_n^{2\alpha})$ by (ii) of Lemma 4.5, which establishes (4.16). The argument (4.17) is a direct result of (4.16). □

LEMMA 4.7. *Assume that $E\varepsilon_1^{4\alpha} < \infty$ for some $\alpha > 1$. Then, for $B \in \mathscr{F}$ and a positive integer M, we have*

$$E \sup_{n \geq M} (\hat{\rho}_n(k) - \rho(k))^2 I(B) \leq K M^{-1} \{P(B)\}^{(\alpha-1)/\alpha} \quad \textit{for } k = 1, \ldots, r,$$

where K is independent of B, M and r.

PROOF. The lemma is established by (i) of Lemma 4.5 and Hölder's inequality. □

Throughout the sequel, we denote $b = c^{-1/2}$, $n_1 = [b^{1/(1+\lambda)}]$ and $n_2 = [(1 - \zeta)n_0]$, $0 < \zeta < 1$.

LEMMA 4.8. *Assume that $E|\varepsilon_1|^{4\alpha} < \infty$, $\alpha > 1$. Then as $c \to 0$,*

$$P(N_c \leq n_2) = O\big(c^{(\alpha - 2\alpha\beta - 1)/2(1+\lambda)}\big).$$

PROOF. Note that $N_c \geq n_1$ by definition. Similar to the arguments in the proof of Fakhre–Zakeri and Lee (1992), Lemma 3, we can write

$$\begin{aligned} P(N_c \leq n_2) &\leq P(|\hat{\tau}_n^2 - \tau^2| \geq \zeta(2-\zeta)\tau^2 \text{ for some } n \in [n_1, n_2]) \\ &\leq \sum_{n=n_1}^{\infty} O(n^{-\alpha} h_n^{2\alpha}) = O\big(n_1^{-(\alpha - 2\alpha\beta - 1)}\big), \end{aligned}$$

where the last step follows from Lemma 4.6 and Lemma 2 of Fakhre–Zakeri and Lee (1992), page 190. This completes the proof. □

LEMMA 4.9. *The family* $\{N_c/n_0\}$ *is uniformly integrable.*

PROOF. The proof is essentially the same as that of Lemma 6 of Fakhre–Zakeri and Lee (1992), page 192, and is omitted for brevity. □

PROOF OF THEOREM 2. By Lemma 4.6, for any $\delta > 0$, we have

$$\sum_{n=1}^{\infty} P(|\hat{\tau}_n^2 - \tau^2| > \delta) = \sum_{n=1}^{\infty} O(n^{-\alpha+2\alpha\beta}) < \infty,$$

so that $\hat{\tau}_n^2 \to \tau^2$ a.s. as $n \to \infty$. Then it follows from the definition of N_c that $N_c/n_0 \to 1$ a.s., which in turn implies (2.17) and (2.18) in view of Lemma 4.9 and Theorem A.1 in the Appendix, respectively. Thus it remains to show (2.19).

Since

$$\frac{R_{N_c}}{R_{n_0}} \simeq \frac{E\sum_{k=1}^{r} A_k(\hat{\rho}_{N_c}(k) - \rho(k))^2 + cEN_c}{2cn_0}$$

and

$$c \simeq \tau^2/n_0^2,$$

it is sufficient, therefore, to show that, for each $k = 1, \ldots, r$,

$$En_0(\hat{\rho}_{N_c}(k) - \rho(k))^2 \to w^2(k) \quad \text{as } c \to 0. \tag{4.18}$$

First, note that, by Theorem A.1 and (2.16), for any $\zeta \in (0, 1)$,

$$n_0^{1/2}(\hat{\rho}_{N_c}(k) - \rho(k))I(N_c \geq (1-\zeta)n_0) \to_D \mathscr{N}(0, w^2(k)) \quad \text{as } c \to 0. \tag{4.19}$$

Now, by Lemma 4.7, we have, for $B \in \mathscr{F}$,

$$\begin{aligned} n_0E(\hat{\rho}_{N_c}(k) &- \rho(k))^2 I(N_c \geq (1-\zeta)n_0)I(B) \\ &\leq n_0E \sup_{n \geq [(1-\zeta)n_0]} (\hat{\rho}_n(k) - \rho(k))^2 I(B) \\ &\leq K(n_0/[(1-\zeta)n_0])\{P(B)\}^{(\alpha-1)/\alpha} \\ &\leq K'P(B)^{(\alpha-1)/\alpha} \quad \text{for some } K' > 0. \end{aligned}$$

Hence it follows that $\{n_0(\hat{\rho}_{N_c}(k) - \rho(k))^2 I(N_c \geq (1-\zeta)n_0)\}$ is uniformly integrable, which together with (4.19) yields

$$En_0(\hat{\rho}_{N_c}(k) - \rho(k))^2 I(N_c \geq (1-\zeta)n_0) \to w^2(k) \quad \text{as } n \to \infty.$$

Therefore, to assert (4.18), we only need to show that

$$En_0(\hat{\rho}_{N_c}(k) - \rho(k))^2 I(N_c \leq [(1-\zeta)n_0]) \to 0 \quad \text{as } c \to 0. \tag{4.20}$$

Putting $n_2 = [(1-\zeta)n_0]$ and $D = (N_c \le n_2)$ and using Lemma 4.7, we obtain

$$\begin{aligned} En_0(\hat{\rho}_{N_c}(k) - \rho(k))^2 I(D) &= En_0 \max_{n_1 \le n \le n_2} (\hat{\rho}_n(k) - \rho(k))^2 I(D) \\ &\le K(n_0/n_1)\{P(D)\}^{(\alpha-1)/\alpha} \\ &= (n_0/n_1) O(c^{\alpha(\alpha-2\alpha\beta-1)/2(\alpha-1)(1+\lambda)}), \end{aligned} \tag{4.21}$$

where the last equality is from Lemma 4.8. Therefore, from the definitions $n_0 \simeq \tau c^{-1/2}$ and $n_1 \simeq c^{-1/2(1+\lambda)}$, (4.21) becomes $O(c^{-\lambda(\alpha-1)+\alpha(\alpha-2\alpha\beta-1)/2(\alpha-1)(1+\lambda)})$. This together with the fact that $\lambda < \alpha(\alpha - 2\alpha\beta - 1)/(\alpha - 1)$ implies (4.18). □

APPENDIX

In this appendix we are concerned with the asymptotic normality of the sample autocorrelation vector $\hat{\boldsymbol{\rho}}_n(r)$ in (2.3) when the sample size itself is random. In Proposition A.1, we deal with the issue in the situation where the sequence $\{a_j\}$ in (1.1) satisfies the condition as in Theorem 3.7 of Phillips and Solo (1993), page 979. Then we consider the general linear process case in Theorem A.1. In the sequel, $\{N_n\}$ denotes a family of positive integer-valued random variables such that $N_n/n \to_P N$ as $n \to \infty$, where $P(0 < N < \infty) = 1$.

PROPOSITION A.1. *Let* $\{X_t\}$ *be the linear process in* (1.1) *with* $E\varepsilon_1^4 < \infty$ *and* $\{a_j\}$ *satisfying the condition*

$$S\colon \sum_{j=0}^{\infty} j a_j^2 < \infty. \tag{A.1}$$

Then we have, as $n \to \infty$,

$$N_n^{1/2}(\hat{\boldsymbol{\rho}}_{N_n}(r) - \boldsymbol{\rho}(r)) \to_D \mathscr{N}(\mathbf{0}, \Gamma), \tag{A.2}$$

where Γ *is the matrix in* (2.18).

THEOREM A.1. *Let* $\{X_t\}$ *be the linear process in* (1.1) *with* $E\varepsilon_1^4 < \infty$. *Then we obtain the random central limit theorem in* (A.2).

To assert Proposition A.1, we need a series of lemmas. Lemma A.1 is a direct result of Gut (1988), Theorem 2.2, page 11, and Lemma A.2 can be proved without difficulties. So, the proofs are omitted for brevity.

LEMMA A.1. *Let* $\{W_n\}$ *be a sequence of random variables that goes to* 0 *with probability* 1. *Then* W_{N_n} *goes to zero in probability.*

LEMMA A.2. *If* W_n *are identically distributed random variables with* $EW_n^2 < \infty$, *it holds that* $W_n/n^{1/2} \to 0$ *a.s. as* $n \to \infty$.

LEMMA A.3. *Assume that* $\{\varepsilon_t\}$ *are iid random variables with mean* 0 *and variance* $\sigma^2 \in (0, \infty)$. *Then for* $M = 1, 2, \dots,$

$$\left(N_n^{-1/2} \sum_{t=1}^{N_n} \varepsilon_t \varepsilon_{t-1}, \dots, N_n^{-1/2} \sum_{t=1}^{N_n} \varepsilon_t \varepsilon_{t-M}\right) \to_D \mathcal{N}(\mathbf{0}, \sigma^4 I_M) \quad \text{as } n \to \infty,$$

where I_M *denotes the* $M \times M$ *identity matrix.*

PROOF. By the Cramér–Wold device, it suffices to show that, for all $\boldsymbol{\theta} = (\theta_1, \dots, \theta_M)'$, $\theta_i \in R$,

$$N_n^{-1/2} \sum_{t=1}^{N_n} \left(\sum_{j=1}^{M} \theta_j \varepsilon_j \varepsilon_{t-j} \right) \to_D \mathcal{N}\left(0, \sum_{i=1}^{m} \theta_i^2 \sigma^4\right) \quad \text{as } n \to \infty.$$

Put $Y_n = n^{-1/2} \sum_{t=1}^{n} \xi_t$, where $\xi_t = \sum_{i=1}^{M} \theta_j \varepsilon_t \varepsilon_{t-j}$. To prove the lemma, we only have to check that Y_n satisfies the conditions in Theorem 5 of Durret and Resnick (1977), page 217. By using Theorems 1 and 2 of Rényi (1960) and Lemma 3 of Blum, Hanson and Rosenblatt (1963), page 391, one can show that Y_n satisfies the conditions. Without detailing the related algebra, we establish the lemma. □

The following lemma can be proved in a similar way.

LEMMA A.4. *Assuming the conditions of Lemma A.*3 *and*, *in addition*, $E\varepsilon_1^4 < \infty$, *we have*, *as* $n \to \infty$,

$$\left(N_n^{-1/2} \sum_{t=1}^{N_n} (\varepsilon_t^2 - \sigma^2), N_n^{-1/2} \sum_{t=1}^{N_n} \varepsilon_t \varepsilon_{t-1}, \dots, N_n^{-1/2} \sum_{t=1}^{N_n} \varepsilon_t \varepsilon_{t-M}\right) \\ \to_D \mathcal{N}(\mathbf{0}, \operatorname{diag}(\operatorname{var} \varepsilon_1^2, \sigma^4 I_M)).$$

LEMMA A.5. *Assume that* $\{\alpha_j\}$ *is a sequence of real numbers with* $\sum_{j=-\infty}^{\infty} |\alpha_j| < \infty$ *and* ε_t *are iid random variables with mean* 0 *and finite variance* $\sigma^2 > 0$. *Then*, *for a positive integer* M *and* $\delta > 0$, *we have*, *for some* $K > 0$,

$$\limsup_{n \to \infty} P\left(\left| \sum_{|j| \ge M} \alpha_j \left\{ N_n^{-1/2} \sum_{t=1}^{N_n} \varepsilon_t \varepsilon_{t-j} \right\} \right| > \delta \right) \le \left(\frac{K\sigma^4}{\delta^2} \right) \left(\sum_{|j| \ge M} |\alpha_j| \right)^2. \tag{A.3}$$

Furthermore, *if*, *in addition*, *we assume* $E\varepsilon_1^4 < \infty$, *then*, *for some* $K > 0$,

$$\limsup_{n \to \infty} P\left(\left| \sum_{|j| \ge M} \alpha_j \left\{ N_n^{-1/2} \sum_{t=1}^{N_n} (\varepsilon_t^2 - \sigma^2) \right\} \right| > \delta \right) \\ \le \left(\frac{K \operatorname{var} \varepsilon_1^2}{\delta^2} \right) \left(\sum_{|j| \ge M} |\alpha_j| \right)^2. \tag{A.4}$$

PROOF. The proof of (A.3) essentially follows the same arguments as in (9)–(14) of Fakhre-Zakeri and Farshidi (1993). As for our concern, replace $S_u(j)$ in their paper (page 94) by $\tilde{S}_u(j) = \sum_{t=1}^{u} \varepsilon_t \varepsilon_{t-j}$. Then use Lemma 3.2 to show that $E \max I_{n,r,l} |\tilde{S}_u(j)|^2$ is bounded by $B_2((l+1)/2^r)\sigma^4$, which finally asserts (A.4) [cf. (13) of Fakhre-Zakeri and Farshidi (1993)]. The proof of (A.4) is similar to that of (A.3). □

Let $\boldsymbol{\rho}_n^*(r)$ be the $r \times 1$ random vector with the kth component being $\rho_n^*(k)$ in (4.2). For $m = 1, 2, \ldots,$ denote $\gamma_{n,m}^*(k) = n^{-1} \sum_{t=1}^{n} X_t(m) X_{t+k}(m)$, where $X_t(m) = \sum_{j=0}^{m} a_j \varepsilon_{t-j}$ and $\gamma_m(k) = \sum_{j=0}^{m-k} a_j a_{j+k} \sigma^2$. In addition, put $\rho_{n,m}^*(k) = \gamma_{n,m}^*(k)/\gamma_{n,m}^*(0)$, $\rho_m(k) = \gamma_m(k)/\gamma_m(0)$, $\boldsymbol{\rho}_{n,m}^*(r) = (\rho_{n,m}^*(1), \ldots, \rho_{n,m}^*(r))'$ and $\boldsymbol{\rho}_m(r) = (\rho_m(1), \ldots, \rho_m(r))'$.

PROOF OF PROPOSITION A.1. Note that, from Lemmas A.1 and A.2,

$$N_n^{1/2}(\hat{\rho}_{N_n}(k) - \rho_{N_n}^*(k)) \to_P 0 \quad \text{as } n \to \infty. \tag{A.5}$$

Hence we can see that (A.2) will hold once

$$N_n^{1/2}(\boldsymbol{\rho}_{N_n}^*(r) - \boldsymbol{\rho}(r)) \to_{\mathscr{D}} \mathscr{N}(\mathbf{0}, \Gamma) \quad \text{as } n \to \infty. \tag{A.6}$$

Following the expression (29) of Phillips and Solo (1993), page 981, we can write

$$n^{1/2}(\rho_n^*(k) - \rho(k)) \sim \left(n^{-1} \sum_{t=1}^{n} X_t^2 \right)^{-1} \sum_{j=1}^{\infty} \left[\alpha(k, j) n^{-1/2} \sum_{t=1}^{n} \varepsilon_t \varepsilon_{t-j} \right], \tag{A.7}$$

where $\alpha(k, j) = f_{k+j}(1) + f_{k-j}(1) - \rho(k)(f_j(1) + f_{-j}(1))$ and $f_j(1)$ are the numbers defined in their paper (see page 980). From Lemma 2.1 and (24)–(28) of their paper (page 972) and Lemmas A.1 and A.2, one can check that under S the argument in (A.7) can be rewritten as

$$n^{1/2}(\rho_n^*(k) - \rho(k)) = \gamma^{-1}(0) \sum_{j=1}^{\infty} \left[\alpha(k, j) n^{-1/2} \sum_{t=1}^{n} \varepsilon_t \varepsilon_{t-j} \right] + \Delta_n(k), \tag{A.8}$$

where $\Delta_n(k) \to 0$ as $n \to \infty$ a.s. Moreover, we have that $\sum_{j=0}^{\infty} |\alpha(k, j)| < \infty$ for each k.

To establish (A.7), consider the random vector

$$\begin{aligned} \mathbf{Z}_n(M) = \gamma^{-1}(0) \Bigg(& \sum_{j=1}^{M} \left[\alpha(1, j) n^{-1/2} \sum_{t=1}^{n} \varepsilon_t \varepsilon_{t-j} \right] \\ & + \Delta_n(1), \ldots, \sum_{j=1}^{M} \left[\alpha(r, j) n^{-1/2} \sum_{t=1}^{n} \varepsilon_t \varepsilon_{t-j} \right] + \Delta_n(r) \Bigg)'. \end{aligned}$$

Note that, from Lemmas A.1 and A.3, $\mathbf{Z}_{N_n}(M) \to_D Z_M \sim \mathscr{N}(\mathbf{0}, \Gamma_M)$, where Γ_M is the $M \times M$ matrix with the (i, j)th entry equal to $\sum_{l=1}^{M} \alpha(i, l)\alpha(l, j)\sigma^4 \gamma^{-2}(0)$, and that $\Gamma_M \to \Gamma$ as $M \to \infty$. Hence, in view of (A.7) and (A.8) and Proposition 6.3.9 of Brockwell and Davis (1990), page 207, it suffices to show that, for $k = 1, \dots, r$,

$$\lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left| \sum_{j\geq M} \left[\alpha(k, j) N_n^{-1/2} \sum_{t=1}^{n} \varepsilon_t \varepsilon_{t-j}\right]\right| > \delta\right) = 0 \quad \text{for all } \delta > 0.$$

Since the above can be proved by Lemma A.5, the proposition is established. □

PROOF OF THEOREM A.1. Note that the sequence $\{a_j;\ 0 \leq j \leq m\}$ satisfies the condition in (A.1). Applying Theorem A.1 to $\{X_t(m);\ t \geq 1\}$, we have, for each m, $N_n^{1/2}(\hat{\boldsymbol{\rho}}_{N_n, m}(r) - \boldsymbol{\rho}_m(r)) \to_D \mathscr{N}(\mathbf{0}, \Gamma_m)$ as $n \to \infty$, where Γ_m is an $r \times r$ matrix whose (k, l)th entry is

$$w_m(k, l) = \sum_{i=1}^{\infty} \{\rho_m(k+i) + \rho_m(k-i) - 2\rho_m(k)\rho_m(i)\}$$
$$\times \{\rho_m(l+i) + \rho_m(l-i) - 2\rho_m(l)\rho_m(i)\}.$$

Thus, by (A.5),

$$N_n^{1/2}(\boldsymbol{\rho}^*_{N_n, m}(r) - \boldsymbol{\rho}_m(r)) \to_D \mathscr{N}(\mathbf{0}, \Gamma_m) \quad \text{as } n \to \infty. \tag{A.9}$$

Since, as $m \to \infty$, $w_m(k, l) \to w(k, l)$, which is defined in (2.20), Γ_m converges to Γ as $m \to \infty$. Therefore, in view of (A.5), (A.9) and Proposition 6.3.9 of Brockwell and Davis (1990), it suffices to show that, for $k = 1, \dots, r$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(N_n^{1/2}|\rho^*_{N_n}(k) - \rho(k) - \rho^*_{N_n,m}(k) + \rho_m(k)| > \delta) = 0$$
$$\text{for all } \delta > 0.$$

In view of (4.8), which is also true for $\rho^*_{n,m}(k)$ and $\gamma^*_{n,m}(k)$, first establish the random central limit theorem for $\gamma^*_{N_n, m}(0), \dots, \gamma^*_{N_n, m}(r)$ by applying the arguments in Remarks 3.9 of Phillips and Solo (1993) to $\{X_t(m)\}$ and using Lemmas A.1. A.2 and A.4 [see Proposition 7.3.3 of Brockwell and Davis (1990) for ordinary central limit theorems]. Then it suffices to check that, for $k = 0, \dots, r$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(N_n^{1/2}|\gamma^*_{N_n}(k) - \gamma(k) - \gamma^*_{N_n,m}(k) + \gamma_m(k)| > \delta) = 0 \quad \text{for all } \delta > 0.$$

The above, however, can be verified by using the argument in (4.4) and Lemma A.5. Without detailing the related algebra, we establish the theorem. □

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